Web-based Supplementary Materials for

Selection Consistency of EBIC for GLIM with Non-canonical Links and Diverging Number of Parameters

by Shan Luo and Zehua Chen

Verification of condition C6 for GLIM with non-canonical link functions

In this supplementary document, we verify condition C6 for some common GLIMs with non-canonical link functions while assuming that σ_i^2 (the variance of response y_i) are bounded away from 0 and from above. For the ease of reference, condition C6 is given below:

C6 The quantities $|x_{ij}|, |h'(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)|, |h''(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)|, i = 1, \ldots, n; j = 1, \ldots, p_n$ are bounded from above, and $\sigma_i^2, i = 1, \ldots, n$ are bounded both from above and below away from zero. Furthermore,

$$\max_{1 \le j \le p_n; 1 \le i \le n} \frac{x_{ij}^2 [h'(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)]^2}{\sum_{i=1}^n \sigma_i^2 x_{ij}^2 [h'(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)]^2} = o(n^{-1/3})$$

$$\max_{1 \le i \le n} \frac{[h''(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)]^2}{\sum_{i=1}^n \sigma_i^2 [h''(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)]^2} = o(n^{-1/3}).$$

The common GLIMs were considered in Wedderburn (1976). In particular, we consider the following exponential families and their corresponding link functions:

(1) Poisson Distribution: $\eta = \ln(\mu)$, μ^γ where $0 < \gamma < 1$;

(2) Binomial Distribution: $\eta = \mu$, $\arcsin(\mu)$, $\ln(\frac{\mu}{1-\mu})$, $\ln(-\ln(1-\mu))$, $\Phi^{-1}(\mu)$.

(3) Gamma Distribution ($G(1, \mu)$): $\eta = \ln \mu$, μ^γ where $-1 \le \gamma < 0$.

The correponding function $\theta = h(\eta)$ for the models above are as follows:

(1) Poisson Distribution: $\theta = \eta$, $\frac{1}{\gamma}\ln\eta$ where $0 < \gamma < 1$;

(2) Binomial Distribution: $\theta = \ln\frac{\eta}{1-\eta}$, $\ln\frac{\sin(\eta)}{1-\sin(\eta)}$, η, $\ln\left(\exp(e^{\eta}) - 1\right)$, $\ln\left(\frac{\Phi(\eta)}{1-\Phi(\eta)}\right)$.

(3) Gamma Distribution: $\theta = -e^{-\eta}$, $-\eta^{-\frac{1}{\gamma}}$.

1 Poisson Distribution

The link $\eta = \mu^{\gamma}$ *where* $0 < \gamma < 1$ *and* $\mu \in [a, b]$. In this situation,

$$h^{'}(\eta) = \frac{1}{\gamma\eta},\ h^{''}(\eta) = -\frac{1}{\gamma\eta^2},\ \sigma^2 = \eta^{\frac{1}{\gamma}}.$$

Hence under the assumption, $\forall 1 \le i \le n$,

$$|h^{'}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)| \in [\frac{1}{\gamma b^{\gamma}}, \frac{1}{\gamma a^{\gamma}}], \sigma_i^2 \in [a, b], |h^{''}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)| \in [\frac{1}{\gamma b^{2\gamma}}, \frac{1}{\gamma a^{2\gamma}}]$$

$$\frac{x_{i,j}^2\left(h^{'}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^{n}\sigma_i^2 x_{i,j}^2\left(h^{'}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)\right)^2} = \frac{b^{2\gamma}}{a^{2\gamma+1}}O\left(\frac{x_{i,j}^2}{\sum_{i=1}^{n}x_{i,j}^2}\right)$$

$$\frac{\left(h^{''}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^{n}\sigma_i^2\left(h^{''}(\boldsymbol{x}_i^{\tau}\boldsymbol{\beta}_0)\right)^2} = \frac{b^{4\gamma}}{a^{4\gamma+1}}O(n^{-1}).$$

when $0 < a < b < +\infty$, C6 is true when $\max_{1\le j\le p_n}\max_{1\le i\le n}\left\{\frac{x_{i,j}^2}{\sum_{i=1}^{n}x_{i,j}^2}\right\} = o(n^{-1/3})$.

2 Binomial Distribution

For binomial distribution, $\sigma_i^2 = \mu_i(1-\mu_i) = \frac{e^{\theta_i}}{(1+e^{\theta_i})^2}$. Here we assume

$$\min_{1\le i\le n}\left(\mu_i \wedge (1-\mu_i)\right) \ge c \text{ where } 0 < c \le 1/2. \tag{1}$$

This implies, $c^2 \le \min_{1\le i\le n} \sigma_i^2 \le \max_{1\le i\le n} \sigma_i^2 \le 1/4$. Therefore,

$$\frac{x_{i,j}^2 \left(h^{'}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^n \sigma_i^2 x_{i,j}^2 \left(h^{'}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2} = O\left(\frac{x_{i,j}^2 \left(h^{'}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^n x_{i,j}^2 \left(h^{'}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}\right)$$

$$\frac{\left(h^{''}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^n \sigma_i^2 \left(h^{''}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2} = O\left(\frac{\left(h^{''}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}{\sum_{i=1}^n \left(h^{''}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)\right)^2}\right).$$

(1) $\mu = \eta, 0 < \eta < 1$:

$$h^{'}(\eta) = \frac{1}{\eta(1-\eta)},\ h^{''}(\eta) = \frac{2\eta - 1}{\eta^2(1-\eta)^2},\ \sigma^2 = \eta(1-\eta).$$

Under assumption (1),

$$4 \le h^{'}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right) \le \frac{1}{c^2};\ \left|h^{''}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right)\right| \le \frac{1-2c}{c^4}$$

for all $1 \le i \le n$. C6 holds when

$$\max_{1\le j\le p_n} \max_{1\le i\le n} \left\{\frac{x_{i,j}^2}{\sum_{i=1}^n x_{i,j}^2}\right\} = o(n^{-1/3}).$$

(2) $\eta = \arcsin \mu$:

$$h^{'}(\eta) = \frac{\cos\eta}{\sin\eta(1-\sin\eta)},\ h^{''}(\eta) = \frac{\sin\eta}{1-\sin\eta} - \frac{\cos^2\eta}{\sin^2\eta},\ \sigma^2 = \sin\eta(1-\sin\eta).$$

Under assumption (1),

$$4\sqrt{2c-c^2} \le \left|h^{'}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right)\right| \le \frac{\sqrt{1-c^2}}{c^2};$$

$$\frac{3c - c^2 - 1}{(1-c)^2 c} \le \left|h^{''}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right)\right| \le \frac{1-c^2-c}{c^2(1-c)}$$

for all $1 \le i \le n$. C6 holds when

$$\max_{1\le j\le p_n} \max_{1\le i\le n} \left\{\frac{x_{i,j}^2}{\sum_{i=1}^n x_{i,j}^2}\right\} = o(n^{-1/3}).$$

(3) $\eta = g(\mu) = \ln\{-\ln(1-\mu)\}$ or $\eta = g(\mu) = \ln\{-\ln(\mu)\}$.

For the first link function, complementary log-log link, we have

$$\theta = \ln(\frac{\mu}{1-\mu}) = h(\eta) = \ln\{\exp(e^{\eta}) - 1\}, \ \sigma^2 = \frac{\exp(e^{\eta}) - 1}{\exp(2e^{\eta})}. \tag{2}$$

Therefore, the first and second order derivatives of $h(\cdot)$ are

$$h^{'}(\eta) = \frac{e^{\eta+e^{\eta}}}{e^{e^{\eta}} - 1}; \ h^{''}(\eta) = \frac{e^{\eta+e^{\eta}}[e^{e^{\eta}} - e^{\eta} - 1]}{\{e^{e^{\eta}} - 1\}^2}. \tag{3}$$

It is easy to see that $e^{\eta} \leq h^{'}(\eta) \leq e^{e^{\eta}}$. Now let us look at $h^{''}(\eta)$. It is straightforward that $|h^{''}(\eta)| \leq |h^{'}(\eta)| \leq e^{e^{\eta}}$. Consider the function $f(x) = \dfrac{e^x(e^x - x - 1)}{(e^x-1)^2}$ on $(0, +\infty)$. Because

$$\lim_{x\to 0} f(x) = \lim_{x\to 1}\frac{x^2/2}{x^2} = \frac{1}{2}; \ \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{1 - \frac{x}{e^x} - \frac{1}{e^x}}{(1-\frac{1}{e^x})^2} = 1. \tag{4}$$

Therefore, there exists a positive constant C_1, C_2 independent of x such that $C_1 \leq f(x) \leq C_2$. That is, $C_1 e^{\eta} \leq h^{''}(\eta) \leq C_2 e^{\eta}$. When $\sigma_i^2 \in [a, b]$ for some $0 < a \leq b \leq 1/4$, for $1 \leq i \leq n$, we have

$$\frac{1+\sqrt{1-4b}}{2b} \leq \exp(e^{\eta_i}) \leq \frac{1+\sqrt{1-4a}}{2a} \text{ or } \frac{1-\sqrt{1-4a}}{2a} \leq \exp(e^{\eta_i}) \leq \frac{1-\sqrt{1-4b}}{2b}.$$

That is, $|h^{'}(\eta_i)|$ and $|h^{''}(\eta_i)|$ are both bounded away from 0 and finite. C6 holds when

$$\max_{1\leq j\leq p_n}\max_{1\leq i\leq n}\left\{\frac{x_{i,j}^2}{\sum_{i=1}^{n} x_{i,j}^2}\right\} = o(n^{-1/3}).$$

The same argument applies to the second link function by changing η to $-\eta$.

(4) $\eta = \Phi^{-1}(\mu)$:

$$h^{'}(\eta) = \frac{f(\eta)}{\Phi(\eta)(1-\Phi(\eta))}, \ h^{''}(\eta) = \frac{f^{'}(\eta)}{\Phi(\eta)(1-\Phi(\eta))} + f^2(\eta)[\frac{1}{(1-\Phi(\eta))^2} - \frac{1}{\Phi^2(\eta)}]$$

$$\sigma^2 = \Phi(\eta)(1-\Phi(\eta))$$

Under assumption (1), $\Phi^{-1}(c) \leq |\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0| \leq \Phi^{-1}(1-c)$. Note that for

$$1 - \Phi(t) \leq \frac{f(t)}{t}, \forall t > 0$$

therefore, we have

$$4c\Phi^{-1}(c) \leq 4f\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right) \leq \left|h^{'}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right)\right| \leq \frac{1}{c^2} f\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right) \leq \frac{1}{\sqrt{2\pi}c^2};$$
$$4f^{'}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right) \leq \left|\frac{f^{'}(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)}{\Phi(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)(1-\Phi(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0))}\right| \leq \frac{1}{c^2} f^{'}\left(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0\right) \leq \frac{\Phi^{-1}(1-c)}{\sqrt{2\pi}c^2};$$
$$\left|f^2(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)[\frac{1}{(1-\Phi(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0))^2} - \frac{1}{\Phi^2(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0)}]\right| \leq \frac{|2c-1|}{c^2(1-c)^2} f^2(\boldsymbol{x}_i^\tau \boldsymbol{\beta}_0) \leq \frac{|2c-1|}{2\pi c^2(1-c)^2}.$$

for all $1 \leq i \leq n$. C6 holds when

$$\max_{1 \leq j \leq p_n} \max_{1 \leq i \leq n} \left\{ \frac{x_{i,j}^2}{\sum_{i=1}^{n} x_{i,j}^2} \right\} = o(n^{-1/3}).$$

3 Gamma Distribution

(1) $\eta = \ln(\mu) : h^{'}(\eta) = e^{-\eta},\ h^{''}(\eta) = -e^{-\eta},\ \sigma^2 = e^{2\eta}$. When σ_i^2 is away from 0 and finite, $|h^{'}|, |h^{''}|$ are bounded. C6 holds when

$$\max_{1 \leq j \leq p_n} \max_{1 \leq i \leq n} \left\{ \frac{x_{i,j}^2}{\sum_{i=1}^{n} x_{i,j}^2} \right\} = o(n^{-1/3}).$$

(2) $\eta = \mu^\gamma$ where $-1 \leq \gamma < 0$. Let $\tilde{\gamma} = -\frac{1}{\gamma}$, then $0 < \tilde{\gamma} \leq 1$. Then

$$h^{'}(\eta) = -\tilde{\gamma}\eta^{\tilde{\gamma}-1},\ h^{''}(\eta) = \tilde{\gamma}(1-\tilde{\gamma})\eta^{\tilde{\gamma}-2},\ \sigma^2 = \eta^{2\tilde{\gamma}}.$$

When σ_i^2 is away from 0 and finite, $|h^{'}|, |h^{''}|$ are bounded. C6 holds when

$$\max_{1 \leq j \leq p_n} \max_{1 \leq i \leq n} \left\{ \frac{x_{i,j}^2}{\sum_{i=1}^{n} x_{i,j}^2} \right\} = o(n^{-1/3}).$$