An extension of max autoregressive models[*]

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To model clustered maxima behaviors in time series analysis, max-autoregressive (MAR) and moving maxima (MM) processes are naturally adapted from linear autoregressive (AR) and moving average (MA) models. Yet, applications of MAR and MM processes are still sparse due to some difficulties of parameter inference and some abnormality of the processes. Basically, some ratios of observations can take constant values in MAR models. The objective of this present work is to introduce a new model that is closely related to the MAR processes and is free of the aforementioned abnormality. A logarithm transformation of the new model leads to time series models with log-positive alpha stable noises and hidden max Gumbel shocks. Theoretical properties of the new models are derived.

1. MOTIVATION

Recordings of daily, weekly or yearly maxima in environmental time series are classically fitted by the Generalized Extreme Value (GEV) distribution that originates from the well established Extreme Value Theory (EVT). To capture temporal dependencies and clustered peak values, linear autoregressive (AR) processes and moving average (MA) processes offer a simple and elegant framework. [3] introduced Max-ARMA processes; [11] generalized Deheuvels' [4] moving minima processes to multivariate maxima of moving maxima (M4) processes. However, applications of the Max-ARMA processes and the M4 processes have been rare due to some inferential difficulties and some abnormality of the processes. Ratios of consecutive observations drawn from either a Max-ARMA process or an M4 process can take constant values. These constant ratios form signature patterns in simulated processes, see [3, 16] for example. These signature patterns have not been observed in real data analysis

even if processes are transformed into Fréchet scales as required in either a Max-ARMA model or an M4 model. As a result, models with more flexible dependence patterns are needed. This paper aims to propose such a kind of models.

One of the simple Max-ARMA processes is the following one

$$(1) \qquad Y_t = \max(aY_{t-1}, \epsilon_t),$$

where $a < 1$ is a positive constant and ϵ_t corresponds to an iid sequence of unit Fréchet random variables, i.e. $P(\epsilon_t < u) = \exp(-1/u)$, $u > 0$. We note that the marginal choice of ϵ_t being unit Fréchet is not essential. Suppose that ϵ_t has the heavy-tailed extreme value distribution $\exp\{-1/u^\tau\}$ for some $\tau > 0$, then one can transform back to the unit Fréchet case by taking $1/\tau$ powers in the Max-ARMA recursion (1); see [3] for detailed arguments.

This simple model can be viewed as an entry point of our discussions and it can also be rewritten as $Y_t = \max_{j=0,1,\ldots}(a^j \epsilon_{t-j})$. Its main drawback is that the random ratio Y_t/Y_{t-1} equals the constant a whenever we have $aY_{t-1} > \epsilon_t$. This implies that the value of a can be estimated exactly. In practice it is highly improbable to find an application with such a behavior. To remove this undesirable effect, we propose the following model

$$(2) \qquad Z_{t,\alpha} = c \max(a(S_{t,\alpha} Z_{t-1,\alpha})^\alpha, \epsilon_t),$$

where $c \in (0, 1]$ is a scale parameter, $a \in (0, 1]$ is an autoregressive parameter, and the sequence $\{S_{t,\alpha}\}$ is independent of the iid sequence $\{\epsilon_t\}$. $\{S_{t,\alpha}\}$ represents an iid sequence of positive α-stable variables defined by its Laplace transform

$$(3) \qquad E(\exp(-uS)) = \exp(-u^\alpha), \text{ for all } u \geq 0,$$

where $\alpha \in (0, 1]$. Here $S_{t,\alpha}$ and $Z_{t-1,\alpha}$ in (2) are also independent.

Recall that a random variable S is said to be stable if for all non-negative real numbers c_1, c_2, there exists a positive real a and a real b such that $c_1 S_1 + c_2 S_2$ is equal in distribution to $aS + b$ where S_1, S_2 are iid copies of S. The case $\alpha = 1$ in (2) means that $S = 1$. This situation corresponds to model (1). Whenever $\alpha < 1$, the ratio $Z_{t,\alpha}/Z_{t-1,\alpha}$ depends on the random variable $S_{t,\alpha}$, and consequently it cannot be equal to a constant. As a result, model (2) does not possess the aforementioned limitation of model (1).
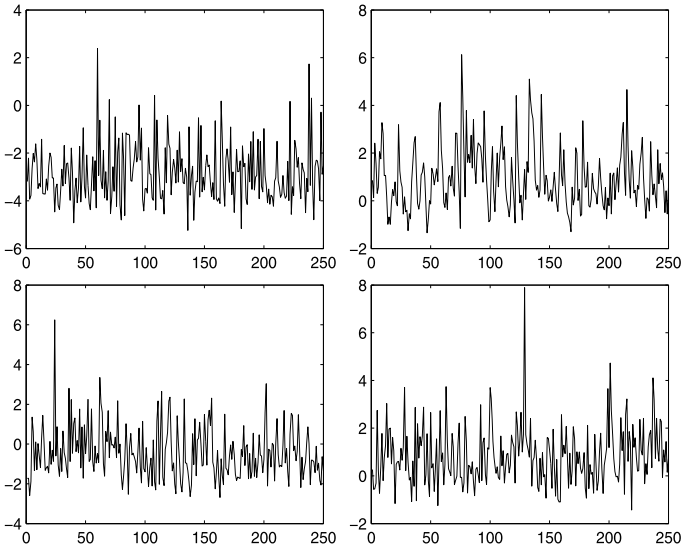
Figure 1. 250 Observations in model (4) with different sets of parameter values of (a, c, α). Respectively, the top-left uses (0.3047, 0.0260, 0.3488), the top-right uses (0.7330, 0.5100, 0.6630), the bottom-left uses (0.5758, 0.2365, 0.1601), and the bottom-right uses (0.7208, 0.8124, 0.2137).

One building block behind (2) is an additive relationship between Gumbel and log positive α-stable variables. If X is Gumbel distributed with parameters μ and σ and is independent of S, a positive α-stable variable, then the sum $X + \sigma \log S$ is also Gumbel distributed with parameters μ and σ/α. Such an additive property has been recently studied by [15] in an environmental context and [5] in a mixture context. In the literature, [2, 7, 13] also worked with such distributions in survival analysis and the modeling of multivariate extremes.

Our proposed model (2) contains three parameters. First, α is a shape parameter. When either $Z_{t-1,\alpha}$ or $S_{t,\alpha}$ is very large, $(S_{t,\alpha} Z_{t-1,\alpha})^\alpha$ has a tendency to become smaller comparing with the original values. Second, a is the parameter that describes the auto-regressive of our model. Third, the parameter c can be viewed as a scale parameter or a location parameter after a log transformation of $Z_{t,\alpha}$. A logarithm transformation of (2) yields the following time series model:

$$(4) \qquad X_{t,\alpha} = \mu + \max\{\gamma + \alpha \log(S_{t,\alpha}) + \alpha X_{t-1,\alpha}, \ \xi_t\},$$

where $\mu = \log(c)$, $\gamma = \log(a)$, $X_{t,\alpha} = \log(Z_{t,\alpha})$, $\xi_t = \log(\epsilon_t)$. In the model, μ is a location parameter, and both $X_{t,\alpha}$ and ξ_t are Gumbel distributed. We can regard (4) as a time series model with log of positive α stable noises $\log(S_{t,\alpha})$ and hidden max Gumbel shocks ξ_t. The idea is as follows. Suppose that $\xi_t = -\infty$ for all t. Then model (4) is a pure autoregressive signal process. Alternatively, suppose that $P(\xi_t > -\infty) = 1$ in (4) at time t. If the signal value of ξ_t is stronger than the signal resulted from the autoregressive signal process, then ξ_t is the new observed signal value,

i.e. the signal process is altered by a hidden (max) Gumbel shock. Figure 1 illustrates four different simulated processes using four different sets of parameter values in (4).

It is clear that model (4) generates skewed data and asymmetry in the upper and lower tails. Model (4) shares the spirit of the seminal nonlinear time series model, i.e. threshold autoregression models introduced by [14]. Model (4) uses a random threshold ξ_t and takes value of the threshold itself as long as the value is larger than the computed autoregressive sum. This new model can also be regarded as a model with an infinite number of change points.

Model (2) and model (4) being basically equivalent, we will only consider (2) in the subsequent sections and derive its theoretical properties.

Before closing this section, we can remark that at least two extensions of model (2) are possible. The following general form

$$(5) \qquad Z_{t,\alpha} = c \max \left(\left[S_{t,\alpha} \max_{1 \le i \le p} \{a_i Z_{t-i,\alpha}\} \right]^\alpha, \epsilon_t \right)$$

corresponds to a max autoregressive process with positive alpha stable moving coefficients and unit Fréchet shocks. We denote this process as $\mathrm{MAP}_\alpha(p)$. Another possible extension for model (2) follows the idea of [5] to take advantage of the $\alpha-$stability. This allows to introduce

$$Z_{t,\alpha} = c \max \left(a \left\{ \left[\sum_{j=0}^{\infty} c_j S_{t-j,\alpha} \right] Z_{t-1,\alpha} \right\}^\alpha, \epsilon_t \right).$$

The rest of the paper is structured as follows. In sections 2 and 3, we present some theoretical properties of model (2), i.e. $\mathrm{MAP}_\alpha(1)$ processes. Section 3.2 deals with statistical estimations of parameters in $\mathrm{MAP}_\alpha(1)$ models with the positive α stable noises being Lévy distributed. Simulation examples are shown in Section 4. In Section 5, we fit a $\mathrm{MAP}_\alpha(1)$ model to weekly maxima of river flow rate data for two rivers: the Eagle River and the Crystal River, located in the mountains of western Colorado in the United States. Section 6 concludes. In Appendix, we present all technical derivations of theoretical results and proofs of propositions.

2. THEORETICAL PROPERTIES OF MAP$_\alpha$(1) PROCESSES

2.1 Stationarity

The following proposition shows that model (2) quickly produces a stationary process as t increases.

Proposition 1. *In the context of (2), suppose that the distribution of $Z_{0,\alpha}$ is Fréchet with scale parameter $b_0 > 0$, i.e. $P(Z_{0,\alpha} \le z) = \exp(-b_0/z)$. Then $Z_{t,\alpha}$ is Fréchet with scale parameter satisfying*

$$(6) \qquad b_t = ac b_{t-1}^\alpha + c$$

for $t = 1, 2, \ldots$, *and* $\{b_t\}$ *has a limit* d *satisfying*

$$(7) \qquad acd^\alpha + c = d$$

as $t \to \infty$.

Furthermore, if $b_0 = d$, we have for any constant values $z_0, z_1, \ldots$,

$$P(Z_{0,\alpha} \leq z_0, \ldots, Z_{t,\alpha} \leq z_t)$$
$$= P(Z_{k,\alpha} \leq z_0, \ldots, Z_{k+t,\alpha} \leq z_t),$$
$$\text{for any } t \geq 0 \text{ and } k \geq 0$$

which implies that the process is stationary.

We note that if $b_0 = d$ satisfies (7), the derivation in the proof of Proposition 1 indicates that model (2) corresponds to a stationary process in which the marginal distribution is Fréchet with the scale parameter being d.

On the other hand, if b_0 is not equal to d, then b_t converges at least linearly, and it has a rate of approximately $(d - c)\alpha/d$. Numerically, except the case of $a = c = \alpha = 1$, in a few iterations ($t < 20$), b_t has already closed to d up to many decimal places. For this reason, we will assume the process generated from (2) is stationary with d as its parameter, i.e. we assume $b_0 = d$.

We note that in model (2), it is not necessary to require $a < 1$. In the case of $a = 1$, we still can have stationary processes, which will be seen in our real data analysis.

2.2 Causality

In Max-ARMA model, [3] called a stationary process causal if it can be expressed as a max-linear process. Following their lead, we call a stationary process *causal* if there exist constants $\beta_j \geq 0$, $j \geq 0$ and

$$(8) \qquad Z_{t,\alpha} = \beta_0 \epsilon_t \vee \left[\bigvee_{j \geq 1} \beta_j \left(\prod_{i=0}^{j-1} S_{t-i}^{\alpha^{i+1}} \right) \epsilon_{t-j}^{\alpha^j} \right] < \infty \text{ a.s.}$$

To simplify notations, the subscript α was dropped in $S_{t,\alpha}$ that simply becomes S_t. We now study the causal properties of model (2). Substituting (8) in our proposed $\text{MAP}_\alpha(1)$ process (2), we have

$$\beta_0 \epsilon_t \vee \beta_1 S_t^\alpha \epsilon_{t-1}^\alpha \vee \left[\bigvee_{j \geq 2} \beta_j \left(\prod_{i=0}^{j-1} S_{t-i}^{\alpha^{i+1}} \right) \epsilon_{t-j}^{\alpha^j} \right]$$

$$= c\epsilon_t \vee ac\beta_0^\alpha S_t^\alpha \epsilon_{t-1}^\alpha \vee \left[\bigvee_{j \geq 1} ac\beta_j^\alpha S_t^\alpha \left(\prod_{i=0}^{j-1} S_{t-1-i}^{\alpha^{i+2}} \right) \epsilon_{t-1-j}^{\alpha^{j+1}} \right]$$

$$= c\epsilon_t \vee ac\beta_0^\alpha S_t^\alpha \epsilon_{t-1}^\alpha \vee \left[\bigvee_{j \geq 2} ac\beta_{j-1}^\alpha S_t^\alpha \left(\prod_{i=0}^{j-2} S_{t-1-i}^{\alpha^{i+2}} \right) \epsilon_{t-j}^{\alpha^j} \right]$$

$$= c\epsilon_t \vee ac\beta_0^\alpha S_t^\alpha \epsilon_{t-1}^\alpha \vee \left[\bigvee_{j \geq 2} ac\beta_{j-1}^\alpha \left(\prod_{i=0}^{j-1} S_{t-i}^{\alpha^{i+1}} \right) \epsilon_{t-j}^{\alpha^j} \right].$$

Comparing all coefficients in the first formula and the last formula from the four formulas above, we have

$$(9) \qquad \beta_0 = c, \ \beta_j = ac\beta_{j-1}^\alpha, \ j \geq 1.$$

It is possible to show (see the Appendix) that the distribution of $Z_{t,\alpha}$ defined by (8) and (9) coincides with the limiting marginal distribution of the $\text{MAP}_\alpha(1)$ process (2) in Proposition 1. We summarize these arguments in Proposition 2.

Proposition 2. *A* $\text{MAP}_\alpha(1)$ *process is causal, i.e.* (8) *holds.*

2.3 Temporal dependence among maxima

To better understand the temporal structure of (2), we compute the following bivariate probability in Result #1.

Result #1.

$$P(Z_{t,\alpha} \leq z_t, \ S_{t,\alpha} Z_{t-1,\alpha} \leq z_{t-1})$$
$$= \exp\left(-\max\left\{ \frac{d}{z_t}, \ \frac{c}{z_t} + \frac{d^\alpha}{z_{t-1}^\alpha} \right\} \right).$$

When $c = \alpha = 1$ and $d = 1/(1-a)$, the formula above can be compared to model (1) from which we have

$$P(Y_t \leq z_t, Y_{t-1} \leq z_{t-1})$$
$$= \exp\left(-\max\left\{ \frac{1}{(1-a)z_t}, \ \frac{1}{z_t} + \frac{1}{(1-a)z_{t-1}} \right\} \right).$$

It can be deduced as

$$\lim_{\alpha \uparrow 1, c \uparrow 1} P(Z_{t,\alpha} \leq z_t, \ S_{t,\alpha} Z_{t-1,\alpha} \leq z_{t-1})$$
$$= P(Y_t \leq z_t, Y_{t-1} \leq z_{t-1}).$$

This equation tells that the joint probability of (Y_t, Y_{t-1}) can arbitrarily closely be approximated by the joint probability of the limit process of $(Z_{t,\alpha}, Z_{t-1,\alpha})$.

The bivariate vector (Y_t, Y_{t-1}) possesses the max-stability property

$$P^u(Y_t \leq uz_t, Y_{t-1} \leq uz_{t-1})$$
$$= P(Y_t \leq z_t, Y_{t-1} \leq z_{t-1}), \text{ for any } u > 0,$$

that represents the foundation of the Extreme Value Theory. Does this fundamental property still hold for the bivariate vector $(Z_{t,\alpha}, Z_{t-1,\alpha})$? We can write that

$$P^u(Z_{t,\alpha} \leq uz_t, S_{t,\alpha} Z_{t-1,\alpha} \leq uz_{t-1})$$
$$= \exp\left(-\max\left\{ \frac{d}{z_t}, \ \frac{c}{z_t} + \frac{d^\alpha u^{1-\alpha}}{z_{t-1}^\alpha} \right\} \right)$$

which implies that

$$\log \frac{P^u(Z_{t,\alpha} \leq uz_t, S_{t,\alpha} Z_{t-1,\alpha} \leq uz_{t-1})}{P(Z_{t,\alpha} \leq z_t, S_{t,\alpha} Z_{t-1,\alpha} \leq z_{t-1})}$$
$$= \max\left[\frac{acd^\alpha}{z_t}, \frac{d^\alpha}{z_{t-1}^\alpha} \right] - \max\left[\frac{acd^\alpha}{z_t}, \frac{d^\alpha u^{1-\alpha}}{z_{t-1}^\alpha} \right].$$
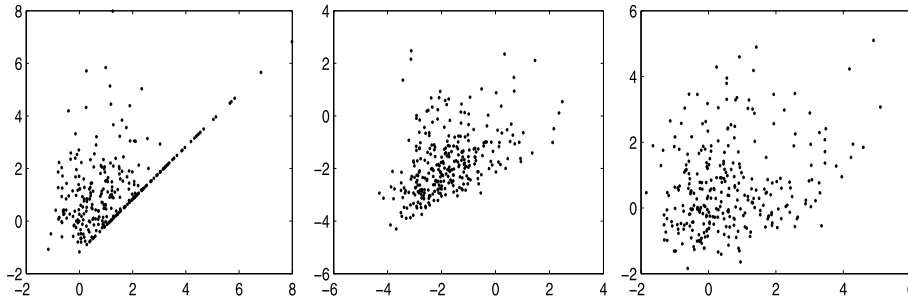
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Figure 2. *Illustrations of asymptotic dependence of* $\log(Y_{t-1})$ *(X-axis) and* $\log(Y_t)$ *(Y-axis) in model (1) in the left panel, and asymptotic independence of* X_{t-1} *(X-axis) and* X_t *(Y-axis) in model (4) with parameter values* $a = 0.3112$, $c = 0.7325$, $\alpha = 0.5020$. *The middle panel is without max-shock noise, i.e.* $\xi_t = -\infty$, *while the right panel is with max-shock noise.*

As $\alpha \uparrow 1, c \uparrow 1$, one can see that the bivariate vector $(Z_{t,\alpha}, Z_{t-1,\alpha})$ becomes more and more max-stable.

To study the joint tail behavior, we examine the so called asymptotic dependence index introduced by [10]. We give the following definition.

Definition 1. Two identically distributed random variables X and Y with distribution function F are called "asymptotically independent", if

(10) $$\lambda = \lim_{u \to x_F} P(Y > u \mid X > u)$$

is 0, where $x_F = \sup\{x \in \mathbb{R} : F(x) < 1\}$. The quantity λ, if it exists, is called the bivariate asymptotic dependence index which quantifies the amount of dependence of the bivariate upper tails. If $\lambda > 0$, then X and Y are called "asymptotically dependent."

The model (1) gives $\lambda = a$, i.e. Y_t and Y_{t-1} are asymptotically dependent. In viewing (4), one can see that X_t and X_{t-1} are more likely to be asymptotically independent mainly because the additive log-positive stable noise in X_t is heavy tailed and is independent of X_{t-1}, and the max-shock noise is also independent of X_{t-1}. To prove the asymptotic (in)dependence of these two random variables, we may need to know the density of S_t. In Section 3, we will prove the asymptotic independence when S_t is a positive α-stable Lévy random variable. In Figure 2, we illustrate three simulated examples which are showing asymptotic (in)dependencies. The left panel depicts $\log(Y_{t-1})$ and $\log(Y_t)$ in model (1) with parameter value $a = 0.3112$. The middle panel is for model (4) with $a = 0.3112$, $c = 0.7325$, $\alpha = 0.5020$, and without max-shock noise, i.e. $\xi_t = -\infty$. The right panel is for model (4) with $a = 0.3112$, $c = 0.7325$, $\alpha = 0.5020$, and with max-shock noises.

From the left panel in Figure 2, one can clearly see the existence of a signature pattern, while the middle panel and the right panel do not show such kind of patterns. One can also see that model (2) can model a wide range of dependence in asymptotic independence.

3. A SPECIAL CASE, THE MAP$_{1/2}$(1) PROCESS

In the literature, Lévy distribution has been widely applied to many real data applications. A Lévy distributed random variable S has the following density function:

$$f_L(s, \delta) = \sqrt{\frac{\delta}{2\pi}} e^{-\frac{\delta}{2s}} \frac{1}{s^{3/2}}, \text{ for all } s > 0$$

where δ is a positive parameter, and we denote $S \sim$ Lévy$(0, \delta)$. The following proposition shows that Lévy$(0, \delta)$ is positive α-stable if and only if $\delta = \frac{1}{2}$.

Proposition 3. *Suppose* $S \sim$ *Lévy*$(0, \delta)$. *Then* S *is positive* α-*stable if and only if* $\delta = \alpha = \frac{1}{2}$.

For notational convenience, we drop the subindex $\alpha = 1/2$ in $Z_{t,1/2}$, i.e. use Z_t, and the same is true for S_t. We also suppose that Z_t is Fréchet distributed with scale parameter d, where d satisfies the equation $acd^{\frac{1}{2}} + c = d$.

3.1 The tail dependence coefficient

To study the dependence among extremes, [8, 9] introduced a very useful concept, the coefficient of tail dependence that provides a finer picture than the parameter λ. We now give a brief description of this coefficient denoted by η characterized in Definition 1. For a broad range of joint distributions with unit Fréchet marginal variables, [8, 9] consider the following model

(11) $$P(X > u, Y > u) \sim L\left(\frac{1}{P(X > u)}\right) P(X > u)^{1/\eta}$$

as $u \to \infty$, where L is a slowly varying function, i.e. $L(tu)/L(u) \to 1$ as $u \to \infty$ for any fixed $t > 0$, and $\eta \in (0, 1]$ is a constant. Using their terminology, the η value effectively determines the decay rate of the joint bivariate survival function evaluated at the same large u, and η is termed as the coefficient of tail dependence. Two marginal variables are called positively associated when $1/2 < \eta \leq 1$; nearly independent when $\eta = 1/2$; and negatively associated when

$0 < \eta < 1/2$. Equation (11) can be expressed as

$$(12) \quad P(Y > u | X > u) \sim \mathrm{L}\left(\frac{1}{P(X > u)}\right) P(X > u)^{1/\eta - 1}$$

as $u \to \infty$, which shows how λ changes with η. It is easy to see that the two variables X and Y are asymptotically dependent when $\eta = 1$ and $\mathrm{L}(u) \not\to 0$ as $u \to \infty$, and are asymptotically independent otherwise.

In our case, we would like to find η of lag-1 tail dependence for stationary $\mathrm{MAP}_{1/2}(1)$ process. Let u be a threshold, we have $P(Z_t \leq u) = P(Z_{t-1} \leq u) = \exp(-d/u)$ and the following joint probabilities in Result #2.

Result #2.

$$(13)$$

$$
\begin{aligned}
&P(Z_t \leq u, \ Z_{t-1} \leq u) \\
&= \exp\left(-\frac{c}{u}\right)\left\{2\exp\left(-\frac{d}{u}\right) - 2\exp\left(-\frac{d}{u}\right)\Phi\left(\frac{ac}{\sqrt{2u}}\right) \right. \\
&\quad - \exp\left(\frac{ac\sqrt{d}}{u}\right)\left[1 - \Phi\left(\frac{ac + 2\sqrt{d}}{\sqrt{2u}}\right)\right] \\
&\quad \left. + \exp\left(-\frac{ac\sqrt{d}}{u}\right)\Phi\left(\frac{ac - 2\sqrt{d}}{\sqrt{2u}}\right)\right\}.
\end{aligned}
$$

Then, we are able to calculate the following joint survival function of (Z_{t-1}, Z_t) in the form of equation (11) in Result #3.

Result #3.

$$(14) \qquad P(Z_t > u, Z_{t-1} > u) \sim \frac{2acd}{\sqrt{\pi}} u^{-\frac{3}{2}}$$
$$\sim \mathrm{cst}\, P(Z_t > u)^{1/\eta}.$$

Since Z_t follows Fréchet distribution with scale parameter d, it is immediately from (14) that the coefficient of lag-1 tail dependence for stationary $\mathrm{MAP}_{1/2}(1)$ process is $\eta = 2/3$ and the corresponding slowly varying function $L(u) = \frac{2ac}{\sqrt{d\pi}}$. With $\eta = 2/3$, we conclude that there exists positive dependence in $\mathrm{MAP}_{1/2}(1)$ processes and an asymptotic independence in the sense that $\lim P(Z_t > u | Z_{t-1} > u) = 0$ as u increases.

3.2 Statistical estimation

Although $\mathrm{MAP}_\alpha(1)$ processes have attractive probabilistic properties, there are difficulties in applying standard statistical estimation methods such as the maximum likelihood. This difficulty has been noted previously for M4 processes, the max-autoregressive moving average or MARMA processes of [3]. [6] got around this difficulty by defining a class of estimators based on empirical processes. [12] applied the generalized method of moment estimation to a class of sparse moving maxima models. The method proposed here for $\mathrm{MAP}_\alpha(1)$ models is similarly motivated.

In this section, we assume that the $\mathrm{MAP}_{1/2}(1)$ process is stationary. We first have the following results.

Result #4.

$$E\left(\exp\left\{-\frac{1}{Z_{t,\frac{1}{2}}}\right\}\right) = \frac{d}{d+1},$$

and

$$E\left(\exp\left\{-\frac{1}{Z_{t,\frac{1}{2}}} - \frac{1}{Z_{t-1,\frac{1}{2}}}\right\}\right) = \frac{cd + acd\sqrt{d+1}}{(d+1)(c+1+ac\sqrt{d+1})}.$$

Using the stationary condition $ac\sqrt{d} + c = d$, we can construct generalized method of moments estimators for all three parameters. Our proposed estimator for $\theta = (d, a, c)$ is

$$
\begin{aligned}
\hat{\theta} = \operatorname*{arg\,min}_{\theta \in \mathbb{R}^+ \times (0,1]^2} &\left[\left(\frac{1}{n}\sum_{i=1}^n \exp\left\{-\frac{1}{Z_{t+i,\frac{1}{2}}}\right\} - \frac{d}{d+1}\right)^2 \right. \\
&+ \left(\frac{1}{n}\sum_{i=1}^n \exp\left\{-\frac{1}{Z_{t+i,\frac{1}{2}}} - \frac{1}{Z_{t-1+i,\frac{1}{2}}}\right\}\right. \\
&\left.\left. - \frac{cd + acd\sqrt{d+1}}{(d+1)(c+1+ac\sqrt{d+1})}\right)^2\right].
\end{aligned}
$$

By ergodic theorem,

$$\frac{1}{n}\sum_{i=1}^n \exp\left\{-\frac{1}{Z_{t+i,\frac{1}{2}}}\right\} \xrightarrow{\text{a.s.}} \frac{d}{d+1},$$

$$\frac{1}{n}\sum_{i=1}^n \exp\left\{-\frac{1}{Z_{t+i,\frac{1}{2}}} - \frac{1}{Z_{t-1+i,\frac{1}{2}}}\right\}$$
$$\xrightarrow{\text{a.s.}} \frac{cd + acd\sqrt{d+1}}{(d+1)(c+1+ac\sqrt{d+1})}.$$

Then by continuous mapping theorem, it is not difficult to see that our proposed estimators are strongly consistent. Due to the complexity of the joint distributions of $\{Z_{t,1/2}\}$, the asymptotics of the estimators do not have an explicit form. In this work, we propose to report Monte Carlo confidence intervals for simulation examples and real data example as well.

4. SIMULATION EXAMPLES

In this section, we simulate the $\mathrm{MAP}_{1/2}(1)$ processes with different choices of parameters a and c. It has been shown that for each set of parameter values, a generated process is a non-stationary process with its limit being stationary. Moreover, the process will reach its stationary limit quickly regardless of the initial value. Motivated from this fact, in each simulation, we discard the first 3,000 values of $Z_{t,\frac{1}{2}}$, and then simply treat the remaining simulated sequence as a stationary $\mathrm{MAP}_{1/2}(1)$ process.

Table 1. Fitted parameter values and Monte Carlo confidence intervals for simulated data

	$a = .8$			$c = .8$			$d = 1.6128$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.4964, 1.0000)	.7890	.7932	(.6838, 1.0000)	.8148	.8035	1.6149
2000	(.5513, 1.0000)	.7965	.7980	(.6980, .9550)	.8077	.8029	1.6139
5000	(.6332, 1.0000)	.8046	.8005	(.7123, .8964)	.8002	.7992	1.6123
10000	(.6783, .9441)	.8029	.8002	(.7329, .8691)	.7999	.7996	1.6127
	$a = .2$			$c = .2$			$d = .2187$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.0000, .5799)	.2124	.1932	(.1660, .2324)	.1998	.1998	.2187
2000	(.0033, .4570)	.2078	.1998	(.1762, .2235)	.1998	.1997	.2187
5000	(.0693, .3536)	.2017	.1999	(.1851, .2154)	.2000	.1999	.2187
10000	(.1043, .3089)	.2008	.1999	(.1894, .2107)	.2000	.2000	.2187
	$a = .1$			$c = .9$			$d = .9895$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.0000, .2418)	.1036	.0967	(.7840, 1.0000)	.8996	.9009	.9894
2000	(.0219, .1938)	.1014	.0991	(.8170, .9876)	.9005	.9002	.9897
5000	(.0473, .1588)	.1000	.0988	(.8465, .9558)	.9010	.9010	.9895
10000	(.0619, .1415)	.1004	.0998	(.8623, .9384)	.9000	.8996	.9896

Table 2. Fitted parameter values and Monte Carlo confidence intervals for simulated data

	$a = .9$			$c = .1$			$d = .1328$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.2628, 1.0000)	.7896	.8999	(.0878, .1247)	.1038	.1028	.1332
2000	(.4061, 1.0000)	.8020	.8999	(.0905, .1176)	.1023	.1015	.1328
5000	(.5830, 1.0000)	.8676	.9000	(.0933, .1113)	.1010	.1006	.1328
10000	(.6702, 1.0000)	.8855	.9000	(.0949, .1077)	.1005	.1001	.1328
	$a = .4$			$c = .6$			$d = .8169$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.1996, .6867)	.4043	.3934	(.4996, .7053)	.6021	.6016	.8170
2000	(.2533, .5803)	.4036	.3983	(.5295, .6747)	.6007	.6000	.8176
5000	(.3045, .5089)	.3998	.3976	(.5562, .6454)	.6007	.6007	.8169
10000	(.3284, .4771)	.4001	.3995	(.5691, .6334)	.6004	.6005	.8171
	$a = .6$			$c = .4$			$d = .5833$
size	95% M.C.I.	mean	median	95% M.C.I.	mean	median	mean
1000	(.3097, 1.0000)	.6088	.5970	(.3320, .4788)	.4012	.4006	.5837
2000	(.3906, .8648)	.6051	.5958	(.3481, .4540)	.4010	.4009	.5843
5000	(.4625, .7609)	.6015	.5982	(.3661, .4343)	.4003	.4003	.5833
10000	(.5004, .7155)	.6021	.6000	(.3762, .4233)	.3998	.3998	.5832

For each set of parameters a, c and each pre-specified sample size, we simulate the process 5,000 times. By using the estimation method discussed in the previous section, we get the estimators $\hat{a}, \hat{c}$ for each simulated sequence. Tables 1 and 2 show the simulation results. In the tables, the size column shows the sample size after trimming the first 3,000 values, the M.C.I. column gives the lower limit and the upper limit of a 95% Monte Carlo confidence interval based on 5,000 estimated values, the mean column shows the mean value of 5,000 estimated values, and similarly the median column shows the median of 5,000 estimated values.

From these two tables, one can see that the mean values and the median values are very close to their corresponding true parameter values, and the 95% Monte Carlo confidence intervals are short in length. These empirical evidences sug-gest that our proposed parameter estimators are consistent, and they well approximate true parameter values.

5. REAL DATA APPLICATION

In this section, we use river flow rate data to demon-strate the applications of our proposed new time series mod-els. Particularly, we fit a $\mathrm{MAP}_{1/2}(1)$ model to deseasonal-ized weekly maxima of river flow rates of the Eagle River which is a tributary of the Colorado River, approximately 70 miles long, in west central Colorado. Our data source is the US Geological Survey (USGS). The site number of the Eagle is 09070000. "The Eagle rises in southeastern Ea-gle County, at the continental divide, and flows northwest past Gilman, Minturn and Avon. Near Wolcott, it turns
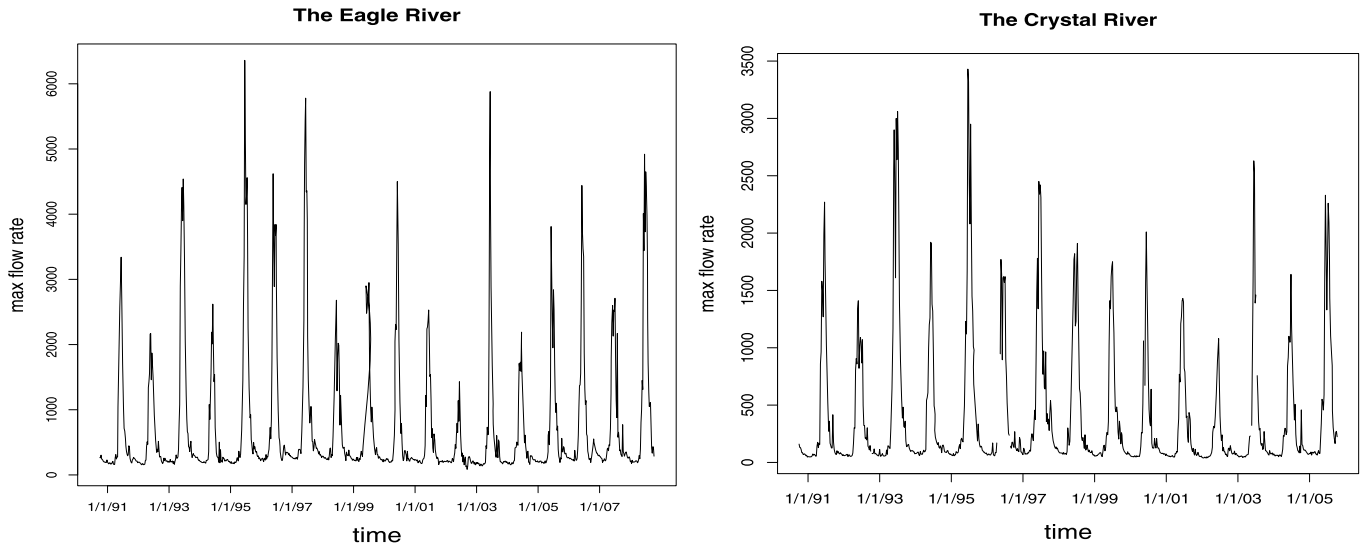
The Eagle River · **The Crystal River**

Figure 3. *Weekly maxima of river flow rates (ft³/sec) in the Eagle River (left panel) and in the Crystal River (right panel).*

Table 3. *Fitted parameter values in $MAP_{1/2}(1)$ with real data and Monte Carlo confidence intervals*

Parameter	The Eagle River		The Crystal River	
	Estimate	M.C.I	Estimate	M.C.I
a	1.0000	(.5594, 1)	1.0000	(.5809, 1)
c	.5903	(.5644, .7864)	.5987	(.5569, .7699)
d	1.2504		1.2745	

west, flowing past Eagle and Gypsum, and joins the Colorado at Dotsero, in western Eagle County. The Eagle is navigable by typical small river craft upstream to Vail in most of the time. Its flow ranges from 200 cfs in late summer of dry years to 7,000 cfs during spring runoff (Wikipedia http://en.wikipedia.org/wiki/Eagle_River_(Colorado))."

The Eagle River flow rates show very strong seasonal effects (Figure 3), most notably, snowmelt in the early summer. This phenomenon is typical within rivers in Colorado geographic area, for example the Crystal River with which [1] analyzed the maximum weekly flow rates using approximated conditional density approach for the deseasonalized data. For comparison purpose, we also include the Crystal River data (the same as used in [1] in this analysis), and we analyze the deseasonalized weekly maxima of flow rates and adopt the same procedure of deseasonalization as done in [1]. Next, we describe the deseasonalization procedure for the Eagle River flow rates.

We first obtain from the USGS Instantaneous Data Archive (IDA) a time series data set (ranging from 1990-10-01 to 2008-09-30 with one reading every 15 minutes). There are about 166 days in which the IDA has not recorded flow rates. Second, from the USGS National Water Information System (NWIS), we obtain daily mean flow rates (ranging from 1946-10-01 to 2008-09-30) with no missing records. Third, for each data point in the IDA time series,

we transform the data using the following three steps: 1) compute the mean value from the NWIS data across all years within a thirteen-day local window which is centered on the day the IDA data point value is recorded; 2) compute the standard deviation from the NWIS data within all the same local windows; 3) standardize the IDA data point value using the mean and the standard deviation obtained in 1) and 2). Fourth, we obtain a deseasonalized daily IDA time series after taking the maximum of all standardized daily point values for each day. Fifth, we use a simple regression model to "recover" the missing daily maxima by regressing the maxima on the means over the period of 1990-10-01 to 2008-09-30. The squared regression coefficient $R^2 = .998$ which shows a very good fit. Finally, we obtain the weekly maxima and plot them in Figures 4 (left panel) and 5 (left panel).

We note that Figure 4 is comparable to Figure 1. The same comparison holds for the Crystal River deseasonalized weekly maxima of river flow rates directly. Fitting model (4) to the deseasonalized weekly maxima, we obtain the following fitted values in Table 3. We note that the estimated values of the parameter a are 1 in both cases. This can be understood by thinking that the parameter a is combined into c, and we still have $ac < 1$, i.e. two processes are stationary.

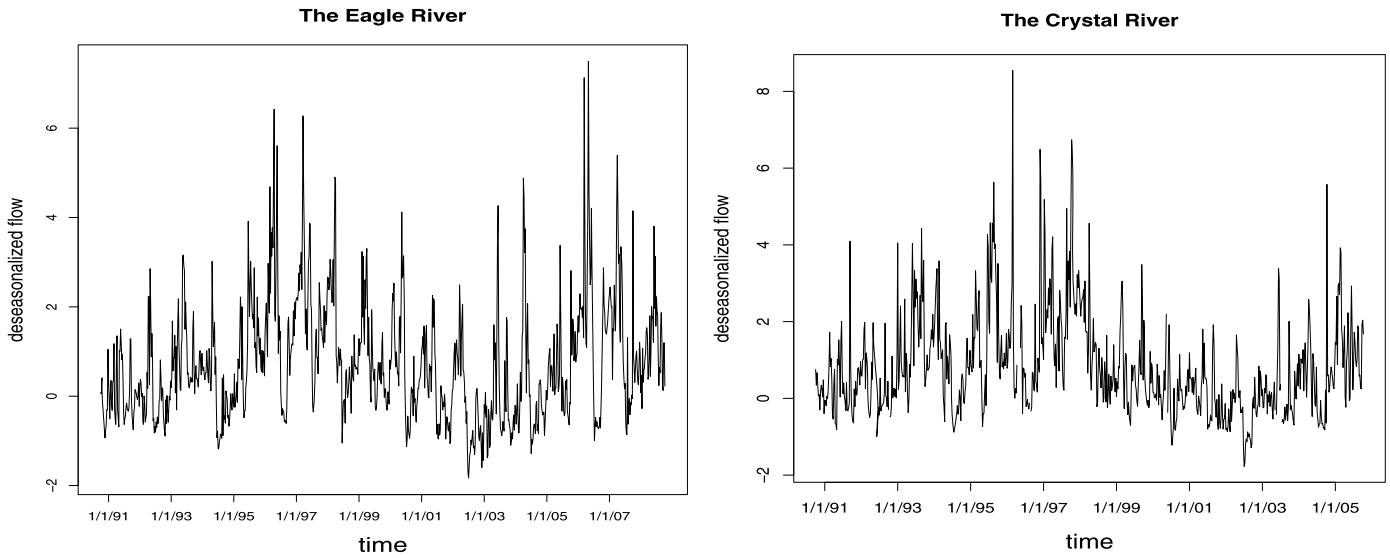Using the estimated values in the table, we simulate weekly maximum deseasonalized time series and compare

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The Eagle River

The Crystal River

Figure 4. Deseasonalized weekly maxima of river flow rates (ft^3/sec) in the Eagle River (left panel) and in the Crystal River (right panel).

The Eagle River

The Crystal River

Figure 5. Deseasonalized weekly maxima of river flow rates (ft^3/sec) in the Eagle River (left panel) and in the Crystal River (right panel).
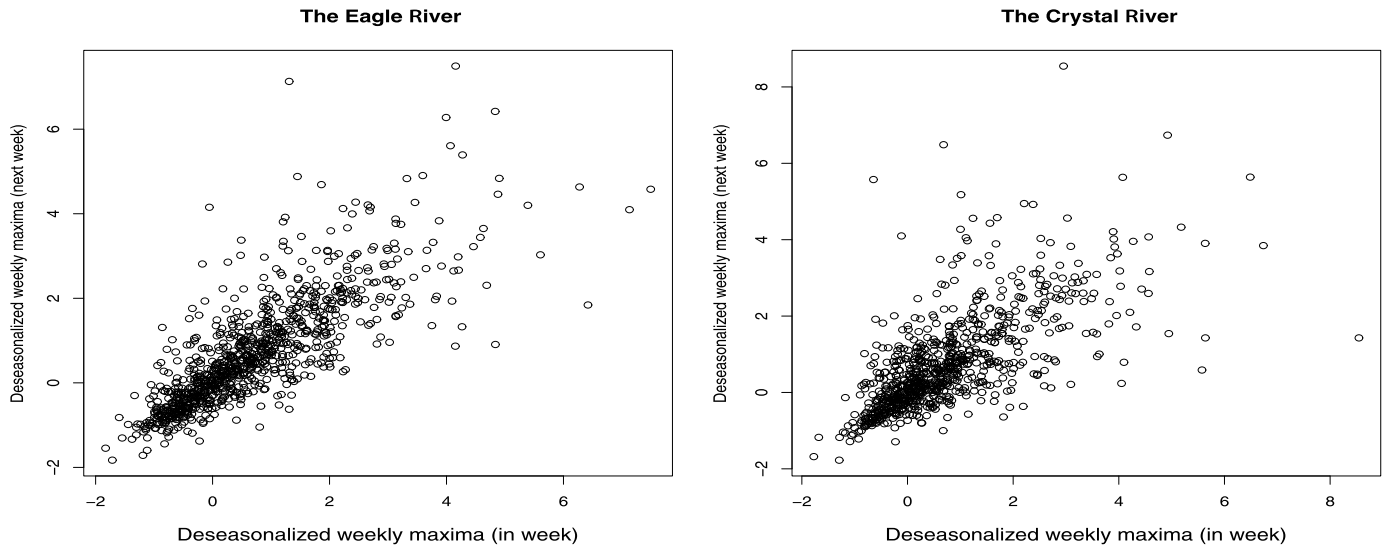
the simulated values with the observed values using quantile-quantile plots. In Figure 6, we can see that our proposed models reasonably well approximate the weekly maxima, especially in the tails.

6. CONCLUSIONS

Modeling extremes in time series is challenging. In this paper, we have generalized the max-autoregressive processes considered in [3] to the max autoregressive processes with positive alpha stable coefficients and unit Fréchet shocks, i.e. $\mathrm{MAP}_\alpha(p)$ models which eliminate the abnormally signature

patterns. We have proved that $\mathrm{MAP}_{1/2}(1)$ models asymptotically independent time series, and empirically demonstrated that $\mathrm{MAP}_\alpha(p)$ models asymptotically independent time series, which enriches models for extremes in time series. Our data examples suggest that $\mathrm{MAP}_{1/2}(1)$ models can directly be applied to some deseasonalized river flow rate data, which is certainly useful in hydrological study and river engineering management.

There are several future emerging research directions related to our present work. As mentioned in the introduction section, there are variants of our proposed models. The major challenges in our proposed models and other potential
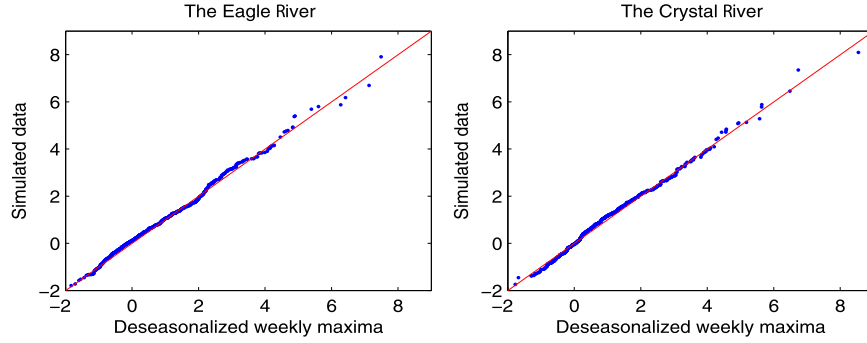
Figure 6. Quantile-quantile plots for deseasonalized weekly maxima of river flow rates (ft^3/sec) in the Eagle River (left panel) and in the Crystal River (right panel).

extensions are to study model reducibility, causality and predictability as they have been extensively studied in ARMA models. Constructing efficient parameter estimation methods is still a challenging task, e.g. when α is also unknown. We expect that there will be more and more theoretical developments in this field. We also expect more and more applications of these new models to different research fields in the near future.

APPENDIX

Proof of Proposition 1. Note that $\log Z_{0,\alpha}$ is Gumbel with $\log(b_0)$ as the location parameter and one as the scale parameter. Then, $\alpha \log S_{1,\alpha} + \alpha \log Z_{0,\alpha}$ follows a Gumbel distribution with $\alpha \log(b_0)$ as the location parameter and one as the scale parameter. We have

$$P(Z_{1,\alpha} \leq z) = P(S_{1,\alpha} Z_{0,\alpha} \leq (z/ac)^{1/\alpha}) P(\epsilon_1 \leq z/c),$$

$$= P\left\{ \alpha \log S_{1,\alpha} + \alpha \log Z_{0,\alpha} \leq \log\left(\frac{z}{ac}\right) \right\} \times \exp\left(-\frac{c}{z}\right),$$

$$= \exp\left\{ -\exp\left[-\log\left(\frac{z}{ac}\right) - \alpha \log(b_0) \right] \right\} \times \exp\left(-\frac{c}{z}\right)$$

$$= \exp\left(-\frac{acb_0^\alpha + c}{z} \right),$$

and similarly we have

$$P(Z_{t,\alpha} \leq z) = \exp\left(-\frac{acb_{t-1}^\alpha + c}{z} \right),$$

for $t = 1, 2, \ldots$. The proof of $\{b_t\}$ having a limit d is divided into three steps.

First, it is easy to see that the function $g(x) = x - acx^\alpha - c$ is decreasing when $0 < x < (ac\alpha)^{1/(1-\alpha)}$, and is increasing when $x > (ac\alpha)^{1/(1-\alpha)}$. In addition, $g(x)$ has a unique zero in $\mathbb{R}^+$, denoted by d, $g(x) > 0$ when $x > d$, and $g(x) < 0$ when $0 < x < d$.

Second, let's assume that for the initial condition b_0 we have $b_0 \geq d$. Then by $g(b_0) \geq 0$,

$$b_0 \geq b_1 = acb_0^\alpha + c \geq acd^\alpha + c = d,$$
$$b_1 \geq b_2 = acb_1^\alpha + c \geq acd^\alpha + c = d.$$

Continuing this procedure, we have that $\{b_t\}$ is a decreasing sequence with lower bound d, and hence it has a limit. Similarly, let's assume that for the initial condition b_0 we have $b_0 < d$. Then by $g(b_0) < 0$,

$$b_0 < b_1 = acb_0^\alpha + c < acd^\alpha + c = d,$$
$$b_1 < b_2 = acb_1^\alpha + c < acd^\alpha + c = d.$$

Continuing this procedure, we have that $\{b_t\}$ is an increasing sequence with upper bound d, and hence it has a limit.

Third, let t tend to infinity on the both sides of $b_t = acb_{t-1}^\alpha + c$. Then,

$$\lim_{t\to\infty} b_t = \lim_{t\to\infty} \{acb_{t-1}^\alpha + c\}$$
$$\Rightarrow b = acb^\alpha + c$$

By the uniqueness of zero of $g(x)$, the limit b must be equal to d, which satisfies $acd^\alpha + c = d$.

For the second part of the proposition, we assume $b_0 = d$. First, when $t = 0$, $P(Z_{0,\alpha} \leq z_0) = P(Z_{k,\alpha} \leq z_0)$ for any $k \geq 0$, because the marginal distribution is identical to Fréchet distribution with scale parameter d.

Next, assume that for $t = n \geq 0$, we have for any constant values $z_0, z_1, \ldots$

$$(15) \qquad P(Z_{0,\alpha} \leq z_0, \ldots, Z_{n,\alpha} \leq z_n)$$
$$= P(Z_{k,\alpha} \leq z_0, \ldots, Z_{k+n,\alpha} \leq z_n)$$

for any $k \geq 0$. Then for $t = n + 1$,

$$P(Z_{0,\alpha} \leq z_0, \ldots, Z_{n,\alpha} \leq z_n, Z_{n+1,\alpha} \leq z_{n+1})$$

$$= P\left(Z_{0,\alpha} \leq z_0, \ldots, Z_{n,\alpha} \leq z_n, Z_{n,\alpha} \leq \frac{z_{n+1}^{1/\alpha}}{(ac)^{1/\alpha} S_{n+1,\alpha}}, \right.$$

$$\left. \epsilon_{n+1} \leq \frac{z_{n+1}}{a} \right)$$

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$$= E\left[P\left(\left\{Z_{0,\alpha} \le z_0, \ldots, Z_{n,\alpha} \le \min\left(z_n, \frac{z_{n+1}^{1/\alpha}}{(ac)^{1/\alpha} S_{n+1,\alpha}}\right)\right\}\middle| S_{n+1,\alpha}\right)\right] P\left(\epsilon_{n+1} \le \frac{z_{n+1}}{a}\right)$$

$$= E\left[P\left(\left\{Z_{k,\alpha} \le z_0, \ldots, \right.\right.\right.$$

$$\left.\left.\left. Z_{k+n,\alpha} \le \min\left(z_n, \frac{z_{n+1}^{1/\alpha}}{(ac)^{1/\alpha} S_{k+n+1,\alpha}}\right)\right\}\middle| S_{k+n+1,\alpha}\right)\right]$$

$$P\left(\epsilon_{k+n+1} \le \frac{z_{n+1}}{a}\right)$$

$$= P\left(Z_{k,\alpha} \le z_0, \ldots, Z_{k+n,\alpha} \le z_n,\right.$$

$$\left. Z_{k+n,\alpha} \le \frac{z_{n+1}^{1/\alpha}}{(ac)^{1/\alpha} S_{k+n+1,\alpha}}, \epsilon_{k+n+1} \le \frac{z_{n+1}}{a}\right)$$

$$= P(Z_{k,\alpha} \le z_0, \ldots, Z_{k+n,\alpha} \le z_n, Z_{k+n+1,\alpha} \le z_{n+1})$$

for any $k \ge 0$, where the third equality comes from (15).

Therefore, if $b_0 = d$, $\mathrm{MAP}_\alpha(1)$ process is stationary by the proof of induction.

Summarizing the above arguments, we conclude that an $\mathrm{MAP}_\alpha(1)$ process reaches stationary regardless what the value of b_0 is. $\square$

Proof of Proposition 2. Define $\mathcal{S}_t = \sigma(S_t, S_{t-1}, S_{t-2}, \ldots)$.

$$P(Z_{t,\alpha} \le z)$$

$$= P\left\{\beta_0 \epsilon_t \vee \left[\bigvee_{j \ge 1} \beta_j \left(\prod_{i=0}^{j-1} S_{t-i}^{\alpha^{i+1}}\right) \epsilon_{t-j}^{\alpha^j}\right] \le z\right\}$$

$$= P\left(\epsilon_t \le \frac{z}{\beta_0}\right) P\left[\bigvee_{j \ge 1} \beta_j \left(\prod_{i=0}^{j-1} S_{t-i}^{\alpha^{i+1}}\right) \epsilon_{t-j}^{\alpha^j} \le z\right]$$

$$= \exp\left(-\frac{\beta_0}{z}\right) E\left[P\left(\epsilon_{t-1} \le \frac{z^{\frac{1}{\alpha}}}{\beta_1^{\frac{1}{\alpha}} S_t}, \epsilon_{t-2} \le \frac{z^{\frac{1}{\alpha^2}}}{\beta_2^{\frac{1}{\alpha^2}} S_t^{\frac{1}{\alpha}} S_{t-1}},\right.\right.$$

$$\left.\left. \epsilon_{t-3} \le \frac{z^{\frac{1}{\alpha^3}}}{\beta_3^{\frac{1}{\alpha^3}} S_t^{\frac{1}{\alpha^2}} S_{t-1}^{\frac{1}{\alpha}} S_{t-2}}, \ldots \middle| \mathcal{S}_t\right)\right]$$

$$= \exp\left(-\frac{\beta_0}{z}\right) E\left[P\left(\epsilon_{t-1} \le \frac{z^{\frac{1}{\alpha}}}{\beta_1^{\frac{1}{\alpha}} S_t}\middle| \mathcal{S}_t\right)\right.$$

$$P\left(\epsilon_{t-2} \le \frac{z^{\frac{1}{\alpha^2}}}{\beta_2^{\frac{1}{\alpha^2}} S_t^{\frac{1}{\alpha}} S_{t-1}}\middle| \mathcal{S}_t\right)$$

$$\left. P\left(\epsilon_{t-3} \le \frac{z^{\frac{1}{\alpha^3}}}{\beta_3^{\frac{1}{\alpha^3}} S_t^{\frac{1}{\alpha^2}} S_{t-1}^{\frac{1}{\alpha}} S_{t-2}}\middle| \mathcal{S}_t\right) \cdots\right]$$

$$= \exp\left(-\frac{\beta_0}{z}\right) E\left[\exp\left(-\frac{\beta_1^{\frac{1}{\alpha}} S_t}{z^{\frac{1}{\alpha}}}\right)\right.$$

$$\left. \exp\left(-\frac{\beta_2^{\frac{1}{\alpha^2}} S_t^{\frac{1}{\alpha}} S_{t-1}}{z^{\frac{1}{\alpha^2}}}\right) \exp\left(-\frac{\beta_3^{\frac{1}{\alpha^3}} S_t^{\frac{1}{\alpha^2}} S_{t-1}^{\frac{1}{\alpha}} S_{t-2}}{z^{\frac{1}{\alpha^3}}}\right) \cdots\right]$$

$$= \exp\left(-\frac{\beta_0}{z}\right) E\left[\lim_{n \to \infty} \prod_{j=1}^{n} \exp\left(-\frac{\beta_j^{\frac{1}{\alpha^j}} \prod_{i=0}^{j-1} S_{t-i}^{\frac{1}{\alpha^{j-i-1}}}}{z^{\frac{1}{\alpha^j}}}\right)\right]$$

$$(16) \quad \equiv \exp\left(-\frac{\beta_0}{z}\right) E\left[\lim_{n \to \infty} \prod_{j=1}^{n} F_j\right].$$

Since $0 \le F_j \le 1, \forall j \ge 1$, $\{\prod_{j=1}^{n} F_j\}$ is a non-increasing random variable sequence which is bounded by 1. By dominated convergence theorem, we have

$$(16) = \exp\left(-\frac{\beta_0}{z}\right) \lim_{n \to \infty} E\left[\prod_{j=1}^{n} F_j\right]$$

$$= \lim_{n \to \infty} \exp\left(-\frac{\beta_0}{z}\right)$$

$$E\left\{E\left[\left(\prod_{j=1}^{n-1} F_j\right) \exp\left(-\frac{\beta_n^{\frac{1}{\alpha^n}} \prod_{i=0}^{n-1} S_{t-i}^{\frac{1}{\alpha^{n-i-1}}}}{z^{\frac{1}{\alpha^n}}}\right)\right.\right.$$

$$\left.\left.\middle| S_t, S_{t-1}, \ldots, S_{t-n+2}\right]\right\}$$

$$= \lim_{n \to \infty} \exp\left(-\frac{\beta_0}{z}\right)$$

$$E\left\{\left(\prod_{j=1}^{n-1} F_j\right) E\left[\exp\left(-\frac{\beta_n^{\frac{1}{\alpha^n}} \prod_{i=0}^{n-1} S_{t-i}^{\frac{1}{\alpha^{n-i-1}}}}{z^{\frac{1}{\alpha^n}}}\right)\right.\right.$$

$$\left.\left.\middle| S_t, S_{t-1}, \ldots, S_{t-n+2}\right]\right\}$$

$$= \lim_{n \to \infty} \exp\left(-\frac{\beta_0}{z}\right)$$

$$E\left\{\left(\prod_{j=1}^{n-2} F_j\right) \exp\left(-\frac{\beta_{n-1}^{\frac{1}{\alpha^{n-1}}} \prod_{i=0}^{n-2} S_{t-i}^{\frac{1}{\alpha^{n-i-2}}}}{z^{\frac{1}{\alpha^{n-1}}}}\right)\right.$$

$$\left. \exp\left(-\frac{\beta_n^{\frac{1}{\alpha^{n-1}}} \prod_{i=0}^{n-2} S_{t-i}^{\frac{1}{\alpha^{n-i-2}}}}{z^{\frac{1}{\alpha^{n-1}}}}\right)\right\}$$

$$= \lim_{n \to \infty} \exp\left(-\frac{\beta_0}{z}\right) E\left\{\left(\prod_{j=1}^{n-2} F_j\right)\right.$$

$$(17) \quad \left. \exp\left(-\frac{(\beta_{n-1}^{\frac{1}{\alpha^{n-1}}} + \beta_n^{\frac{1}{\alpha^{n-1}}}) \prod_{i=0}^{n-2} S_{t-i}^{\frac{1}{\alpha^{n-i-2}}}}{z^{\frac{1}{\alpha^{n-1}}}}\right)\right\}.$$

Using this conditional expectation argument repeatedly, we have

$$(17) = \lim_{n \to \infty} \exp\left(-\frac{\beta_0}{z}\right) E\left\{\left(\prod_{j=1}^{n-3} F_j\right) \exp\left(-z^{-\frac{1}{\alpha^{n-2}}}\right.\right.$$

$$\left.\left. [\beta_{n-2}^{\frac{1}{\alpha^{n-2}}} + (\beta_{n-1}^{\frac{1}{\alpha^{n-1}}} + \beta_n^{\frac{1}{\alpha^{n-1}}})^\alpha] \prod_{i=0}^{n-3} S_{t-i}^{\frac{1}{\alpha^{n-i-3}}}\right)\right\}$$

$$= \cdots$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \exp\left(-\frac{\beta_0}{z}\right) E\big[\exp(-z^{-\frac{1}{\alpha}}\{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} + \{\beta_3^{\frac{1}{\alpha^3}} \\
&\quad + \cdots \{\beta_{n-2}^{\frac{1}{\alpha^{n-2}}} + \{\beta_{n-1}^{\frac{1}{\alpha^{n-1}}} + \beta_n^{\frac{1}{\alpha^{n-1}}}\}^\alpha\}^\alpha \cdots\}^\alpha\}^\alpha\}S_t)\big] \\
&= \lim_{n\to\infty} \exp\left(-\frac{1}{z}\big[\beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} + \{\beta_3^{\frac{1}{\alpha^3}} \right. \\
&\quad \left. + \cdots \{\beta_{n-2}^{\frac{1}{\alpha^{n-2}}} + \{\beta_{n-1}^{\frac{1}{\alpha^{n-1}}} + \beta_n^{\frac{1}{\alpha^{n-1}}}\}^\alpha\}^\alpha \cdots\}^\alpha\}^\alpha\}^\alpha\big]\right) \\
&\equiv \lim_{n\to\infty} \exp\left(-\frac{d_n}{z}\right) = \exp\left(-\frac{\lim_{n\to\infty} d_n}{z}\right).
\end{aligned}
$$

Using the recursions (9) that $\{\beta_j\}$ satisfies, we can show that $\{d_n\}$ is a non-decreasing sequence with finite limit d. Hence, $Z_{t,\alpha}$ follows Fréchet distribution with scale parameter $d = \beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} + \{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha$, which satisfies

$$
\begin{aligned}
acd^\alpha & + c \\
&= c + ac\{\beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} \\
&\quad + \{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha\}^\alpha \\
&= \beta_0 + \{a^{\frac{1}{\alpha}} c^{\frac{1}{\alpha}}\beta_0 + a^{\frac{1}{\alpha}} c^{\frac{1}{\alpha}}\{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} \\
&\quad + \{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha\}^\alpha \\
&= \beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{a^{\frac{1}{\alpha^2}} c^{\frac{1}{\alpha^2}}\beta_1^{\frac{1}{\alpha}} + a^{\frac{1}{\alpha^2}} c^{\frac{1}{\alpha^2}}\{\beta_2^{\frac{1}{\alpha^2}} \\
&\quad + \{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha\}^\alpha \\
&= \beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} + \{a^{\frac{1}{\alpha^3}} c^{\frac{1}{\alpha^3}}\beta_2^{\frac{1}{\alpha^2}} \\
&\quad + a^{\frac{1}{\alpha^3}} c^{\frac{1}{\alpha^3}}\{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha\}^\alpha \\
&= \cdots \\
&= \beta_0 + \{\beta_1^{\frac{1}{\alpha}} + \{\beta_2^{\frac{1}{\alpha^2}} + \{\beta_3^{\frac{1}{\alpha^3}} + \cdots\}^\alpha\}^\alpha\}^\alpha \\
&= d.
\end{aligned}
$$

Actually, we have shown that the distribution of $Z_{t,\alpha}$ coincides with the limiting marginal distribution of the $\mathrm{MAP}_\alpha(1)$ process (2) in Proposition 1, which implies that the relation (8) is true. $\qquad\square$

Proof of Result #1.

$$
\begin{aligned}
P(Z_{t,\alpha} &\leq z_t, \ S_{t,\alpha}Z_{t-1,\alpha} \leq z_{t-1}) \\
&= P(\epsilon_t \leq z_t/c)P(S_{t,\alpha}Z_{t-1,\alpha} \leq m_t), \\
&\quad \text{with } m_t = \min((z_t/ac)^{1/\alpha}, z_{t-1}), \\
&= \exp(-c/z_t)\int P(Z_{t-1} \leq m_t/s)f_S(s)ds, \\
&\quad \text{with } f_S \text{ the density of } S_t, \\
&= \exp(-c/z_t)\int \exp(-sd/m_t)f_S(s)ds, \\
&\quad \text{because } Z_{t-1} \text{ is Fréchet}, \\
&= \exp(-c/z_t)E[\exp(-S_t d/m_t)],
\end{aligned}
$$

$$
\begin{aligned}
&= \exp(-c/z_t)\exp[-(d/m_t)^\alpha], \text{ because of (3),} \\
&= \exp(-c/z_t)\exp[-d^\alpha \max\{ac/z_t, 1/z_{t-1}^\alpha\}], \\
&= \exp\left(-\max\left\{\frac{acd^\alpha + c}{z_t}, \frac{c}{z_t} + \frac{d^\alpha}{z_{t-1}^\alpha}\right\}\right), \\
&= \exp\left(-\max\left\{\frac{d}{z_t}, \frac{c}{z_t} + \frac{d^\alpha}{z_{t-1}^\alpha}\right\}\right).
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of Proposition 3. Taking the Laplace transform of S, we have

$$
(18) \qquad E(\exp(-uS)) = \sqrt{\frac{c}{2\pi}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{1}{s^{3/2}}ds
$$

for all $u \geq 0$. In order to calculate the integral in the right-hand side of (18), let $w = \sqrt{us} + \sqrt{\frac{c}{2s}}$ and $v = \sqrt{us} - \sqrt{\frac{c}{2s}}$. Then,

$$
\begin{aligned}
\int_0^\infty & e^{-us-\frac{c}{2s}}\frac{1}{s^{3/2}}ds \\
&= -\frac{2}{\sqrt{c/2}}\int_0^\infty e^{-us-\frac{c}{2s}}\left(-\frac{\sqrt{c/2}}{2s^{3/2}} + \frac{\sqrt{u}}{2\sqrt{s}}\right)ds \\
&\quad + \frac{2}{\sqrt{c/2}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{\sqrt{u}}{2\sqrt{s}}ds \\
&= -\frac{2}{\sqrt{c/2}}\int_\infty^\infty e^{-w^2+\sqrt{2uc}}dw + \sqrt{\frac{2u}{c}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{1}{\sqrt{s}}ds \\
&= \sqrt{\frac{2u}{c}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{1}{\sqrt{s}}ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\infty & e^{-us-\frac{c}{2s}}\frac{1}{s^{3/2}}ds \\
&= \frac{2}{\sqrt{c/2}}\int_0^\infty e^{-us-\frac{c}{2s}}\left(\frac{\sqrt{c/2}}{2s^{3/2}} + \frac{\sqrt{u}}{2\sqrt{s}}\right)ds \\
&\quad - \frac{2}{\sqrt{c/2}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{\sqrt{u}}{2\sqrt{s}}ds \\
&= \frac{2}{\sqrt{c/2}}\int_{-\infty}^\infty e^{-v^2-\sqrt{2uc}}dv - \sqrt{\frac{2u}{c}}\int_0^\infty e^{-us-\frac{c}{2s}}\frac{1}{\sqrt{s}}ds
\end{aligned}
$$

which gives

$$
(19) \qquad
\begin{aligned}
\int_0^\infty e^{-us-\frac{c}{2s}}\frac{1}{s^{3/2}}ds &= \frac{1}{\sqrt{c/2}}\int_{-\infty}^\infty e^{-v^2-\sqrt{2uc}}dv \\
&= \frac{e^{-\sqrt{2uc}}}{\sqrt{c/2}}\int_{-\infty}^\infty e^{-v^2}dv \\
&= \sqrt{\frac{2\pi}{c}}e^{-\sqrt{2uc}}.
\end{aligned}
$$

Plugging (19) into the right-hand side of (18), we have

$$E(\exp(-uS)) = \sqrt{\frac{c}{2\pi}} \int_0^\infty e^{-us - \frac{c}{2s}} \frac{1}{s^{3/2}} ds$$

$$= \sqrt{\frac{c}{2\pi}} \sqrt{\frac{2\pi}{c}} e^{-\sqrt{2uc}} = e^{-\sqrt{2uc}}.$$

Hence, S is positive α-stable if and only if $c = \frac{1}{2}$, which implies $\alpha = \frac{1}{2}$. $\square$

Proof of Result #2. Recall that S_t is Lévy$(0, \frac{1}{2})$ distributed and it is possible to show that

$$(20) \qquad P(S_t \leq v) = 2\left[1 - \Phi\left(\frac{1}{\sqrt{2v}}\right)\right],$$

where $\Phi(\cdot)$ is the standard normal cdf of N$(0,1)$. Then

$$P(Z_t \leq u, \ Z_{t-1} \leq u)$$

$$= P\left(\max\left\{a(S_t Z_{t-1})^{0.5}, \epsilon_t\right\} \leq \frac{u}{c}, \ Z_{t-1} \leq u\right)$$

$$= P\left(S_t Z_{t-1} \leq \left(\frac{u}{ac}\right)^2, \epsilon_t \leq \frac{u}{c}, Z_{t-1} \leq u\right)$$

$$= P\left(\epsilon_t \leq \frac{u}{c}\right) P\left(S_t Z_{t-1} \leq \left(\frac{u}{ac}\right)^2, Z_{t-1} \leq u\right)$$

$$= \exp\left(-\frac{c}{u}\right) E\left[P\left(S_t Z_{t-1} \leq \left(\frac{u}{ac}\right)^2, Z_{t-1} \leq u | Z_{t-1}\right)\right]$$

$$= \exp\left(-\frac{c}{u}\right) \int_0^u P\left(S_t \leq \frac{(u/ac)^2}{y}\right) g_{Z_{t-1}}(y) dy,$$

with $g_{Z_{t-1}}(y) = \frac{d}{y^2} \exp\left(-\frac{d}{y}\right)$

$$= 2\exp\left(-\frac{c}{u}\right) \int_0^u \left[1 - \Phi\left(\frac{ac\sqrt{y}}{u\sqrt{2}}\right)\right] g_{Z_{t-1}}(y) dy,$$

by (20),

$$= 2\exp\left(-\frac{c}{u}\right) \exp\left(-\frac{d}{u}\right)$$

$$(21) \quad -2\exp\left(-\frac{c}{u}\right) \int_0^u \Phi\left(\frac{ac\sqrt{y}}{u\sqrt{2}}\right) \frac{d}{y^2} \exp\left(-\frac{d}{y}\right) dy.$$

The integral in equation (21) can be expressed as

$$\int_0^u \Phi\left(\frac{ac\sqrt{y}}{u\sqrt{2}}\right) \frac{d}{y^2} \exp\left(-\frac{d}{y}\right) dy$$

$$= \Phi\left(\frac{ac}{\sqrt{2u}}\right) \exp\left(-\frac{d}{u}\right)$$

$$- \int_0^u \exp\left(-\frac{d}{y}\right) \Phi'\left(\frac{ac\sqrt{y}}{u\sqrt{2}}\right) \frac{ac}{2u\sqrt{2y}} dy$$

$$= \Phi\left(\frac{ac}{\sqrt{2u}}\right) \exp\left(-\frac{d}{u}\right)$$

$$(22) \qquad - \frac{ac}{4u\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{y}} \exp\left(-\frac{d}{y} - \frac{a^2 c^2 y}{4u^2}\right) dy.$$

Using integration method similar to that applied in the proof of Proposition 3, we can show

$$\frac{ac}{4u\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{y}} \exp\left(-\frac{d}{y} - \frac{a^2 c^2 y}{4u^2}\right) dy$$

$$= -\frac{1}{2} \exp\left(\frac{ac\sqrt{d}}{u}\right) \left[1 - \Phi\left(\frac{ac + 2\sqrt{d}}{\sqrt{2u}}\right)\right]$$

$$(23) \qquad + \frac{1}{2} \exp\left(-\frac{ac\sqrt{d}}{u}\right) \Phi\left(\frac{ac - 2\sqrt{d}}{\sqrt{2u}}\right).$$

Hence,

$$(22) = \Phi\left(\frac{ac}{\sqrt{2u}}\right) \exp\left(-\frac{d}{u}\right) + \frac{1}{2} \exp\left(\frac{ac\sqrt{d}}{u}\right)$$

$$\left[1 - \Phi\left(\frac{ac + 2\sqrt{d}}{\sqrt{2u}}\right)\right]$$

$$(24) \qquad - \frac{1}{2} \exp\left(-\frac{ac\sqrt{d}}{u}\right) \Phi\left(\frac{ac - 2\sqrt{d}}{\sqrt{2u}}\right).$$

Substituting (24) for integral in (21), we get the desired result. $\square$

Proof of Result #3.

$$P(Z_t > u, Z_{t-1} > u)$$

$$= 1 - P(Z_t \leq u) - P(Z_{t-1} \leq u) + P(Z_t \leq u, \ Z_{t-1} \leq u)$$

$$= 1 - \exp\left(-\frac{d}{u}\right) - \exp\left(-\frac{d}{u}\right) + \exp\left(-\frac{c}{u}\right)\left\{2\exp\left(-\frac{d}{u}\right)\right.$$

$$- 2\exp\left(-\frac{d}{u}\right) \Phi\left(\frac{ac}{\sqrt{2u}}\right)$$

$$- \exp\left(\frac{ac\sqrt{d}}{u}\right)\left[1 - \Phi\left(\frac{ac + 2\sqrt{d}}{\sqrt{2u}}\right)\right]$$

$$+ \exp\left(-\frac{ac\sqrt{d}}{u}\right) \Phi\left(\frac{ac - 2\sqrt{d}}{\sqrt{2u}}\right)\right\} \text{ by } (13),$$

$$= \exp\left(-\frac{c}{u}\right)\left\{\exp\left(\frac{c}{u}\right) - 2\exp\left(-\frac{d-c}{u}\right) + 2\exp\left(-\frac{d}{u}\right)\right.$$

$$- 2\exp\left(-\frac{d}{u}\right) \Phi\left(\frac{ac}{\sqrt{2u}}\right)$$

$$- \exp\left(\frac{ac\sqrt{d}}{u}\right)\left[1 - \Phi\left(\frac{ac + 2\sqrt{d}}{\sqrt{2u}}\right)\right]$$

$$(25) \quad + \exp\left(-\frac{ac\sqrt{d}}{u}\right) \Phi\left(\frac{ac - 2\sqrt{d}}{\sqrt{2u}}\right)\right\}.$$

Expanding (25) to the order $o(\frac{1}{u^2})$, we get

$$(25) = \left[1 - \frac{c}{u} + \frac{c^2}{2u^2} + o\left(\frac{1}{u^2}\right)\right]\left\{1 + \frac{c}{u} + \frac{c^2}{2u^2} + o\left(\frac{1}{u^2}\right)\right.$$

$$- 2\left[1 - \frac{d-c}{u} + \frac{(d-c)^2}{2u^2} + o\left(\frac{1}{u^2}\right)\right]$$

$$+ 2\left[1 - \frac{d}{u} + \frac{d^2}{2u^2} + o\left(\frac{1}{u^2}\right)\right] - 2\left[1 - \frac{d}{u} + \frac{d^2}{2u^2}\right.$$

$$+ o\left(\frac{1}{u^2}\right)\right]\left[\frac{1}{2} + \frac{ac}{2\sqrt{u\pi}} - \frac{a^3 c^3}{24u^{3/2}\sqrt{\pi}} + o\left(\frac{1}{u^2}\right)\right]$$

$$-\left[1+\frac{ac\sqrt{d}}{u}+\frac{a^2c^2d}{2u^2}+o\left(\frac{1}{u^2}\right)\right]\left[\frac{1}{2}\right.$$
$$\left.-\frac{ac+2\sqrt{d}}{2\sqrt{u\pi}}+\frac{(ac+2\sqrt{d})^3}{24u^{3/2}\sqrt{\pi}}+o\left(\frac{1}{u^2}\right)\right]$$
$$+\left[1-\frac{ac\sqrt{d}}{u}+\frac{a^2c^2d}{2u^2}+o\left(\frac{1}{u^2}\right)\right]$$
$$\left[\frac{1}{2}+\frac{ac-2\sqrt{d}}{2\sqrt{u\pi}}-\frac{(ac-2\sqrt{d})^3}{24u^{3/2}\sqrt{\pi}}+o\left(\frac{1}{u^2}\right)\right]\right\}$$
$$=\left[1-\frac{c}{u}+\frac{c^2}{2u^2}+o\left(\frac{1}{u^2}\right)\right]$$
$$\left[\frac{2acd}{u^{3/2}\sqrt{\pi}}-\frac{c^2-4cd+d^2}{2u^2}+o\left(\frac{1}{u^2}\right)\right]$$
$$\sim\frac{2acd}{\sqrt{\pi}}u^{-\frac{3}{2}},$$
$$\sim \operatorname{cst}P(Z_t>u)^{1/\eta},\ \text{because}\ P(Z_t>u)\sim d\,u^{-1}.$$
$$\square$$

Proof of Result #4.

$$E\left(\exp\left\{-\frac{1}{Z_{t,\frac{1}{2}}}\right\}\right)=\int_0^\infty\exp\left(-\frac{1}{z}\right)\exp\left(-\frac{d}{z}\right)\frac{d}{z^2}dz$$
$$=\frac{d}{d+1}\int_0^\infty\exp\left(-\frac{d+1}{z}\right)\frac{d+1}{z^2}dz$$
$$=\frac{d}{d+1}.$$

Before calculating $E(\exp\{-\frac{1}{Z_{t,\frac{1}{2}}}-\frac{1}{Z_{t-1,\frac{1}{2}}}\})$, we find the joint density $f(x,y)$ of (Z_t,Z_{t-1}). Similar to (21),

$$P(Z_{t,\frac{1}{2}}\le x,Z_{t-1,\frac{1}{2}}\le y)$$
$$=2\exp\left(-\frac{c}{x}\right)\exp\left(-\frac{d}{y}\right)$$
$$-2\exp\left(-\frac{c}{x}\right)\int_0^y\Phi\left(\frac{ac\sqrt{w}}{x\sqrt{2}}\right)\frac{d}{w^2}\exp\left(-\frac{d}{w}\right)dw.$$

Then

$$f(x,y)=2\left[\frac{c}{x^2}\exp\left(-\frac{c}{x}\right)\frac{d}{y^2}\exp\left(-\frac{d}{y}\right)\right.$$
$$-\frac{c}{x^2}\exp\left(-\frac{c}{x}\right)\frac{d}{y^2}\exp\left(-\frac{d}{y}\right)\Phi\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)$$
$$\left.+\exp\left(-\frac{c}{x}\right)\frac{d}{y^2}\exp\left(-\frac{d}{y}\right)\Phi'\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)\frac{ac\sqrt{y}}{x^2\sqrt{2}}\right]$$
$$(26)\qquad \equiv 2[I_1-I_2+I_3].$$

Then

$$E\left(\exp\left\{-\frac{1}{Z_{t,\frac{1}{2}}}-\frac{1}{Z_{t-1,\frac{1}{2}}}\right\}\right)$$
$$(27)\qquad =2\int_0^\infty\int_0^\infty\exp\left(-\frac{1}{x}-\frac{1}{y}\right)[I_1-I_2+I_3]dxdy$$

and

$$\int_0^\infty\int_0^\infty\exp\left(-\frac{1}{x}-\frac{1}{y}\right)I_1dxdy$$
$$=\int_0^\infty\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}dx\int_0^\infty\exp\left(-\frac{d+1}{y}\right)\frac{d}{y^2}dy$$
$$(28)\qquad =\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right),$$

$$\int_0^\infty\int_0^\infty\exp\left(-\frac{1}{x}-\frac{1}{y}\right)I_2dxdy$$
$$=\int_0^\infty\left\{\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}\int_0^\infty\exp\left(-\frac{d+1}{d}\right)\frac{d}{y^2}\right.$$
$$\left.\Phi\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)dy\right\}dx$$
$$=\int_0^\infty\left\{\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}\frac{d}{d+1}\int_0^\infty\Phi\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)\right.$$
$$\left.d\exp\left(-\frac{d+1}{y}\right)\right\}dx$$
$$=\int_0^\infty\left\{\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}\frac{d}{d+1}\left[1-\int_0^\infty\exp\left(-\frac{d+1}{y}\right)\right.\right.$$
$$\left.\left.\Phi'\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)\frac{ac}{2x\sqrt{2y}}dy\right]\right\}dx$$
$$=\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\frac{d}{d+1}\int_0^\infty\int_0^\infty\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}$$
$$\exp\left(-\frac{d+1}{y}\right)\Phi'\left(\frac{ac\sqrt{y}}{x\sqrt{2}}\right)\frac{ac}{2x\sqrt{2y}}dydx$$
$$=\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\frac{d}{d+1}\int_0^\infty\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}\frac{ac}{4x\sqrt{\pi}}$$
$$\int_0^\infty\frac{1}{\sqrt{y}}\exp\left(-\frac{d+1}{y}-\frac{a^2c^2y}{4x^2}\right)dydx$$
$$=\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\frac{d}{d+1}\int_0^\infty\exp\left(-\frac{c+1}{x}\right)\frac{c}{x^2}\frac{1}{2}$$
$$\exp\left(-\frac{ac\sqrt{d+1}}{x}\right)dx$$
$$=\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\frac{1}{2}\left(\frac{d}{d+1}\right)$$
$$\int_0^\infty\exp\left(-\frac{c+1+ac\sqrt{d+1}}{x}\right)\frac{c}{x^2}dx$$
$$(29)=\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\frac{1}{2}\left(\frac{d}{d+1}\right)\left(\frac{c}{c+1+ac\sqrt{d+1}}\right),$$

where the sixth equality is obtained by a calculation similar to that applied in obtaining equation (23).

$$\int_0^\infty\int_0^\infty\exp\left(-\frac{1}{x}-\frac{1}{y}\right)I_3dxdy$$
$$=\int_0^\infty\int_0^\infty\exp\left(-\frac{c+1}{x}\right)\exp\left(-\frac{d+1}{y}\right)\frac{d}{y^{3/2}}\frac{ac}{2x^2\sqrt{\pi}}$$

$$\exp\left(-\frac{a^2c^2y}{4x^2}\right)dxdy$$

$$= \int_0^\infty \exp\left(-\frac{c+1}{x}\right)\frac{acd}{2x^2\sqrt{\pi}}\int_0^\infty \frac{1}{y^{3/2}}\exp\left(-\frac{d+1}{y}\right.$$
$$\left.-\frac{a^2c^2y}{4x^2}\right)dydx$$

$$= \int_0^\infty \exp\left(-\frac{c+1}{x}\right)\frac{acd}{2x^2}\frac{1}{\sqrt{d+1}}\exp\left(-\frac{ac\sqrt{d+1}}{x}\right)dx$$

$$= \frac{1}{2}\left(\frac{acd}{\sqrt{d+1}}\right)\int_0^\infty \frac{1}{x^2}\exp\left(-\frac{c+1+ac\sqrt{d+1}}{x}\right)dx$$

$$(30) \qquad = \frac{1}{2}\frac{acd}{\sqrt{d+1}(c+1+ac\sqrt{d+1})},$$

where the third equality is obtained by a calculation similar to that applied in the proof of Proposition 3.

Combining the results of (27), (28), (29), (30), we get

$$E\left(\exp\left\{-\frac{1}{Z_{t,\frac{1}{2}}}-\frac{1}{Z_{t-1,\frac{1}{2}}}\right\}\right)$$
$$= 2\left[\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)-\left(\frac{c}{c+1}\right)\left(\frac{d}{d+1}\right)\right.$$
$$+\frac{1}{2}\left(\frac{d}{d+1}\right)\left(\frac{c}{c+1+ac\sqrt{d+1}}\right)$$
$$\left.+\frac{1}{2}\frac{acd}{\sqrt{d+1}(c+1+ac\sqrt{d+1})}\right]$$
$$= \frac{cd+acd\sqrt{d+1}}{(d+1)(c+1+ac\sqrt{d+1})}.$$

$\square$

Received 28 May 2011

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