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Remarks suggested by the paper of H. Tong

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This note remarks on H. Tong's 1980 paper and associated aspects of chaotic systems.

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1. INTRODUCTION

In his paper "Threshold models in time series analysis — 30 years on", Howell Tong discussed the background of his 1980 paper on threshold autoregressions, their application and the extensive research the 1980 paper led to in the following 30 years. In this brief note, I will mention some remarks of W. B. Wu that can be applied to get some insight into some of these models [9]. But the bulk of the few remarks I make will focus on the motivation and development of what Tong referred to as chaos and what is now sometimes called "chaos theory".

2. FIRST ORDER SCHEME

Assume that the ϵ_j are independent, identically distributed random variables. Consider the recursive scheme

$$x_n = R(x_{n-1}, \epsilon_n) \tag{1}$$

with R a measurable function. First order threshold autoregressive schemes are a particular special case. Let

$$L_\epsilon = \sup_{x \neq x'} \frac{|R(x,\epsilon) - R(x',\epsilon)|}{|x - x'|}.$$

If there are $\alpha > 0$ and x_0 such that

$$\mathbb{E}(\log L_\epsilon) < 0 \text{ and } L_{\epsilon_0} + |R(x_0, \epsilon_0)| \in L^\alpha, \tag{2}$$

one can show that there is a unique stationary distribution for (1). Further, under these conditions one can show that the corresponding stationary process is geometrically ergodic.

It is also of interest to consider the recursive scheme when the driving process ϵ_n is, say, a stationary Markov process. One would expect possible results to also depend on the character of the Markov process.

It is also worthwhile mentioning that when the conditions (2) are satisfied, the stationary solution of (1) has a one-sided (or causal) representation

$$x_n = g(\epsilon_n, \epsilon_{n-1}, \ldots)$$

in terms of the iid random variables ϵ_j.

3. CHAOS

It was quite a surprise when the phenomenon of chaos was increasingly found to arise in the analysis of a variety of dynamical deterministic systems of an applied character. We can think of it either in the discrete

$$x_n = f(x_{n-1}) \tag{3}$$

or continuous context

$$x_t = f_t(x_0),\ t \geq 0, \tag{4}$$

with f a continuous function on the state space, or f_t a continuous function of t and the state space variable (perhaps arising as a solution of a differential system of equations).

Classically if one wishes only to have a small global perturbation of a solution

$$|f_t(x_0) - f_t(x_0')| < \delta,\ \delta > 0,\ t \geq 0, \tag{5}$$

it is believed one can accomplish this by a sufficiently small displacement

$$|x_0 - x_0'| < \varepsilon(\delta) > 0. \tag{6}$$

However, it was already clear for over 100 years since the time of Poincaré (see the discussion in the book of J. Moser [4]) in the attempt to resolve questions relating to the stability of orbits that this would not generally be the case. And that asymptotic behavior with a limiting fixed point or a periodic orbit might not be the case. So here we have the 3-body problem or the n-body problem concerned with the stability of planetary orbits.

One found that there were simple dynamical systems for which there was incredible sensitivity to initial conditions such that orbits initially close diverged at an exponential rate and appeared to have chaotic trajectories. In fact a critical question was often that of distinguishing between data of a random character and that generated possibly by a deterministic chaotic system. A simple example of such a chaotic dynamical system is given by the logistic map

$$x_n = ax_{n-1}(1 - x_{n-1}) \tag{7}$$

with the constant a greater than or equal to 4. A measure of the rate of divergence of trajectories is given by the Lyapunov exponent of the system.

E. Lorenz considered the Rayleigh-Benard convection in which a layer of uniform depth is kept at a constant temperature difference, lower temperature at the upper level and higher temperature at the lower level. A Fourier expansion was made of the solution with the first three terms of the expansion kept. This led him to the following system of differential equations for the three terms (see [2])

$$
\begin{aligned}
\dot{x} &= -\sigma x + \sigma y, \\
\dot{y} &= -xz + rx - y, \\
\dot{z} &= xy - \rho z.
\end{aligned}
\tag{8}
$$

The dotted variables are derivatives with respect to t while σ, r, ρ are constants. With an appropriate choice of the constants σ, r, ρ, Lorenz was led to what appeared to be a chaotic system from the trajectories he computed. This suggested that there are limits to time ranges for which one could carry out effective weather prediction, perhaps two weeks.

4. ATTRACTORS

Results on dynamical systems often show asymptotic behavior in the appropriate phase space in which the orbits converge densely, on an attractor set as time progresses (see Ruelle [5]). The simplest cases are those in which the attractor is a single point or a periodic attractor. However, the attractor set may have a much more complicated structure in which case it is often referred to as a "strange attractor". The attractor in the case of the Lorenz set of equations as a chaotic system is one of the earliest shown to be a strange attractor. Smale [6] considered some economic models as dynamical systems. May [3] has considered chaotic models and their utility in a biological and ecological context. Ruelle [5] proposes a dynamical system approach to the problem of turbulence. In the approach of Kolmogorov to a theory of turbulence, there is a concept of transfer of kinetic energy from low wave number to high wave number. In [1], an attempt at a rigorous argument confirming the approach of Kolmogorov is sketched.

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