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Bayesian non-randomized response models for surveys with sensitive questions

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Three non-randomized response (NRR) models (namely, the non-randomized triangular, crosswise and hidden sensitivity models) have recently been developed for analyzing dichotomous sensitive questions. Unlike existing randomized response (RR) models, no randomizing device is required for NRR models. This helps to reduce the cost, increase the efficiency, ensure the reproducibility, widen the applicability and encourage the cooperation. However, in applications (e.g., estimating the proportion of a rare sensitive attribute in a population) with highly skewed likelihood functions, classical asymptotic methods based on maximum likelihood estimates and their asymptotic standard errors may not be adequate. The purposes of this article are two folds. First, we develop Bayesian approaches for analyzing dichotomous sensitive questions based on the aforementioned NRR models. For both the non-randomized triangular and crosswise models, we obtain the exact posterior distribution and its explicit posterior moments, derive posterior mode via the EM algorithm and provide procedure for generating i.i.d. posterior samples. For the hidden sensitivity model, we consider Bayesian analysis under the commonly used conjugate Dirichlet prior. Second, noting that the covariance structure associated with the Dirichlet distribution is completely nonpositive, we propose three new joint priors for modeling independence structure with restrictions, negative correlation structure and positive correlation structure, respectively. A new hierarchical modeling strategy is provided. Importance sampling and data augmentation algorithm are employed to compute posterior moments and generate posterior samples. Three data sets from a sensitive sexual behavior study, an induced abortion study and a HIV study are used to illustrate the proposed methodologies.

Keywords and phrases: Bayesian method, DA algorithm, EM algorithm, Non-randomized response models, Randomized response technique, Sensitive questions.

1. INTRODUCTION

Asking people questions and collecting their responses is an important source of information that informs decision making in many medical studies, public health policies and social issues. However, asking sensitive questions is generally seen as problematic in survey research due to concerns about information privacy. Because of these concerns, some respondents might intentionally give false information or simply refuse to divulge any information at all. As a result, directly asking sensitive questions is prone to error and bias. In order to increase the reliability and validity of responses, a number of strategies have been developed to minimize the likelihood of such error and bias.

The *randomized response* (RR) technique proposed by Warner (1965) is perhaps the first attempt to obtain more reliable information for estimating the proportion of a sensitive attribute in a population without revealing any respondent's actual status. However, a *randomizing device* (RD) is necessary and must be provided to each respondent to determine whether he/she needs to answer the sensitive question directly (with probability p) or the complement of the sensitive question (with probability $1-p$). Possible RDs include spinner with an arrow pointer, colored plastic balls/beads, coins, dice and poker chips. The requirement of RDs almost restricts the applicability of Warner RR model to face-to-face interview only; otherwise, inevitably increases the cost of the survey. Even worse, Warner model does not work for $p = 1/2$, which is a fatal limitation for obtaining trust from interviewees. In addition, Warner model is usually criticized by its inefficiency. To overcome some of the above limitations, Horvitz et al. (1967) and Greenberg et al. (1969) developed an unrelated question RR model. Subsequently, other authors (Kuk, 1990; Mangat & Singh, 1990; Mangat, 1994; Chang & Liang, 1996; Zou, 1997; Gjestvang & Singh, 2006) suggested various modified RR models. Nonetheless, all these RR techniques heavily rely on those interviewer-controlled RDs, resulting in high cost, low cooperation and lack of reproductivity.

To overcome these drawbacks associated with RR models, two *non-randomized response* (NRR) models, namely the triangular and crosswise models, were developed recently by Yu et al. (2008) for a single sensitive dichotomous question. Tian et al. (2007) proposed a non-randomized *hidden sensitivity* (HS) model for analyzing the association between two sensitive dichotomous questions. Unlike traditional RR models, the NRR models utilize an independent (or unrelated) non-sensitive question (e.g., season of birth) in the survey to indirectly obtain a respondent's answer to a sensitive question. In general, the non-randomized triangular

Table 1. The triangular model and the corresponding cell probabilities

Categories	$W=0$	$W=1$
$Y=0$	○	•
$Y=1$	•	•

Categories	$W=0$	$W=1$	Total
$Y=0$	$(1-\pi)(1-p)$	$(1-\pi)p$	$1-\pi$
$Y=1$	$\pi(1-p)$	πp	π
Total	$1-p$	p	1

Respondent: Please truthfully put a tick in the circle or in the triangle formed by the three dots.

design is more efficient than the randomized Warner design. Most importantly, all NRR designs do not require any RDs and hence substantially reduce the cost, improve the cooperation, and ensure the reproductivity.

Although the aforementioned NRR models have been shown to be very useful to surveys involving sensitive questions, existing classical analysis methods based on maximum likelihood estimates and their asymptotic standard errors may not be adequate in applications (e.g., estimating the proportion of a rare sensitive attribute in a population) with highly skewed likelihood function.[1] In addition, when investigators have some knowledge about the parameters of interest before they obtain the data, Bayesian estimation methods may be more appealing. For instance, Greenberg et al. (1969) suggested that if a membership in the sensitive group really possesses a socially disapproved attribute it is reasonable to assume that the corresponding proportion should be in neighborhood of 0.05 and 0.10. Winkler & Frankin (1979), Pitz (1980) and Spurrier & Padgett (1980) presented Bayesian approaches for the Warner and unrelated question RR models, respectively, using parametric models of prior information. Using the Bayes linear estimator, similar results were obtained by O'Hagan (1987) from a nonparametric model. Migon & Tachibana (1997) considered Bayesian approximation in RR model. Using the Gibbs sampler, Unnikrisknan & Kunte (1999) developed an unified model for RR strategies of which the Warner, unrelated question RR as well as polychotomous models are special cases. Bar-Lev et al. (2003) presented a common Bayesian approach to four RR models. DiPietro (2004) described a data analysis project in the Bayesian framework. Kim et al. (2006) provided Bayesian methods for Mangat's (1994) RR model.

The purposes of this article are two-fold. First, we develop Bayesian approaches for the aforementioned three NRR models under the commonly used beta or Dirichlet priors. The main advantage of adopting the conjugate prior is for its mathematical and computational simplicity. It is well known that the covariance structure associated with the Dirichlet distribution is completely non-positive. For those cases that require positive covariance structures, the Dirichlet distribution is not appropriate. Hence, it is the second purpose of this paper to propose three new priors for modeling independence structure with restrictions, negative correlation structure and positive correlation structure, respectively. A new hierarchical modeling strategy is also proposed.

The rest of the paper are organized as follows. We first briefly review the three NRR models and discuss some of their advantages over the RR models in Section 2. In Sections 3 and 4, we obtain the exact posterior distributions and their explicit posterior moments, derive posterior modes via the EM algorithm and provide approach to generate i.i.d. posterior samples for both the triangular and crosswise models, respectively. In Section 5, we present the Bayesian analysis for the HS model under a conjugate Dirichlet prior. We then propose three new priors for different covariance structures. Importance sampling and *data augmentation* (DA) algorithms are employed to compute the posterior moments and generate posterior samples under the positive covariance structure. Three data sets from a sensitive sexual behavior study, an induced abortion study and a HIV study are used to illustrate the proposed methods in Section 6. We finally conclude in Section 7.

2. NON-RANDOMIZED RESPONSE MODELS

2.1 The non-randomized triangular model for one sensitive question

Let $Y=1$ denote the class of people with a sensitive characteristic (e.g., drug-taking) and $Y=0$ the complementary class. Let W be a dichotomous variate associated with a non-sensitive question and independent of Y. For instance, $W=1$ may represent the class of people who were born between July and December and $W=0$ represents the corresponding complementary class. The survey designer should choose an appropriate W in such a way that the proportion $p=\Pr(W=1)$ is either known or can be estimated easily. Without loss of generality, let p be known. The purpose is to estimate the proportion $\pi=\Pr(Y=1)$.

For a face-to-face personal interview, the survey designer may replace the sensitive question by the tabular form presented on the left-hand side of Table 1 and ask the respondent to put a tick in the circle or in the triangle formed by the three dots according to his/her truthful status. It is noteworthy that respondents to $\{Y=0, W=0\}$ are simply non-drug users born between January and June. In other words, $\{Y=0, W=0\}$ represents a non-sensitive subclass. On the other hand, a tick in the triangle indicates the respondent can be either a drug user or a non-drug user born between

[1]See Figure 1(b) and (c) in §3. We note that the likelihood function is entirely identical to the posterior distribution when the prior is a uniform distribution.

Table 2. The crosswise model (Yu et al., 2008)

Categories	$W=0$	$W=1$
$Y=0$	○	•
$Y=1$	•	○

Categories	$W=0$	$W=1$	Total
$Y=0$	$(1-\pi)(1-p)$	$(1-\pi)p$	$1-\pi$
$Y=1$	$\pi(1-p)$	πp	π
Total	$1-p$	p	1

Respondent: Please truthfully put a tick in the diagonal with the two circles or the off-diagonal with the two dots.

Table 3. Questionnaire for the hidden sensitivity (HS) model

Categories	$W=1$	$W=2$	$W=3$	$W=4$
I: $\{X=0, Y=0\}$	Block 1: ______	Block 2: ______	Block 3: ______	Block 4: ______
Ⅱ: $\{X=0, Y=1\}$	Category Ⅱ: please put a tick in Block 2			
Ⅲ: $\{X=1, Y=0\}$	Category Ⅲ: please put a tick in Block 3			
Ⅳ: $\{X=1, Y=1\}$	Category Ⅳ: please put a tick in Block 4			

July and December. Therefore, $\{Y=1\} \cup \{Y=0, W=1\}$ can be regarded as a non-sensitive subclass as well. Such camouflage would presumably encourage respondents to not only participate in the survey but also provide truthful responses. In all subsequent discussion, this is called the *triangular model*.

2.2 The crosswise model for one sensitive question

Besides the triangular model, we can consider the following so-called *crosswise model* for analyzing a single sensitive question. Let Y and W be defined in §2.1, $p = \Pr(W=1)$ and $\pi = \Pr(Y=1)$. The interviewer may reformulate the sensitive question in the format as shown on the left-hand side of Table 2 and ask the interviewee to truthfully put a tick in either the diagonal with two circles or the off-diagonal with two dots. It is important to notice that both $\{Y=0, W=0\} \cup \{Y=1, W=1\}$ and $\{Y=0, W=1\} \cup \{Y=1, W=0\}$ are non-sensitive subclasses. Thus, whether an interviewee possesses the sensitive attribute will not be exposed. Yu et al. (2008) showed that this crosswise model is a non-randomized version of the original Warner model.

2.3 The hidden sensitivity model for two sensitive questions

Consider two binary sensitive variates X and Y. For the sensitive variate X, let $X=1$ denote the sensitive attribute of a respondent (e.g., taking drug), and $X=0$ the non-sensitive one (e.g., not taking drug). For the sensitive variate Y, let $Y=1$ be the other sensitive attribute (e.g., HIV+) and $Y=0$ the non-sensitive one (e.g., HIV−). Let $\theta_x = \Pr(X=1)$, $\theta_y = \Pr(Y=1)$, $\theta_1 = \Pr(X=0, Y=0)$, $\theta_2 = \Pr(X=0, Y=1)$, $\theta_3 = \Pr(X=1, Y=0)$ and $\theta_4 = \Pr(X=1, Y=1)$. Hence, $\theta_x = \theta_3 + \theta_4$ and $\theta_y = \theta_2 + \theta_4$. A commonly used measure of association is the odds ratio defined as $\delta = \theta_1\theta_4/(\theta_2\theta_3)$. The objective is to estimate θ_x, θ_y, θ_is and δ.

To obtain reliable responses from respondents, we introduce a non-sensitive variate W, which is independent of (X, Y), with four mutually exclusive categories. Let $p_i = \Pr(W=i)$ for $i = 1, \ldots, 4$. Like the triangular and crosswise models, the variate W should be chosen in such a way that all p_is can be obtained or estimated easily. Therefore, we assume that p_is are known. For example, let $\{W=i\}$ denote that a respondent was born in the i-th quarter and we can thus assume that p_is are approximately all equal to 1/4.

Instead of directly answering the sensitive question, each respondent is asked to answer a new question as shown in Table 3. Since $\{X=0, Y=0\}$ represents an non-sensitive subclass, we have reason to believe that a respondent will put a tick in Block i ($i = 1, \ldots, 4$) according to his/her truthful status if (s)he belongs to this category. The other categories (i.e., Blocks Ⅱ to Ⅳ), however, are sensitive to the respondent. If the respondent belongs to Block Ⅱ (Ⅲ or Ⅳ), (s)he is requested/forced to put a tick in Block 2 (3 or 4) so that his/her privacy is somehow protected. This technique is simply called the *hidden sensitivity* model in the sense that the sensitive attribute of a respondent is being hidden. Table 4 shows the cell probabilities θ_is and the observed frequencies n_is. Let n_1 denote the observed frequency of respondents putting a tick in Block 1. n_2 represents the sum of the frequencies of respondents belonging to Block 2 and Block Ⅱ. We can interpret n_3 and n_4 similarly.

Unlike the popular randomized response models, it is noteworthy that all the aforementioned NRR models have the following advantages: (i) they do not require any RDs and the study cost is thus reduced; (ii) the results can be reproducible; (iii) they can be easily operated for both interviewers and interviewees; and (iv) they can be applied to both face-to-face personal interviews and mail questionnaires.

Table 4. Cell probabilities, observed and unobservable frequencies for the HS model

Categories	$W=1$	$W=2$	$W=3$	$W=4$	Total
I: $\{X=0, Y=0\}$	$p_1\theta_1$	$p_2\theta_1$	$p_3\theta_1$	$p_4\theta_1$	θ_1 (Z_1)
II: $\{X=0, Y=1\}$					θ_2 (Z_2)
III: $\{X=1, Y=0\}$					θ_3 (Z_3)
IV: $\{X=1, Y=1\}$					θ_4 (Z_4)
Total	p_1 (n_1)	p_2 (n_2)	p_3 (n_3)	p_4 (n_4)	1 (n)

Note: $n=\sum_{i=1}^4 n_i$, $Z_1 = n-(Z_2+Z_3+Z_4)$, where (Z_2, Z_3, Z_4) are unobservable.

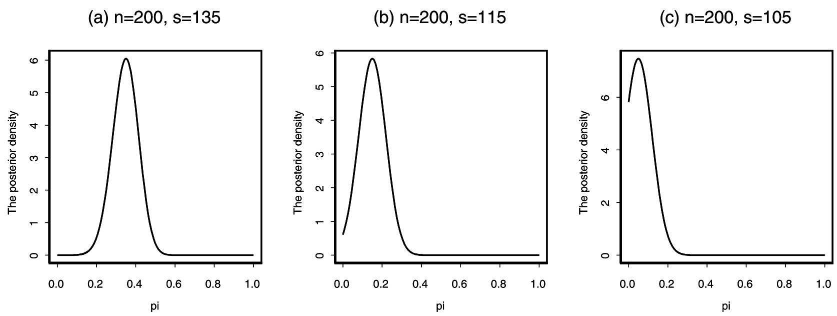

Figure 1. Posterior distributions for π under the uniform prior (i.e., $a=b=1$) and $p=\Pr(W=1)=0.5$ for the non-randomized triangular model. (a) $n=200$ and $s=135$; (b) $n=200$ and $s=115$; (c) $n=200$ and $s=105$.

3. BAYESIAN METHODS FOR THE NON-RANDOMIZED TRIANGULAR MODEL

In this section, we first derive the exact posterior distribution of π and its explicit posterior moments. We then derive the posterior mode via the EM algorithm (Dempster et al., 1977) when the posterior distribution of π is highly skewed. Finally, we utilize the exact *inverse Bayes formulae* (IBF) sampler (Tian, Tan & Ng, 2007) to generate i.i.d. posterior samples.

3.1 Posterior moments in closed-form

For the triangular model given in Table 1, we define a 'hidden' variable $Y^{\rm HT}$ as follows:

$$Y^{\rm HT} = \begin{cases} 1, & \text{with probability } \pi+(1-\pi)p, \\ & \quad \text{if a tick is put in the triangle,} \\ 0, & \text{with probability } (1-\pi)(1-p), \\ & \quad \text{if a tick is put in the circle.} \end{cases}$$

Let $Y_{\rm obs} = \{y_i^{\rm HT}: i=1,\ldots,n\}$ denote the observed data for the n respondents with $y_i^{\rm HT}=1$ if the i-th respondent puts a tick in the triangle; $=0$ otherwise. The likelihood function for π is then given by

$$\begin{aligned} L_{\scriptscriptstyle T}(\pi|Y_{\rm obs}) &= \prod_{i=1}^n [\pi+(1-\pi)p]^{y_i^{\rm HT}}[(1-\pi)(1-p)]^{1-y_i^{\rm HT}} \\ &= [\pi+(1-\pi)p]^s[(1-\pi)(1-p)]^{n-s}, \ 0\le\pi\le1, \end{aligned}$$

where $s \,\hat{=}\, \sum_{i=1}^n y_i^{\rm HT}$. If we choose the beta distribution Beta(a,b) to be the prior distribution of π, then the posterior distribution of π takes the following closed-form expression:

$$\text{(3.1)} \quad \begin{aligned} f(\pi|Y_{\rm obs}) &= c_{\scriptscriptstyle T}^{-1}(a,b;s,n-s) \\ &\quad \times \pi^{a-1}(1-\pi)^{b+n-s-1}[\pi+(1-\pi)p]^s, \end{aligned}$$

where the normalizing constant is given by

$$\text{(3.2)} \quad c_{\scriptscriptstyle T}(a,b;s,n-s) = \sum_{j=0}^s \binom{s}{j} p^{s-j} B(a+j, b+n-j).$$

For $a=b=1$ and $p=\Pr(W=1)=0.5$, Figure 1 shows the posterior distributions of π for three different combinations of n and s.

When $f(\pi|Y_{\rm obs})$ is fairly symmetric, the first two posterior moments are good enough to describe the location and discrepancy of the posterior distribution. From (3.1), the t-th posterior moment of π has the following explicit expression:

$$\text{(3.3)} \qquad E(\pi^t|Y_{\rm obs}) = \frac{c_{\scriptscriptstyle T}(a+t,b;s,n-s)}{c_{\scriptscriptstyle T}(a,b;s,n-s)}, \quad t\ge1.$$

3.2 Calculation of the posterior mode via the EM algorithm

When $f(\pi|Y_{\rm obs})$ is highly skewed (for instance, see Figure 1(c)), the posterior mode is usually adopted for describing the location. To derive the mode, we first introduce an unobservable variable Z, which denotes the number of respondents with the sensitive attribute. Obviously, the number of respondents without the sensitive characteristic is $n-Z$. Thus, the complete-data is $Y_{\rm com}=\{Y_{\rm obs}, Z\}$. The complete-data posterior distribution and the conditional predictive distribution are given by

$$f(\pi|Y_{\rm obs}, Z) = {\rm Beta}(\pi|a+Z, b+n-Z), \tag{3.4}$$

and

$$f(Z|Y_{\rm obs}, \pi) = {\rm Binomial}\,(Z|s, \pi/[\pi+(1-\pi)p]), \tag{3.5}$$

respectively. Using the EM algorithm, the M-step computes the complete-data posterior mode as

$$\tilde\pi_{_T} = \frac{a+Z-1}{a+b+n-2} \tag{3.6}$$

and the E-step is to replace Z by its conditional expectation

$$E(Z|Y_{\rm obs}, \pi) = \frac{s\,\pi}{\pi+(1-\pi)p}. \tag{3.7}$$

3.3 Generation of i.i.d. posterior samples via the exact IBF sampling

We re-write (3.4) and (3.5) as $f(\pi|Y_{\rm obs}, Z) = {\rm Beta}(\pi|a+Z, b+n-Z)$ and $f(Z|Y_{\rm obs}, \pi) = {\rm Binomial}\,(Z|s, \pi/[\pi+(1-\pi)p])$. According to the exact IBF algorithm presented in Appendix A, to generate i.i.d. posterior samples we simply need to identify $\mathcal{S}_{(Z|Y_{\rm obs})}$ and calculate $\{\omega_k\}_{k=1}^K$. Obviously,

$$\mathcal{S}_{(Z|Y_{\rm obs})} = \mathcal{S}_{(Z|Y_{\rm obs},\pi)} = \{z_1,\ldots,z_K\} = \{0,1,\ldots,s\}$$

and $K=s+1$. Setting $\pi_0=0.5$, from (A.2) and (A.3), we obtain

$$\begin{aligned} q_k(0.5) &= \frac{\binom{s}{z_k}p^{s-z_k}/(1+p)^s}{0.5^{a+b+n}/B(a+z_k, b+n-z_k)} \\ &\propto \binom{s}{z_k}\frac{\Gamma(a+z_k)\Gamma(b+n-z_k)}{p^{z_k}}, \end{aligned}$$

for $k=1,\ldots,K$, and $\omega_k = q_k(0.5)/\sum_{k'=1}^K q_{k'}(0.5)$.

4. BAYESIAN METHODS FOR THE CROSSWISE MODEL

4.1 Posterior moments in closed-form

Let $Y_{\rm obs}=\{y_i^{\rm HW}: i=1,\ldots,n\}$ denote the observed data for the n respondents, where $y_i^{\rm HW}=1$ if the i-th respondent puts a tick in the main diagonal with the two circles; $=0$ otherwise. The likelihood function for π is then given by

$$\begin{aligned} L_{_W}(\pi|Y_{\rm obs}) &= \prod_{i=1}^n [\pi p+(1-\pi)(1-p)]^{y_i^{\rm HW}} \\ &\quad\cdot[\pi(1-p)+(1-\pi)p]^{1-y_i^{\rm HW}} \\ &= [\pi p+(1-\pi)(1-p)]^r \\ &\quad\cdot[\pi(1-p)+(1-\pi)p]^{n-r}, \quad 0\le\pi\le 1, \end{aligned}$$

where $r \,\hat{=}\, \sum_{i=1}^n y_i^{\rm HW}$. Assume that $\pi\sim{\rm Beta}(a,b)$. Thus, the posterior distribution of π is

$$f(\pi|Y_{\rm obs}) = \frac{\pi^{a-1}(1-\pi)^{b-1}L_{_W}(\pi|Y_{\rm obs})}{c_{_W}(a,b;r,n-r)}, \tag{4.1}$$

where the normalizing constant $c_{_W}(a,b;r,n-r) \,\hat{=}\, \frac{p^{n-r}(1-p)^r}{\Gamma(a+b+n)}c^*_{_W}(a,b;r,n-r)$ and

$$\begin{aligned} c^*_{_W}(a,b;r,n-r) = \sum_{j_1=0}^{r}\sum_{j_2=0}^{n-r}\binom{r}{j_1}\binom{n-r}{j_2}\left(\frac{p}{1-p}\right)^{j_1-j_2} \\ \cdot\Gamma(a+j_1+j_2)\Gamma(b+n-j_1-j_2). \end{aligned}$$

Therefore, the t-th posterior moment of π is given by

$$E(\pi^t|{\rm data}) = \frac{c^*_{_W}(a+t,b;r,n-r)}{c^*_{_W}(a,b;r,n-r)}\cdot\frac{\Gamma(a+b+n)}{\Gamma(a+t+b+n)},\ t\ge 1. \tag{4.2}$$

4.2 Calculation of the posterior mode via the EM algorithm

To derive the posterior mode, we first introduce two unobservable variables $Z=(Z_1,Z_2)$, where Z_1 and Z_2 are respectively the counts of cell-$(1,1)$ and cell-$(1,0)$ in Table 2. Thus, the complete-data posterior distribution and the conditional predictive distribution are given by

$$f(\pi|Y_{\rm obs}, Z) = {\rm Beta}(\pi|a+Z_1+Z_2, b+n-Z_1-Z_2), \quad \text{and} \tag{4.3}$$

$$\begin{aligned} f(Z|Y_{\rm obs},\pi) &= f(Z_1|Y_{\rm obs},\pi)\cdot f(Z_2|Y_{\rm obs},\pi) \\ &= {\rm Binomial}\left(Z_1\middle|r,\ \frac{\pi p}{\pi p+(1-\pi)(1-p)}\right) \\ &\quad\cdot{\rm Binomial}\left(Z_2\middle|n-r,\ \frac{\pi(1-p)}{\pi(1-p)+(1-\pi)p}\right), \end{aligned} \tag{4.4}$$

respectively. Using the EM algorithm, the M-step yields the following complete-data posterior mode

$$\tilde\pi_{_W} = \frac{a+Z_1+Z_2-1}{a+b+n-2} \tag{4.5}$$

while the E-step is to replace Z_1+Z_2 by its conditional expectation

$$\begin{aligned} E(Z_1+Z_2|Y_{\rm obs},\pi) &= \frac{r\pi p}{\pi p+(1-\pi)(1-p)} \\ &\quad+\frac{(n-r)\pi(1-p)}{\pi(1-p)+(1-\pi)p}, \quad p\neq 1/2. \end{aligned} \tag{4.6}$$

In fact, when $p=1/2$, (4.6) becomes $E(Z_1+Z_2|Y_{\rm obs},\pi) = n\pi$, which does not depend on the observed data $Y_{\rm obs}$ nor r.

In this case, the EM algorithm in (4.5) and (4.6) converges in one step and we have $\tilde{\pi}_W = (a-1)/(a+b-2)$, which is actually the mode of the prior distribution Beta(a, b).

4.3 Generation of i.i.d. posterior samples via the exact IBF sampling

To apply the exact IBF algorithm to the present model, we simply need to identify the conditional support of $Z|(Y_{\text{obs}}, \pi)$. From (4.4), we have

$$\mathcal{S}_{(Z|Y_{\text{obs}})} = \mathcal{S}_{(Z|Y_{\text{obs}},\pi)} = \{z_1, \dots, z_K\} = \left\{ \begin{array}{cccc} (0,0) & (0,1) & \cdots & (0,n-r) \\ (1,0) & (1,1) & \cdots & (1,n-r) \\ \vdots & \vdots & \ddots & \vdots \\ (r,0) & (r,1) & \cdots & (r,n-r) \end{array} \right\},$$

where $K = (r+1)(n-r+1)$. We then calculate $\{\omega_k\}_{k=1}^K$ according to (A.2) and (A.3) with $\pi_0 = 0.5$.

5. BAYESIAN METHODS FOR THE HIDDEN SENSITIVITY MODEL

5.1 Bayesian inferences under Dirichlet prior

Assume that there are totally n respondents with n_i ticks being put in Block i for $i = 1, \dots, 4$ (see Table 4). Let $Y_{\text{obs}} = \{n; n_1, \dots, n_4\}$ denote the observed frequencies with $n = \sum_{i=1}^4 n_i$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_4)^\top \in \mathbb{T}_4$ be the cell probability vector, where $\mathbb{T}_4 \hat{=} \{(\theta_1, \dots, \theta_4)^\top : \theta_i \geq 0, \sum_{i=1}^4 \theta_i = 1\}$. The observed-data likelihood function for $\boldsymbol{\theta}$ is then

$$L_H(\boldsymbol{\theta}|Y_{\text{obs}}) = \theta_1^{n_1} \prod_{i=2}^4 (p_i\theta_1 + \theta_i)^{n_i}, \quad \boldsymbol{\theta} \in \mathbb{T}_4, \tag{5.1}$$

where $p_i = \Pr(W = i)$, $i = 1, \dots, 4$, are assumed to be known constants.

5.1.1. Posterior moments in closed-form

The natural prior for $\boldsymbol{\theta}$ is the Dirichlet distribution Dirichlet$(\mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_4)^\top$. Thus, the posterior distribution of $\boldsymbol{\theta}$ has the following closed-form expression:

$$f(\boldsymbol{\theta}|Y_{\text{obs}}) = c_H^{-1}(\mathbf{a}, \mathbf{n}) \cdot \prod_{i=1}^4 \theta_k^{a_i - 1} \cdot L_H(\boldsymbol{\theta}|Y_{\text{obs}}), \quad \boldsymbol{\theta} \in \mathbb{T}_4, \tag{5.2}$$

where the normalizing constant $c_H(\mathbf{a}, \mathbf{n}) = c_H^*(\mathbf{a}, \mathbf{n}) / \Gamma(\sum_{i=1}^4 a_i + n)$ and

$$c_H^*(\mathbf{a}, \mathbf{n}) = \sum_{j_2=0}^{n_2} \sum_{j_3=0}^{n_3} \sum_{j_4=0}^{n_4} \Big\{ \Gamma(a_1 + n_1 + j_2 + j_3 + j_4) \cdot \prod_{\ell=2}^4 \binom{n_\ell}{j_\ell} \Gamma(a_\ell + n_\ell - j_\ell) p_\ell^{j_\ell} \Big\}.$$

The posterior moment of $\boldsymbol{\theta}$ is given by

$$E\Big(\theta_1^{t_1}\theta_2^{t_2}\theta_3^{t_3}\theta_4^{t_4} \Big| Y_{\text{obs}}\Big) = \frac{c_H^*(\mathbf{a}+\mathbf{t}, \mathbf{n})}{c_H^*(\mathbf{a}, \mathbf{n})} \cdot \frac{\Gamma(\sum_{i=1}^4 a_i + n)}{\Gamma(\sum_{i=1}^4 (a_i + t_i) + n)}.$$

Hence, the posterior moments of θ_x, θ_y and δ can be readily expressed as

$$\begin{aligned} E(\theta_x|Y_{\text{obs}}) &= E(\theta_3|Y_{\text{obs}}) + E(\theta_4|Y_{\text{obs}}), \\ E(\theta_x^2|Y_{\text{obs}}) &= E(\theta_3^2|Y_{\text{obs}}) + 2E(\theta_3\theta_4|Y_{\text{obs}}) + E(\theta_4^2|Y_{\text{obs}}), \\ E(\theta_y|Y_{\text{obs}}) &= E(\theta_2|Y_{\text{obs}}) + E(\theta_4|Y_{\text{obs}}), \\ E(\theta_y^2|Y_{\text{obs}}) &= E(\theta_2^2|Y_{\text{obs}}) + 2E(\theta_2\theta_4|Y_{\text{obs}}) + E(\theta_4^2|Y_{\text{obs}}), \\ E(\delta|Y_{\text{obs}}) &= E(\theta_1\theta_2^{-1}\theta_3^{-1}\theta_4|Y_{\text{obs}}), \quad \text{and} \\ E(\delta^2|Y_{\text{obs}}) &= E(\theta_1^2\theta_2^{-2}\theta_3^{-2}\theta_4^2|Y_{\text{obs}}). \end{aligned}$$

5.1.2. Calculation of the posterior mode via the EM algorithm

To derive the posterior mode of $\boldsymbol{\theta}$, we treat the observed frequencies n_2, n_3 and n_4 as incomplete data and the frequencies Z_2, Z_3 and Z_4 as missing data (see Table 4). Let $Z = (Z_2, Z_3, Z_4)^\top$ with $Z_1 = n - Z_2 - Z_3 - Z_4$. Thus, the complete-data posterior distribution and the conditional predictive distribution are given by

$$f(\boldsymbol{\theta}|Y_{\text{obs}}, Z) = \text{Dirichlet}\,(\boldsymbol{\theta}|a_1 + Z_1, \dots, a_4 + Z_4), \tag{5.3}$$

and

$$f(Z|Y_{\text{obs}}, \boldsymbol{\theta}) = \prod_{i=2}^4 \text{Binomial}\,(Z_i|n_i,\ \theta_i/[p_i\theta_1 + \theta_i]), \tag{5.4}$$

respectively. Based on the EM algorithm, the M-step calculates the complete-data posterior mode by

$$\tilde{\theta}_i = \frac{a_i + Z_i - 1}{\sum_{\ell=1}^4 a_\ell + n - 4}, \quad i = 2, 3, 4,\ \tilde{\theta}_1 = 1 - \tilde{\theta}_2 - \tilde{\theta}_3 - \tilde{\theta}_4, \tag{5.5}$$

and the E-step is to replace $\{Z_i\}$ by their conditional expectations

$$E(Z_i|Y_{\text{obs}}, \boldsymbol{\theta}) = \frac{n_i\theta_i}{p_i\theta_1 + \theta_i}, \quad i = 2, 3, 4. \tag{5.6}$$

5.1.3. Generation of posterior samples via the DA algorithm

Based on (5.3) and (5.4), we can use the DA algorithm (Tanner & Wong, 1987) to generate posterior samples of $\boldsymbol{\theta}$. We may choose $\boldsymbol{\theta}_0 = (0.25, \dots, 0.25)^\top$ as the initial value.

5.2 Bayesian inferences under other priors

In the previous section, we consider the Dirichlet distribution as the prior of $\boldsymbol{\theta}$. It is well known that the covariance structure associated with the Dirichlet distribution is completely non-positive. Obviously, those cases that possess, for instance, positive covariance structures cannot be modeled

Table 5. Parameter spaces Θ and Θ_x and their cell probabilities

Categories	$Y=0$	$Y=1$	Marginal	Categories	$Y=0$	$Y=1$	Marginal
$X=0$	θ_1	θ_2	$\theta_1+\theta_2$	$X=0$	$(1-\theta_x)\xi$	$(1-\theta_x)(1-\xi)$	$1-\theta_x$
$X=1$	θ_3	θ_4	$\theta_3+\theta_4$	$X=1$	$\theta_x(1-\eta)$	$\theta_x\eta$	θ_x
Space Θ			1	Space Θ_x			1

by the Dirichlet prior. To explore the essence of the Dirichlet prior, we first transform the original parameter space $\Theta = \{\theta_1, \theta_2, \theta_3\}$ into an orthogonal parameter space, say $\Theta_x = \{\theta_x, \xi, \eta\}$, and present an equivalent prior distribution for $(\theta_x, \xi, \eta)^\top$ when $(\theta_1, \theta_2, \theta_3)^\top$ follows a Dirichlet distribution. Next, we develop three new joint priors for $(\theta_x, \xi, \eta)^\top$ for modeling (i) independence structure with restrictions, (ii) negative correlation structure, and (iii) positive correlation structures. Finally, for the positive correlation structure, we derive the corresponding posterior moments for the parameters of interest via the importance sampling and generate posterior samples via the DA algorithm.

5.2.1. Orthogonal parameter space

Let $\theta_x = \Pr(X=1)$ denote the marginal probability of X, $\xi = \Pr(Y=0|X=0)$ and $\eta = \Pr(Y=1|X=1)$ be the corresponding conditional probabilities. Table 5 illustrates the fundamental relationship between the two parameter spaces $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and $\Theta_x = \{\theta_x, \xi, \eta\}$.

It is noteworthy that the following one-to-one transformation

$$\theta_x = \theta_3 + \theta_4, \quad \xi = \frac{\theta_1}{\theta_1+\theta_2}, \quad \text{and} \quad \eta = \frac{1-\theta_1-\theta_2-\theta_3}{1-\theta_1-\theta_2} \tag{5.7}$$

maps the original parameter space Θ into the orthogonal parameter space (i.e., an unit cube in $\mathbb{R}^3$) Θ_x. The corresponding Jacobian is given by

$$|J| = \left|\frac{\partial(\theta_1, \theta_2, \theta_3)}{\partial(\theta_x, \xi, \eta)}\right| = \theta_x(1-\theta_x).$$

We have the following result.

Theorem 1. *If $(\theta_1, \theta_2, \theta_3)^\top$ follows the Dirichlet prior Dirichlet$(a_1, a_2, a_3; a_4)$ in space Θ, the equivalent prior distribution of $(\theta_x, \xi, \eta)^\top$ in Θ_x is given by*

$$\begin{aligned} &\theta_x \sim \text{Beta}(a_3+a_4, a_1+a_2), \\ &\xi \sim \text{Beta}(a_1, a_2), \quad \text{and} \\ &\eta \sim \text{Beta}(a_4, a_3), \quad \text{or} \quad 1-\eta \sim \text{Beta}(a_3, a_4), \end{aligned} \tag{5.8}$$

where θ_x, ξ and η are mutually independent.

5.2.2. Joint prior for modeling independence with restrictions

Sometimes, prior information on θ_x, ξ and η are available in the form of restrictions. For example, let $X=1$ if a person has annual income being greater than or equal to \$100,000; $=0$ otherwise, and $Y=1$ if a person travels at least once every year; $=0$ otherwise. Thus, we have $\eta = \Pr$(a person travels once every year| the annual income $\geq \$100,000$) and $\xi = \Pr$(a person does not travel at all every year| the annual income $\geq \$100,000$). In general, the possibility of traveling every year is positively related to annual income. Therefore, it is reasonable to impose the following restrictions on η and ξ,

$$\eta \geq 1-\eta, \quad \text{and} \quad \xi \geq 1-\xi, \tag{5.9}$$

i.e., $\eta \geq 0.5$, $\xi \geq 0.5$. Let TBeta$(a, b; L, U)$ denote the truncated beta distribution defined on the interval $[L, U]$. Hence, an alternative joint prior for $(\theta_x, \xi, \eta)^\top$ to (5.8) is

$$\begin{aligned} &\theta_x \sim \text{Beta}(a_3+a_4, a_1+a_2), \\ &\xi \sim \text{TBeta}(a_1, a_2; 0.5, 1), \quad \text{and} \\ &\eta \sim \text{TBeta}(a_4, a_3; 0.5, 1), \end{aligned} \tag{5.10}$$

where θ_x, ξ and η are independent. In other words, the joint prior (5.10) is adequate for modeling the assumption of independence between ξ and η with restrictions (5.9).

5.2.3. Joint prior for modeling negative correlation structure

In some applications, the assumption of independence between ξ and η may not be adequate while the negative correlation structure appears to be more practical. One possible way for modeling negative correlation structure is to consider the following inequality constraint:

$$\xi \geq 1-\eta.$$

Define $\xi^* = 1-\xi$, $\eta^* = 1-\eta$. From this inequality constraint, we obtain

$$\xi^* \geq 0, \qquad \eta^* \geq 0, \quad \text{and} \quad \xi^* + \eta^* \leq 1. \tag{5.11}$$

Naturally, a Dirichlet prior can be assigned to $(\xi^*, \eta^*)^\top$. It is well known that the components of a Dirichlet random vector are negatively correlated, and so are ξ and η.

5.2.4. Joint prior for modeling positive correlation structure

Now, we consider the case that ξ and η are positively correlated. The first problem is to identify an appropriate prior distribution. The second problem is to compute the corresponding posterior moments for the parameters of interest. Here, we propose a positively correlated bivariate-beta distribution as the joint prior of (ξ, η). We then employ the importance sampling to calculate the posterior moments and the DA algorithm to obtain posterior samples.

5.2.4.1. Positively correlated bivariate-beta distribution. A two-dimensional random vector $\mathbf{w} = (w_1, w_2)^\top$ is said to follow a *positively correlated bivariate-beta distribution*, denoted as $\mathbf{w} \sim \text{PCBBeta}(\gamma_1, \gamma_2; a, b)$, if the conditional distributions of w_1 and w_2 given τ are independent, and

$$w_i | \tau \sim \text{Beta}(\gamma_i \tau, \gamma_i(1-\tau)), \quad \gamma_i > 0, \ i = 1, 2,$$

where $\tau \sim \text{Beta}(a, b)$, $a > 0$, $b > 0$ (see, e.g., Albert and Gupta, 1983; 1985). Theorem 2 below gives the joint density of $\mathbf{w}$ and an algorithm for generating the random vector $\mathbf{w}$.

Theorem 2. *If $\mathbf{w} = (w_1, w_2)^\top \sim PCBBeta(\gamma_1, \gamma_2; a, b)$, then (i) the density of $\mathbf{w}$ is*

(5.12)

$$f(\mathbf{w}) = \int_0^1 \frac{\tau^{a-1}(1-\tau)^{b-1}}{B(a,b)} \prod_{i=1}^{2} \frac{w_i^{\gamma_i \tau - 1}(1-w_i)^{\gamma_i(1-\tau)-1}}{B(\gamma_i \tau, \gamma_i(1-\tau))} \, d\tau.$$

(ii) Samples of $\mathbf{w}$ can be generated as follows: First generate a Beta(a, b) random variate τ, and then independently generate w_i from Beta$(\gamma_i \tau, \gamma_i(1-\tau))$ for $i = 1, 2$.

We obtain the correlation coefficient between w_1 and w_2 in the following theorem.

Theorem 3. *If $\mathbf{w} = (w_1, w_2)^\top \sim PCBBeta(\gamma_1, \gamma_2; a, b)$, then the correlation coefficient between w_1 and w_2 is*

$$\rho(w_1, w_2) = \sqrt{\frac{(\gamma_1+1)(\gamma_2+1)}{(\gamma_1+a+b+1)(\gamma_2+a+b+1)}}. \tag{5.13}$$

The proof of Theorem 3 is given in Appendix B. Theorem 3 shows that the PCBBeta distribution can be used to quantify the positive association between two random variables. In particular, if $\gamma_1 = \gamma_2 = \gamma$ and $a = b = 1$, then

$$\rho(w_1, w_2) = \frac{\gamma+1}{\gamma+3} = 1 - \frac{2}{\gamma+3},$$

which is an increasing function of γ. The larger the prespecified value of γ, the stronger the association between w_1 and w_2.

5.2.4.2. Computing posterior moments via importance sampling. Substituting (5.7) into (5.1), we can rewrite the likelihood function as

$$L_H(\theta_x, \xi, \eta | Y_{\text{obs}}) = (1-\theta_x)^{n_1+n_2} \cdot h(\theta_x, \xi, \eta), \tag{5.14}$$

where

$$h(\theta_x, \xi, \eta) = \xi^{n_1}[p_2\xi + (1-\xi)]^{n_2}[p_3(1-\theta_x)\xi + \theta_x(1-\eta)]^{n_3} \cdot [p_4(1-\theta_x)\xi + \theta_x\eta]^{n_4}.$$

Motivated by (5.8) and (5.12), we may consider the following distributions as the joint prior for $(\theta_x, \xi, \eta)^\top$ if ξ and η are believed to be positively correlated

(5.15) $\theta_x \sim \text{Beta}(\alpha, \beta)$, $\quad (\xi, \eta)^\top \sim \text{PCBBeta}(\gamma, \gamma; 1, 1)$ and they are independent.

The resultant posterior distribution can be shown to be

$$f(\theta_x, \xi, \eta | \text{data}) = c_H^{-1}(\alpha, \beta, \gamma, \mathbf{n}) \cdot \theta_x^{\alpha-1}(1-\theta_x)^{n_1+n_2+\beta-1} \cdot f(\xi, \eta) \cdot h(\theta_x, \xi, \eta).$$

By importance sampling, we obtain

$$c_H(\alpha, \beta, \gamma, \mathbf{n}) \doteq \frac{B(\alpha, n_1+n_2+\beta)}{L} \sum_{\ell=1}^{L} h(\theta_x^{(\ell)}, \xi^{(\ell)}, \eta^{(\ell)}),$$

where $\{\theta_x^{(\ell)}\}_{\ell=1}^L$ are a sample of size L from Beta$(\alpha, n_1 + n_2 + \beta)$ and $\{\xi^{(\ell)}, \eta^{(\ell)}\}_{\ell=1}^L$ are a sample of size L from PCBBeta$(\gamma, \gamma; 1, 1)$ via Theorem 2(ii).

The parameters of interest can be expressed as

$$\theta_x = \theta_x, \qquad \theta_y = (1-\theta_x)(1-\xi) + \theta_x\eta, \quad \text{and} \quad \delta = \frac{\xi\eta}{(1-\xi)(1-\eta)}.$$

Therefore, the posterior moments of θ_x can be readily calculated via the importance sampling. However, calculations of posterior moments such as $E(\theta_y^t|\text{data})$ and $E(\delta^t|\text{data})$ require the evaluation of the expression $((1-\theta_x)(1-\xi)+\theta_x\eta)^t \cdot h(\theta_x, \xi, \eta)$ L times at $(\theta_x^{(\ell)}, \xi^{(\ell)}, \eta^{(\ell)})$ for $\ell = 1, \ldots, L$. To this end, the DA algorithm can be adopted for this purpose and we discuss the algorithm as follows.

5.2.4.3. Generating posterior samples via the DA algorithm. The likelihood function (5.14) and the prior assumption (5.15) can be re-formulated in terms of a hierarchical model with three stages. In the first stage, we can augment the observed data Y_{obs} with three latent variates $\{Z_2, Z_3, Z_4\} = Y_{\text{mis}}$ so that the complete-data likelihood is given by

$$L_H(\theta_x, \xi, \eta | Y_{\text{obs}}, Y_{\text{mis}}) = \theta_x^{Z_3+Z_4}(1-\theta_x)^{n-Z_3-Z_4} \cdot \xi^{Z_1}(1-\xi)^{Z_2} \cdot \eta^{Z_4}(1-\eta)^{Z_3},$$

where $Z_1 = n - Z_2 - Z_3 - Z_4$. Similar to (5.4), the conditional predictive distribution is

(5.16)

$$\begin{aligned} f(Y_{\text{mis}} | Y_{\text{obs}}, \theta_x, \xi, \eta) &= \text{Binomial}\left(Z_2 | n_2, \ (1-\xi)/[p_2\xi + 1 - \xi]\right) \\ &\quad \cdot \text{Binomial}\left(Z_3 | n_3, \ \theta_x(1-\eta)/[p_3(1-\theta_x)\xi + \theta_x(1-\eta)]\right) \\ &\quad \cdot \text{Binomial}\left(Z_4 | n_4, \ \theta_x\eta/[p_4(1-\theta_x)\xi + \theta_x\eta]\right). \end{aligned}$$

In the second stage, given a hyperparameter τ, the joint prior is a product of independent beta distributions:

$$\begin{aligned} f(\theta_x, \xi, \eta | \tau) &= f(\theta_x) \cdot f(\xi|\tau) \cdot f(\eta|\tau) \\ &= \text{Beta}(\theta_x|\alpha, \beta) \cdot \text{Beta}(\xi|\gamma\tau, \gamma(1-\tau)) \\ &\quad \cdot \text{Beta}(\eta|\gamma\tau, \gamma(1-\tau)). \end{aligned}$$

In the third stage, we assign τ an uniform prior, namely, $f(\tau) = I_{(0,1)}(\tau)$, where $I_D(\cdot)$ represents the indicator function of the set D. We write the joint distribution of the

Table 6. Sensitive sexual behavior data

Item	Self-report ($N = 102$)			UCT ($N = 244$)		
($N = 346$)	Rate	Yes	No	Rate	Yes	No
1. Sex without a condom	0.59	60	42	0.70	171	73
2. Drank until intoxication	0.77	79	23	0.70	171	73
3. Sex after drinking	0.48	49	53	0.49	120	124
4. Sex without a condom after drinking	0.36	37	65	0.65	159	85
5. Had sex	0.74	75	27	0.84	205	39

Source: LaBrie & Earleywine (2000).

complete-data and parameters as $L_H(\theta_x, \xi, \eta|Y_{\text{obs}}, Y_{\text{mis}}) \cdot f(\theta_x, \xi, \eta|\tau) \cdot f(\tau)$. Thus, we have

$$(5.17)\quad f(\theta_x, \xi, \eta|Y_{\text{obs}}, Y_{\text{mis}}, \tau) = \text{Beta}(\theta_x|Z_3 + Z_4 + \alpha, n - Z_3 - Z_4 + \beta) \cdot \text{Beta}(\xi|Z_1 + \gamma\tau,\ Z_2 + \gamma[1-\tau]) \cdot \text{Beta}(\eta|Z_4 + \gamma\tau,\ Z_3 + \gamma[1-\tau])$$

$$(5.18)\quad f(\tau|Y_{\text{obs}}, \xi, \eta) \propto \frac{(\xi\eta)^{\gamma\tau}[(1-\xi)(1-\eta)]^{\gamma(1-\tau)}}{[B(\gamma\tau, \gamma(1-\tau))]^2}, \quad \tau \in (0,1).$$

Sampling from (5.16) and (5.17) is pretty straightforward. Note that (5.18) is an un-normalized one-dimensional density function defined on $(0,1)$. The grid points method (see, e.g., Gelmen et al., 1995, p. 302) can be used to generate random samples from this distribution. The implementation of the Gibbs sampling and the calculation of arbitrary expectations of interest were presented thoroughly in Gelfand & Smith (1990) and Arnold (1993), and are hence omitted here.

6. NUMERICAL ILLUSTRATIONS

6.1 Sensitive sexual behavior data

Most studies of sexual behaviors employ conventional self-report surveys. Researchers have long criticized the validity of these self-reports since sexual behavior is often highly private. An alternative approach, called *unmatched-count technique* (UCT), provides participants a chance to answer sensitive items without directly admit to the sensitive behavior (Wimbush & Dalton, 1997). In the UCT method, half of the participants will receive a set of, for instance, five questions (in which all questions are non-sensitive) while the other half will receive a set of six questions (in which one of them is the sensitive question). It should be noted that the five non-sensitive questions are common to all respondents. At the end of the survey, respondents simply indicate the number of statements that are true for them. The base rate estimate for the sensitive item is determined through random assignment of participants and comparisons between the two samples. All samples were obtained via simple random sampling. The main feature of the UCT is that participants do not respond directly to the sensitive item(s).

LaBrie & Earleywine (2000) used an anonymous self-report questionnaire and the UCT to estimate the base rates for some sexual risk behaviors (e.g., having sex without a condom and having sex without a condom after drinking). Three hundred forty-six college students were randomly divided into three groups. Group 1 (102 subjects) received a true/false conventional self-report survey. Groups 2 (122 subjects) and 3 (122 subjects) were UCT protocol groups, with Group 2 receiving Form A and Group 3 receiving Form B (see Appendix B in LaBrie & Earleywine, 2000, for more details). Their findings are reported in Table 6. For example, 36% of the respondents receiving the conventional survey endorsed having sex without a condom after consuming alcohol while the UCT protocol revealed a base rate estimate of 65% for the same behavior. Thus, the anonymous self-report questionnaire revealed only half the percentage of persons engaging in risky sexual behavior after drinking reported by the UCT protocol.

To illustrate the proposed methods in Section 3, for the third sensitive item, we combine the numbers of "Yes" and "No" with those for the two survey methods, resulting in $n = 346$, $n_{\text{yes}} = 49 + 120 = 169$ and $n_{\text{no}} = 53 + 124 = 177$. In the triangular model, we further let $\pi = \Pr(Y = 1) = \Pr(\text{having sex after drinking})$ and $p = \Pr(W = 1) = 0.5$. For the ideal situation (i.e., no sampling errors), the observed counts in the triangle would be $s = n_{\text{no}}/2 + n_{\text{yes}} \approx 258$. Therefore, we obtain the observed data $Y_{\text{obs}} = \{n, s, n-s\} = \{346, 258, 88\}$. Using (3.3), we have $E(\pi|Y_{\text{obs}}) = 0.488506$ and $E(\pi^2|Y_{\text{obs}}) = 0.240819$. Thus, the 95% Bayesian *credible interval* (CI) for π based on normality approximation is given by $[0.396960, 0.580052]$.

Using $\pi^{(0)} = 0.5$ as an initial value, the EM algorithm (3.6) and (3.7) converges in 28 iterations. The resultant posterior mode of π is $\tilde{\pi}_T = 0.49133$, which is very close to the Bayesian mean $E(\pi|Y_{\text{obs}})$. Using the exact IBF sampling described in Section 3.3, we generate $L = 20{,}000$ i.i.d. posterior samples from $f(\pi|Y_{\text{obs}})$. The histogram based on these samples is plotted in Figure 2(b), which shows that the exact IBF sampling can recover the density completely. The corresponding posterior mean, standard error and 95% Bayesian CI for π are 0.488805, 0.0468918 and [0.395190, 0.577961].

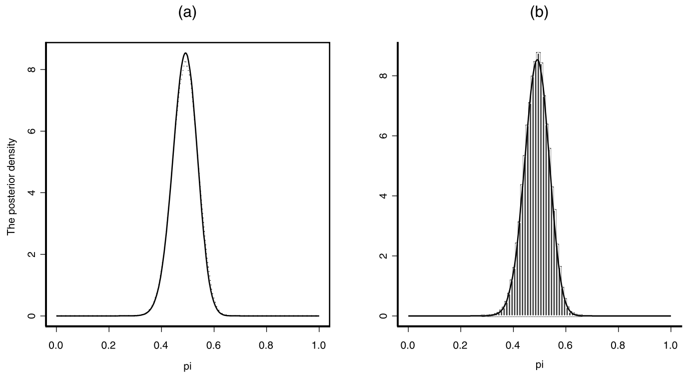

Figure 2. Posterior distribution of $\pi = \Pr($having sex after drinking$)$ under the uniform prior (i.e., $a = b = 1$) and $p = \Pr(W = 1) = 0.5$ for the non-randomized triangular model for $n = 346$ and $s = 258$. (a) The comparison between the posterior distribution (solid curve) exactly given by (3.1) with the dotted curve estimated by a kernel density smoother based on i.i.d. posterior samples generated via the exact IBF sampling. (b) The histogram based on i.i.d. posterior samples generated via the exact IBF sampling.

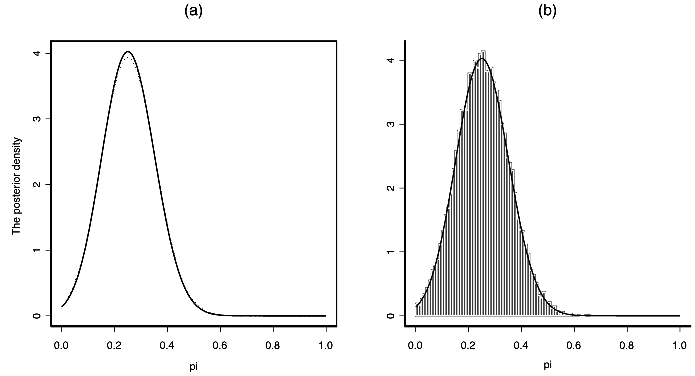

Figure 3. Posterior distribution of $\pi = \Pr($having induced abortion$)$ under the uniform prior (i.e., $a = b = 1$) and $p = \Pr(W = 1) = 0.3$ for the non-randomized cross model for $n = 150$ and $r = 90$. (a) The comparison between the posterior distribution (solid curve) exactly given by (4.1) with the dotted curve estimated by a kernel density smoother based on i.i.d. posterior samples generated via the exact IBF sampling. (b) The histogram based on i.i.d. posterior samples generated via the exact IBF sampling.

6.2 Induced abortion data

Liu & Chow (1976) considered an induced abortion study in Taichung City and Taoyuan County, Taiwan (see also Winkler and Franklin, 1979). They adopted the multiple-trial version of the Warner model to increase the efficiency of estimation. Since the present paper only discusses the single-trial Warner model with the crosswise model as its non-randomized version, we simply use the data from the first trial of each respondent. The target population of interest in this study is those married women of age 20 to 44 in the South District of Taichung City, Taiwan. The investigators would like to estimate the incidence rate of induced abortions in the target population. With $p = 0.3$, the survey yielded 90 "Yes" answers (i.e., $r = \sum_{i=1}^{n} y_i^{\rm HW} = 90$ in (4.1)) and 60 "No" answers (i.e., $n = 150$). Using likelihood-based method, the proportion of married women of child-bearing age who have had induced abortion is estimated to be $\hat{\pi}_W = 0.25$ with estimated variance being $\widehat{\rm Var}(\hat{\pi}_W) = 0.01$ (Migon & Tachibana, 1997, p. 406). The resultant 95% CI of π is $[0.25 - 1.96\sqrt{0.01},\ 0.25 + 1.96\sqrt{0.01}\,] = [0.054, 0.446]$.

To illustrate the proposed methods in Section 4, we consider the uniform prior (i.e., $a = b = 1$). Note that $p = 0.3$ and the observed data $Y_{\rm obs} = \{n, r, n-r\} = \{150, 90, 60\}$. Using (4.2), we obtain $E(\pi|Y_{\rm obs}) = 0.2544$ and $E(\pi^2|Y_{\rm obs}) = 0.0742$. Thus, ${\rm Var}(\pi|Y_{\rm obs}) = 0.0095$ so that the 95% Bayesian CI for π based on normality approximation is $[0.0632, 0.4457]$.

Using $\pi^{(0)} = 0.5$ as an initial value, the EM algorithm (4.5) and (4.6) converges in 96 iterations. The posterior mode of π is $\tilde{\pi}_W = 0.25$, which is same as the MLE $\hat{\pi}_W$. Using the exact IBF sampling described in Section 4.3, we generate $L = 20,000$ i.i.d. posterior samples from $f(\pi|Y_{\rm obs})$. The histogram based on these samples is plotted in Figure 3(b), which shows that the exact IBF sampling recovers the density completely. The corresponding posterior mean, standard error and 95% Bayesian CI for π are 0.2546, 0.0973 and [0.0680, 0.4490].

6.3 HIV data

Strauss et al. (2001) reported an HIV data set which examined the relationship between self-reported HIV status and history of sex exchange for drugs and money. All participants were drug dependent women offenders who were mandated to treatment through the criminal justice system of New York City. The data were collected as part of an evaluation study of four drug treatment programs, respectively classified as prison-based, jailed-based, community-based residential and community-based outpatient. The data reflected baseline responses from 325 clients interviewed at the four treatment programs from May, 1995 through December, 1996. Notice that there are incomplete data for 83 subjects. Table 7 gives the cross-classification of history of sex exchange (no or yes, denoted by $X = 0$ or $X = 1$) and HIV status (negative or positive, denoted by $Y = 0$ or $Y = 1$) as reported by the women. The objective is to examine if association exists between sex exchange and HIV status. Obviously, both questions (i.e., sex history and HIV status) are highly sensitive questions to respondents.

To illustrate the proposed methods in Section 5.1, we let $W = i$ if the respondent was born in the k-th quarter, and it

Table 7. HIV data from Strauss et al. (2001)

History of sex exchange	HIV status		
	$Y = 0$ (HIV−)	$Y = 1$ (HIV+)	missing
$X = 0$ (no)	108 (m_1, θ_1)	18 (m_2, θ_2)	44
$X = 1$ (yes)	93 (m_3, θ_3)	23 (m_4, θ_4)	39

Note: X denotes history of sex exchange and Y denotes HIV status.

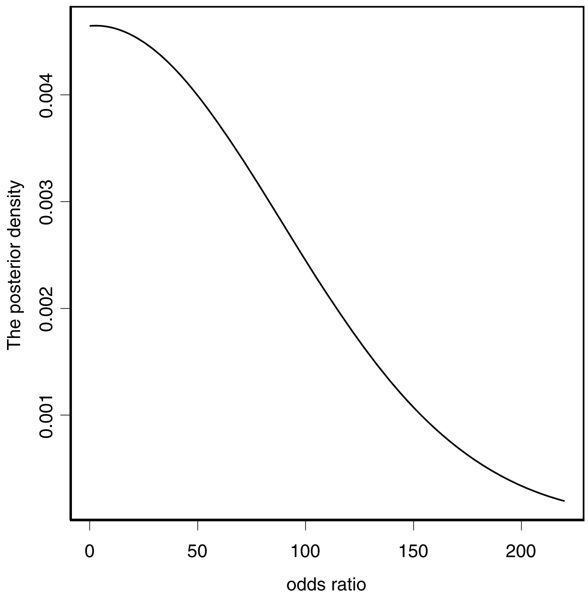

Figure 4. The posterior density of the odds ratio δ estimated by a kernel density smoother based on the last 20,000 posterior samples generated by the DA algorithm.

is thus reasonable to assume that $p_i = \Pr\{W = i\} = 0.25$ for $i = 1, \ldots, 4$, and W is independent of the two sensitive questions. For the ideal situation (i.e., no sampling errors), the observed counts would be $n_1 = m_1/4 = 27$, $n_2 = 27 + m_2 = 45$, $n_3 = 27 + m_3 = 120$ and $n_4 = 27 + m_4 = 50$ if the missing data in Table 7 are ignored. To consider the situation with sampling errors, we first generate 50 i.i.d. samples from Multinomial$(108; (0.25, \ldots, 0.25)^\top)$, then calculate the average of these counts, and finally yield $(n'_1, \ldots, n'_4) = (28, 26, 26, 28)^\top$. Therefore, we obtain the observed frequencies $Y_{\text{obs}} = \{n; n_1, \ldots, n_4\} = \{242, 28, 44, 119, 51\}$.

Using $\theta^{(0)} = (0.25, 0.25, 0.25, 0.25)^\top$ as initial values, the EM algorithm (5.5) and (5.6) converges in 100 iterations. The posterior modes of $\boldsymbol{\theta}$ and odds ratio δ are listed in the second column of Table 8. Based on (5.3) and (5.4), we employ the DA algorithm to generate 40,000 posterior samples and only use the second half of the samples. The Bayes estimates of $\boldsymbol{\theta}$ and δ are given in Table 8. Since the Bayes CIs include the value of 1, we have reason to believe that there is no association between sex exchange and HIV status. Figure 4 shows the posterior density of the odds ratio δ estimated by a kernel density smoother based on the last 20,000 posterior samples generated by the DA algorithm.

7. DISCUSSION

Yu et al. (2008) and Tian et al. (2007) studied the survey designs for the triangular, crosswise and hidden sensitivity models, respectively. They investigated these models from a frequentist perspective. In this article, we on the other hand focus on the analysis of these models in the Bayesian framework. The Bayesian framework provides a natural way to study these models when only partial information are available. Furthermore, the Bayesian approach is particularly appealing when sample information are relatively limited.

For the HS model, the resultant MLE $\hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_4)^\top$ with explicit form is possibly invalid, i.e., it does not satisfy: $\hat{\theta}_i \geq 0$, $i = 1, \ldots, 4$, $\sum_{i=1}^4 \hat{\theta}_i = 1$. However, we do not have this issue in the Bayesian framework.

In practice, the selection of prior is very important to researchers. For the HS model, the choice of a Dirichlet prior is equivalent to a product of three independent beta distributions (see (5.8)). In some cases, the assumption of independence between ξ and η may not be appropriate. For modeling independence with restrictions and negative correlation structure, we suggest two distinct priors (see (5.10) and (5.11)). By implementing a very simple sampling algorithm, the PCBBeta distribution becomes desirable for modeling positive correlation structure. The control of similarity can be fine tuned by selecting appropriate value of the parameter γ. Based on the PCBBeta prior, we employ the importance sampling to calculate the corresponding posterior moments.

The DA algorithm is a special Gibbs sampler. In order to implement the DA algorithm, we first formulate the original problem into a hierarchical models with three stages. We then derived the full conditional distributions (5.16)–(5.18) with simple sampling methods.

Table 8. Posterior modes and estimates of parameters for HIV data

Parameters	Posterior mode	Bayes mean	Bayes std	95% Bayes CI
θ_1	0.4628	0.4635	0.0757	[0.3208, 0.6172]
θ_2	0.0661	0.0680	0.0317	[0.0105, 0.1339]
θ_3	0.3760	0.3723	0.0426	[0.2877, 0.4538]
θ_4	0.0950	0.0960	0.0346	[0.0300, 0.1648]
$\delta = \theta_1\theta_4/\theta_2\theta_3$	1.7692	3.6939	31.431	[0.5733, 11.932]

APPENDIX A. THE EXACT IBF SAMPLING

Suppose that both the complete-data posterior distribution $f(\pi|Y_{\rm obs}, Z)$ and the conditional predictive distribution $f(Z|Y_{\rm obs}, \pi)$ are available. The fundamental conditional sampling principle states that: If we could obtain independent samples $\{Z^{(\ell)}\}_{\ell=1}^L$ from $f(Z|Y_{\rm obs})$ and generate $\pi^{(\ell)} \sim f(\pi|Y_{\rm obs}, Z^{(\ell)})$ for $\ell = 1,\ldots,L$, then $\{\pi^{(\ell)}\}_1^L$ are i.i.d. samples from the observed posterior distribution $f(\pi|Y_{\rm obs})$. In other words, the key issue is to generate independent samples from $f(Z|Y_{\rm obs})$.

Let $\mathcal{S}_{(\pi|Y_{\rm obs})}$ and $\mathcal{S}_{(Z|Y_{\rm obs})}$ denote the conditional supports of $\pi|Y_{\rm obs}$ and $Z|Y_{\rm obs}$, respectively. The sampling-wise IBF states that (Tan et al., 2003)

(A.1)

$$f(Z|Y_{\rm obs}) \propto \frac{f(Z|Y_{\rm obs}, \pi_0)}{f(\pi_0|Y_{\rm obs}, Z)}, \quad \text{for any arbitrary } \pi_0 \in \mathcal{S}(\pi|Y_{\rm obs}) \text{ and all } Z \in \mathcal{S}(Z|Y_{\rm obs}).$$

When Z is a discrete random variable/vector taking finite values on the domain, we denote the conditional support of $Z|(Y_{\rm obs}, \pi)$ by $\mathcal{S}_{(Z|Y_{\rm obs},\pi)} = \{z_1,\ldots,z_K\}$. Since $f(Z|Y_{\rm obs}, \pi)$ is available, we can first directly identify $\{z_k\}_1^K$ from the model specification and all $\{z_k\}_1^K$ become known. Noting that $\{z_k\}_1^K$ generally do not depend on π, we have $\mathcal{S}_{(Z|Y_{\rm obs})} = \mathcal{S}_{(Z|Y_{\rm obs},\pi)} = \{z_1,\ldots,z_K\}$. Due to the discreteness of Z, the notation $f(z_k|Y_{\rm obs})$ will be used to denote the probability mass function, i.e., $f(z_k|Y_{\rm obs}) = \Pr\{Z = z_k|Y_{\rm obs}\}$. Therefore, it suffices to find $\omega_k = f(z_k|Y_{\rm obs})$ for $k = 1,\ldots,K$. For any $\pi_0 \in \mathcal{S}(\pi|Y_{\rm obs})$, let

(A.2)

$$q_k(\pi_0) = \Pr\{Z = z_k|Y_{\rm obs}, \pi_0\}/f(\pi_0|Y_{\rm obs}, z_k), \quad k = 1,\ldots,K.$$

From the sampling-wise IBF (A.1), we immediately obtain

$$\text{(A.3)} \qquad \omega_k = q_k(\pi_0)/\Sigma_{k'=1}^K q_{k'}(\pi_0), \quad k = 1,\ldots,K$$

and $\{\omega_k\}_1^K$ are independent of π_0. Thus, it is easy to sample from $f(Z|Y_{\rm obs})$ since it is a discrete distribution with probability ω_k on z_k for $k = 1,\ldots,K$. We summarize the algorithm as follows (Tian, Tan & Ng, 2007).

THE EXACT IBF SAMPLING:

(i) Identify $\mathcal{S}_{(Z|Y_{\rm obs})} = \mathcal{S}_{(Z|Y_{\rm obs},\pi)} = \{z_1,\ldots,z_K\}$ from $f(Z|Y_{\rm obs}, \pi)$ and calculate $\{\omega_k\}_1^K$ according to (A.3) and (A.2);

(ii) Generate i.i.d. samples $\{Z^{(\ell)}\}_{\ell=1}^L$ of Z from the probability mass function $f(Z|Y_{\rm obs})$ with probabilities $\{\omega_k\}_1^K$ on $\{z_k\}_1^K$;

(iii) Generate $\pi^{(\ell)} \sim f(\pi|Y_{\rm obs}, Z^{(\ell)})$ for $\ell = 1,\ldots,L$, then $\{\pi^{(\ell)}\}_1^L$ are i.i.d. samples from the observed posterior distribution $f(\pi|Y_{\rm obs})$.

APPENDIX B. THE PROOF OF THEOREM 3

Let μ and σ^2 denote the mean and variance of τ, respectively. We have $\mu = a/(a+b)$ and $\sigma^2 = ab/\{(a+b+1)(a+b)^2\}$. Using the rule of conditional expectation, we obtain

$$\begin{aligned} E(w_i) &= E\{E(w_i|\tau)\} = E(\tau) = \mu, \quad i = 1,2, \\ E(w_i^2) &= E\{E(w_i^2|\tau)\} = E\Big(\frac{\gamma_i\tau^2+\tau}{\gamma_i+1}\Big) \\ &= \frac{\gamma_i(\sigma^2+\mu^2)+\mu}{\gamma_i+1}, \quad i = 1,2, \\ E(w_1w_2) &= E\{E(w_1w_2|\tau)\} = E(\tau^2) = \sigma^2+\mu^2, \\ \mathrm{Var}(w_i) &= \frac{\gamma_i\sigma^2+\mu-\mu^2}{\gamma_i+1}, \quad i = 1,2. \end{aligned}$$

Hence,

$$\begin{aligned} \rho(w_1, w_2) &= \frac{E(w_1w_2) - E(w_1)E(w_2)}{\sqrt{\mathrm{Var}(w_1)\cdot\mathrm{Var}(w_2)}} \\ &= \sqrt{\frac{(\gamma_1+1)(\gamma_2+1)}{(\gamma_1+(\mu-\mu^2)/\sigma^2)(\gamma_2+(\mu-\mu^2)/\sigma^2)}}. \end{aligned}$$

Noting that $(\mu-\mu^2)/\sigma^2 = a+b+1$, we obtain (5.13) immediately. □

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