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A new quantile function based model for modeling price behaviors in financial markets

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This paper uses a class of quantile functions to develop a new time series model for studying financial price behaviors through the tail properties of the price instead of the volatilities (variances) of the price. The model takes the updated information into account and characterizes the price behaviors using a tail order measure which helps forecast how volatile the prices will be, and a tail balance measure that helps estimate whether an investment tends to gain or tends to lose. The model parameters can be estimated using the method of maximum likelihood, and two real data sets are analyzed to show the potential usefulness of our proposed model.

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1. INTRODUCTION

Price behaviors of securities have long received attention from researchers and practitioners, and forecasting future prices using financial time series models is of importance in the fields of risk management and portfolio selection. Researchers have made significant efforts to characterize price behaviors of various securities in order to provide valuable advice for investors to make investment and financial decisions. Recently, generalized autoregressive conditionally heteroskedastic (GARCH) process based models have been well accepted for financial modeling.

The literature on GARCH models roots in the seminal work [4] on autoregressive conditionally heteroskedastic models and those developed in [1, 10]. Studies along this line have focused on modeling the effects of stochastic volatility in financial data. Reference [7] summarizes that GARCH models provide a reasonably good fit to real financial data with a relatively small number of parameters.

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In price behavior forecasting, however, some of the commonly used assumptions on the historical price behaviors may be inadequate. For GARCH models this includes the stationarity assumption ([8]) and the assumptions about the distributions of financial time series observations ([7]). In practice, price behaviors and the performance of financial securities show features such as volatility clustering, fat tails and changing tail orders empirically ([5, 9, 3, 2]), and the distributions in some parametric GARCH models cannot fit the tail behaviors very well over time ([3]). On the other hand, the nonparametric GARCH models are rather time consuming to implement, and the trade-off between forecasting accuracy and computational efficiency related to model complexity is a well known issue and a major concern in the literature.

This paper presents a novel approach to model financial time series. Instead of modeling volatilities (variances) as in GARCH models, we propose to model the tail properties of the financial time series using a class of quantile functions that have been shown to be capable of representing real price behaviors well ([6, 3]). There are several features in our approach. First, our approach offers an alternative way to understand price behaviors. While the usual volatility approach uses highs and lows and the switching among the highs and the lows in the observations to help people read the price behaviors, our approach uses a tail order measure to convey the volatility information in terms of fat tail (more volatile) and thin tail (less volatile) in distribution, and uses a balance measure to describe the degree of tendency for the average prices to move up or move down. For investors, our approach provides useful information that the volatility approach does not provide or does not provide directly. Second, our approach is straightforward to implement and easy to connect to related modeling efforts because of the availability of closed form quantile, density and distribution functions. Third, our approach seems to be robust and stable for financial time series from different financial markets.

The rest of the paper is organized as below. Section 2 introduces the class of quantile functions to be used. Section 3 develops our quantile function based time series model and discusses the estimation of the model parameters. Section 4 illustrates the use of our proposed model through modeling two real data sets, and Section 5 draws some conclusions and offers a discussion on future research questions.

2. THE QUANTILE FUNCTIONS

To find alternative characterization of price behaviors of financial securities, two families of distributions were introduced in [6] through two classes of quantile functions. In this paper we make use of the Class I quantile functions.

A generic member $q(y;\alpha,\beta,\delta,\mu)$ of the Class I quantile functions is defined by

$$q(y;\alpha,\beta,\delta,\mu) = \delta^{\frac{1}{\alpha}} \left\{ \log \frac{y^\beta}{1-y^\beta} \right\}^{(\frac{1}{\alpha})} + \mu, \tag{1}$$

where $y \in (0,1)$ and the parameters α, β, δ and μ satisfy $\alpha \in (0,\infty)$, $\beta \in (0,\infty)$, $\delta \in (0,\infty)$ and $\mu \in (-\infty,\infty)$, and the superscript '(α)' represents the operation

$$x^{(\alpha)} = \begin{cases} x^\alpha & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -(-x)^\alpha & \text{if } x < 0. \end{cases}$$

As discussed in [6] (page 78), μ is the location parameter and δ is the scale parameter. As to the other two parameters α and β, α controls the tail order (how fat or thin the tail is) while β serves as a tail balance adjuster. If we think of the returns or log returns of the prices of a stock that follow the probability law induced from $q(y;\alpha,\beta,\delta,\mu)$, then the tail order α describes the volatility of the price movement because a smaller value of α leads to a fatter tail in the distribution or a bigger probability of getting values far from the center, which means more volatile. The tail balance adjuster β, on the other hand, indicates the probability of making profit relative to that of losing. When $\beta = 1$ which gives a balanced distribution, the probabilities of making and losing money are equal. If $\beta < 1$, the right tail is fatter than the left tail, which means that it is more likely to make a profit. If $\beta > 1$, the left tail is fatter than the right tail, which means that it is more likely to lose.

For a given quantile function $q(y;\alpha,\beta,\delta,\mu)$, the corresponding distribution function $F(x;\alpha,\beta,\delta,\mu)$ and density

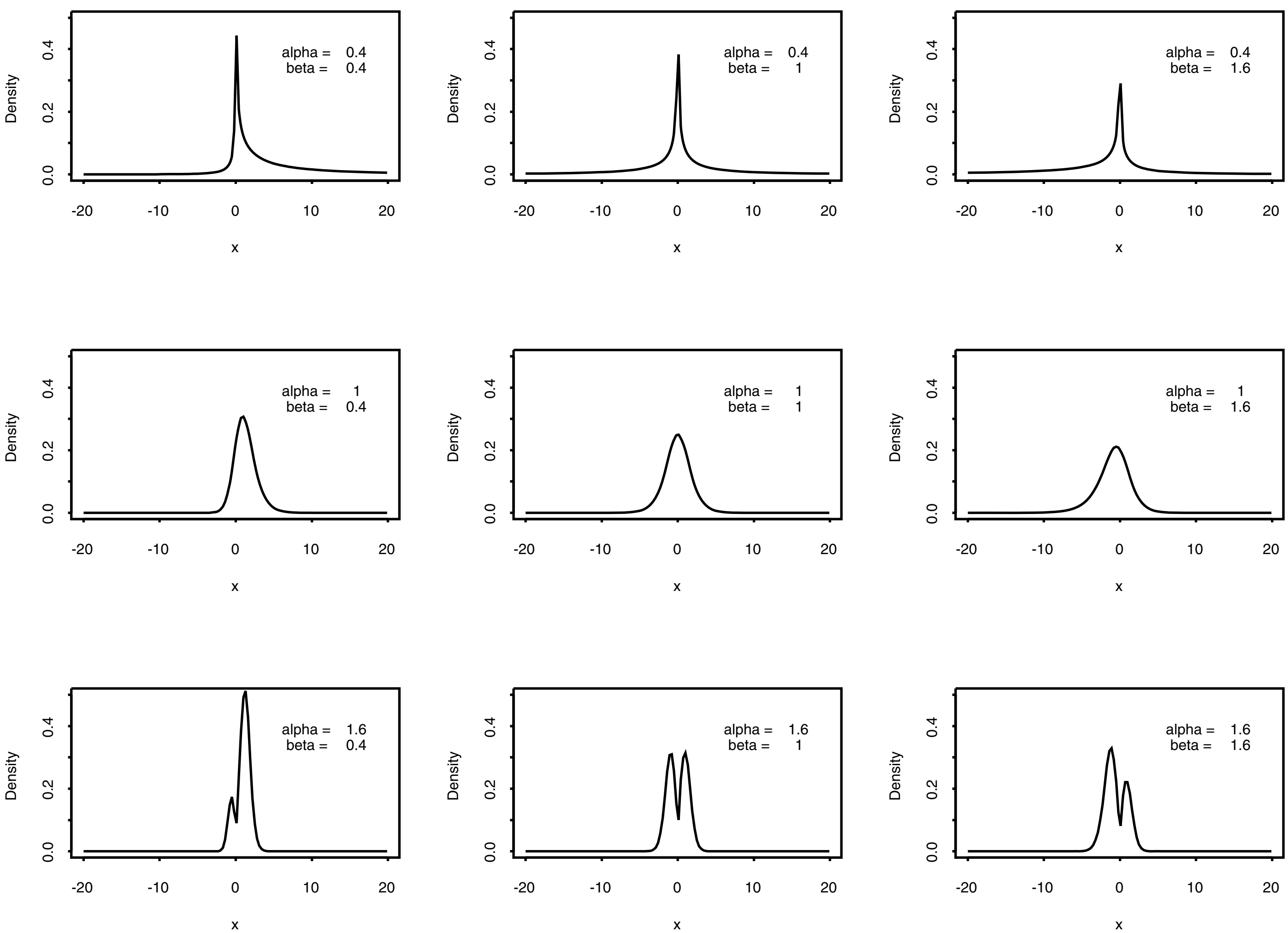

Figure 1. Density functions.

function $p(x;\alpha,\beta,\delta,\mu)$ also have closed forms and are given by

$$F(x;\alpha,\beta,\delta,\mu) = \left\{\frac{1}{1+\mathrm{e}^{-\frac{1}{\delta}(x-\mu)^{(\alpha)}}}\right\}^{\frac{1}{\beta}},$$

$$p(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\cdot\frac{(x-\mu)^{(\alpha)}}{(x-\mu)}e^{-\frac{1}{\delta}(x-\mu)^{(\alpha)}}}{\delta\beta\cdot(1+e^{-\frac{1}{\delta}(x-\mu)^{(\alpha)}})^{1+\frac{1}{\beta}}},$$

for $x\in(-\infty,+\infty)$ but $x\neq\mu$. These closed-form formulae are good to have when it comes to estimate the model parameters and use the model in financial applications. Figure 1 displays the various possible shapes of the density function ($\delta=1$ and $\mu=0$).

3. A QUANTILE FUNCTION BASED MODEL

GARCH models assume that the returns or the log-returns X_t satisfy

$$X_t = \sigma_t Z_t,$$

where $\{\sigma_t\}$ is the volatility process, $\{Z_t\}$ is the noise process consisting of i.i.d. random variables with zero mean and unit variance, and σ_t and Z_t are independent. The focus is on σ_t, which is modeled as a function of some past X_t values and past σ_t values.

We propose to model tail properties of the return or log-return distributions and define our quantile function based model for the returns or the log-returns X_t to be

$$X_t = \delta_t^{\frac{1}{\alpha_t}}\left\{\log\frac{U_t^{\beta_t}}{1-U_t^{\beta_t}}\right\}^{(\frac{1}{\alpha_t})} + \mu_t, \tag{2}$$

where $\{U_t\}$ are i.i.d. random variables over interval (0, 1). In principle we can allow each of α_t, β_t, δ_t and μ_t to depend on past X_t values and past values of itself. However, we will focus on the following special case in this paper, namely,

$$X_t = \delta^{\frac{1}{\alpha_t}}\left\{\log\frac{U_t^{\beta_t}}{1-U_t^{\beta_t}}\right\}^{(\frac{1}{\alpha_t})} + \mu, \tag{3}$$

$$\alpha_t = \sum_{i=1}^{p} a_i(\log(2+X_{t-i}^2))^{-1} + \sum_{j=1}^{q} b_j\alpha_{t-j}, \tag{4}$$

$$\beta_t = \sum_{l=1}^{r} c_l\left(\log(1+\mathrm{e}^{X_{t-l}^{(\alpha_{t-l})}}) - X_{t-l}^{(\alpha_{t-l})}\right) + \sum_{m=1}^{s} d_m\beta_{t-m}, \tag{5}$$

where p, q, r and s are non-negative integers, and δ, μ, $a_1,\ldots,a_p$, $b_1,\ldots,b_q$, $c_1,\ldots,c_r$ and $d_1,\ldots,d_s$ are the unknown model parameters. The two profiles created by the two tail property measures α_t and β_t convey useful information for investors. The tail order measure α_t helps investors tell whether the price behaviors will be more volatile or relatively quiet, while the tail balance measure β_t helps investors tell whether an investment tends to gain or tends to lose. The function $(\log(2+x^2))^{-1}$ used in defining α_t is decreasing as $|x|$ increases in $(0,+\infty)$ and the function $\log(1+\mathrm{e}^{x^{(\alpha)}}) - x^{(\alpha)}$ used in defining β_t is decreasing as x increases in $(-\infty,+\infty)$. These functions are chosen in this way to implement the discussion in Section 2 and keep the α_t and β_t profiles positive.

To estimate the parameters for model (3), we employ the maximum likelihood method. Let Θ denote the parameter space for model (3) (p, q, r and s are fixed) and let θ_0 be the unknown parameter under which $X=(X_1,\ldots,X_n)$ are generated. The log likelihood function for $\theta\in\Theta$, conditioned on $X_t=\bar{X}$ (the sample mean), $\alpha_t=0.5$, and $\beta_t=1$, for $t=0,-1,-2,\ldots$ (finite and no more than $-\max(p,q,r,s)$) is given by

$$\begin{aligned} l(\theta) &= \sum_{t=1}^{n}\ln(p(X_t;\alpha_t,\beta_t,\delta,\mu)) \\ &= \sum_{t=1}^{n}\left\{\ln\left(\frac{\alpha_t}{\delta\beta_t}\right) + \ln\left(\frac{(X_t-\mu)^{(\alpha_t)}}{X_t-\mu}\right) - \frac{1}{\delta}(X_t-\mu)^{(\alpha_t)} \right. \\ &\quad \left. -\left(1+\frac{1}{\beta_t}\right)\ln\left[1+\exp\left(-\frac{1}{\delta}(X_t-\mu)^{(\alpha_t)}\right)\right]\right\}. \end{aligned} \tag{6}$$

The value of θ that maximizes equation (6) is the maximum likelihood estimate of θ_0, denoted by $\hat{\theta}_0$.

4. APPLICATIONS

In this section, we illustrate the usefulness of our quantile function based model by fitting the model to two stock price data sets: the IBM and the Wal-Mart stock prices from January 4, 1999 to November 12, 2007. Let $p=1$, $q=1$, $r=1$ and $s=1$ in model (3), the maximum likelihood estimates for $\theta_0=(a_1,b_1,c_1,d_1,\delta,\mu)$ are given in Table 1. In Figure 2 and Figure 3, we plot the observed stock series, the α profile and the β profile for IBM and Wal-Mart, respectively.

We see from Figure 2 that for the IBM stock, the $\hat{\alpha}_t$ profile matches the volatility movements seen in the observed prices very well, and that the $\hat{\beta}_t$ profile summarizes the profit making/losing opportunities over the time. Except in the three periods of late September and October in 2002, May in 2005 and July in 2002, in which the IBM stock tended to lose, for the rest of the time in the time period considered, the IBM stock tended to make a profit.

Table 1. Parameter estimates of our model for the IBM and Wal-Mart stock data sets

	$\hat{a}_1$	$\hat{b}_1$	$\hat{c}_1$	$\hat{d}_1$	$\hat{\delta}$	$\hat{\mu}$
IBM	6.838	0.013	1.546	0.078	1.012	74.585
Wal-Mart	0.392	0.949	0.818	0.011	0.956	34.308

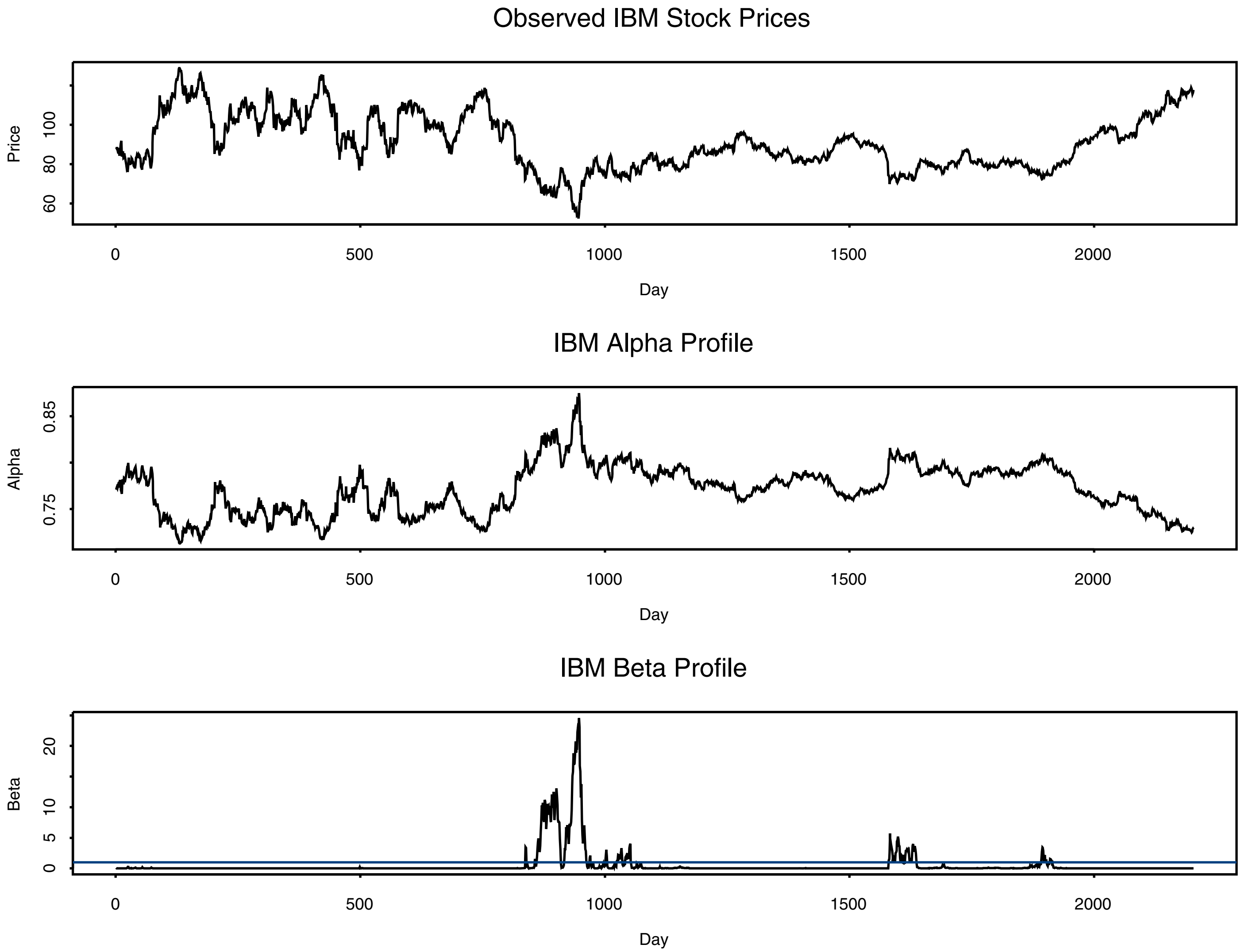

Figure 2. Modeling IBM stock prices.

From Figure 3 we see that for the Wal-Mart prices, the $\hat{\alpha}_t$ profile mostly stays high and flat, and the $\hat{\beta}_t$ profile is below one initially and way below one afterward for the entire period, implying that the Wal-Mart stock was relatively stable and tended to make a profit all the time.

The initial parts of the $\hat{\alpha}_t$ profiles and the $\hat{\beta}_t$ profiles in Figures 2 and 3 are affected by the conditions used to set up the conditional likelihood (6). We have tried different starting conditions and found that for the data sets we used, the impact of the starting conditions is very little, and certainly does not alter the shapes of the $\hat{\alpha}_t$ and $\hat{\beta}_t$ profiles after a few days into the price series.

5. CONCLUSIONS AND DISCUSSION

In this paper we have proposed to study price behaviors through modeling the tail properties of the return or log-return distributions instead of the usual approach of modeling the standard deviations of the return or log-return distributions. We have developed a specific model to do so using a class of quantile functions and shown that our new model is capable of fitting various financial time series. A feature of our new model is that it offers intuitively clear and useful information to help investors understand the price behaviors better.

We have however only started a research direction in this paper. Other parameter estimation methods should be explored, standard errors for the parameter estimates are needed, a large sample theory that can back up our approach will be helpful, and more data sets must be tried to gain more experience in using our new model.

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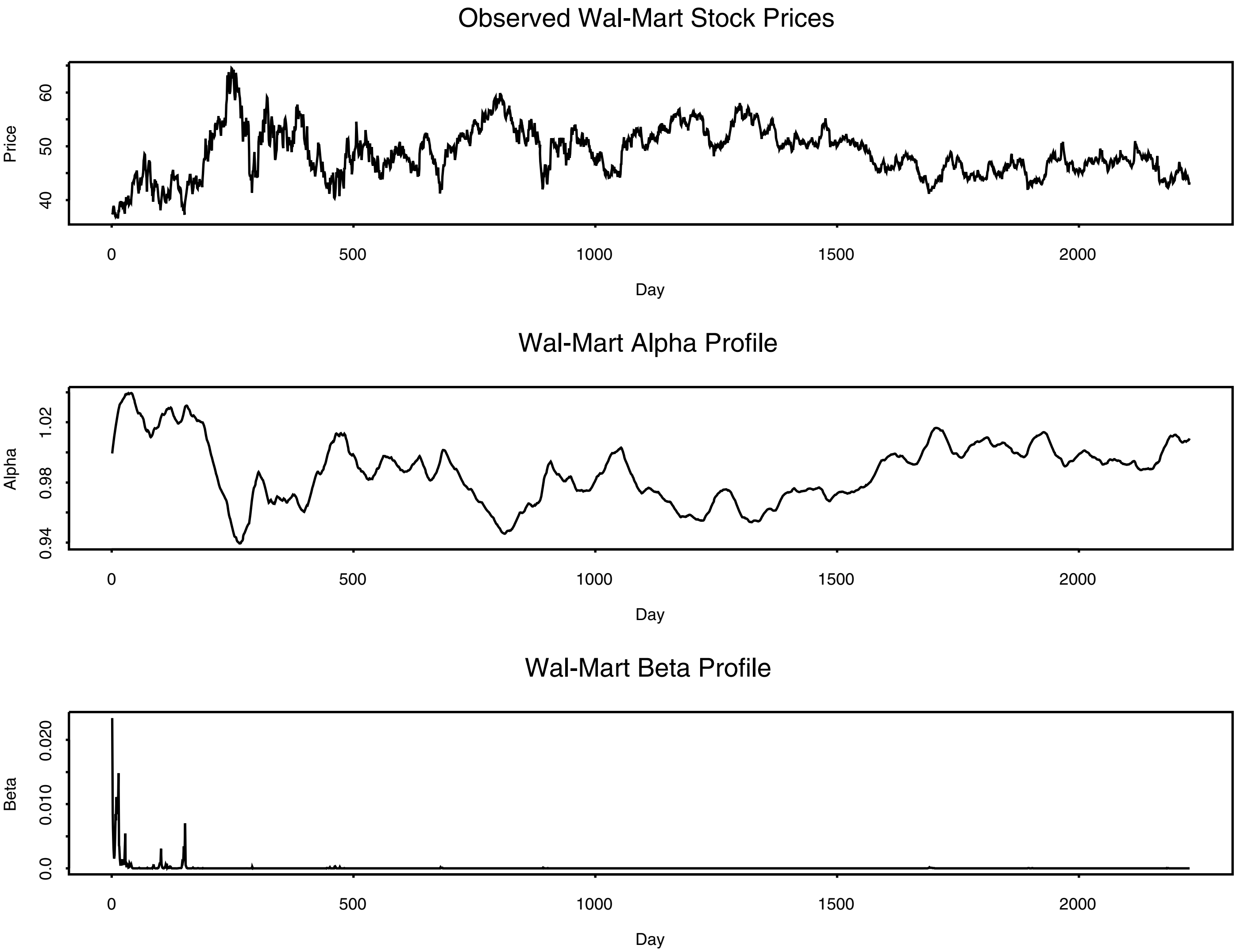

Figure 3. Modeling Wal-Mart stock prices.

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