Boundedness and moduli of algebraic varieties

Caucher Birkar

ABSTRACT. In this short note we give a survey of some of the recent developments in moduli theory from the point of view of birational geometry.

1. Introduction

This text is based on a talk given at a conference celebrating the 110th anniversary of the birth of Shing-Shen Chern, held at Tsinghua University in October 2021.

We will work over an algebraically closed field k of characteristic zero. We will aim to explain some of the recent advances on boundedness of various families of algebraic varieties and related results in moduli theory from the point of view of birational geometry. We do not attempt to go too much into the history of the subject.

Boundedness of a class of varieties roughly means that the varieties in the class are parametrised, not necessarily in a one-to-one way, by points of a scheme of finite type. There are many reasons for trying to show that certain classes of algebraic varieties form bounded families. We list a few:

- Boundedness is often one of the first steps towards construction of finite type moduli spaces.
- Boundedness is used in the proof of certain fundamental results which are not primarily concerned with boundedness of varieties, e.g. construction of complements on Fano varieties.
- Numerical data of varieties in a bounded family are usually finite, hence one is reduced to checking questions against a finite set of data.

Moduli spaces on the other hand parametrise collections of algebraic varieties in a more precise form. Roughly speaking, this means that each point of the moduli space corresponds to exactly one member of the given collection. These spaces are not viewed just as sets but with their own algebraic geometric structure.

There are many reasons for studying moduli spaces. Here we list a few:

- Constructing moduli spaces is a natural step towards classification of the varieties under consideration.
- Moduli theory provides highly non-trivial examples of varieties and schemes.
- Moduli spaces are used to solve other fundamental problems in algebraic geometry, other areas of mathematics, and mathematical physics.

Moduli theory has a long history going back at least as far as 19th century. For example, Riemann studied moduli spaces of curves of fixed genus. But moduli was given a new meaning in the Grothendieck school of algebraic geometry where the emphasis was not only on constructing a space that parametrises a given class of objects but also that the moduli space should give information about families of such objects. Equipped with tools developed by Grothendieck, Mumford elevated the theory to a new level via the introduction of the concept of stability and the development of Geometric Invariant Theory (GIT) which is a general approach to construction of moduli spaces. He successfully constructed moduli spaces of stable curves and abelian varieties and paved the way for the development of the theory for other classes of varieties. GIT has also been successfully applied to the moduli theory of sheaves on algebraic varieties.

As long as one is concerned with moduli of smooth varieties, GIT often works well. Indeed, Gieseker constructed moduli of surfaces of general type and Viehweg constructed moduli of good minimal models X; good means K_X is semi-ample. In general these moduli spaces are only quasi-projective and not compact. To obtain compact moduli spaces, it is usually necessary to incorporate singular varieties in the family.

It is strongly desirable to construct projective moduli spaces. One reason is that it is easier to study the geometry of compact spaces. Another reason is that projective schemes are of finite type, hence one deals with finitely many numerical data. Yet another reason is that projectivity ensures that one does not encounter weird properties exhibited by some non-projective schemes. To construct projective moduli spaces, one is forced to work with families that are bounded. It turns out that ensuring this boundedness is often one of the hardest steps in the process.

Deligne and Mumford already compactified moduli spaces of smooth projective curves of fixed genus by adding nodal stable curves. Construction of moduli spaces of singular varieties on the one hand and compactification of moduli spaces of smooth varieties on the other hand motivated the search for alternatives to GIT. In the 80's, Kollár and Shepherd-Barron [20] initiated the KSBA moduli theory for surfaces of general type which was later completed by Alexeev [2]. This approach was also applied by Alexeev to construct compact moduli of abelian varieties [1]. After four decades and with contributions by many people, the KSBA theory was recently completed resulting in the construction of compactified moduli spaces of varieties of general type, see Kollár [21]. More recently, Birkar [4] has applied tools from the KSBA theory to construct compactified moduli spaces of good minimal models. On the other hand, a large number of people have contributed to the construction of moduli spaces for K-stable Fano varieties.

The construction of the moduli spaces in the last paragraph rely heavily on fundamental developments in birational geometry, in particular, on the machinery of the minimal model program [12] [19] [11] and on various boundedness results and related topics proved in [16] [17] [18] [9] [8] [5] [7] [6] [13] [4].

Acknowledgements. I would like to thank the referee for the corrections.

2. Moduli problems and Hilbert schemes

So what is a moduli space? Suppose we are given a family $\mathcal{P}$ of projective varieties or schemes. We can think of a moduli space for $\mathcal{P}$ as a space parametrising elements of $\mathcal{P}$ and giving information about families of such elements. To define a moduli problem, we need to keep in mind two important points:

- How we consider the elements of $\mathcal{P}$, e.g. up to isomorphism.
- Which families are allowed.

For example, one might try to construct a moduli space for smooth projective curves X of genus g in $\mathbb{P}^2$. We may count two such X as one only if they are the same as subsets of $\mathbb{P}^2$; so if X, X' are isomorphic in the abstract sense but sit in different places in $\mathbb{P}^2$, then we count them as different. On the other hand, one may want to take a different approach and count X, X' as one if there is an isomorphism $\mathbb{P}^2 \to \mathbb{P}^2$ mapping X onto X'. The two different approaches lead to different moduli problems. So it is important to make it clear how we differentiate between the elements of $\mathcal{P}$.

Allowing arbitrary families of the elements of $\mathcal{P}$ usually runs into trouble. So one is forced to put some restrictions on the kind of families allowed.

Before going further, let's consider some examples of moduli problems and moduli spaces. We start with a classical example.

Example 2.1 (Hypersurfaces). Fix natural numbers n, r and consider

$$\begin{aligned}\mathcal{P} &= \{\text{degree } r \text{ hypersurfaces } \ X \subset \mathbb{P}^n\} \\ &= \{\text{closed subschemes } X \subset \mathbb{P}^n \text{ defined by a degree } r \text{ polynomial}\}.\end{aligned}$$

This is in 1-1 correspondence with

$$\{\text{degree } r \text{ homog. polynomials } 0 \neq F \in k[t_0, \dots, t_n] \text{ up to scaling}\}.$$

This is in turn in 1-1 correspondence with closed points of $\mathbb{P}^N$ where $N = \binom{n+r}{r} - 1$. So we see that the closed points of $\mathbb{P}^N$ parametrise the elements of $\mathcal{P}$ in a natural way.

Example 2.2 (Hilbert schemes). Pick a polynomial $\Phi \in \mathbb{Q}[t]$ so that $\Phi(\mathbb{Z}) \subset \mathbb{Z}$. Each closed subscheme $X \subset \mathbb{P}^n$ has a so-called Hilbert polynomial Φ_X satisfying $\Phi_X(m) = \mathcal{X}(\mathcal{O}_X(m))$ for $m \in \mathbb{Z}$. Consider

$$\mathcal{P} = \{\text{closed subschemes } X \subset \mathbb{P}^n \text{ with } \Phi_X = \Phi\}.$$

Grothendieck proved that

- there is a projective scheme H such that
$$\mathcal{P} \overset{1-1}{\longleftrightarrow} \text{closed points of } H,$$
- there is a universal family $\mathcal{H} \to H$ such that for each scheme S/k and for each closed subscheme $Z \subset \mathbb{P}^n \times S$ flat over S with $\Phi_{Z_s} = \Phi$ for fibres Z_s, there is a morphism $S \to H$ so that $Z \to S$ is the pullback of $\mathcal{H} \to H$ via $S \to H$.

The **Hilbert scheme** H is the moduli space of $\mathcal{P}$ in the nicest possible sense, that is, it is a fine moduli space. Being a fine moduli space essentially means that there is a universal family as in the example so that all allowable families are pullbacks of the universal family. Moduli spaces of varieties are often not fine.

In modern terms, defining a **moduli problem** for a class of varieties or schemes $\mathcal{P}$ means defining a contravariant functor $\mathfrak{F} : \mathcal{C} \to \mathcal{S}$ from a category of schemes $\mathcal{C}$ to the category of sets $\mathcal{S}$ where $\mathfrak{F}(C)$ consists of certain morphisms over C whose fibres are elements of $\mathcal{P}$ and such that for a morphism of schemes $C \to D$ in $\mathcal{C}$, $\mathfrak{F}(D) \to \mathfrak{F}(C)$ is given by pullback of families.

If there is a scheme M in $\mathcal{C}$ representing the functor, then we say M is a **fine** moduli space for $\mathcal{P}$ or more precisely for $\mathfrak{F}$. On the other hand, a scheme M with the following properties is called a **coarse** moduli space:

- we have a natural transformation of functors: $\mathfrak{F} \to \mathrm{Hom}_{\mathcal{C}}(-, M)$,
- $\mathfrak{F}(\mathrm{Spec}\, k) \to \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spec}\, k, M)$ is one-to-one,
- for any scheme N in $\mathcal{C}$, any natural transformation $\mathfrak{F} \to \mathrm{Hom}_{\mathcal{C}}(-, N)$ is factored as $\mathfrak{F} \to \mathrm{Hom}_{\mathcal{C}}(-, M)$ composed with a unique natural transformation $\mathrm{Hom}_{\mathcal{C}}(-, M) \to \mathrm{Hom}_{\mathcal{C}}(-, N)$.

Giving precise descriptions of functors of the moduli spaces considered below will be too technical and take much space. We will then often suppress this in the following sections and usually focus on the given class of the varieties and schemes.

3. General strategy for construction of moduli spaces

Let $\mathcal{P}$ be a set of projective varieties or schemes and $\mathfrak{F}$ be a moduli problem for $\mathcal{P}$. Assume $\mathcal{P}$ is bounded which by definition means that there is a projective morphism $V \to T$ of schemes of finite type so that each $X \in \mathcal{P}$ is isomorphic to the fibre of $V \to T$ over some closed point t. Then there exists n such that each $X \in \mathcal{P}$ has a embedding $X \subseteq \mathbb{P}^n$ with bounded degree (in practice, one puts some stronger assumptions on this embedding). In particular,

$$\{\text{Hilbert polynomial } \Phi_X \mid X \in \mathcal{P}\}$$

is finite. We can partition $\mathcal{P}$ so that we can assume $\Phi_X = \Phi$ is fixed. Each $X \in \mathcal{P}$ determines a point $[X] \in H$ in the Hilbert scheme H for $\mathbb{P}^n, \Phi$, but not in a unique way because $X \hookrightarrow \mathbb{P}^n$ is not unique.

Assume there exists a locally closed subscheme $R \subseteq H$ such that

$$\text{closed points of } R = \{[X] \in H \mid X \hookrightarrow \mathbb{P}^n, X \in \mathcal{P}\}$$

where $X \hookrightarrow \mathbb{P}^n$ runs through the possible embeddings mentioned above. Then

$$M = R/\mathrm{Aut}(\mathbb{P}^n) \quad \text{(if it exists)}$$

gives a coarse moduli space for $\mathcal{P}$.

The basic idea is this: if $X \hookrightarrow \mathbb{P}^n$ and if $\alpha \in \mathrm{Aut}(\mathbb{P}^n)$, then α induces another embedding

$$X \hookrightarrow \mathbb{P}^n \xrightarrow{\alpha} \mathbb{P}^n.$$

But taking the quotient means that the two embeddings correspond to the same point in the quotient M. Thus if we can ensure that all the embeddings of X are given by the above automorphisms, then taking the quotient tackles the non-uniqueness issue of the embeddings.

This discussion makes it clear that Hilbert schemes are an important tool in the story of moduli spaces.

4. Moduli of curves

Fix $g \geq 0$. Consider

$$\mathcal{P}_g = \{\text{smooth projective curves of genus } g\text{, up to isomorphism}\}.$$

Riemann showed that there is a variety M_g so that

$$\mathcal{P}_g \quad \overset{1-1}{\longleftrightarrow} \quad \text{closed points of } M_g.$$

For example, M_0 is just a point as $\mathcal{P}_0 = \{\mathbb{P}^1\}$. But $M_1 = \mathbb{A}^1$ via

$$\mathcal{P}_1 = \{\text{elliptic curves}\} \quad \overset{1-1}{\longleftrightarrow} \quad \text{closed points of } \mathbb{A}^1$$

which sends an elliptic curve to its j-invariant.

Mumford proved that M_g is a moduli space in the modern sense. Deligne-Mumford showed that for $g \geq 2$, there is a meaningful compactification $\overline{M}_g$:

$$\overline{\mathcal{P}}_g = \{\text{stable curves of genus } g\} \overset{1-1}{\longleftrightarrow} \text{closed points of } \overline{M}_g$$

by enlarging the class $\mathcal{P}_g$ to that of stable curves of genus g. Knudsen showed that $\overline{M}_g$ is projective.

A **stable curve** is a connected projective curve X with

- at worst nodal singularities,
- genus g, i.e. $h^1(\mathcal{O}_X) = g$,
- K_X ample.

If C is a rational irreducible component of X, then the last condition ensures that C^ν contains at least 3 points of $\nu^{-1}X_{sing}$ where $\nu\colon X^\nu \to X$ denotes normalisation.

We said that $\overline{M}_g$ is a meaningful compactification of M_g. That is because it is not an arbitrary compactification but rather itself is a moduli space for an enlarged class of curves. On the other hand, both $M_g, \overline{M}_g$ are not fine moduli spaces, i.e. there is no universal family.

It is not difficult to see that M_0 is not a fine moduli space. Indeed, assume otherwise, say M_0 has a universal family $\mathcal{M}_0 \to M_0$. Then $\mathcal{M}_0 = \mathbb{P}^1$. But then consider a $\mathbb{P}^1$-bundle $X \to S$ such that $X \not\simeq S \times \mathbb{P}^1$. The moduli map $S \to M_0$ is constant but $X \to S$ cannot be pullback of $\mathcal{M}_0 \to M_0$. Similar considerations show that $M_g, \overline{M}_g$ are not fine moduli spaces.

Moduli of elliptic curves behaves better when one fixes an identity element of the elliptic curve, that is, by marking a point on the curve. More generally, it is very fruitful to construct moduli of marked curves as markings often naturally appear in applications.

An ***n*-marked stable curve** of genus g is a projective connected curve X with n marked points $x_1, \dots, x_n$ on X_{smooth} such that

- X has at worst nodal singularities,
- genus g, i.e. $h^1(\mathcal{O}_X) = g$,
- $K_X + \sum x_i$ is ample.

Knudsen and Mumford proved that there is a projective coarse moduli space $\overline{M}_{g,n}$ for n-marked stable curves of genus g.

How about moduli of higher dimensional varieties? We will discuss this in detail in the following sections.

5. Birational geometry of pairs

Moduli of varieties is perhaps best understood in the context of birational geometry. Indeed, we will see below that many advances in moduli theory rely on techniques and results of birational geometry. In this section, we recall some very basic notions of birational geometry.

A **pair** (X, B) consists of a normal variety X and a boundary divisor B with rational coefficients in $[0, 1]$ so that $K_X + B$ is $\mathbb{Q}$-Cartier. Singularities of (X, B) are defined by taking a log resolution $\phi\colon W \to X$ and writing

$$K_W + B_W = \phi^*(K_X + B).$$

We say (X, B) is lc (resp. klt) if every coefficient of B_W is ≤ 1 (resp. < 1).

Given a projective lc pair (X, B), standard conjectures of birational geometry predict that there is a birational transformation

$$(X, B) \dashrightarrow (X', B')$$

such that either

- (X', B') admits a **Mori-Fano** fibration, or
- (X', B') is a **good minimal model**.

The latter means $m(K_{X'} + B')$ is generated by global sections for some $m \in \mathbb{N}$, defining a contraction $X' \to Z$. In particular, this gives a Calabi-Yau fibration $(X', B') \to Z$ (note that it is possible for $X' \to Z$ to be birational or even the identity).

To construct moduli spaces one usually restricts attention to good minimal models. Because presence of negativity, say for $K_X + B$, usually prevents having a good moduli theory. It is then not surprising that much of the literature on moduli theory is focused on construction of moduli spaces for certain classes of good minimal models, e.g. curves, K3 surfaces, abelian varieties, canonical models. From now on then we will mainly work with good minimal models.

Given a good minimal model (X, B), its **Kodaira dimension** $\kappa(X, B)$ is the dimension of the base Z of the corresponding contraction $X \to Z$ associated to $K_X + B$. Obviously,

$$\kappa(X, B) \in \{0, 1, \dots, \dim X\}.$$

In dimension one, (X, B) being a good minimal model means $\deg(K_X + B) \geq 0$. Then

- $\kappa(X, B) = 0$ iff $\deg(K_X + B) = 0$,
- $\kappa(X, B) = 1$ iff $\deg(K_X + B) > 0$.

In dimension two, for a good minimal model $(X, B) \to Z$ we have:

- $\kappa(X, B) = 0$ iff $K_X + B \equiv 0$ iff (X, B) is Calabi-Yau,
- $\kappa(X, B) = 1$ iff $X \to Z$ is a fibration over a curve,
- $\kappa(X, B) = 2$ iff $X \to Z$ is birational.

For the purposes of moduli theory one needs to consider a more general kind of pair in which the underlying scheme may not be irreducible. This is necessary for compactification of moduli spaces because limits of varieties are often not normal. We have already seen such pairs in dimension one: nodal curves.

A **semi-log canonical (slc)** pair (X, B) consists of a reduced quasi-projective scheme X of pure dimension and a divisor $B \geq 0$ on X with rational coefficients in $[0, 1]$ satisfying the following conditions:

- X is S_2 with nodal codimension one singularities,
- no component of $\operatorname{Supp} B$ is contained in the singular locus of X,
- $K_X + B$ is $\mathbb{Q}$-Cartier,
- if $\pi \colon X^\nu \to X$ is the normalisation of X and B^ν is the sum of the birational transform of B and the conductor divisor of π, then (X^ν, B^ν) is lc.

6. Moduli of KSBA stable pairs of general type

Until recently, compact moduli theory was mainly focused on lower dimensional varieties (dimensions one and two) and very special varieties in higher dimension, e.g. abelian varieties. But this has changed in recent years thanks to advances in birational geometry. Already in the 90's Viehweg constructed moduli spaces of smooth good minimal models. But these moduli spaces are only quasi-projective. One can of course compactify them in an arbitrary sense but such compactifications are not interesting. One would really like to get a "meaningful" compactification in the sense that the compactified space is also a moduli space for an enlarged class of algebraic varieties or schemes.

During the last four decades, Kollár, Alexeev, Shepherd-Barron, and others have developed a theory of moduli of varieties and pairs of general type, i.e. of maximal Kodaira dimension, which can handle schemes with slc singularities leading to projective moduli spaces.

To get finite type moduli spaces, one needs to fix certain invariants. In the case of curves, it is enough to fix the genus to get a finite type moduli space but in higher dimensions one fixes the volume rather than genus.

Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. A (d, c, v)-**KSBA-stable pair** is a connected projective pure dimensional pair (X, B) such that

- (X, B) is slc of dimension d,
- $B = cD$ for some integral divisor D,
- $K_X + B$ is ample with volume $\mathrm{vol}(K_X + B) := (K_X + B)^d = v$.

When X is normal, then (X, B) is a good minimal model with the corresponding contraction $X \to Z$ being the identity morphism.

It takes more work to define (d, c, v)-stable families $(X, B) \to S$. We do not recall the definition as it is quite technical. It suffices to say that very roughly speaking, such a family is a flat projective family with (d, c, v)-stable log fibres.

Example 6.1. Assume $(X, B = \sum x_i)$ is an n-marked stable curve of genus g. Then (X, B) is a $(1, 1, v)$-stable pair with $v = \deg(K_X + B)$.

Example 6.2. Assume $X \subset \mathbb{P}^{d+1}$ is a hypersurface of degree r and $B \subset X$ be a general hyperplane section. Then (X, B) is a $(d, 1, v)$-stable pair with $v = (r - d - 1)^d r$.

The next result is an important step in the construction of projective moduli. Its proof relies on many other results in birational geometry, in particular, [**12**] [**18**] [**17**].

Theorem 6.3 (Hacon, M$^{\mathrm{c}}$Kernan, Xu [**16**]). *Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. Then (d, c, v)-KSBA stable pairs form a bounded family.*

Next is the higher dimensional version of Deligne-Mumford theorem for curves, which is proved in [**21**].

Theorem 6.4 (Kollár, Alexeev, et al). *Fix $d \in \mathbb{N}$ and $c, v \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, v)-KSBA-stable pairs.*

Besides moduli considerations such as definition of stable families and the boundedness result mentioned above, the theorem relies on many other results and techniques of birational geometry.

7. Moduli of stable Calabi-Yau pairs

Next we treat moduli of Calabi-Yau varieties and more generally pairs. In birational geometry, a projective lc (or slc) pair (X, B) is Calabi-Yau if $K_X + B \sim_{\mathbb{Q}} 0$. In particular, this class includes K3 surfaces, abelian varieties, and smooth Calabi-Yau varieties that appear in other contexts. Moduli of Calabi-Yau varieties also has a long history.

Unlike varieties and pairs of general type, in general Calabi-Yau pairs do not carry any natural polarisation that is, ample divisor, so we need to pick one. Moduli spaces of smooth Calabi-Yau varieties have been successfully constructed and well-studied. However, one of the main issues has been to construct meaningful compactifications of these moduli spaces, and to treat the corresponding singular cases. This requires to establish boundedness of certain classes of Calabi-Yau pairs which is by now resolved and discussed below.

We start with the definition of stable Calabi-Yau pairs which was gradually defined in work of Alexeev, Hacking, Kollár-Xu, and others.

Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. A (d, c, u)**-stable Calabi-Yau pair** $(X, B), A$ is defined by the data:

- (X, B) is projective slc of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$,
- $B = cD$ for some integral divisor $D \geq 0$,
- $A \geq 0$ is an ample integral divisor with volume $\mathrm{vol}(A) = u$,
- $(X, B + tA)$ is slc for some $t \in \mathbb{Q}^{>0}$,

Again it takes more work to define stable families and the corresponding moduli functor but this is done similarly to the KSBA stable case.

Example 7.1. Assume X is an elliptic curve, $B = 0$, and $A \in X$ is a point. Then $(X, B), A$ is a $(1, 1, 1)$-stable Calabi-Yau pair.

Example 7.2. Assume $X = \mathbb{P}^2$, $B \subset X$ an elliptic curve, and $A \subset X$ is a conic. Then $(X, B), A$ is a $(2, 1, 4)$-stable Calabi-Yau pair.

Example 7.3. Consider a smooth projective curve $C \subset \mathbb{P}^2$ of degree $r \geq 4$. Let $B = \frac{3}{r}C$ and $A = C$. Then $(X, B), A$ is a stable Calabi-Yau pair. Hacking [**15**] constructed compactification of the moduli space of such curves C by compactifying the moduli space of the corresponding $(X, B), A$.

By definition, for each (d, c, u)-stable Calabi-Yau pair $(X, B), A$, there is $t > 0$ so that $(X, B + tA)$ is KSBA stable. However, t is a priori not fixed, so it is not clear how it may depend on the initial data d, c, u. Therefore, we cannot apply boundedness of KSBA stable pairs (6.3). The desired boundedness of stable Calabi-Yau pairs takes a lot more work and this is done in [**7**]. In addition to work already mentioned, the proof crucially relies on boundedness of Fano varieties [**8**] [**9**] [**5**] and the techniques of its proof.

Theorem 7.4 (Birkar [**7**]). *Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. Then the (d, c, u)-stable Calabi-Yau pairs form a bounded family.*

Given a (d, c, u)-stable Calabi-Yau pair $(X, B), A$, the boundedness implies $(X, B + tA)$ is a KSBA stable pair, for some fixed $t \in \mathbb{Q}^{>0}$ depending only on d, c, u. From this then one derives the next result which was first published in the first arxiv version of [**7**] but then incorporated into the more general framework of [**4**].

Theorem 7.5 (Birkar). *Fix $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$. There is a projective coarse moduli space for (d, c, u)-stable Calabi-Yau pairs.*

Restricting the family of (d, c, u)-stable Calabi-Yau pairs to special situations gives many interesting examples of moduli spaces, e.g. Fano varieties polarised by certain anti-pluricanonical divisors, or K3 surfaces polarised by effective ample divisors.

8. Moduli of stable minimal models

We discussed moduli of good minimal models (X, B) of maximal Kodaira dimension (KSBA stable) and minimal Kodaira dimension (stable Calabi-Yau). How about other Kodaira dimensions? Recall that the Kodaira dimension ranges from 0 to $\dim X$, so large classes of pairs of intermediate Kodaira dimension, that is, Kodaira dimension $1, \dots, \dim X - 1$ remain to be treated (for example, in dimension two, the remaining case is Kodaira dimension 1; such a minimal model comes with an elliptic fibration or conic bundle structure $X \to Z$). There seems to be few results in the literature regarding compact moduli of such varieties and pairs. Strictly speaking one should also consider models (X, B) with maximal Kodaira dimension $\dim X$ where $K_X + B$ is not necessarily ample because this is not covered by the KSBA stable case.

In [**4**], we have developed a moduli theory of stable minimal models of arbitrary Kodaira dimension. First we recall the definition of stable minimal models without fixing invariants. A **stable minimal model** $(X, B), A$ consists of a connected projective pair (X, B) and a divisor $A \geq 0$ such that

- (X, B) is slc,
- $K_X + B$ is semi-ample defining a contraction $f\colon X \to Z$,
- $K_X + B + tA$ is ample for some $t > 0$, and
- $(X, B + tA)$ is slc for some $t > 0$.

The first and second conditions say that (X, B) is a good minimal model. The third condition just says that A is ample over Z. The fourth condition says that A should not contain any component of the locus where (X, B) is not klt, e.g. in dimension one this means A should not contain any node nor any component of B with coefficient one.

Example 8.1. Any KSBA stable pair (X, B) is a stable minimal model with $A = 0$ and $X \to Z$ the identity morphism.

Example 8.2. Any stable Calabi-Yau $(X, B), A$ is a stable minimal model with $X \to Z$ the constant morphism.

EXAMPLE 8.3. If $(X, B) \to Z$ is a klt good minimal model and $A \geq 0$ is an ample over Z divisor, then $(X, B), A$ is a stable minimal model.

EXAMPLE 8.4. Suppose $X \to Z$ is a smooth good minimal model of dimension 2, $\kappa(X) = 1$, and $A \geq 0$ is a multi-section. Then $(X, B), A$ is a stable minimal model.

Many more examples can be constructed either as limits of families of normal stable minimal models or as structures associated to singularities, see [4].

To get a good moduli theory we need to fix more invariants compared to the KSBA and Calabi-Yau cases. This is not surprising as the minimal model case in general is more complex.

Let $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. A (d, c, u, σ)-stable minimal model is a stable minimal model $(X, B), A$ such that

- $\dim X = d$,
- $\frac{1}{c}B$ and $\frac{1}{c}A$ are integral,
- $\mathrm{vol}(A|_F) = u$ where F is a general fibre of $f\colon X \to Z$ over any component of Z, and
- $\mathrm{vol}(K_X + B + tA) = \sigma(t)$ for sufficiently small $t > 0$.

A more general setup is presented in [4] where one can fix finitely many values for u. This is important for applications because on some stable minimal models, the volumes $\mathrm{vol}(A|_F)$ take different values over different irreducible components of Z.

EXAMPLE 8.5. Assume X is a smooth Calabi-Yau variety of dimension d, $f\colon X \to Z$ is an elliptic fibration, $B = f^*H$ for a general hyperplane section H on Z, and $A \geq 0$ is a multi-section of degree u. Then $(X, B), A$ is a $(d, 1, u, \sigma)$-stable minimal model where σ is the polynomial determined by

$$\sigma(t) := (K_X + B + tA)^d$$

for sufficiently small t.

EXAMPLE 8.6. A (d, c, u, σ)-stable minimal model $(X, B), A$ is a KSBA-stable pair iff $\sigma = \mathrm{vol}(K_X + B)$ is constant.

EXAMPLE 8.7. A (d, c, u, σ)-stable minimal model $(X, B), A$ is a stable Calabi-Yau pair iff $\sigma(t) = \mathrm{vol}(A)t^d$ is a monomial of degree $d = \dim X$.

As before, to get a projective moduli space we first need to deal with the boundedness problem which is settled by the next result. Indeed, similar to the Calabi-Yau case, the number t in the definition of a (d, c, u, σ)-stable minimal model is a priori not fixed and the hardest part of the process is to show that we can choose t universally depending only on d, c, u, σ. The proof of this heavily relies, among other things, on the theory of generalised pairs [14] [6] [13] as well as all the previous boundedness results mentioned above (for general type, Fano, and Calabi-Yau pairs).

Theorem 8.8 (Birkar [4]). *Let $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. Then the (d, c, u, σ)-stable minimal models form a bounded family.*

Theorem 8.9 (Birkar [4]). *Let $d \in \mathbb{N}$ and $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$. Then there is a projective coarse moduli space for the (d, c, u, σ)-stable minimal models.*

The theorem expands compact moduli theory to settings that have previously been largely unexplored, e.g. intermediate Kodaira dimension case. Little is known about the geometry of these moduli spaces $M_{d,c,u,\sigma}$. One can ask many questions in this direction, for example:

- Under what conditions is $M_{d,c,u,\sigma}$ non-empty?
- What kind of singularities does $M_{d,c,u,\sigma}$ have? In general, singularities can be arbitrarily bad but in specific settings, one might get reasonable singularities.
- What is the Kodaira dimension of the components of $M_{d,c,u,\sigma}$?
- For explicit choices of d, c, u, σ, describe the (d, c, u, σ)-stable minimal models and then describe the moduli space $M_{d,c,u,\sigma}$. For example, consider $d = 3$, $c = 1$, $u = 1$, $\sigma = 3t^2 + t^3$.

9. Fano varieties

Fano varieties do not behave as well as minimal models with respect to moduli spaces. This is in particular due to the fact that in general Fano varieties and Fano fibrations tend to have too many regular and birational automorphisms.

One way to remedy the situation is to consider polarised Fano varieties and then use ideas in the construction of moduli of stable minimal models. A **stable Fano pair** is of the form $(X, \Lambda), A$ where $(X, \Lambda + A), A$ is a stable Calabi-Yau pair and $\Lambda \geq 0$. Since $-(K_X + \Lambda) \sim_{\mathbb{Q}} A$, the pair (X, Λ) is indeed Fano (with slc singularities) which is polarised by A.

Let $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, c, u, σ)-stable Fano pair is a stable Fano pair $(X, \Lambda), A$ where $(X, \Lambda + A), A$ is a (d, c, u, σ)-stable Calabi-Yau pair.

Theorem 9.1 (Birkar [4]). *There is a projective coarse moduli space for the (d, c, u, σ)-stable Fano pairs.*

A nice aspect of this approach is that it also works for Fano fibrations. A **stable log Fano fibration** is of the form $(X, \Lambda), A \to Z$ where $(X, \Lambda + A), A$ is a stable minimal model and $\Lambda \geq 0$, and $X \to Z$ is the contraction defined by $K_X + \Lambda + A$. Then $-(K_X + \Lambda) \sim_{\mathbb{Q}} A/Z$, so we can see that $(X, \Lambda) \to Z$ is indeed a log Fano fibration (with slc singularities) which is polarised by A.

Let $d \in \mathbb{N}$, $c, u \in \mathbb{Q}^{>0}$, and $\sigma \in \mathbb{Q}[t]$ be a polynomial. A (d, c, u, σ)-stable log Fano fibration is a stable log Fano fibration $(X, \Lambda), A \to Z$ where $(X, \Lambda + A), A$ is a (d, c, u, σ)-stable minimal model.

Theorem 9.2 (Birkar [4]). *There is a projective coarse moduli space for the (d, c, u, σ)-stable log Fano fibrations.*

On the other hand, the geometry of log Fano fibrations is treated in [10] [5] [3] from the point of view of boundedness, singularities, etc. These works go much beyond moduli considerations.

A different approach to moduli of Fano varieties is motivated by the existence of special metrics and originates in differential geometry. In this approach, one restricts attention to the class of so-called K-stable Fano varieties (and its variants) rather than treating all Fano varieties. This approach has been taken up by a large number of people that is too long to list here. The singular case also heavily relies on results of birational geometry including [8] [9].

10. Polarisation by non-effective divisors

In the above definition of stable Calabi-Yau and stable minimal models, the polarisation was given by an effective divisor $A \geq 0$. The reason for taking effective divisors is that one can then construct projective moduli spaces although it usually leads to larger moduli spaces. Another possibility that has often been used traditionally is to take a line bundle or a Cartier divisor class for the polarisation. For example, Viehweg takes such a polarisation to treat moduli of smooth good minimal models [23].

One may define a **traditional stable minimal model** $(X, B), A$ to consist of a projective pair (X, B) and a Cartier divisor A (not necessarily effective) such that

- (X, B) is klt,
- $K_X + B$ is semi-ample defining a contraction $f\colon X \to Z$, and
- $K_X + B + tA$ is ample for some $t > 0$.

Fixing appropriate numerical invariants, one can show that the corresponding models form a bounded family, using the results of [7] [4] (one can even consider not necessarily Cartier divisor classes for the polarisation). Note that the singularities are klt by assumption. Allowing lc or slc singularities is problematic.

One can then perhaps construct the moduli space of such minimal models. But the problem is that such a moduli space will usually be only quasi-projective. It is difficult to "meaningfully" compactify these moduli spaces without putting some further strong assumptions. One could try to enlarge these moduli spaces as much as possible without insisting on getting a projective moduli space, that is, to construct a partial meaningful compactification. For example, see [22] and the references therein for the Calabi-Yau case. What is clear is that no matter what approach is taken (effective or not necessarily effective polarisations), the above boundedness results and relevant results play an important role in order to compactify or partially compactify the moduli spaces.

References

[1] V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. of Math. (2), **155** (2002), no. 3, 611–708. MR 1923963

[2] V. Alexeev, *Moduli spaces $M_{g,n}(W)$ for surfaces*, Higher-dimensional complex varieties (Trento, 1994), de Gruyter, Berlin, 1996, pp. 1–22. MR 1463171

[3] C. Birkar, *Singularities on Fano fibrations and beyond*, arXiv.2305.18770.

[4] C. Birkar, *Moduli of algebraic varieties*, arXiv:2211.11237.

[5] C. Birkar, *Boundedness of Fano type fibrations*, To appear in Ann. Sci. ENS, arXiv:2209.08797.

[6] C. Birkar, *Boundedness and volume of generalised pairs*, arXiv:2103.14935v2.

[7] C. Birkar, *Geometry of polarised varieties*, Pub. Math. IHES, **137** (2023), 47–105. MR 4588595

[8] C. Birkar, *Singularities of linear systems and boundedness of Fano varieties*, Ann. of Math., **193** (2021), no. 2, 347–405. MR 4224714

[9] C. Birkar, *Anti-pluricanonical systems on Fano varieties*, Ann. of Math., **190** (2019), no. 2, 345–463. MR 3997127

[10] C. Birkar, *Singularities on the base of a Fano type fibration*, J. Reine Angew Math., **715** (2016), 125–142. MR 3507921

[11] C. Birkar, *Existence of log canonical flips and a special LMMP*, Pub. Math. IHES., **115** (2012), 325–368. MR 2929730

[12] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc., **23** (2010), 405–468. MR 2601039

[13] C. Birkar, C. D. Hacon, *Variations of generalised pairs.* arxiv:2204.10456v1.

[14] C. Birkar, D-Q. Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Pub. Math. IHES., **123** (2016), 283–331. MR 3502099

[15] P. Hacking, *Compact moduli of plane curves*, Duke Math. J., **124** (2004), no. 2, 213–257. MR 2078368

[16] C. D. Hacon, J. McKernan, C. Xu, *Boundedness of moduli of varieties of general type*, J. Eur. Math. Soc. (JEMS), **20** (2018), no. 4, 865–901. MR 3779687

[17] C. D. Hacon, J. McKernan, C. Xu, *ACC for log canonical thresholds*, Ann. of Math. (2), **180** (2014), no. 2, 523–571. MR 3224718

[18] C. D. Hacon, J. McKernan, C. Xu, *On the birational automorphisms of varieties of general type*, Ann. of Math. (2), **177** (2013), no. 3, 1077–1111. MR 3034294

[19] C. D. Hacon, C. Xu, *Existence of log canonical closures*, Invent. Math., **192** (2013), no. 1, 161–195. MR 3032329

[20] J. Kollár, N. I. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math., **91** (1988), no. 2, 299–338. MR 0922803

[21] J. Kollár, *Families of varieties of general type*, Cambridge University Press, 2023. MR 4566297

[22] Y. Odaka, *On log minimality of weak K-moduli compactifications of Calabi-Yau varieties*, arXiv:2108.03832. MR 4498830

[23] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 30, Springer-Verlag, Berlin, 1995. MR 1368632

Yau Mathematical Sciences Center, JingZhai, Tsinghua University, Hai Dian District, Beijing, China 100084

Email address: `birkar@tsinghua.edu.cn`