A survey on Yau's uniformization conjecture

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Abstract. This is a survey on Yau's uniformization conjecture which states that any complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. The accomplishments of the conjecture during the past decades and recent developments, from the view point of geometry, will be reviewed.

1. Introduction

The classical uniformization theorem says that a simply connected Riemann surface is biholomorphic to one of the sphere S^2, the complex plane $\mathbb{C}$, or the disc D. Its higher dimensional generalization forms one of the themes of the current geometry and topology. From the view point of differential geometry, one may investigate the problem by imposing certain geometric conditions, which can be roughly divided into elliptic, parabolic or hyperbolic cases. The current paper is concerned with the parabolic case. For this case, there is a well-known conjecture due to Yau [**47**]:

Conjecture 1.1. *A complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to* $\mathbb{C}^n$.

The analogous elliptic case is called Frankel conjecture, which says that a compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to the complex projective space $\mathbb{C}P^n$. The Frankel conjecture has been completely solved by Siu-Yau [**42**]. A stronger algebraic geometric version, the Hartshone conjecture, was also true (see Mori [**35**]).

During the past decades, there have been much important progress for Yau's conjecture, especially for maximal volume growth case, but the full conjecture still remains to be answered so far. The purpose of this paper, based on [**17**], is to give a survey on these accomplishments and some recent developments.

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Historically, the first result in this direction should be the following theorem due to Mok-Siu-Yau [**33**] in 1981:

Theorem 1.1 (Mok-Siu-Yau [**33**]). *Let M be a complete noncompact Kähler manifold of nonnegative holomorphic bisectional curvature of complex dimension $n \geq 2$. Suppose there exist positive constants C_1, C_2 such that for a fixed base point x_0 and some $\epsilon > 0$,*

(i) $Vol(B(x_0, r)) \geq C_1 r^{2n}$, $0 \leq r < +\infty$,
(ii) $R(x) \leq C_2/d(x_0, x)^{2+\epsilon}$ *on M,*

where $Vol(B(x_0, r))$ denotes the volume of the geodesic ball $B(x_0, r)$ centered at x_0 with radius r, $R(x)$ denotes the scalar curvature and $d(x_0, x)$ denotes the geodesic distance between x_0 and x. Then, M is isometrically biholomorphic to $\mathbb{C}^n$ with the flat metric.

The above theorem is also called a gap theorem, which roughly says that if the curvature is nonnegative and decays faster than quadratic, then the curvature is actually identically equal to zero. For this reason, in the above theorem, the curvature can only be assumed to be nonnegative, not positive instead. Besides the first result in this direction, the above theorem also indicates what kind of conditions should be considered for the geometry of the discussed manifolds, namely,

i) the volume growth;
ii) the curvature decay.

It will be seen later that these two conditions are not independent, they are relevant to each other.

The starting point in the proof of Theorem 1.1 is to solve the Poincaré-Lelong equation:

$$\sqrt{-1}\partial\bar{\partial}u = Ric, \tag{1.1}$$

with the purpose to find a pluri-subharmonic function (psh) u. The conditions (i) and (ii) in Theorem 1.1 ensure that the equation (1.1) can be solved and the solution u is bounded. The equation (1.1) is an overdetermined system with only one unknown function u. In solving (1.1), nonnegative bisectional curvature condition plays a key role in applying a Bochner type formula. The philosophy of the proof of Theorem 1.1 is that a complete Kähler manifold with nonnegative bisectional curvature can not support nontrivial bounded psh functions, except constants.

Of course, one may also ask the analogous question to the uniformization conjecture by relaxing the positive curvature condition to nonnegative. For compact Kähler manifolds, this is the so-called generalized Frankel conjecture, which was completely solved by Mok [**32**].

The parabolic analogue of [**32**] is the following:

THEOREM 1.2 (Cao-Chen-Zhu). *Let M be a complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature. Then one of the following holds:*

(i) *M admits a Kähler metric with bounded and positive bisectional curvature;*

(ii) *The universal cover $\hat{M}$ of M splits holomorphically and isometrically as*

$$\hat{M} = \mathbb{C}^k \times M_1 \times \cdots M_{l_1} \times N_1 \times \cdots \times N_{l_2}$$

where k, l_1, l_2 are nonnegative integers, $\mathbb{C}^k$ is the complex Euclidean space with flat metric, M_i, $1 \leq i \leq l_1$, are complete (compact or noncompact) Kähler manifolds with bounded and nonnegative bisectional curvature admitting a Kähler metric with bounded and positive bisectional curvature, $N_j, 1 \leq j \leq l_2$, are irreducible compact Hermitian symmetric spaces of rank ≥ 2 with the canonical metrics.

This theorem reduces the nonnegative bisectional curvature case to positive bisectional curvature case when the curvature is bounded. We remark that in the above theorem, the bounded curvature condition can be replaced by assuming the volumes of all unit balls are non-collapsed. The proof of Theorem 1.2 is a maximum principle argument for the Ricci flow. This argument was used previously by Gu [**21**] in compact case to give an alternative and transcendental proof of Mok's theorem [**32**].

There are several approaches in literatures [**33**], [**30**], [**40**], [**13**], [**8**], [**27**] tackling Yau's uniformization Conjecture 1.1. We list them in the following:

a) construct a biholomorphic map from M to $\mathbb{C}^n$;
b) construct a complete flat Kähler metric on M;
c) construct a biholomorphic map F from M to itself such that the basin of attraction of some point P, i.e. $\Omega = \{x \in M^n : lim_{i\to\infty} F^i(x) = P\}$ contains an open neighborhood of P.

For a), we need to construct holomorphic functions on the manifold. A conservative strategy is first to embed M to $\mathbb{C}^N$ or $\mathbb{CP}^N$ as an algebraic subvariety for some large N, then proceed to prove this subvariety is just $\mathbb{C}^n$. The major tool in analysis for the former step is the L^2- estimate of $\bar{\partial}$-operator (see [**2**], [**24**]):

THEOREM 1.3 (Andreotti-Vesentini, Hömander). *On a complete Kähler manifold (M, ω), suppose we have a function φ, a Hermitian holomorphic line bundle $\tilde{L}$ with curvature $(1,1)-$ form $C_1(\tilde{L})$ such that*

$$\sqrt{-1}\partial\bar{\partial}\varphi + C_1(\tilde{L}) + Ric \geq c(x)\omega \tag{1.2}$$

where $c(x)$ is a positive function on M, ω is the Kähler form; suppose we also have a $\bar{\partial}-$ closed $\tilde{L}-$ valued $(0,1)$ form f on M such that

$$\int_M \frac{\|f\|^2}{c} e^{-\varphi} < \infty. \tag{1.3}$$

Then the equation

$$\bar{\partial}\xi = f \tag{1.4}$$

admits a smooth solution ξ (section of $\tilde{L}$) such that

$$\int_M \|\xi\|^2 e^{-\varphi} \leq \int_M \frac{\|f\|^2}{c} e^{-\varphi}. \tag{1.5}$$

To construct holomorphic functions by using Theorem 1.3, the key point is to find a strictly psh function u of suitable growth. As in the work of [**33**], solving Poincaré-Lelong equation (1.1) is one method to obtain a psh function. Another natural method to produce psh functions is using distance functions and comparison theorems in Riemannian geometry (see [**22**]). Usually, the psh functions obtained from comparison theorem are only Lipschitz continuous. One can use the heat equation

$$\frac{\partial u}{\partial t} = \triangle u \tag{1.6}$$

to smooth it. The point is that the Levi form $\sqrt{-1}\partial\bar{\partial}u$ satisfies a parabolic Lichnerowicz equation:

$$\frac{\partial}{\partial t} u_{\alpha\bar{\beta}} = \triangle u_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}\xi\bar{\eta}} u_{\eta\bar{\xi}} - \frac{1}{2}(R_{\alpha\bar{\xi}} u_{\xi\bar{\beta}} + R_{\xi\bar{\beta}} u_{\alpha\bar{\xi}}). \tag{1.7}$$

The solvability of equation (1.6) can be guaranteed under very mild conditions, for example, $|u| \leq Ce^{Cd(x_0,x)^2}$. In this case, the maximum principle argument can be applied to (1.7). That is to say, $\sqrt{-1}\partial\bar{\partial}u \geq 0$ is preserved under the heat equation, and it becomes strictly positive for $t > 0$, unless the universal cover of the manifold splits isometrically as a product $M^n = N^k \times \mathbb{C}^{n-k}$, see [**37**]. If the bisectional curvature or Ricci curvature is strictly positive, the splitting can not happen, and the heat equation deformation will give a smooth strictly psh function $u(\cdot, t)$ for $t > 0$.

Now we talk about b). One naive approach is to deform the initial metric using the Ricci flow:

$$\frac{\partial}{\partial t} g = -2Ric. \tag{1.8}$$

Hopefully, a suitable scaled limit as $t \to \infty$ will produce a flat Kähler metric. This approach was initiated by Shi [**40**].

c) is one characterization of $\mathbb{C}^n$ due to [**39**] and [**45**]. Nevertheless, it is difficult to construct such a biholomorphic map F. Using Ricci flow, in [**8**], a generalized version of c) has been successfully developed to give a proof of Yau's uniformization conjecture under certain conditions.

2. Geometry of bisectional curvature

In this section, we are mainly concerned with the volume growth and curvature decay properties. The above Theorem 1.1 indicates that we should study the geometry at infinity of complete noncompact Kähler manifolds with positive bisectional curvature. These results can be compared with that of complete Riemannian manifold with positive Ricci or sectional curvature.

THEOREM 2.1 (Chen-Zhu [15]). *Let M^n be a complex n-dimensional complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then the volume growth of M satisfies*

$$Vol(B(x_0, r)) \geq c(1+r)^n$$

for all $0 \leq r < \infty$, where c is some positive constant depending on x_0.

THEOREM 2.2 (Chen-Zhu [15]). *Let M^n be a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then for any $x_0 \in M^n$, there exists a positive constant C such that*

$$\frac{1}{vol(B(x_0, r))} \int_{B(x_0,r)} R(x)dv \leq \frac{C}{1+r}$$

for all $0 \leq r < \infty$, where $R(x)$ is the scalar curvature of M^n.

Since positive bisectional curvature implies positive Ricci curvature, by Bishop-Gromov volume comparison theorem,

$$r \to \frac{vol(B(x, r))}{r^{2n}}$$

is a nonincreasing function. Theorem 2.1 says that the volumes of geodesic balls grow at order between n and $2n$. Therefore, if there exist $x \in M^n$ and $C > 0$ such that

$$vol(B(x, r)) \geq C^{-1} r^{2n}$$

for all $r > 0$, we say the manifold has maximal volume growth. On the other hand, if

$$vol(B(x, r)) \leq C(1+r)^n$$

for all $r > 0$, we say the manifold has minimal volume growth. Remark that an example of Klembeck and Cao's static Kähler Ricci soliton with positive bisectional curvature has volume growth of order n:

$$vol(B(x_0, r)) \approx (1+r)^n, \tag{2.1}$$

and the curvature decays linearly

$$R \approx \frac{1}{1+r}. \tag{2.2}$$

That means the results in Theorem 2.1 and 2.2 are sharp. Note also that examples with intermediate growth order

$$vol(B(x_0, r)) \approx (1+r)^{n(1+\epsilon)}, 0 < \epsilon < 1 \tag{2.3}$$

have been constructed in literatures, see [46].

On the other hand, the gap theorem 1.1 and Theorem 2.2 also suggest that the curvature decays from linear r^{-1} to quadratic r^{-2}. The examples of various intermediate decay orders $r^{-(1+\epsilon)}$ can also be constructed [46].

We discuss a little on the proof of Theorems 2.1 and 2.2.

Let $\gamma : [0,\infty) \to M^n$ be a ray, and

$$b_\gamma(x) = \lim_{t\to\infty}(t - d(x,\gamma(t))) \tag{2.4}$$

be the Busemann function associated to γ, which is strictly psh by comparison theorem. One can smooth b_γ a little by using the heat equation (1.6) to obtain a smooth strictly psh function u of linear growth

$$|u(x)| \le C(1 + d(x,x_0)) \tag{2.5}$$

and bounded gradient

$$|\nabla u| \le C. \tag{2.6}$$

Let $0 \le \xi \le 1$ be a cut-off function such that $\xi\mid_{B(x_0,r)}= 1$, $\xi\mid_{M^n\setminus B(x_0,2r)}= 0$, $|\nabla\xi| \le c/r$. Integrating by parts gives,

$$\begin{aligned}\int_{B(x_0,2r)} \xi^n(\sqrt{-1}\partial\bar\partial u)^n &\le \frac{C}{r}\int_{B(x_0,2r)} \xi^{n-1}(\sqrt{-1}\partial\bar\partial u)^{n-1}\wedge\omega \\ &\cdots \\ &\le \frac{C}{r^n}\int_{B(x_0,2r)}\omega^n.\end{aligned} \tag{2.7}$$

Theorem 2.1 follows. For Theorem 2.2, choosing a suitable weight function based on u, using Theorem 1.3, one can construct a nontrivial holomorphic section S of the canonical line bundle K, and S has at most exponential growth in geodesic distances:

$$|S|(x) \le Ce^{C(1+d(x,x_0))}. \tag{2.8}$$

The Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar\partial\log|S|^2 = [S = 0] + Ric \tag{2.9}$$

implies

$$\triangle\log|S|^2 \ge R \tag{2.10}$$

which holds in weak sense. Theorem 2.2 follows by integrating $\triangle\log|S|^2$ over geodesic balls.

We remark that Theorem 2.2 plays a crucial role in the Ricci flow, which ensures the maximal volume growth condition can be preserved.

3. Maximal volume growth

The maximal volume growth case has been thoroughly studied in literatures. One possible reason is that in some sense, this condition ensures that the geometry of the manifold at infinity is already close to that of the Euclidean space. Mok-Siu-Yau [33] says that the curvature can not decay faster than quadratic. But we do have many examples whose curvature decays exactly in a quadratic manner. After the seminal work of [33], the first essential breakthrough was made by Mok [30].

THEOREM 3.1 (Mok [30]). *Let M be a complete noncompact Kähler manifold of complex dimension n with positive holomorphic bisectional curvature. Suppose there exist positive constants C_1, C_2 such that for a fixed base point x_0,*

$$\text{(i)}\quad vol(B(x_0, r)) \geq C_1 r^{2n}, \quad 0 \leq r < \infty$$
$$\text{(ii)}\quad 0 < R(x) \leq C_2/(1 + d^2(x_0, x)) \tag{3.1}$$

then M is biholomorphic to an affine algebraic variety. Moreover, if in addition the complex dimension $n = 2$ and

(iii) *the Riemannian sectional curvature of M is positive,*

then M is biholomorphic to $\mathbb{C}^2$.

The proof of Theorem 3.1 is a profound generalization of Kodaria's embedding theorem. The Poincaré-Lelong equation (1.1) gives a psh function u of logarithmic growth. The L^2- estimate implies that the algebra $P(X)$ of holomorphic functions of polynomial growth and its quotient field $R(X)$ have plentiful functions. The main tool is a multiplicity estimate for functions in $P(X)$, i.e., for any function $f \in P(X)$, $x \in X$,

$$mult_x(f = 0) \leq C deg(f) \tag{3.2}$$

where $deg(f)$ is the growth order of f w.r.t. the geodesic distance, C is a constant independent of f. Multiplicity estimate immediately implies that the field of rational functions $R(X)$ is a finite extension of a purely transcendental extension of $\mathbb{C}$ of degree n. More precisely, $R(X) = \mathbb{C}(f_1, \cdots, f_n, g/h)$, $f_i, g, h \in P(X)$. To prove the map $F = (f_1.....f_n, g, h)$ of X into an affine algebraic variety misses only finite number of subvarieties, one can use Skoda's L^2-estimate for the ideal problem. Finally, by adding finite number of holomorphic functions of polynomial growth, F can be desingularized and completed to a proper embedding.

In his Ph.D thesis [40], W.X. Shi initiated the Ricci flow approach to solve Yau's uniformization conjecture. He replaced the pointwise quadratic curvature decay condition in Mok's theorem with an averaged one:

THEOREM 3.2 (Shi [40]). *Let M be a complete noncompact Kähler manifold of complex dimension n with bounded and positive holomorphic bisectional curvature. Suppose there exist positive constants C_1, C_2 such that for a fixed base point x_0,*

$$\text{(i)}\ vol(B(x_0, r)) \geq C_1 r^{2n}, \quad 0 \leq r < \infty$$
$$\text{(ii)}\ \frac{1}{vol(B(x, a))} \int_{B(x,a)} R dv \leq C_2/(1 + a^2), \tag{3.3}$$

then M is biholomorphic to a pseudo-convex domain in $\mathbb{C}^n$.

Shi [40] claimed that the manifold is biholomorphic to $\mathbb{C}^n$. The proof contains a gap which can be explained in the following. First of all, the Kähler condition, the positivity of bisectional curvature and the maximal volume growth are all preserved by the Ricci flow. Shi developed an elegant a priori estimate, from which he proved that the solution exists for all $t > 0$ and the scalar curvature satisfies

$$R(x,t) \leq \frac{C}{1+t}[\log(2+t)]^3. \tag{3.4}$$

Fix a point $P \in M$, and a nonzero vector $v \in T_P M$, Shi proved that $\hat{g}(x,t) = \frac{1}{|v|^2_{g_t}} g(x,t)$ converges in C^∞_{loc}-topology to a flat Kähler metric as $t \to \infty$. The regret is that we do not know whether the limit metric is complete or not. But it is enough to conclude that the manifold is biholomorphic to a pseudo-convex domain.

The quadratic curvature decay condition was further removed in complex dimension 2 in [**13**]. More precisely, we proved that

Theorem 3.3 (Chen-Tang-Zhu [**13**]). *Let M be a complete noncompact Kähler manifold of complex dimension 2 with bounded and positive holomorphic bisectional curvature. Suppose there exist a positive constant C_1 such that for a fixed base point x_0,*

$$vol(B(x_0,r)) \geq C_1 r^4, \quad 0 \leq r < \infty, \tag{3.5}$$

then M is biholomorphic to $\mathbb{C}^2$.

This is a result in this direction for the first time that there is no any curvature decay assumption. For the proof of Theorem 3.3, we should answer the question in several levels. The proof consists of three parts. In the first part, we answer the question in topological category, i.e. we prove the manifold is homeomorphic to $\mathbb{R}^4$. The argument is the following. By Theorem 2.2, the maximal volume growth condition is preserved (with the same asymptotic volume ratio) under the Ricci flow. Via a blow up and blow down argument, all possible singularity models can be excluded except the Type III singularities in Hamilton's classification [**23**], i.e., the solution of the Ricci flow exists for all time $t > 0$ and the scalar curvature satisfies

$$R(x,t) \leq \frac{C}{1+t}. \tag{3.6}$$

Then the injectivity radius of the evolving metric at any point is greater than $c\sqrt{1+t}$ and any geodesic ball of radius $c\sqrt{1+t}$ is pseudo-convex. This implies that M is homeomorphic to $\mathbb{R}^4$ from the generalized Poincaré conjecture, moreover, M is Stein as a complex manifold.

In the second part, in order to find a psh function, we try to solve the Poincaré-Lelong equation (1.1). Note that we do not assume any curvature decay condition in advance. The idea is to transform the decay estimate in time (3.6) of the evolving metrics to the decay estimate in space at $t = 0$, via a priori estimates. More precisely, by combining the equation

$$R(\cdot,0) = \triangle_0 F(x,t) + g^{\alpha\bar{\beta}}(x,0) R_{\alpha\bar{\beta}}(x,t) \tag{3.7}$$

where $F(x,t) = \log \frac{det(g_{\alpha\bar{\beta}}(x,t))}{det(g_{\alpha\bar{\beta}}(x,0))}$, and the estimate (3.6), we deduced that

$$\int_{B(x,a)} R(y)d(x,y)^{-2}dv(y) \le C\log(2+a), \tag{3.8}$$

for all $a > 0$.

This estimate is enough to solve the Poincaré-Lelong equation (1.1) to find a strictly psh function of logarithmic growth:

$$|u|(x) \le C\log(2 + d(x, x_0)). \tag{3.9}$$

In the third part, we basically followed the approach of Mok [**30**] to construct a biholomorphic map from M to a quasi-affine algebraic variety. Combining with a classical theorem of Ramanujam [**38**], we concluded that M is biholomorphic to $\mathbb{C}^2$.

The dimension restriction in Theorem 3.3 was further removed by Chau-Tam [**6**]. They proved the following theorem:

Theorem 3.4 (Chau-Tam [**6**]). *Let M be a complete noncompact Kähler manifold of complex dimension n with bounded and nonnegative holomorphic bisectional curvature. Suppose there exists a positive constant C_1 such that for a fixed base point x_0,*

$$vol(B(x_0, r)) \ge C_1 r^{2n}, \quad 0 \le r < \infty, \tag{3.10}$$

then M is biholomorphic to $\mathbb{C}^n$.

The proof is sketched as follows. The Ricci flow was also used to deform the initial metric, the Kählerity, nonnegativity of bisectional curvature, and maximal volume growth are all preserved as before. The linear curvature decay (3.6) was extended to all dimensions by Ni [**36**]. When curvature operator is nonnegative, the result can be proved using the standard blowing up and blowing down argument, see [**15**]. Let $\tilde{g}(t) = e^{-t}g(e^t)$, then

$$\frac{\partial}{\partial t}\tilde{g}_{i\bar{j}}(x,t) = -\tilde{R}_{i\bar{j}}(x,t) - \tilde{g}_{ij}(x,t) \tag{3.11}$$

for all $t \in (-\infty, \infty)$. One may expect the solution will be close to an expanding Kähler Ricci soliton. For the expanding Kähler Ricci soliton, there is a biholomorphic map verifying the condition in Rosay-Rudin-Varolin's result [**39**], [**45**].

Fix a point $P \in M$, since the injectivity radius of $\tilde{g}(t)$ is uniformly bounded from below, using Theorem 1.3, there exists a $r > 0$ such that for each $i \in \mathbb{N}$, one can construct a biholomorphic map $\Psi_i : B(r) = \{|z| < r\} \to \Psi_i(B(r)) \subset M$, such that $\Psi_i(B(r))$ contains a geodesic ball of radius $r/2$ around P of the metric $\tilde{g}_i$. Moreover, $\Psi_i^*\tilde{g}_i$ is close to the Euclidean metric. Fix a large N, let $F_i = \Psi^{-1}_{(i+1)N} \circ \Psi^{-1}_{(i)N}$ be a biholomorphic map from $B(r)$ to its image. The key point is to prove that the map F_i is asymptotically close to a single map. Cao's Li-Yau-Hamilton inequality [**3**] plays an important role in the argument. For any $t_k \to \infty$, the solution $\tilde{g}(t + t_k)$ behaves close to an expanding Kähler Ricci soliton, this ensures that F_i is asymptotically close to a single map F, see [6] for details.

Recently, the bounded curvature condition in Chau-Tam's theorem was further removed by Liu [27]. Some new ideas from Cheeger-Colding theory to this problem were introduced.

THEOREM 3.5 (Liu [27]). *Let M be a complete non-compact Kähler manifold of complex dimension n with nonnegative holomorphic bisectional curvature. Suppose there exists a positive constant C_1 such that for a fixed base point x_0,*

$$vol(B(x_0, r)) \geq C_1 r^{2n}, \quad 0 \leq r < \infty, \tag{3.12}$$

then M is biholomorphic to $\mathbb{C}^n$.

Liu [27] studied the Gromov-Hausdorff limit of Kähler manifolds whose holomorphic bisectional curvatures are bounded from below and volumes are non-collapsed. The celebrated three circle theorem [29] and Cheeger-Colding theory ensured that $\bar{\partial}$-equation on the holomorphic tangent bundle can be solved. This eventually constructed a global integrable holomorphic vector field retracting to a point, and provided the desired biholomorphic map from M to $\mathbb{C}^n$. In a previous paper [28], Liu proved that the ring of holomorphic functions of polynomial growth on a complete manifold with nonnegative bisectional curvature is finitely generated, confirming another conjecture of Yau [47].

We remark that in [26], Lee-Tam proved that on a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth, the Kähler Ricci flow can always be solved for a short time, and it preserves all these good conditions, moreover, the curvature becomes bounded instantly for $t > 0$. Combining with Theorem 3.4, this provides an alternative proof of Theorem 3.5.

4. Non-maximal volume growth

Yau's uniformization conjecture is true in the maximal volume growth case, this gives us a strong confidence that the conjecture should always be true. The purpose of this section is to review some progress and ideas for the non-maximal volume growth case, for which it is more difficult since the geometry of the manifolds at infinity is more complicated.

We should mention the following result of Shi [41]:

THEOREM 4.1. *Suppose M^n is a complete noncompact Kähler manifold with bounded and positive sectional curvature. Suppose there exist constants $0 < \epsilon$, $C_1 < \infty$ such that*

$$\int_{B(x_0,r)} R(x)dx \leq \frac{C_1}{(1+r)^{1+\epsilon}} vol(B(x_0, r)), \forall x_0 \in M, 0 < r < \infty. \tag{4.1}$$

Then M^n is biholomorphic to a pseudo-convex domain of $\mathbb{C}^n$.

Shi [41] used the Ricci flow to deform the initial metric and proved that the solution exists for all time $t > 0$ and the scalar curvature satisfies

$$R(x,t) \leq C(1+t)^{-\frac{2\epsilon}{1+\epsilon}}. \tag{4.2}$$

Using L^2-estimate for $\bar{\partial}$-operator, there exists a smooth family of holomorphic maps $\tilde{\phi}_t : \{|z| < c(1+t)^{\frac{\epsilon}{1+\epsilon}}\} \to M^n$, $\tilde{\phi}_t(0) = P$, $\tilde{\phi}_t$ is a local diffeomorphism and essentially close to the exponential map of the metric g_t at P. The inverse $\tilde{\phi}_t^{-1}$ can be defined on any fixed compact set for all large t, since M admits a sequence of bounded convex exhausting domains $P \in \Omega_1 \subset \Omega_2 \cdots$. More precisely, there exists a sequence of $t_1 < t_2 < \cdots$ such that $\Phi_t = \tilde{\phi}_t^{-1}$ is a biholomorphic map from Ω_k into $\mathbb{C}^n$ for all $t \geq t_k$. Using the results of Anderson-Lempert [1] and Forstneric-Rosay [20], with the help of the family Φ_{t_k}, for each $k \geq 3$, one can construct a family of biholomorphic maps $\Phi_{k,t}$ such that

1) $\Phi_{k,t} = \Phi_t$, for $t \geq t_k$,
2) for $t \leq t_k$, $\Phi_{k,t}$ is defined at least on Ω_k and very close to $\Phi_{k-1,t}$ on Ω_{k-2}.

The limit $\Phi = \lim_{k\to\infty} \Phi_{k,t_3}$ is a biholomorphic map from M^n to $\mathbb{C}^n$ such that $\Phi(M)$ is pseudo-convex domain in $\mathbb{C}^n$.

Note that the positivity of sectional curvature already implies the manifold is Stein and diffeomorphic to the Euclidean space. This is still not known for bisectional curvature up to now. We remark that if the bisectional curvature is nonnegative, bounded, and decays uniformly and linearly in the average sense:

$$\int_{B(x_0,r)} R(x)dx \leq \frac{C_1}{(1+r)} vol(B(x_0,r)), \forall x_0 \in M, 0 < r < \infty, \tag{4.3}$$

Chau-Tam [9] proved that the manifold is holomorphically covered by a pseudoconvex domain (in $\mathbb{C}^n$), which is homeomorphic to $\mathbb{R}^{2n}$. The following result obtained by the first author in [12] is a partial extension of the above Theorem 4.1, replacing sectional curvature by holomorphic bisectional curvature.

THEOREM 4.2. *Let M be a complete n-dimensional Kähler manifold with bounded and positive holomorphic bisectional curvature such that there exist $\frac{n}{n+1} < \epsilon < 1$, and $C > 0$ such that*

$$\frac{1}{vol(B(y,a))} \int_{B(y,a)} R(x)dv \leq \frac{C}{a^{1+\epsilon}}, \quad \forall y \in M, 0 \leq a < \infty. \tag{4.4}$$

Then M is homeomorphic to $\mathbb{R}^{2n}$ and biholomorphic to a pseudo-convex domain in $\mathbb{C}^n$.

In general, if there are constants $0 < \delta < \epsilon < 1$, and $C > 0$ such that

$$\begin{aligned} & vol(B(x_0, a)) \geq C^{-1} a^{2n-\delta}, \forall a \geq 1 \\ & \frac{1}{vol(B(y,a))} \int_{B(y,a)} R(x) dv \leq \frac{C}{a^{1+\epsilon}}, \quad \forall y \in M, 0 \leq a < \infty, \end{aligned} \tag{4.5}$$

then the conclusion of Theorem 4.2 holds.

The $1+\epsilon$-order curvature decay implies that the Poincaré-Lelong equation (1.1) can be solved and solution u grows in $1-\epsilon$ order. This in turn yields that the volume growth order is at least $n(1+\epsilon)$. The condition $\epsilon > \frac{n}{n+1}$ ensures that (4.5) holds for $\delta = n(1-\epsilon)$.

The proof of the result is also using the Ricci flow.

The key point is to prove that the injectivity radius $inj(M, g_t) \to \infty$ as $t \to \infty$. On the one hand, the scalar curvature of the solution decays as follows:

$$R(x,t) \leq C(1+t)^{-\frac{2\epsilon}{1+\epsilon}}. \tag{4.6}$$

By estimating the volume element, one can prove

$$\frac{vol_t(B_t(x_0,r))}{vol_0(B(x_0,r))} \geq 1 - C\frac{t}{r^{1+\epsilon}}\left[1 + \left(\frac{t}{r^{1+\epsilon}}\right)^{\frac{1-\epsilon}{1+\epsilon}}\right]. \tag{4.7}$$

This gives

$$vol_t(B_t(x_0,r)) \geq \frac{1}{2} vol_0(B(x_0,r)) \tag{4.8}$$

for $r = C^{-1} t^{\frac{1}{1+\epsilon}}$. By Bishop-Gromov volume comparison theorem, we have

$$vol_t(B_t(x_0,a)) \geq \left(\frac{a}{r}\right)^{2n} vol_t(B_t(x_0,r)) \tag{4.9}$$

for all $a \leq r$.

Note that, the injectivity radius at x_0 can be bounded from below by

$$inj_t(M,x_0) \geq C^{-1} a \frac{vol_t(B_t(x_0,a))}{vol_t(B_t(x_0,a)) + vol(B_{-t^{-\frac{2\epsilon}{1+\epsilon}}}(a))} \tag{4.10}$$

where $a = C^{-1} t^{\frac{\epsilon}{1+\epsilon}}$, $vol(B_{-t^{-\frac{2\epsilon}{1+\epsilon}}}(a))$ is the volume of geodesic ball of radius a in the space form of constant curvature $-t^{-\frac{2\epsilon}{1+\epsilon}}$, see [**11**] and [**18**]. Combining with (4.5), (4.8) and (4.9), we have

$$\begin{aligned} inj_t(M,x_0) &\geq C^{-1} t^{\frac{\epsilon}{1+\epsilon}} \frac{(\frac{a}{r})^{2n} r^{2n-\delta}}{(\frac{a}{r})^{2n} r^{2n-\delta} + a^{2n}} \\ &\geq C^{-1} t^{\frac{\epsilon-\delta}{1+\epsilon}} \to \infty \end{aligned} \tag{4.11}$$

as $t \to \infty$, since we have assumed $\epsilon - \delta > 0$. Note that the metric is shrinking under the Ricci flow, the ball of radius $C^{-1} t^{\frac{\epsilon-\delta}{2(1+\epsilon)}}$ at time t is a very large region.

By assuming the finiteness of the first Chern number C_1^n and a mixture of the curvature decay and volume growth, To [44] proved the following compactification theorem:

THEOREM 4.3 (To [44]). *Let X be an n-dimensional noncompact complete Kähler manifold of positive Ricci curvature and of finite topological type. Suppose for some base point $x_0 \in X$ that there exist positive constants k_1, k_2, k_3 and a positive real number p such that*

$$\begin{aligned} &\text{(i) } \textit{Scalar Curvature } < k_1/(d(x_0;x))^p \\ &\text{(ii) } \int_X Ric^n < \infty, \\ &\text{(iii) } \int_{B(x_0,a)} \omega^n/(1+d(x_0;x))^{np} < k_2 \log a, \forall a \geq 1, \\ &\text{(iv) } |Bisect(v,w)| < k_3 Ric(v,v) \end{aligned} \tag{4.12}$$

for all unit tangent vectors $v, w \in T'(X)$ and $x \in X$. Then X is biholomorphic to a quasi-projective variety. Moreover, if $p \geq 2$, the theorem is valid without assuming condition (iii).

The embedding map was constructed by using holomorphic sections of pluri-anticanonical line bundles. This method has been previously systematically developed in [43], [31], [34]. The finiteness of Chern numbers C_1^n (i.e. the above condition (ii)) is a natural condition to derive the "Bezout estimate" for divisors of holomorphic sections.

THEOREM 4.4 (Chen-Zhu [16]). *Let M^n be a complete noncompact Kähler manifold with bounded and positive sectional curvature, and*

$$\int_{M^n} Ric^n < \infty.$$

Then M^n is biholomorphic to a quasi-projective variety. In the case of complex dimension $n = 2$, M^2 is biholomorphic to $\mathbb{C}^2$.

We conjecture that the Chern number $\int Ric^n$ is always finite for a complete noncompact Kähler manifold with positive bisectional curvature.

A partial result in this direction is the following:

THEOREM 4.5 (Chen-Zhu [17]). *Let M^n be a complete Kähler manifold with bounded sectional curvature and positive Ricci curvature. Suppose M^n admits a Hermitian holomorphic line bundle L such that the curvature $C_1(L)$ of L is positive and bounded and satisfies*

$$\int_{M^n} C_1(L)^n < \infty.$$

Then

$$\int_{M^n} Ric^n < \infty.$$

Corollary 4.6. *Let M^n be a complete Kähler manifold with bounded sectional curvature and positive Ricci curvature. Suppose M^n admits a strictly psh φ such that $\sqrt{-1}\partial\bar{\partial}\varphi$ is bounded and*

$$\int_{M^n}(\sqrt{-1}\partial\bar{\partial}\varphi)^n < \infty.$$

Then

$$\int_{M^n} Ric^n < \infty.$$

Suppose we have proved M^n is biholomorphic to a pseudo-convex domain of $\mathbb{C}^n$ (as in Theorem 3.1), let $z_1, z_2, \cdots z_n$ be coordinates in $\mathbb{C}^n$, and set $\phi = \log(1+|z|^2)$, then $\int_{M^n}(\sqrt{-1}\partial\bar{\partial}\phi)^n < \infty$. Thus, to derive the finiteness of the Chern number, it is reduced to find a strictly psh function ϕ with finite Monge-Ampere measure and bounded Levi-form. Recall that we say M^n has minimal volume growth (in the sense of Theorem 2.1), if there exist $x_0 \in M^n$ and $C > 0$ such that

$$vol(B(x_0, r)) \le C(1+r)^n,$$

for all $r > 0$, where n is the complex dimension of the manifold. The following result gives an affirmative answer in the case of minimal volume growth.

Proposition 4.7. *Let M^n be a complete Kähler manifold with positive bisectional curvature. Then there exists a smooth strictly psh function ϕ satisfying*

$$\begin{aligned} &\text{(i)} \quad |\phi(x)| \le C(1+d(x,x_0)); \\ &\text{(ii)} \quad |\nabla\phi|(x) + |\partial\bar{\partial}\phi|(x) \le C; \end{aligned} \tag{4.13}$$

and if M^n has minimal volume growth, we have

$$\text{(iii)} \quad \int_{M^n}(\sqrt{-1}\partial\bar{\partial}\phi)^n < \infty. \tag{4.14}$$

Proof. Fix a point $P \in M^n$, let

$$b(x) = \sup_{\gamma} \lim_{t\to\infty}(t - d(x, \gamma(t))) \tag{4.15}$$

where the supremum is taken over all geodesic rays departing from P. Then $b(x)$ is Lipschitz continuous with Lipschitz constant 1. By comparison theorem in Riemannian geometry, $b(x)$ is strictly psh. Consider the heat equation

$$\begin{cases} \left(\frac{\partial}{\partial t} - \triangle\right)u = 0, \\ u\mid_{t=0} = b(x) \end{cases}$$

where $\triangle u = g^{\alpha\bar{\beta}}u_{\alpha\bar{\beta}}$. As mentioned in Section 1, the Levi form $\sqrt{-1}\partial\bar{\partial}u$ satisfies a parabolic Lichnerowicz equation:

$$\frac{\partial}{\partial t}u_{\alpha\bar{\beta}} = \triangle u_{\alpha\bar{\beta}} + R_{\alpha\bar{\beta}\xi\bar{\eta}}u_{\eta\bar{\xi}} - \frac{1}{2}(R_{\alpha\bar{\xi}}u_{\xi\bar{\beta}} + R_{\xi\bar{\beta}}u_{\alpha\bar{\xi}}),$$

which implies by maximum principle that $u(x,t)$ is strictly psh for any $t>0$.

By direct computations, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\triangle\right)|u_\alpha|^2=-|u_{\alpha\bar\beta}|^2-|u_{\alpha\beta}|^2-R_{\alpha\bar\beta}u_\beta u_{\bar\alpha}\\&\left(\frac{\partial}{\partial t}-\triangle\right)|u_{\alpha\bar\beta}|^2=-|u_{\alpha\bar\beta\gamma}|^2-|u_{\alpha\bar\beta\bar\gamma}|^2-\sum_{\alpha,\beta}R_{\alpha\bar\alpha\beta\bar\beta}(\lambda_\alpha-\lambda_\beta)^2,\end{aligned}\tag{4.16}$$

where $u_{\alpha\bar\beta}=\lambda_\alpha\delta_{\alpha\beta}$ is diagonalized at a point. Combining the two equations in (4.16), and using the curvature condition, we have

$$\left(\frac{\partial}{\partial t}-\triangle\right)(|u_\alpha|^2+t|u_{\alpha\bar\beta}|^2)\le 0.\tag{4.17}$$

Applying the maximum principle on (4.17) implies

$$\begin{aligned}|\nabla u|^2=2|u_\alpha|^2\le 1,\\|u_{\alpha\bar\beta}|^2(x,t)\le\frac{1}{2t}.\end{aligned}\tag{4.18}$$

Set $\phi(x)=u(x,1)$, (i) and (ii) in (4.13) follow from (4.18). To prove (iii), as in (2.7), we have

$$\int_{B(x_0,2r)}\xi^n(\sqrt{-1}\partial\bar\partial\phi)^n\le\frac{C}{r^n}\int_{B(x_0,2r)}\omega^n,\tag{4.19}$$

where $\xi_{B(x_0,r)}\equiv 1$, $\xi_{M^n\setminus B(x_0,2r)}=0$, $0\le\xi\le 1$, $|\nabla\xi|\le\frac{C}{r}$. The minimal volume growth condition implies that the right hand side of (4.19) is bounded, hence there is a constant C independent of r such that

$$\int_{B(x_0,r)}(\sqrt{-1}\partial\bar\partial\phi)^n\le C.\tag{4.20}$$

Let $r\to\infty$, (4.14) follows. Then ϕ fulfills all requirements of Proposition 4.7. □

Thus the combination of Theorem 4.4, Corollary 4.6 and Proposition 4.7 gives a partial answer to Yau's uniformization conjecture in the case of minimal volume growth. Finally, we state a new result by removing the condition of bounded curvature.

THEOREM 4.8. *Let M^n be a complete noncompact Kähler manifold with positive sectional curvature. Suppose M^n has minimal volume growth. Then M^n is biholomorphic to an affine algebraic variety. Moreover, if $n=2$, then M is biholomorphic to $\mathbb{C}^2$.*

The proof has recourse to a theorem of Demailly on characterizing affine algebraic varieties.

THEOREM 4.9 ([**19**]). *Let X^n be a complex manifold of complex dimension n. Then X^n is biholomorphic to an affine variety if and only if X^n possesses a smooth strictly psh proper function ϕ with the following properties:*

(a) $\int_{X^n}(\sqrt{-1}\partial\bar{\partial}\phi)^n < \infty$;

(b) *The Ricci curvature of the Kähler metric* $\beta = \sqrt{-1}\partial\bar{\partial}e^{\phi}$ *satisfies*

$$Ricci(\beta) \geq -\sqrt{-1}\partial\bar{\partial}\psi \tag{4.21}$$

for some C^0 *function* $\psi \leq A\phi + B$, *where* A, B *are positive constants;*

(c) $dim_{\mathbb{R}} H^{2q}(X^n, \mathbb{R}) < \infty$, *for all* $q \geq 0$.

Proof. of Theorem 4.8. Let $b(x)$ be the Busemann function in (4.15). Since the sectional curvature is positive, by comparison theorem, there is a constant $C_1 > 0$ such that

$$\frac{1}{2}d(x,P) - C_1 \leq b(x) \leq d(x,P). \tag{4.22}$$

As in Proposition 4.7, let $u(x,t)$ be the heat equation deformation of $b(x)$, and set $\phi(x) = u(x,1)$. Then ϕ satisfies

$$\frac{1}{3}d(x,P) - C_2 \leq \phi(x) \leq d(x,P) + C_2. \tag{4.23}$$

In particular, ϕ is proper and satisfies (i)(ii)(iii) in Proposition 4.7. Now, (c) is trivial from a theorem of Gromoll and Meyer. It remains to check (b).

From

$$\beta_{i\bar{j}} \triangleq \partial_i\partial_{\bar{j}}(e^{\phi}) = e^{\phi}(\phi_{i\bar{j}} + \phi_i\phi_{\bar{j}})$$

we know

(4.24)

$$\begin{aligned} Ric(\beta)_{i\bar{j}} &= -(\log det(\beta_{k\bar{l}}))_{i\bar{j}} = Ric(g)_{i\bar{j}} - n\phi_{i\bar{j}} - (\log \frac{det(\phi_{k\bar{l}} + \phi_k\phi_{\bar{l}})}{det(g_{k\bar{l}})})_{i\bar{j}} \\ &\geq -\psi_{i\bar{j}} \end{aligned}$$

where $\psi = n\phi + \log \frac{det(\phi_{k\bar{l}}+\phi_k\phi_{\bar{l}})}{det(g_{k\bar{l}})}$. Combining with (4.13), we obtain

$$\psi \leq n\phi + C$$

which verifies (b). The proof is completed. $\square$

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