Revisiting S.S. Chern's article "The geometry of G-structures", 55 years after its publication

Jean-Pierre Bourguignon

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Abstract. Professor Chern Shiing-Shen contributed to Geometry in many different ways, proving important results and, on many occasions, providing a view on how the field could move forward. One of his articles that both presented a remarkable synthesis of new developments and offered his vision for future ones is *"The Geometry of G-structures"* published in 1966.

In the present text, I propose an analysis of the article's main features. It also contains a testimony by Chern himself on how some of his main contributions came about and his view on what may come next, that proved remarkably correct.

Introduction

For this lecture given on the occasion of the celebration of S.S. Chern's 110$^{\text{th}}$ birthday, I found it appropriate to discuss his article *"The Geometry of G-Structures"* (cf. [**19**]) that appeared in 1966, i.e. $55\,(=\frac{110}{2})$ years ago. This article has been a landmark contribution to the development and the recognition of Differential Geometry as a major field.

One should be aware that the very central role that the field now enjoys in Mathematics and Theoretical Physics is the result of a progressive transformation of the internal architecture of Mathematics that developed all along the second half of the century. Among others, giants such as Chern, Sir Michael Atiyah and Isadore M. Singer greatly contributed to this transformation. Note that, at the time, Chern was also interested in other topics as the announcement of a lecture he gave at IHÉS in 1964 shows: it was entitled *"Minimal Surfaces Embedded in n-space"*.

Before looking at the article in some more detail, I would like to draw a parallel with another comprehensive document produced this one by Élie

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Cartan, his *"Notice des travaux scientifiques"* (cf. [11]). It was written in 1931 in the context of Cartan's candidacy to the Académie des Sciences de Paris. It is well known that Chern often presented Élie Cartan as his main mentor as far as his approach to Geometry goes. The 33-pages long obituary of Élie Cartan that Chern and Claude Chevalley published (cf. [20]) in the Bulletin of the American Mathematical Society is a remarkable tribute to Élie Cartan underlying the importance of his work.

Coming back to the very special historical moments that the 1940s and the 1950s have been in the history of Mathematics, one needs to emphasise the major impact the introduction of new concepts, such as fibre bundles, had on Geometry, as it forced to rethink a number of basic approaches and to develop new tools. On several occasions, Chern stated unambiguously that Élie Cartan understood Fibre Bundle Theory before it was formally defined by Hasler Whitney (cf. [46], [47] and [48]) and Charles Ehresmann (cf. [23], [24] and [25]) among others. The book [43] became later a standard reference on this topic and the synthesis by Daniel Bernard (cf. [5]) can also be mentioned.

At the same time, the coming together of several branches of Mathematics (Topology, Geometry, and the theory of several complex variables) changed considerably the landscape of the field. This is something that Chern's article illustrates remarkably.

1. An overview of the article

Here is what Chern says about the topic of the article: *"A unifying idea is the notion of a G-structure, which is the modern version of a local equivalence problem first emphasized and exploited in its various special cases by Élie Cartan."*

The article is an expanded version of an AMS Colloquium lecture Chern gave back in 1960. Already in 1950, in the lecture entitled *"Differential Geometry of Fibre Bundles"* he gave at the International Congress of Mathematicians in Harvard (cf. [17]) Chern develops a very complete view of the impact the newly introduced Fibre Bundle Theory will have on Geometry.

In the first section of the article, he describes the general approach he takes as follows: *"Two general problems are of importance:*

- *I. Existence or nonexistence of certain structures on a manifold;*
- *II. Local and global properties of a given structure."*

Chern then specifies: *"We will emphasize simple and concrete problems, at the expense of generality."*

Here is the table of contents of the article:

(1) Introduction
(2) Riemannian structure
(3) Connections
(4) G-structure
(5) Harmonic forms

(6) Leaved structure
(7) Complex structure
(8) Sheaves
(9) Characteristic classes
(10) Riemann-Roch, Hirzebruch, Grothendieck, and Atiyah-Singer theorems
(11) Holomorphic mappings of complex analytic manifolds (I. Compact manifolds; II. Equidistribution)
(12) Isometric mappings of Riemannian manifolds
(13) General theory of G-structures

Let G be a Lie group. Here is what Chern says in Section 4 about G-structures. He works with n-dimensional differentiable manifolds M. In this context, there is a natural bundle to consider, namely the tangent bundle. One can consider of course the vector bundle $TM \longrightarrow M$ of tangent vectors to M, or equivalently the principal Gl_n-bundle of linear frames, that we denote by $GlM \longrightarrow M$. For G a subgroup of Gl_n, a G-structure on an n-dimensional manifold is a reduction of the structure group of the tangent bundle to G.

Here is what it means technically. Let M be a manifold with an atlas of charts (U_i, x^i). The tangent bundle over M inherits a natural atlas of charts (x^i, X^i) over TU_i: at a vector v in TM, the $(X^i(v))$ are the components of v in the natural basis $(\partial/\partial x^i)$. This construction also allows to trivialise locally the bundle of linear frames $GlM \longrightarrow M$.

To reduce the tangent bundle of M to a group G means choosing a G-subbundle of the bundle of linear frames GlM. Since G is a subgroup of Gl_n, for any frame in M, one can consider its orbit under G.

In Section 4, Chern introduces examples of G-structures:

- An O_n-structure determines a Riemannian metric and vice-versa;
- When $n = 2m$, one can consider the subgroup $Gl_m\mathbb{C}$ of linear maps that preserve the complex structure for an identification of $\mathbb{R}^{2m}$ with $\mathbb{C}^m$; a $Gl_m\mathbb{C}$-structure is an almost complex structure;
- He discusses conformal structures and structures determined by a connection, whose automorphism group is automatically a Lie group;
- He also considers structures involving elements of contact of higher order, e.g. in relation with projective geometry.

He makes a brief presentation of the equivalence problem. The general problem of equivalence has been formulated as a problem on exterior differential systems of a certain type. In the real analytic case, Élie Cartan (cf. [**10**]) and Masatake Kuranishi (cf. [**34**]) proved that, in a finite number of steps, the system can be prolonged either to a system without solution or to a system in involution, hence solvable.

He mentions two instances where the problem of equivalence is not solved:

(1) Almost complex structures, except for $m = 2$ by a theorem of Paulette Libermann (cf. [36]);
(2) Symplectic structures; note that this is the only place in the article where they are mentioned.

2. Riemannian structures

Example 3 in Section 4 deals with $G = SO_2 (= S^1)$. This is of course the case of 2-dimensional Riemannian metrics on an oriented surface. If one uses an angle α to parametrise the circle and if one introduces the 1-forms ϕ_1 and ϕ_2 defined as:

$$\phi_1 = \cos\alpha\,\theta_1 + \sin\alpha\,\theta_2\ ,$$
$$\phi_2 = -\sin\alpha\,\theta_1 + \cos\alpha\,\theta_2$$

for a coframe (θ_1, θ_2), one can uniquely determine a 1-form

$$\pi = d\alpha + \text{lin. comb.}(\theta_1, \theta_2)$$

by the conditions $d\phi_1 = \pi \wedge \phi_2$, $d\phi_2 = -\pi \wedge \phi_1$. Then the key formula is: $d\pi = K\,\phi_1 \wedge \phi_2$, where K is the Gaussian curvature.

It is indeed inspired by this *transgression formula* that Chern found his new proof of the Gauss-Bonnet formula (cf. [15]) as he explained in the interview he gave me at the AMS Summer Institute held at UCLA in July 1990[1] (cf [9] for the full text of the interview in French).

To my question:*"When did you realise that you had put your hands on something important?"*, Chern replies: *"Oh, right at the beginning. You see, everything began with my proof of the Gauss-Bonnet formula. That year, 1943, I went to the Institute for Advanced Study in Princeton. There I met André Weil. He had just finished his article with Carl. B. Allendoerfer on the Gauss-Bonnet formula (cf. [2]), and the way they cut a Riemannian manifold into manifolds with boundary, glue them together and so on. André told me: "why could not there be a direct proof?" I therefore considered the simplest case, that of a surface, and I realised that, in this case the proof was given by the transgression formula."*

To my comment: *"By going to the frame bundle?"*, Chern replies: *"Yes, and the transgression formula contains not only the proof of the Gauss-Bonnet formula for surfaces with boundary, but also the proof of the Theorema Egregium of Gauss. All this comes from this single formula. I was pleased to have discovered that, even in the case of a 2-dimensional manifold, this is something people like Gauss and Darboux did not have. It is Heinz Hopf who called the attention to the higher dimensional case: in a survey article entitled* "Differentialgeometrie und Topologische Gestalt" *published in the Jahresberichte der Deutschen Mathematiker Vereinigung, he says that it is one of the most important and difficult problems of Differential Geometry. In this case, one needs to transgress to the bundle of*

[1]The video of the interview shot by Anthony Philips, finally entitled *"If possible, do nothing"*, is available at https://www.youtube.com/watch?v=vConuqi5vT0.

unit vectors and not any more to the frame bundle. It took me some time to master all this. Then, naturally, if one does something for the Euler class, one should also do it for the Stiefel-Whitney classes. I had difficulties with torsion classes, namely homology classes for which a multiple bounds. Soon, I realised that one must complexify. Then things became much simpler (cf. [**16**]*). One should note that, at this time, the atmosphere in Princeton was dominated by Topology; one did not speak of cochains but of relative cohomology. It took some time for complex characteristic classes to become useful. But I was very pleased with this proof of the Gauss-Bonnet theorem.*

I should also mention that, while at the Institute, I showed the manuscript to Hermann WEYL *who congratulated me. Before that, I had written an article entitled* "The Geometry of Isotropic Surfaces" *that got published in the Annals of Mathematics of which Hermann* WEYL *had been the rapporteur. When I sent this article from China to* LEFSCHETZ*, who was then the editor-in-chief of the Annals, I had received a letter saying* "we got so many articles that it would be good if you could withdraw yours"*. I did not reply to this letter, but about a month later I got another letter from him saying* "your article has been examined by a rapporteur who warmly recommended it, and we will be pleased to publish it in the Annals"*. The rapporteur wrote a long report of almost ten pages, I think. After arriving in Princeton, one day Hermann* WEYL *asked me:* "CHERN, do you know who was the rapporteur of your article?"*. That was him. He had heard about this work. We had several contacts, and* WEYL *appreciated a lot my proof of the Gauss-Bonnet formula."*

To my question: *"All in all, you felt quite sure that you were on the right track both because of your personal conviction and thanks to the reactions of people around you?"*, CHERN's reply is a clear *"Yes."*

In Section 9 CHERN discusses a systematic theory of characteristic classes. He starts with the universal bundle theorem transferring the issue of classifying G-bundles to the study of homotopy classes of mappings of a manifold into the classifying space for G. One of the first questions he considers is the determination of obstructions for the reduction of the structure group of a bundle to a subgroup, a key question in the theory of G-structures. He then looks at some examples, focusing in particular on the question of existence of an almost complex structure on a manifold, mentioning other obstructions of a cohomological nature.

He of course also discusses the expression of characteristic classes using the curvature of a connection. He recalls the theorem of André WEIL stating that the cohomology class determined by the image of symmetric multilinear functions on the Lie algebra of G, when substituting the curvature 2-form of a G-connection, does not change. This has later been coined as the *Chern-Weil Theory* (cf. [**18**]).

When applied to $G = U_m$, this provides the curvature expression for the CHERN classes. When applied to $G = SO_{2m}$ CHERN discusses some divisibility properties in relation with the Euler class giving a new proof of the Borel-Serre Theorem (cf. [**6**]) on the existence of almost complex structures on S^{2m}, that leaves, besides the Riemann sphere S^2, only S^6 as a sphere potentially bearing a complex structure.

It is worth pointing out that, in 1938, appeared in the Annals of Mathematics an article by Cornelius Lanczos (cf. [**35**]) who was considering possible quadratic Lagrangians on 4-manifolds in an effort to generalise General Relativity. He noticed that a specific combination of the three natural quadratic forms on the Riemann curvature tensor did not lead to any field equations. A bad news for a physicist looking for a generalisation of the Einstein equation! But a good news for mathematicians as this means that this Lagrangian had the potential to give a topological invariant of the manifold. It indeed provided the expression of the Euler characteristic that was discovered some years later by Chern.

The notion of *holonomy* of a covariant derivative is (briefly) discussed in Section 5 under the heading of harmonic forms. Chern considers a group G which is a subgroup of SO_n, hence for which there is natural Riemannian metric attached to it. In the case G is the holonomy group of the Riemannian connection, he is interested in the refinement in the real cohomology that one can derive from its representation via harmonic forms. When applied to $G = U_m$, this leads to the Hodge decomposition of the cohomology, and to consequences on the Betti numbers.

3. Complex structures

Section 7 is dedicated to the question of existence of a complex structure on a manifold, *"a nontrivial fact"* as he says.

For this question too, he starts considering some examples, namely submanifolds of the obvious complex manifolds $\mathbb{C}^m$ and $\mathbb{C}P^m$ and also quotients thereof. He discusses complex tori and Hopf manifolds, topologically $S^1 \times S^{2m-1}$ that give examples of non algebraic complex manifolds. He also discusses at length more complicated examples coming from blowing up a known complex manifold or from ruled surfaces, recalling Hirzebruch?s result that $S^2 \times S^2$ has an infinite number of (inequivalent) complex structures (cf. [**30**] and [**31**]).

The specific question of the existence of a complex structure on a manifold is taken up further. The Newlander-Nirenberg theorem (cf. [**37**]) takes care of the local theory provided the appropriate tensor measuring the integrability of the candidate holomorphic distribution vanishes. Chern stresses that, for manifolds admitting an almost complex structure, no general method to solve the global problem is available. He continues saying: *"In this respect the question whether S^6 has a complex structure remains one of the most urgent problems on complex manifolds"*. It is still an open question!

He mentions that the example just announced by Antonius Van de Ven (cf. [**45**]) uses the Riemann-Roch-Hirzebruch formula (cf. [**29**] and [**3**]) and therefore is a consequence of the Atiyah-Singer Index Theorem (cf. [**4**]).

Of course of equal importance is the *"determination of all complex structures on a manifold"*. Beyond the example of $S^2 \times S^2$ discussed previously, the rigidity of the projective spaces $\mathbb{C}P^m$ for $m = 2$ or odd is mentioned and also for manifolds for which the 1-dimensional cohomology with values in the sheaf of germs of holomorphic vector fields vanishes. The very important question of structuring the moduli space of complex structures is also touched upon with the mention of the work of Kunihiko KODAIRA and Donald SPENCER (cf. [**33**]).

The question of moduli spaces of complex vector bundles, that will become so important in the future not only in Mathematics but also in Theoretical Physics, is not discussed.

Many questions related to complex geometry are discussed in Section 8 entitled "Sheaves". The section starts with general considerations on the meaning of the cohomology with values in a sheaf with DE RHAM?'s theorem (cf. [**42**]) as an example. He states that *"it is in complex manifolds that the sheaf theory is most useful"* starting with the example of the Dolbeault Theorem (cf. [**22**].) He moves on discussing the Cousin Problem, the notion of "coherent sheaves" and their role in the study of Stein manifolds, as well as the theorem by Henri CARTAN and Jean-Pierre SERRE on the finiteness of the cohomology of analytic coherent sheaves over compact complex manifolds (cf. [**12**]).

One should also mention the work Raoul BOTT and CHERN did on the equidistribution of the zeroes of holomorphic sections of Hermitian bundles (cf. [**7**]).

4. Beyond the differential context

Such a comprehensive survey comes at a moment when moving from local issues to more global ones led to call the field "Differential Geometry in the large" ("Differentialgeometrie im Großen" for the Oberwolfach regular Tagung that CHERN and Wilhelm KLINGENBERG organised for a number of years). Still, we know that new trends came up. This is with this in mind that I rebounded on a statement CHERN made about *"a new dress for Geometry"*. Here is the precise question I asked him when I interviewed him: *"In an article published in the "American Mathematical Monthly", you establish a parallel between the development of Geometry and that of clothing, the concept of a manifold being put in correspondence with modern men and the way they dress themselves. According to you what will be the new clothing of Geometry after manifolds?"*

Here is his reply: *"That was for fun. Geometry should not be limited to so smooth objects. I think that researchers in Differential Geometry should consider singular spaces. It is remarkable that smoothness plays such an important role in Geometry. We would like to believe that continuity should be sufficient, as it is in Topology. But smoothness gets rid of a lot of uninteresting stuff while keeping still interesting and important things. It is* *unbelievable to think that functions have to be twice differentiable to be able to do Differential Geometry. Consider the Calculus of Variations: one needs to be able to speak about critical points, and it is much nicer if functions are differentiable. One must work a bit more to get the second variation and isolate the index. Even if smoothness makes the tools of Algebra and Analysis available, some of the most important properties are nevertheless concentrated at the critical points. One should pay more attention to the concept of stratified manifolds or a variation of this concept. After all, almost all important theorems in Differential Geometry have some discrete or combinatorial equivalents. My view is that smooth manifolds are only one stage. We must consider a more general situation, including going to infinite dimensions. It often happens that the most interesting properties are preserved when one takes this step. One must know how to keep important things and get rid of unimportant ones. This requires work and a lot of insight."*

In the second half of the 20$^{\text{th}}$ century, one of the central developments in Differential Geometry has been the key role played by Non-Linear Analysis, hence the denomination "Global Analysis" that broadened further the field. Indeed, a number of key problems in the field have been resolved thanks to new analytical developments using estimates inspired by geometric considerations.

This is of course the case for the solution in 1976 of a central question in Kählerian Geometry, namely the Calabi conjecture, by Yau Shing Tung (cf. [**49**]), among many other achievements.

To relate more directly to the need to study singularities suggested by Chern, one can cite:

- The 1982 seminal work by Karen Uhlenbeck on *"Removable singularities in Yang-Mills Fields"* (cf. [**44**]);
- The existence of a local flow for the (opposite of the) Ricci curvature, when viewed as a vector field in the space of Riemannian metrics on a compact manifold; the result has been obtained by Denis DeTurck (cf. [**21**]) solving a problem I suggested in 1979 (cf. [**8**]); using the so-called *Ricci flow* to evolve metrics on a manifold led to many beautiful applications by Richard Hamilton (see e.g. [**27**] for the first one) where geometric assumptions are made to avoid the occurence of singularities;
- The control of singularities that may occur in the Ricci flow by Grigori Perelman (cf. [**38**], [**39**] and [**40**]) leading to his solution of the Poincaré Conjecture.

The context of Differential Geometry has been considerably broadened in the last 50 years with the importance given to general Metric Spaces, in particular by Mikhail Gromov. Here are some of the paths followed.

There has been attempts by Alexander Alexandrov (cf. [**1**]) to develop a notion of bounded curvature in the context of metric spaces with enough geodesics. Gromov systematically used a metric in the space of compact metric spaces (cf. [26]), that is now called the Gromov-Hausdorff distance, to derive very powerful convergence theorems impacting Riemannian Geometry but also Group Theory (and also Image Analysis).

Metric Measured Spaces with a control on the volume of balls lie in an intermediate situation between differentiable manifolds and general metric spaces. Many efforts have been dedicated in recent years to extend to them some of the classical tools. The pioneering work of Jeff Cheeger (cf. [13]) shows that a good part of the differential calculus can be extended to them. This led in particular to a deeper understanding of the Poincaré Inequality, that is so important in establishing estimates in Analysis, but also to introducing a new notion of Lipschitz dimension for metric spaces by Cheeger and Bruce Kleiner (cf. [14]).

5. Some conclusions

The link between Geometry and Groups goes back to Hermann von Helmholtz (cf. [28]), Felix Klein and his Erlangen Programme (cf. [32]), Henri Poincaré (cf. [41]) and Élie Cartan.

The article *"The Geometry of G-Structures?* extends this tradition. It shows how, in the context of Bundle Theory, one can study various Geometries and connect them to important developments in Topology and in particular in Algebraic Topology.

The comprehensive approach taken by Professor Chern is very typical of his very broad command on Mathematics at large and has inspired many geometers in the second part of the 20$^{\text{th}}$ century, and continues to be fruitful, hence worth being celebrated.

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INSTITUT DES HAUTES ÉTUDES SCIENTIFIQUES, 35, ROUTE DE CHARTRES, F-91440 BURES-SUR-YVETTE (FRANCE)

Email address: `JPB@ihes.fr`