Cannon-Thurston maps in Kleinian groups and geometric group theory

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Abstract. We give a survey account of Cannon-Thurston maps, both in the original context of Kleinian groups, as well as in the more general context of Geometric Group Theory. Some of the principal applications are mentioned.

Contents

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1. Introduction

A discrete subgroup Γ of $PSL_2(\mathbb{C})$, also called a **Kleinian group**, can be seen from a number of viewpoints:

(1) As a discrete faithful *representation* of an abstract group into $PSL_2(\mathbb{C})$.
(2) *Geometrically*, as acting properly discontinuously by isometries on $\mathbf{H}^3$, or equivalently, as the fundamental group of the 3-manifold $\mathbf{H}^3/\Gamma$.
(3) *Dynamically*, as acting by complex analytic automorphisms of the Riemann sphere $\hat{\mathbb{C}}$.

The last, complex analytic, point of view was taken by in their study of quasiconformal deformations of Fuchsian groups in the 60's, giving rise to the rich theory of quasi-Fuchsian groups. The geometric viewpoint was developed and popularized by Thurston in the 70's and 80's. The relationships between these different points of view have often turned out to be deep. In [**Thu82**], Thurston posed a number of questions that established a conjectural picture connecting and relating them.

We introduce some basic terminology first. The **limit set** of the Kleinian group Γ, denoted by Λ_Γ, is the collection of accumulation points of a Γ-orbit $\Gamma\cdot z$ for some $z \in \hat{\mathbb{C}}$. Λ_Γ is independent of z. It may be thought of as the locus of chaotic dynamics of Γ on $\hat{\mathbb{C}}$, i.e. for Γ non-elementary and any $z \in \Lambda_\Gamma$, $\Gamma \cdot z$ is dense in Λ_Γ. We shall identify the Riemann sphere $\hat{\mathbb{C}}$ with the sphere at infinity S^2 of $\mathbf{H}^3$. The complement of the limit set $\hat{\mathbb{C}} \setminus \Lambda_\Gamma$ is called the domain of discontinuity Ω_Γ of Γ. If the Kleinian group Γ is torsion-free, it acts freely and properly discontinuously on Ω_Γ with a Riemann surface quotient. In [**Thu82**, Problem 14], Thurston raised the following question, which is at the heart of the work that is surveyed in this paper:

Question 1.1. *Suppose Γ has the property that $(\mathbf{H}^3 \cup D_\Gamma)/\Gamma$ is compact. Then is it true that the limit set of any other Kleinian group Γ' isomorphic to Γ is the continuous image of the limit set of Γ, by a continuous map taking the fixed points of an element γ to the fixed points of the corresponding element γ'?*

A special case of Question 1.1 had been answered affirmatively in seminal work of Cannon and Thurston [**CT85, CT07**]:

Theorem 1.2 ([**CT07**]). *Let M be a closed hyperbolic 3-manifold fibering over the circle with fiber Σ. Let $\widetilde{\Sigma}$ and $\widetilde{M}$ denote the universal covers of Σ and M respectively. After identifying $\widetilde{\Sigma}$ (resp. $\widetilde{M}$) with $\mathbf{H}^2$ (resp. $\mathbf{H}^3$), we obtain the compactification $\mathbb{D}^2 = \mathbf{H}^2 \cup S^1$ (resp. $\mathbb{D}^3 = \mathbf{H}^3 \cup S^2$) by attaching the circle S^1 (resp. the sphere S^2) at infinity. Let $i : \Sigma \to M$ denote the inclusion map of the fiber and $\widetilde{i} : \widetilde{\Sigma} \to \widetilde{M}$ the lift to the universal cover. Then $\widetilde{i}$ extends to a continuous map $\hat{i} : \mathbb{D}^2 \to \mathbb{D}^3$.*

A version of Question 1.1 was raised by Cannon and Thurston in the context of closed surface Kleinian groups:

Question 1.3. ([**CT07**, Section 6]) *Suppose that a closed surface group $\pi_1(S)$ acts freely and properly discontinuously on $\mathbf{H}^3$ by isometries such that the quotient manifold has no accidental parabolics. Does the inclusion $\tilde{i} : \widetilde{S} \to \mathbf{H}^3$ extend continuously to the boundary?*

Continuous boundary extensions as in Question 1.3, if they exist, are called **Cannon-Thurston maps**. Question 1.3 is intimately related to a much older question asking if limit sets are locally connected:

Question 1.4. *Let Γ be a finitely generated Kleinian group such that the limit set Λ_Γ is connected. Is Λ_Γ locally connected?*

It is shown in [**CT07**] that for simply degenerate surface Kleinian groups, Questions 1.3 and 1.4 are equivalent, via the Caratheodory extension Theorem. The connection of Question 1.3 to the larger question of connecting dynamics on $\hat{\mathbb{C}}$ to the geometry of $\mathbf{H}^3/\Gamma$ via the celebrated Ending Lamination Theorem of Brock-Canary-Minsky [**BCM12**] is explicated in Section 6.2.

1.1. History of the problem. We give below a brief historical account of the steps that led to the resolution of the above questions.

(1) In the 70's it was believed [**Abi76**] that Question 1.4 had a negative answer for simply degenerate Kleinian groups.
(2) In 1980, Floyd [**Flo80**] proved that the analogous problem for non-cocompact geometrically finite Kleinian groups has an affirmative answer.
(3) In their major breakthrough, Cannon and Thurston [**CT85**] proved Theorem 1.2. This paper was published more than two decades later [**CT07**]!

(4) Minsky [Min94] extended the techniques and result of [CT85] to closed surface groups of bounded geometry (see [Mj10] for a different proof).
(5) The author [Mit98b], and, independently, Klarreich [Kla99] using very different methods, extended Minsky's result to hyperbolic 3-manifolds of bounded geometry with incompressible core and without parabolics.
(6) Alperin-Dicks-Porti [ADP99] gave an elementary proof for fibers of the figure eight knot complement, thought of as a punctured surface bundle over the circle. This has led to a development in a very different direction [CD02, CD06, DS10, DW12]. This was the first extension to a geometrically infinite case with parabolics.
(7) Using the model geometry for punctured torus Kleinian groups proven by Minsky [Min99], McMullen [McM01] proved the existence of Cannon-Thurston maps for punctured torus groups.
(8) Bowditch [Bow13, Bow07] proved the existence of Cannon-Thurston maps for punctured surface groups of bounded geometry (see also [Mj10] for a different proof).
(9) Bowditch's result was extended to pared manifolds with incompressible (relative) core by the author [Mj09].
(10) Miyachi [Miy02] (see also [Sou06]) proved the existence of Cannon-Thurston maps for handlebody groups of bounded geometry.
(11) In [Mj11a, Mj16] the author proved the existence of Cannon-Thurston maps for certain special unbounded geometries.
(12) The general surface group case was accomplished in [Mj14a] and the general Kleinian group case in [Mj17b].

1.2. Geometric group theory. After the introduction of hyperbolic metric spaces by Gromov [Gro85], Question 1.3 was extended by the author [Mit97b, Bes04, Mit98c] to the context of a hyperbolic group H acting freely and properly discontinuously by isometries on a hyperbolic metric space X. Let Γ_H denote a Cayley graph of H with respect to a finite generating set. There is a natural map $i : \Gamma_H \to X$, sending vertices of Γ_H to the $H-$orbit of a point $x \in X$, and connecting images of adjacent vertices by geodesic segments in X. Let $\widehat{\Gamma_H}, \widehat{X}$ denote the Gromov compactification of Γ_H, X respectively. The analog of Question 1.3 is the following:

QUESTION 1.5. *Does $i : \Gamma_H \to X$ extend continuously to a map $\hat{i} : \widehat{\Gamma_H} \to \widehat{X}$?*

Continuous extensions as in Question 1.5 are also referred to as **Cannon-Thurston maps** and make sense when Γ_H is replaced by an arbitrary hyperbolic metric space Y. A simple and basic criterion for the existence of Cannon-Thurston maps was established in [Mit98a, Mit98b]:

LEMMA 1.6. *Let $i : (Y, y) \to (X, x)$ be a proper map between (based) Gromov-hyperbolic spaces. A Cannon-Thurston map $\hat{i} : \widehat{Y} \to \widehat{X}$ exists if and only if the following holds:*

There exists a non-negative proper function $M : \mathbb{N} \to \mathbb{N}$, such that if $\lambda = [a, b]_Y$ is a geodesic lying outside an N-ball around y, then any geodesic segment $[i(a), i(b)]_X$ in X joining $i(a)$, $i(b)$ lies outside the $M(N)$-ball around $x = i(y)$.

2. Closed 3-manifolds

2.1. 3-manifolds fibering over the circle. We start by giving a sketch of the proof of Theorem 1.2, following the original Cannon-Thurston paper [**CT85**]. We use the notation of Theorem 1.2.

1) Let ϕ be the pseudo-Anosov homeomorphism giving the monodromy of M as a $\Sigma-$bundle over S^1. Let $\mathcal{F}_s$ and $\mathcal{F}_u$ denote the stable and unstable singular measured foliations of ϕ on Σ. Equip Σ with a singular Euclidean metric d_0 using $\mathcal{F}_s$ and $\mathcal{F}_u$. Thus d_0 has the local expression $dx^2 + dy^2$ away from finitely many singularities on Σ. Lift this metric to $\widetilde{\Sigma}$ and identify the $x-$ (resp. $y-$) direction with $\mathcal{F}_s$ (resp. $\mathcal{F}_u$). Then, in local charts U, in the complement of the singularities in $\widetilde{\Sigma}$, d_0 has again an expression $dx^2 + dy^2$. Denote the 'flow' direction by the $t-$co-ordinate. Then $U \times \mathbb{R} (\subset \widetilde{M})$ is equipped with a Sol-type metric with local form $ds^2 = dt^2 + e^t dx^2 + e^{-t} dy^2$. Pasting these metrics together, in particular along the flow lines through the singularities, we get a singular Sol-type metric d_s on $\widetilde{M}$.

2) Let l be a leaf (i.e. a connected component) of $\mathcal{F}_s$ (or $\mathcal{F}_u$). By flowing l in the $t-$direction we obtain H_l homeomorphic to $\mathbb{R}^2$, equipped with the metric $dt^2 + e^t dx^2$ (or $dt^2 + e^{-t} dy^2$). Thus the intrinsic metric on H_l is (quasi-)isometric to that on $\mathbf{H}^2$. Since we are forgetting the y (or x) co-ordinate in the process, any such H_l is totally geodesic in $(\widetilde{M}, d_s)$.

3) Compactify $\widetilde{M}$ by adjoining a 'can'

$$\partial_c \widetilde{M} = \partial\widetilde{\Sigma} \times [-\infty, \infty] \cup \widetilde{\Sigma} \times \{-\infty, \infty\}.$$

4) We get a natural cell-like decomposition of $\partial_c \widetilde{M}$ consisting of the following collection $\mathcal{G}$ of contractible cell-like subsets:

(1) $(l_0 \times \infty) \cup (\partial l_0 \times [-\infty, \infty])$ for l_0 a (possibly singular) leaf of $\mathcal{F}_s$.
(2) $(l_0 \times -\infty) \cup (\partial l_0 \times [-\infty, \infty])$ for l_0 a (possibly singular) leaf of $\mathcal{F}_u$.
(3) $\{t\} \times [-\infty, \infty]$ for $t \in \partial\widetilde{\Sigma} \setminus (\partial\mathcal{F}_s \cup \partial\mathcal{F}_u)$.

5) Collapsing each of these contractible cell-like subsets we obtain the quotient space ∂M, which is homeomorphic to S^2 by Moore's theorem [**Moo25**] below.

THEOREM 2.1. *Let $\mathcal{G}$ denote a cell-like decomposition of Z homeomorphic to S^2 (as above). Then the quotient space $Z/\mathcal{G}$ is homeomorphic to S^2.*

6) Since each cell-like subset in $\mathcal{G}$ intersects $\partial\Sigma$ (identified with $\partial\Sigma \times \{0\}$ say) in isolated points (in any case, finitely many) and since this is

$\pi_1(\Sigma)-$equivariant, we thus have a $\pi_1(\Sigma)-$equivariant quotient map $\partial i : \partial\widetilde{\Sigma} \to \partial\widetilde{M}$.

7) It remains to show that ∂i continuously extends $\widetilde{i} : \widetilde{\Sigma} \to \widetilde{M}$, where $\widetilde{\Sigma}$, $\widetilde{M}$ are identified with $\mathbf{H}^2$, $\mathbf{H}^3$ and hence their boundaries with S^1, S^2 respectively. To show this it suffices to construct a system of neighborhoods of $p \in S^1$ and prove that they are mapped into a system of neighborhoods of $\partial i(p)$.

These neighborhoods are constructed as follows. For any $p \in S^1$, there exist a sequence of bi-infinite geodesics l_n contained in leaves (connected components) of $\mathcal{F}_s \cup \mathcal{F}_u$ such that $l_n \to p$ and so does a component K_n of $S^1 \setminus \partial l_n$ containing p. Then K_n is the boundary of a small neighborhood U_n of p and the family $\{U_n\}$ forms a neighborhood system. Flowing U_n in the $t-$direction, we obtain a convex set V_n (identifiable with $U_n \times \mathbb{R}$) bounded by the totally geodesic plane H_{l_n}. It is easy to check that V_n can again be chosen to form a neighborhood system of $\partial i(p)$. This completes the proof.

8) The above proof also gives the structure of the **Cannon-Thurston map** ∂i. We note that $\partial i(p) = \partial i(q)$ for $p \neq q$ if and only if p, q are end-points of a leaf of either $\mathcal{F}_s$ or $\mathcal{F}_u$.

2.2. Quasiconvexity. The structure of Cannon-Thurston maps in Section 2.1 can be used to establish quasiconvexity of certain subgroups of $\pi_1(M)$. Let $H \subset \pi_1(\Sigma)$ be a finitely generated infinite index subgroup of the fiber group. Then, due to the LERF property for surface subgroups [**Sco78**], there is a finite sheeted cover of Σ where H is geometric, i.e. it is carried by a proper embedded subsurface of (a finite sheeted cover of) Σ. But such a proper subsurface cannot carry a leaf of $\mathcal{F}_s$ or $\mathcal{F}_u$ since $\mathcal{F}_s$ and $\mathcal{F}_u$ are arational. This gives us the following Theorem of Scott and Swarup:

THEOREM 2.2 ([**SS90**]). *Let $H \subset \pi_1(\Sigma)$ be a finitely generated infinite index subgroup of the fiber group in $\pi_1(M)$. Then H is quasiconvex in $\pi_1(M)$.*

A somewhat more sophisticated theorem was proved by Cooper-Long-Reid [**CLR94**]. Let us denote the suspension flow on M by $\mathcal{F}$. Let $f : S \to M$ be an immersion transverse to the flow. Let $\tilde{f} : \widetilde{S} \to \widetilde{M}$ be a lift to the universal cover. Also fix a lift $\widetilde{\Sigma}$ of Σ. Each flow line intersects $\widetilde{\Sigma}$ exactly once. This gives a map $\Pi : \widetilde{M} \to \widetilde{\Sigma}$ sending flow lines to intersection points with $\widetilde{\Sigma}$. Identify $\widetilde{\Sigma}$ with $\mathbf{H}^2$. Let $A(S)$ denote $\Pi \circ \tilde{f}(\widetilde{S})$. Cooper-Long-Reid establish the following necessary and sufficient condition for geometric finiteness of S.

THEOREM 2.3 ([**CLR94**]). *$A(S)$ is a (quasi)convex subset of $\widetilde{\Sigma}(= \mathbf{H}^2)$. $f(S)$ is quasi-Fuchsian if and only if $A(S)$ is a proper subset of $\widetilde{\Sigma}$.*

In [**CLR94**] it is also shown that $A(S)$ is an infinite sided polygon in $\mathbf{H}^2$ whose sides correspond to leaves of the stable and unstable laminations.

2.3. Quasigeodesic flows. In this section, we discuss generalizations of the Cannon-Thurston theorem to the context of flows by Calegari [**CD03,**

Cal00, Cal06], Fenley [**Fen12**] and Frankel [**Fra15**]. We therefore turn to flows on arbitrary closed hyperbolic 3-manifolds M. All flows $\mathcal{F}$ will be non-singular, i.e. the associated vector field has no zeros Such a flow $\mathcal{F}$ is called **quasigeodesic** if each flow line of $\mathcal{F}$ lifts to a quasigeodesic in the universal cover $\widetilde{M}$. The lifted flow in $\widetilde{M}$ is denoted $\widetilde{\mathcal{F}}$. The **flow space** P is the space of flow lines equipped with the quotient topology. Thus we have a quotient map $\Pi : \widetilde{M} \to P$. P is homeomorphic to a plane if M is hyperbolic [**Cal06**]. Calegari shows that P also admits a natural compactification by adjoining a certain 'universal' circle S^1_u. In fact, Calegari first constructs two universal circles, which parameterize the action of $\pi_1(M)$ at infinity and then combines them together to get S^1_u. Moreover, he obtains a pair of laminations $\Lambda^\pm_u$ of S^1_u preserved by $\pi_1(M)$. As in the case of Theorem 1.2, there is a compactification $\overline{P} = P \cup S^1_u$ called the **ends compactification**.

Next, since any flow line σ of $\mathcal{F}$ is a quasigeodesic, there exist natural maps $\Phi^\pm : P \to S^2_\infty$ sending each σ to $\sigma(\pm\infty)$. Frankel proves the following analog of Theorem 1.2 in this context:

THEOREM 2.4 ([**Fra15**]). *Let $\mathcal{F}$ be a quasigeodesic flow on a closed hyperbolic 3-manifold M. There are unique continuous extensions of the endpoint maps $\Phi^\pm$ to the compactified orbit space $\overline{P}$, and Φ^+ agrees with Φ^- on the boundary.*

Next, let $s : P \to \mathbf{H}^3$ be a transversal section to the lifted flow $\widetilde{\mathcal{F}}$. Then there exists a natural compactification $P \cup S^1_u$ of P inheriting a $\pi_1(M)$ action and a unique continuous extension $\bar{s} : P \cup S^1_u \to \mathbf{H}^3 \cup S^2_\infty$. The boundary map $\partial s : S^1_u \to S^2$ is $\pi_1(M)$-equivariant surjective.

2.3.1. *Pseudo-anosov flows.* We now turn to pseudo-Anosov flows $\mathcal{F}$ on closed 3-manifolds M following Fenley [**Fen12**]. A key point is that he does not assume M to be hyperbolic. The flow space of $\widetilde{\mathcal{F}}$ is again a plane P. He constructs a natural compactification of P with an ideal circle boundary S^1_I. He shows that if there are no perfect fits between stable and unstable leaves of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ is not topologically conjugate to a suspension Anosov flow, then S^1_I has a natural sphere quotient S^2_d (the Cannon-Thurston property). The sphere S^2_d is a dynamical systems ideal boundary for a compactification of $\widetilde{M}$. Fenley shows the following using a theorem due to Bowditch [**Bow98**] characterizing hyperbolic groups.

THEOREM 2.5 ([**Fen12**]). *$\pi_1(M)$ acts on S^2_d is as a uniform convergence group. Hence $\pi_1(M)$ is Gromov hyperbolic. Further, the action of $\pi_1(M)$ on S^2_d is topologically conjugate to the action of $\pi_1(M)$ on the Gromov boundary $\partial\pi_1(M)$.*

Fenley also points out that pseudo-Anosov flows without perfect fits are quasigeodesic flows. He then obtains an application to the case of an $\mathbb{R}$-covered foliation $\mathcal{F}$ in an aspherical, atoroidal M. There is a transversely oriented foliation $\mathcal{F}_2$ associated to $\mathcal{F}$ and a pseudo-Anosov flow Φ transverse to $\mathcal{F}_2$. He proves that there are no perfect fits for this flow, nor is it conjugate to a suspension Anosov flow. Hence Φ is quasigeodesic and $\mathcal{F}_2$ satisfies the Cannon-Thurston property.

2.4. Punctured torus bundles. In this subsection, we discuss some work on Cannon-Thurston maps for punctured torus bundles. We shall return to the general case of punctured surfaces later. In [**ADP99**], a direct and elementary proof of the existence of Cannon-Thurston maps was established for the fiber group of the figure eight knot complement group. They also show the following. Let Γ_2 denote the Cayley graph of the fiber subgroup which is a free group on 2 generators. If an edge is removed from Γ_2, the authors show that the complementary subtrees give rise to a decomposition of S^2 into two disks bounded by a common Jordan curve c. The curve c can be further cut into two arcs with common endpoints, so that cusps are dense in each.

This approach was developed extensively in the context of hyperbolic punctured torus bundles by Cannon, Dicks, Sakuma and Wright in [**CD02, CD06, DS10, DW12**] to obtain a number of special features that arise in the punctured torus bundle case:

(1) A generalization of the [**ADP99**] result above to all punctured torus bundles was obtained in [**CD02**]. In particular, a Cannon-Thurston map $i_{CT} : \partial\widetilde{\Sigma} \to S^2$ exists, where Σ is the fiber punctured torus and $\widetilde{\Sigma}$ its universal cover.
(2) Identify $\partial\widetilde{\Sigma}$ with S^1. Let $q \in S^1$ correspond to a cusp. Then $S^1 \setminus i_{CT}^{-1} \circ i_{CT}(q)$ is a countable union of intervals. The images of these intervals under i_{CT} give a bi-infinite sequence of fractal subsets of the plane which tessellate it. This is the content of [**CD06**].
(3) In [**DS10**], the above tessellation is related to a tessellation of the plane arising from Jørgensen's canonical triangulation of a punctured torus bundle.
(4) The fractal curves of [**ADP99**] and [**CD02**] are shown to be codified by the classical Adler-Weiss automata in [**DW12**].

3. Gromov-hyperbolic groups

3.1. Preliminaries. We turn now to Cannon-Thurston maps in the context of geometric group theory and furnish a different proof of Theorem 1.2. We recall a couple of basic Lemmata we shall be needing from [**Mit98b**]. The following says that nearest point projection onto a geodesic in a hyperbolic space is coarsely Lipschitz.

LEMMA 3.1. *Let (X, d) be a δ-hyperbolic metric space. Then there exists a constant $C \geq 1$ such that the following holds:*

Let $\lambda \subset X$ be a geodesic segment and let $\Pi : X \to \lambda$ be a nearest point projection to λ. Then $d(\Pi(x), \Pi(y)) \leq Cd(x, y)$ for all $x, y \in X$.

The next Lemma says that nearest point projections and quasi-isometries almost commute.

LEMMA 3.2. *Let (X,d) be a δ-hyperbolic metric space. Given $K \geq 0$, $\epsilon \geq 0$, there exists C such that the following holds:*

Let $\lambda = [a,b]$ be a geodesic segment in X. Let $p \in X$ be arbitrary and let q be a nearest point projection of p onto λ. Let ϕ be a $(K,\epsilon)-$quasi-isometry from X to itself and let $\Phi(\lambda) = [\phi(a), \phi(b)]$ be a geodesic segment in X joining $\phi(a)$, $\phi(b)$. Let r be a nearest point projection of $\phi(p)$ to $\Phi(\lambda)$. Then $d(r, \phi(q)) \leq C$.

SKETCH OF PROOF. The proof of Lemma 3.2 follows from the fact that a geodesic tripod T (built from $[a,b]$ and $[p,q]$) is quasiconvex in hyperbolic space and a quasi-isometric image $\phi(T)$ of T lies close to a geodesic tripod T' built from $[\phi(a), \phi(b)]$ and $[\phi(p), r]$. Hence the image $\phi(q)$ of the centroid q of T lies close to the centroid r of T'. $\square$

3.2. The key tool: hyperbolic ladder. The key idea behind the proof of Theorem 1.2 and its generalizations in [**Mit98a, Mit98b**] is the construction of a **hyperbolic ladder** $\mathcal{L}_\lambda \subset \widetilde{M}$ for *any* geodesic in $\widetilde{\Sigma}$. Since the context is geometric group theory, we replace $\widetilde{\Sigma}$ and $\widetilde{M}$ by quasi-isometric models in the form of Cayley graphs $\Gamma_{\pi_1(\Sigma)}$ and $\Gamma_{\pi_1(M)}$ respectively. Let us denote $\Gamma_{\pi_1(\Sigma)}$ by Y and $\Gamma_{\pi_1(M)}$ by X. Then X can be thought of as (is quasi-isometric to) a tree T of spaces, where

(1) T is the simplicial tree with underlying space $\mathbb{R}$ and vertices at $\mathbb{Z}$.
(2) All the vertex and edge spaces are (intrinsically) isometric to Y.
(3) The edge space to vertex space inclusions are uniform quasi-isometries (and not just qi-embeddings).
(4) It follows from the assumptions above that (Y, d_Y) is properly embedded in (X, d_X).

Thus X is a tree T of spaces satisfying the qi-embedded condition [**BF92**].

Given a geodesic $\lambda = \lambda_0 \subset Y$, we now sketch the promised construction of the **ladder** $\mathcal{L}_\lambda \subset X$ containing λ. Index the vertices by $n \in \mathbb{Z}$. Since the edge-to-vertex inclusions are quasi-isometries, this induces a quasi-isometry ϕ_n from the vertex space Y_n to the vertex space Y_{n+1} for $n \geq 0$. A similar quasi-isometry ϕ_{-n} exists from Y_{-n} to the vertex space $Y_{-(n+1)}$. These quasi-isometries are defined on the vertex sets of Y_n, $n \in \mathbb{Z}$. ϕ_n induces a map Φ_n from geodesic segments in Y_n to geodesic segments in Y_{n+1} for $n \geq 0$ by sending a geodesic in Y_n joining a, b to a geodesic in Y_{n+1} joining $\phi_n(a)$, $\phi_n(b)$. Similarly, for $n \leq 0$. Inductively define:

- $\lambda_{j+1} = \Phi_j(\lambda_j)$ for $j \geq 0$,
- $\lambda_{-j-1} = \Phi_{-j}(\lambda_{-j})$ for $j \geq 0$,
- $\mathcal{L}_\lambda = \bigcup_j \lambda_j$.

$\mathcal{L}_\lambda$ turns out to be quasiconvex in X. To prove this, we construct a coarsely Lipschitz retraction $\Pi_\lambda : \bigcup_j Y_j \to \mathcal{L}_\lambda$ as follows.

On Y_j define $\Pi_j(y)$ to be a nearest-point projection of y to λ_j and define

$$\Pi_\lambda(y) = \pi_j(y),\ for\, y \in Y_j.$$

The following theorem asserts that Π_λ is coarsely Lipschitz.

THEOREM 3.3 ([**Mit98a**, **Mit98b**, **Mj10**]). *There exists $C \geq 1$ such that for any geodesic $\lambda \subset Y$,*

$$d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C d_X(x, y)$$

for $x, y \in \bigcup_i Y_i$.

SKETCH OF PROOF. The proof requires only the hyperbolicity of Y, but not that of X. It suffices to show that for $d_X(x, y) = 1$, $d_X(\Pi_\lambda(x), \Pi_\lambda(y)) \leq C$. Thus x, y may be thought of as

(1) either lying in the same Y_j. This case follows directly from Lemma 3.1.

(2) or lying vertically one just above the other. Then (up to a bounded amount of error), we can assume without loss of generality, that $y = \phi_j(x)$. This case now follows from Lemma 3.2. $\square$

Since a coarse Lipschitz retract of a hyperbolic metric space is quasiconvex, we immediately have:

COROLLARY 3.4. *If (X, d_X) is hyperbolic, there exists $C \geq 1$ such that for any λ, $\mathcal{L}_\lambda$ is $C-$quasiconvex.*

Note here that we have not used any feature of Y except its hyperbolicity. In particular, we do not need the specific condition that $Y = \widetilde{\Sigma}$. We are now in a position to prove a generalization of Theorem 1.2.

THEOREM 3.5 ([**Mj10**]). *Let (X, d) be a hyperbolic tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition, where T is $\mathbb{R}$ or $[0, \infty)$ with vertex and edge sets Y_j as above, $j \in \mathbb{Z}$. Assume (as above) that the edge-to-vertex inclusions are, moreover, uniform quasi-isometries (and not just qi-embeddings). For $i : Y_0 \to X$ there is a Cannon-Thurston map $\hat{i} : \widehat{Y_0} \to \widehat{X}$.*

PROOF. Fix a basepoint $y_0 \in Y_0$. By Lemma 1.6 and quasiconvexity of $\mathcal{L}_\lambda$ (Corollary 3.4), it suffices to show that for all $M \geq 0$ there exists $N \geq 0$ such that if a geodesic segment λ lies outside the N-ball about $y_0 \in Y_0$, then $\mathcal{L}_\lambda$ lies outside the M-ball around $y_0 \in X$. Equivalently, we need a proper function $M(N) : \mathbb{N} \to \mathbb{N}$.

Since Y_0 is properly embedded in X, there exists a proper function $g : \mathbb{N} \to \mathbb{N}$ such that λ lies outside the $g(N)$-ball about $y_0 \in X$.

Let p be any point on $\mathcal{L}_\lambda$. Then $p = p_j \in Y_j$ for some j. Assume without loss of generality that $j \geq 0$. It is not hard to see that there exists C_0, depending only on X, such that for any such p_j, there exists $p_{j-1} \in Y_{j-1}$ with

$d(p_j, p_{j-1}) \leq C_0$. It follows inductively that there exists $y \in \lambda = \lambda_0$ such that $d_X(y, p) \leq C_0 j$. Hence, by the triangle inequality, $d_X(y_0, p) \geq g(N) - C_0 j$.

Next, looking at the 'vertical direction', $d_X(y_0, p) \geq j$ and hence

$$d_X(y_0, p) \geq \max(g(N) - C_0 j, j) \geq \frac{g(N)}{C_0 + 1}.$$

Defining $M(N) = \frac{g(N)}{C_0+1}$, we see that $M(N)$ is a proper function of N and we are done. □

3.3. Applications and generalizations.

3.3.1. *Normal subgroups and trees.* The ladder construction of Section 3.2 has been generalized considerably. We work in the context of an exact sequence $1 \to N \to G \to Q$, with N hyperbolic and G finitely presented. We observe that for the proof of Theorem 3.5 to go through it suffices to have a qi-section of Q into G to provide a 'coarse transversal' to flow along. Such a qi-section was shown to exist by Mosher [**Mos96**]. We then obtain the following.

THEOREM 3.6 ([**Mit98a**]). *Let G be a hyperbolic group and let H be a hyperbolic normal subgroup that is normal in G. Then the inclusion of Cayley graphs $i : \Gamma_H \to \Gamma_G$ gives a Cannon-Thurston map $\hat{i} : \widehat{\Gamma_H} \to \widehat{\Gamma_G}$.*

Theorem 3.6 was generalized by the author and Sardar to a purely coarse geometric context, where no group action is present. The relevant notion is that of a metric bundle for which we refer the reader to [**MS12**]. Roughly speaking, the data of a metric bundle consists of vertex and edge spaces as in the case of a tree of spaces, with two notable changes:

(1) The base T is replaced by an arbitrary graph B.
(2) All edge-space to vertex space maps are quasi-isometries rather than just quasi-isometric embeddings.

With these modifications in place we have the following generalizations of Mosher's qi-section Lemma [**Mos96**] and Theorem 3.6:

THEOREM 3.7 ([**MS12**]). *Suppose $p : X \to B$ is a metric graph bundle satisfying the following:*

(1) *B is a Gromov hyperbolic graph.*
(2) *Each fiber F_b, for b a vertex of B is δ-hyperbolic (for some $\delta > 0$) with respect to the path metric induced from X.*
(3) *The barycenter maps $\partial^3 F_b \to F_b$, $b \in B$, sending a triple of distinct points on the boundary ∂F_b to their centroid, are (uniformly, independent of b) coarsely surjective.*
(4) *X is hyperbolic.*

Then there is a qi-section $B \to X$. The inclusion map $i_b : F_b \to X$ gives a Cannon-Thurston map $\hat{i} : \widehat{F_b} \to \widehat{X}$.

The ladder construction can also be generalized to the general framework of a tree of hyperbolic metric spaces.

Theorem 3.8 ([**Mit98b**]). *Let (X, d) be a tree (T) of hyperbolic metric spaces satisfying the qi-embedded condition. Let v be a vertex of T and (X_v, d_v) be the vertex space corresponding to v. If X is hyperbolic then the inclusion $i : X_v \to X$ gives a Cannon-Thurston map $\hat{i} : \widehat{X_v} \to \widehat{X}$.*

3.3.2. *Hydra groups.* **Hyperbolic hydra** groups were introduced by Brady, Dison and Riley in [**BDR11**]. They constructed a sequence Γ_k, $k = 1, 2, \cdots$ of hyperbolic groups with finite-rank free subgroups H_k such that the distortion function of H_k in Γ_k is at least A_k, the $k-$th term of the Ackermann function (an example of a totally computable function that is not primitive recursive). Baker and Riley prove the following:

Theorem 3.9 ([**BR13**]). *Hyperbolic hydra have Cannon–Thurston maps for all k.*

3.3.3. *Arcs in the boundary.* Recall the situation in Section 3.2 where we had geodesics $\lambda \subset Y$ and ladders $\mathcal{L}_\lambda \subset X$ constructed out of them. It can be easily seen that $\mathcal{L}_\lambda$ is (uniformly, independent of λ) quasi-isometric to $\mathbf{H}^2$. It follows that there is an embedding $\partial\mathcal{L}_\lambda \to \partial X$, where X is the relevant tree (ray or bi-infinite geodesic) of hyperbolic spaces. This allows us to join any pair of points in ∂X by quasi-arcs which may be thought of as boundaries of ladders $\mathcal{L}_\lambda$. In the special case of a simply degenerate surface Kleinian group, the limit set ∂X is a dendrite of Hausdorff dimension 2. Since the quasi-arcs $\partial\mathcal{L}_\lambda \to \partial X$ have Hausdorff dimension uniformly bounded away from 2, and all cut-points of ∂X lie within such quasi-arcs, we have the following Theorem due to Bowditch:

Theorem 3.10 ([**Bow05**]). *Let Γ be a simply degenerate Kleinian group of bounded geometry. Let Λ_Γ be its dendritic limit set. Then the set of cut points of Λ_Γ has Hausdorff dimension strictly less than 2.*

3.3.4. *A non-proper Cannon-Thurston map.* It was observed in Section 3.2 that the construction of the ladder $\mathcal{L}_\lambda$ and Theorem 3.3 require hyperbolicity of the space Y (the fiber in Theorem 1.2) but not that of the total space X. In particular, this goes through for the Birman exact sequence

$$1 \to \pi_1(\Sigma) \to Mod(\Sigma, *) \to Mod(\Sigma) \to 1,$$

where $Mod(\Sigma)$ (resp. $Mod(\Sigma, *)$) refers to the mapping class group of a closed surface Σ (resp. Σ with a marked point $*$). In [**LMS11**], Leininger, the author and Schleimer constructed a Cannon-Thurston map in a closely related, non-proper, setting.

Fix a hyperbolic metric on Σ and a basepoint $* \in \Sigma$. Let $p : \mathbf{H}^2 \to \Sigma$ be the universal cover and fix a lift $z \in p^{-1}(*)$. With this normalization, we obtain an isomorphism between $\pi_1(\Sigma, *)$ and the deck transformation group $\pi_1(\Sigma)$ of $p : \mathbf{H}^2 \to \Sigma$.

Let $\mathcal{C}(\Sigma)$ and $\mathcal{C}(\Sigma, *)$ be the relevant curve complexes. There exists a natural 'forgetful' projection $\Pi : \mathcal{C}(\Sigma, *) \to \mathcal{C}(\Sigma)$. The fiber $\Pi^{-1}(v)$, for v a vertex of $\mathcal{C}(\Sigma)$, was identified by Kent-Leininger-Schleimer [**KLS09**] to be

$\pi_1(\Sigma)-$equivariantly isomorphic to the Bass-Serre tree T_v of the splitting of $\pi_1(\Sigma)$ given by splitting it along the simple curve v. $\pi_1(\Sigma)$ acts on $\mathcal{C}(\Sigma,*)$ via the point-pushing map. There is a natural map

$$\Phi : \mathcal{C}(\Sigma) \times \mathbf{H}^2 \to \mathcal{C}(\Sigma,*),$$

sending $\{v\} \times \mathbf{H}^2$ $\pi_1(\Sigma)-$equivariantly to $T_v(\subset \mathcal{C}(\Sigma,*))$ and extending linearly over simplices. Let Φ_v denote the restriction of Φ to $\{v\}\times\mathbf{H}^2$. We thus think of Φ_v as a map $\Phi_v : \mathbf{H}^2 \to \mathcal{C}(\Sigma,*)$.

We shall next identify a collection of geodesic rays $\mathcal{ATF} \subset \mathbf{H}^2$. It is not hard to see [**LMS11**] that if σ is a geodesic ray in Σ lying eventually inside a component of a pre-image of a proper essential subsurface of Σ, then $\Phi_v(\Sigma)$ has finite diameter in $\mathcal{C}(\Sigma,*)$. The remaining set of geodesic rays is denoted as $\mathcal{ATF}$. Identifying elements of $\mathcal{ATF}$ with their end-points in $\partial\mathbf{H}^2$ we obtain a set of full Lebesgue measure. The following establishes the existence of a Universal Cannon–Thurston map in this setup.

THEOREM 3.11 ([**LMS11**]). *For v a vertex of $\mathcal{C}(\Sigma)$, $\Phi_v : \mathbf{H}^2 \to \mathcal{C}(\Sigma,*)$ has a unique continuous $\pi_1(\Sigma)-$equivariant extension*

$$\overline{\Phi_v} : \mathbf{H}^2 \cup \mathcal{ATF} \to \overline{\mathcal{C}(\Sigma,*)}.$$

The boundary map $\partial\Phi = \overline{\Phi_v|_{\mathcal{ATF}}}$ does not depend on v and is a quotient map onto $\partial\mathcal{C}(\Sigma,*)$.

Point-preimages: *Given distinct points $x, y \in \mathcal{ATF}$, $\partial\Phi(x) = \partial\Phi(y)$ if and only if x and y are ideal endpoints of a leaf or ideal vertices of a complementary polygon of the lift of an arational lamination on Σ.*

As a consequence of Theorem 3.11, we find that the boundary of the curve complex $\mathcal{C}(\Sigma,*)$ is locally connected.

3.4. A counterexample. An explicit counterexample to Question 1.5 was found by Baker and Riley [**BR13**] in the context of hyperbolic subgroups of hyperbolic groups. The counterexample is in the spirit of small cancellation theory and uses Lemma 1.6 to rule out the existence of Cannon-Thurston maps.

The authors of [**BR13**] start with the free group on 5 generators $F_5 = \langle a, b, c_1, c_2, d_1, d_2\rangle$. They introduce relators on F_5 to obtain a hyperbolic group G such that

(1) The subgroup $H = \langle b, d_1, d_2\rangle$ is a free subgroup of rank 3.
(2) The pair (H, G) *does not* have a Cannon-Thurston map.

A word about the relators used in constructing G is in order. The authors choose positive words C, C_1, C_2 on c_1, c_2 and D_1, D_2, D_{11}, D_{12}, D_{21}, D_{22} on d_1, d_2 such the relators are given by

(1) $a^{-1}b^{-1}ab = C$
(2) $b^{-1}c_ib = C_i$, $i = 1, 2$
(3) $(ab)^{-1}d_i(ab) = D_i$, $i = 1, 2$
(4) $c_i^{-1}d_jc_i = D_{ij}$, $i, j = 1, 2$

Matsuda and Oguni [**MO14**] used the construction of Baker and Riley above to show that for any non-elementary hyperbolic group H, there is a hyperbolic group G containing H such that the pair (H, G) has no Cannon-Thurston map. They generalize this to relatively hyperbolic groups too.

3.5. Point pre-images: laminations. In Section 2.1, it was pointed out that the Cannon-Thurston map $\hat{i}$ identifies $p, q \in S^1$ if and only if p, q are end-points of a leaf or ideal end-points of a complementary ideal polygon of the stable or unstable lamination.

In [**Mit97a**] an algebraic theory of ending laminations was developed based on Thurston's theory [**Thu80**]. The theory was developed in the context of a normal hyperbolic subgroup of a hyperbolic group G and used to give an explicit structure for the Cannon-Thurston map in Theorem 3.6.

Definition 3.12 ([**BFH97**, **CHL07**, **CHL08a**, **CHL08b**, **KL10**, **KL15**, **Mit97a**]). *An* **algebraic lamination** *for a hyperbolic group H is an H-invariant, flip invariant, closed subset $\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H \setminus \Delta)/\sim$, where $(x, y) \sim (y, x)$ denotes the flip and Δ the diagonal in $\partial H \times \partial H$.*

Let

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of finitely presented groups with H, G hyperbolic. It follows by work of Mosher [**Mos96**] that there is a qi section $\sigma : Q \to G$ and hence Q is hyperbolic. In [**Mit97a**], we construct algebraic ending laminations naturally parametrized by points in the boundary ∂Q. We describe the construction now.

Every element $g \in G$ gives an automorphism of H sending h to $g^{-1}hg$ for all $h \in H$. Let $\phi_g : \mathcal{V}(\Gamma_H) \to \mathcal{V}(\Gamma_H)$ be the resulting bijection of the vertex set $\mathcal{V}(\Gamma_H)$ of Γ_H. This induces a map Φ_g sending an edge $[a, b] \subset \Gamma_H$ to a geodesic segment joining $\phi_g(a)$, $\phi_g(b)$.

For some (any) $z \in \partial\Gamma_Q$ we shall describe an algebraic ending lamination Λ_z. Fix such a z and let

(1) $[1, z) \subset \Gamma_Q$ be a geodesic ray starting at 1 and converging to $z \in \partial\Gamma_Q$.
(2) $\sigma : Q \to G$ be a qi section [**Mos96**].
(3) z_n be the vertex on $[1, z)$ such that $d_Q(1, z_n) = n$.
(4) $g_n = \sigma(z_n)$.

For $h \in H$, let $\mathcal{S}_n^h$ be the H–invariant collection of all free homotopy representatives (or equivalently, shortest representatives in the same conjugacy class) of $\phi_{g_n^{-1}}(h)$ in Γ_H. Identifying equivalent geodesics in $\mathcal{S}_n^h$ one obtains a subset $\mathbb{S}_n^h$ of unordered pairs of points in $\widehat{\Gamma_H}$. The intersection with $\partial^2 H$ of the union of all subsequential limits (in the Hausdorff topology) of $\{\mathbb{S}_n^h\}$ is denoted by Λ_z^h.

DEFINITION 3.13. *The set of* **algebraic ending laminations corresponding to** $z \in \partial\Gamma_Q$ *is given by*

$$\Lambda_{EL}(z) = \bigcup_{h \in H} \Lambda_z^h.$$

DEFINITION 3.14. *The set* Λ *of all algebraic ending laminations is defined by*

$$\Lambda_{EL} = \bigcup_{z \in \partial\Gamma_Q} \Lambda_{EL}(z).$$

The following was shown in [**Mit97a**]:

THEOREM 3.15. *The Cannon-Thurston map* $\hat{i}$ *of Theorem 3.6 identifies* p, q *if* $(p,q) \in \Lambda_{EL}$. *Conversely, if* $\hat{i}(p) = \hat{i}(q)$ *for* $p \neq q \in \partial\Gamma_H$, *then* $(p,q) \in \Lambda_{EL}$.

3.5.1. *Finite-to-one.* The classical Cannon-Thurston map of Theorem 1.2 is finite-to-one. Swarup asked (cf. Bestvina's Geometric Group Theory problem list [**Bes04**, Prolem 1.20]) if the Cannon-Thurston maps of Theorem 3.6 are also finite-to-one. I. Kapovich and Lustig answered this in the affirmative in the following case.

THEOREM 3.16 ([**KL15**]). *Let* $\phi \in Out(F_N)$ *be a fully irreducible hyperbolic automorphism, i.e. an irreducible hyperbolic automorphism all whose powers are irreducible. Let* $G_\phi = F_N \rtimes_\phi \mathbb{Z}$ *be the associated mapping torus group. Let* ∂i *denote the Cannon-Thurston map of Theorem 3.6 in this case. Then for every* $z \in \partial G_\phi$, *the cardinality of* $(\partial i)^{-1}(z)$ *is at most* $2N$.

Bestvina-Feighn-Handel [**BFH97**] define a closely related set Λ_{BFH} of algebraic laminations in the case covered by Theorem 3.16 using train-track representatives of free group automorphisms. Any algebraic lamination $\mathcal{L}$ defines a relation $\mathcal{R}_\mathcal{L}$ on ∂F_N by $a\mathcal{R}_\mathcal{L}b$ if $(a,b) \in \mathcal{L}$. The transitive closure of $\mathcal{L}$ will be called its diagonal closure. In [**KL15**], Kapovich and Lustig further show that in the case covered by Theorem 3.16, Λ_{EL} precisely equals the diagonal closure of Λ_{BFH}.

3.6. Quasiconvexity. This subsection deals with a theme explored in [**DKL14, MR18, DT16**]. We follow the treatment in [**MR18**]. Developing the method of Scott-Swarup's Theorem 2.2 and using Kapovich-Lustig's work [**KL15**], we obtain an analogous theorem for finitely generated infinite index subgroups of the fiber group of Theorem 3.16 in [**Mit99**]. This approach is further developed in [**MR18**], where the relationship between the following different kinds of algebraic laminations are examined. We are in the setup of Theorem 3.6.

(1) For an action of H on an $\mathbb{R}-$tree, there is a dual lamination $\Lambda_\mathbb{R}$ [**Thu80, BFH97, CHL07, CHL08a, KL10**].
(2) Thurston's ending laminations [**Thu80**] and the algebraic ending lamination Λ_{EL} of [**Mit97a**].

(3) The Cannon-Thurston lamination Λ_{CT} arising in the context of the existence of a Cannon-Thurston map [**CT07**] and Theorem 3.6.

The following quasiconvexity result is a consequence of a close examination of the above laminations and their inter-relationships. We refer the reader to [**FM02**] and [**DT17**] for the notions of convex cocompact subgroups of Teichmuller space or Outer Space.

THEOREM 3.17. [**MR18**] *Let*

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of hyperbolic groups, where H is either a free group or a closed surface group and Q is convex cocompact in Outer Space or Teichmuller space respectively. For the free group case, we assume further that Q is purely hyperbolic. Let K be a finitely generated infinite index subgroup of H. Then K is quasiconvex in G.

The case where H is a closed surface group in Theorem 3.17 was obtained by Dowdall, Kent and Leininger [**DKL14**] by different methods. In [**DT16**], Dowdall and Taylor use the methods of their earlier work [**DT17**] on convex cocompact purely hyperbolic subgroups of $Out(F_n)$ to give a different proof of Theorem 3.17 when H is free.

3.7. Relative hyperbolicity. The notion of a Cannon-Thurston map can be extended to the context of relative hyperbolicity. This was done in [**MP11**]. Let X and Y be relatively hyperbolic spaces, hyperbolic relative to the collections $\mathcal{H}_X$ and $\mathcal{H}_Y$ of 'horosphere-like sets' respectively. For $H \in \mathcal{H}_X$ (or $\mathcal{H}_Y$), equipped with metric d_H, the *horoballification* H^h of H is obtained by equipping $H \times \mathbb{R}_+$ with a metric that is Euclidean in the $\mathbb{R}_+$ factor while $H \times \{t\}$ is equipped with the exponentially scaled metric $e^{-t}d_H$. Gluing such horoballifications H^h to H in X (resp. Y) for all H in X (resp. Y) we obtain the *horoballifications* $\mathcal{G}(X, \mathcal{H}_X)$ (resp. $\mathcal{G}(Y, \mathcal{H}_Y)$) of X (resp. Y) with respect to $\mathcal{H}_X$ (resp. $\mathcal{H}_Y$) (see [**Bow12**] for details). Note that $\mathcal{G}(X, \mathcal{H}_X)$, $\mathcal{G}(Y, \mathcal{H}_Y)$ are hyperbolic. The electrifications will be denoted as $\mathcal{E}(X, \mathcal{H}_X)$, $\mathcal{E}(Y, \mathcal{H}_Y)$.

DEFINITION 3.18. *A map $i : Y \to X$ is strictly type-preserving if for all $H_Y \in \mathcal{H}_Y$ there exists $H_X \in \mathcal{H}_X$ such that $i(H_Y) \subset H_X$ and images of distinct horospheres-like sets in Y lie in distinct horosphere-like sets in X.*

Let $i : Y \to X$ be a strictly type-preserving proper embedding. Then i induces a proper embedding $i_h \colon \mathcal{G}(Y, \mathcal{H}_Y) \to \mathcal{G}(X, \mathcal{H}_X)$.

DEFINITION 3.19. *A Cannon-Thurston map exist for a strictly type-preserving inclusion $i : Y \to X$ of relatively hyperbolic spaces X, Y if a Cannon-Thurston map exists for the induced map $i_h \colon \mathcal{G}(Y, \mathcal{H}_Y) \to \mathcal{G}(X, \mathcal{H}_X)$.*

Lemma 1.6 generalizes to the following.

LEMMA 3.20. *A Cannon-Thurston map for $i : Y \to X$ exists if and only if there exists a non-negative proper function $M : \mathbb{N} \to \mathbb{N}$ such that the following holds:*

Fix a base-point $y_0 \in Y$. Let $\hat{\lambda}$ in $\mathcal{E}(Y, \mathcal{H}_Y)$ be an electric geodesic segment starting and ending outside horospheres. If $\lambda^b = \hat{\lambda} \setminus \bigcup_{K \in \mathcal{H}_Y} K$ lies outside $B_N(y_0) \subset Y$, then for any electric quasigeodesic $\hat{\beta}$ joining the end points of $\hat{i}(\hat{\lambda})$ in $\mathcal{E}(X, \mathcal{H}_X)$, $\beta^b = \hat{\beta} \setminus \bigcup_{H \in \mathcal{H}_X} H$ lies outside $B_{M(N)}(i(y_0)) \subset X$.

Theorem 3.8 then generalizes to:

THEOREM 3.21 ([**MP11**]). *Let $P\colon X \to T$ be a tree of relatively hyperbolic spaces satisfying the qi-embedded condition. Assume that*

(1) *the inclusion maps of edge-spaces into vertex spaces are strictly type-preserving*
(2) *the induced tree of electrified (coned-off) spaces continues to satisfy the qi-embedded condition*
(3) *X is strongly hyperbolic relative to the family $\mathcal{C}$ of maximal cone-subtrees of horosphere-like sets.*

Then a Cannon-Thurston map exists for the inclusion of relatively hyperbolic spaces $i\colon X_v \to X$, where (X_v, d_{X_v}) is the relatively hyperbolic vertex space corresponding to v.

3.8. Punctured surface bundles. We illustrate Theorem 3.21 by sketching a proof in the classical case of a hyperbolic 3-manifold fibering over the circle with fiber a punctured surface Σ^h. This case was first established by Bowditch [**Bow07**] by a somewhat different proof. We will work with the cover N^h corresponding to the fiber subgroup. We lose nothing and gain greater generality by working with punctured surface Kleinian groups of bounded geometry. So N^h will henceforth denote *the convex core of* a simply or doubly degenerate manifold of bounded geometry corresponding to a punctured surface Kleinian group.

Excise the cusps of N^h leaving us a manifold N that has one or two ends. Then, by work of Minsky [**Min92**], N is quasi-isometric to the universal curve over a Lipschitz path in Teichmuller space from which cusps have been removed. Identify Σ^h with a base pleated surface in N^h. Excising cusps from Σ^h we obtain a base surface Σ in N. The universal cover $\widetilde{\Sigma}$ is quasi-isometric to a tree. The inclusion $i : \widetilde{\Sigma} \subset \widetilde{N}$ is a special case of the setup of Theorem 3.21.

Let $\lambda = [a, b]$ be a geodesic segment in $\widetilde{\Sigma}$. As in Section 3.2, we obtain a **ladder** $\mathcal{L}_\lambda$. Theorem 3.3 ensures that there is a coarsely Lipschitz retraction from $\widetilde{N}$ to $\mathcal{L}_\lambda$. Hence, in particular, $\mathcal{L}_\lambda$ is quasi-isometrically embedded in $\widetilde{N}$.

Let λ^h be a geodesic in $\widetilde{\Sigma^h}$ lying outside a large ball around p. Let $\lambda \subset \widetilde{\Sigma}$ be the ambient geodesic in $\widetilde{\Sigma}$ (i.e. the geodesic in the intrinsic metric on $\widetilde{\Sigma}$) joining the same pair of points. It follows from the hypothesis on λ^h

that λ lies outside a large ball *modulo horodisks corresponding to lifts of the cusps*. Also, outside of horodisks, λ^h and λ track each other [**Far98**]. Since $\mathcal{L}_\lambda$ is qi-embedded in $\widetilde{N}$, there exists a quasigeodesic $\mu \subset \widetilde{N}$ lying in a bounded neighborhood of $\mathcal{L}_\lambda$ joining the end-points of λ. Modulo horoballs in $\widetilde{N^h}$, corresponding to lifts of cusps, μ lies outside a large ball around p. Let μ^h be the hyperbolic geodesic joining the end points of μ. Away from horoballs, μ and μ^h track each other [**Far98**]. Away from horoballs therefore, μ^h lies outside large balls about p. The points at which μ^h enters and leaves a particular horoball therefore lie outside large balls about p. But then the hyperbolic segment joining them must do the same. This shows that μ^h must itself lie outside large balls around p. The existence of a Cannon-Thurston map follows (Lemmas 1.6 and 3.20 for instance).

Recall that there is a universal bundle over Teichmüller space $Teich(\Sigma^h)$ where the fiber over $x \in Teich(\Sigma^h)$ is, tautologically, the Riemann surface s. For r a geodesic in $Teich(\Sigma^h)$, the pullback bundle over r can be equipped with a metric that is infinitesimally a product of the Euclidean metric on r with the hyperbolic metrics on $x \in r \subset Teich(\Sigma^h)$. The universal cover of the resulting bundle is called the *universal metric bundle* over r. Theorem 3.17 now generalizes to the following:

THEOREM 3.22 ([**MR18**]). *Let H be $\pi_1(\Sigma^h)$ for Σ^h a non-compact hyperbolic surface of finite volume and K a finitely generated infinite index subgroup of H. Let r be a thick geodesic ray in the Teichmuller space $Teich(\Sigma^h)$ and let $r_\infty \in \partial Teich(\Sigma^h)$ be the limiting ending lamination. Let X denote the universal metric bundle over r minus a small neighborhood of the cusps and let $\mathcal{H}$ denote the horosphere boundary components. Then any orbit of K in X is relatively quasiconvex in $(X, \mathcal{H})$.*

We also obtain the following:

THEOREM 3.23 ([**MS12, MR18**]).
Let

1. *$H = \pi_1(\Sigma^h)$ be the fundamental group of a surface with finitely many punctures*
2. *$H_1, \cdots, H_n$ be the peripheral subgroups of H*
3. *Q be a convex cocompact subgroup of the pure mapping class group of S^h*
4. *K be a finitely generated infinite index subgroup of H.*

Let

- *$1 \to H \to G \to Q \to 1$, and*
- *$1 \to H_i \to N_G(H_i) \to Q \to 1$*

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}, i = 1, \cdots, n$. and K is relatively quasiconvex in G.

4. Kleinian surface groups

In this section and the next we shall describe a sequence of models for degenerate ends of 3-manifolds following [**Min01**, **Min94**, **Mj10**, **Mj11a**, **Mj16**] (in this section) and [**Min10**, **BCM12**, **Mj14a**] (in the next section) and indicate how to generalize the ladder construction of Section 3.2 incorporating electric geometry. Using a process of recovering hyperbolic geodesics from electric geodesics, we shall establish the existence of Cannon-Thurston maps. We shall focus on closed surfaces and follow the summary in [**LM18**] for the exposition.

The topology of each building block is simple: it is homeomorphic to $S\times[0,1]$, where S is a closed surface of genus greater than one. Geometrically, the top and bottom boundary components in the first three model geometries are uniformly bi-Lipschitz to a fixed hyperbolic structure on S. In fact it is inconvenient to consider them to be equipped with a fixed hyperbolic structure. We do so henceforth. The different types of geometries of the blocks make for different model geometries of ends.

Definition 4.1. *A model E_m is said to be built up of blocks of some prescribed geometries* **glued end to end**, *if*

(1) *E_m is homeomorphic to $S \times [0, \infty)$*
(2) *There exists $L \geq 1$ such that $S \times [i, i+1]$ is $L-$bi-Lipschitz to a block of one of the three prescribed geometries: bounded, i-bounded or amalgamated.*

$S \times [i, i+1]$ will be called the $(i+1)$**th block** *of the model E_m.*

The **thickness** *of the* $(i+1)$**th block is the length of the shortest path between $S \times \{i\}$ and $S \times \{i+1\}$ in $S \times [i, i+1] (\subset E_m)$.**

4.1. Bounded geometry. Minsky [**Min01**, **Min94**] calls an end E of a hyperbolic 3-manifold to be of bounded geometry if there are no arbitrarily short closed geodesics in E.

Definition 4.2. *Let $B_0 = S \times [0, 1]$ be given the product metric. If B is $L-$biLipschitz homeomorphic to B_0, it is called an $L-$***thick block**.

An end E is said to have a model of bounded geometry if there exists L such that E is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing $L-$thick blocks end-to-end.

It follows from work of Minsky [**Min92**] that if E is of bounded geometry, it has a model of bounded geometry. The existence of Cannon-Thurston maps in this setup is then a replica of the proof of Theorem 3.5 (See also Section 3.8).

4.2. i-bounded geometry.

Definition 4.3 ([**Mj11a**]). *An end E of a hyperbolic 3-manifold M has* **i-bounded geometry** *if the boundary torus of every Margulis tube in E has bounded diameter.*

We give an alternate description. Start with a fixed closed hyperbolic surface S and a finite collection $\mathcal{C}$ of disjoint simple closed geodesics on S. Let $N_\epsilon(\sigma_i)$ denote an ϵ neighborhood of σ_i, $\sigma_i \in \mathcal{C}$, were ϵ is small enough to ensure that lifts of $N_\epsilon(\sigma_i)$ to $\widetilde{S}$ are disjoint.

DEFINITION 4.4. *Let $I = [0,3]$. Equip $S \times I$ with the product metric. Let $B^c = (S \times I - \cup_i N_\epsilon(\sigma_i) \times [1,2])$, equipped with the induced path-metric. Perform Dehn filling on some $(1,n)$ curve on each resultant torus component of the boundary of B^c (the integers n are quite arbitrary and may vary). We call n the* **twist coefficient**. *Foliate the relevant torus boundary component of B^c by translates of $(1,n)$ curves. Glue in a solid torus Θ, which we refer to as a* **Margulis tube**, *with a hyperbolic metric foliated by totally geodesic disks bounding the $(1,n)$ curves.*

The resulting copy of $S \times I$ thus obtained, equipped with the above metric is called a **thin block**.

DEFINITION 4.5. *A model manifold E_m of* **i-bounded geometry** *is built out of gluing thick and thin blocks end-to-end*

It follows from work in [**Mj11a**] that

PROPOSITION 4.6. *An end E of a hyperbolic 3-manifold M has* **i-bounded geometry** *if and only if it is bi-Lipschitz homeomorphic to a model manifold E_m of i-bounded geometry.*

We give a brief indication of the construction of $\mathcal{L}_\lambda$ and the proof of the existence of Cannon-Thurston maps in this case. First electrify all the Margulis tubes. This ensures that in the resulting electric geometry, each block is of bounded geometry. More precisely, there is a (metric) product structure on $S \times [0,3]$ such that each $\{x\} \times [0,3]$ has uniformly bounded length in the electric metric.

Further, since the curves in $\mathcal{C}$ are electrified in a block, Dehn twists about components of $\mathcal{C}$ are isometries from $S \times \{1\}$ to $S \times \{2\}$ in a thin block. This allows the construction of $\mathcal{L}_\lambda$ to go through as before and ensures that it is quasiconvex in the resulting electric metric.

Finally given an electric geodesic lying outside large balls modulo Margulis tubes one can recover a genuine hyperbolic geodesic tracking it outside Margulis tubes. The criterion of Lemma 1.6 can now be satisfied using the same technique as the one we used in Section 3.8 to handle horoballs. The existence of Cannon-Thurston maps follows in this case.

4.3. Amalgamation geometry. Again, as in Definition 4.4, start with a fixed closed hyperbolic surface S, a collection of disjoint simple closed curves $\mathcal{C}$ and set $I = [0,3]$. Perform Dehn surgeries on the Margulis tubes corresponding to $\mathcal{C}$ as before. Let $K = S \times [1,2] \subset S \times [0,3]$ and let $K^c = (S \times I - \cup_i N_\epsilon(\sigma_i) \times [1,2])$. Instead of fixing the product metric on the complement K^c of Margulis tubes in K, allow these complementary components to have *arbitrary geometry* subject only to the restriction that the geometries of

$S \times \{1,2\}$ are fixed. Equip $S \times [0,1]$ and $S \times [3,4]$ with the product metrics. The resulting block is said to be a **block of amalgamation geometry**. After lifting to the universal cover, complements of Margulis tubes in the lifts $\widetilde{S} \times [1,2]$ are termed **amalgamation components**.

DEFINITION 4.7. *An end E of a hyperbolic 3-manifold M has* **amalgamated geometry** *if*

(1) *it is bi-Lipschitz homeomorphic to a model manifold E_m consisting of gluing thick and amalgamation geometry blocks end-to-end.*
(2) *Amalgamation components are (uniformly) quasiconvex in $\widetilde{E}_m$.*

To construct the ladder $\mathcal{L}_\lambda$ we electrify amalgamation components as well as Margulis tubes. This ensures that in the electric metric,

(1) Each amalgamation block has bounded geometry
(2) The mapping class element taking $S \times \{1\}$ to $S \times \{2\}$ induces an isometry of the electrified metrics.

Quasiconvexity of $\mathcal{L}_\lambda$ in the electric metric now follows as before. To recover the data of hyperbolic geodesics from quasigeodesics lying close to $\mathcal{L}_\lambda$, we use (uniform) quasiconvexity of amalgamation components and existence of Cannon-Thurston maps follows.

5. Split geometry

We need to relax the assumption of the previous section that the boundary components of model blocks are of (uniformly) bounded geometry. Roughly speaking, split geometry is a generalization of amalgamation geometry where

(1) A Margulis tube is allowed to travel through contiguous blocks and split them.
(2) The complementary pieces, now called split components, are quasiconvex in a somewhat weaker sense.

One of the main points of [**Mj14a**] is to show than any degenerate end has split geometry. Below, we shall invert this logic and start with the Minsky model [**Min10**] of degenerate ends and come up with the appropriate notion of split geometry. We shall follow the exposition of [**Mj14a**].

First, some basic notions and notation. The complexity of a compact surface $S = S_{g,b}$ of genus g and b boundary components is denoted by $\xi(S_{g,b}) = 3g + b$. For an essential subsurface Y of S, the curve complex and pants complex are denoted by $\mathcal{C}(Y)$ and $\mathcal{P}(Y)$ respectively. For a simplex $\alpha \in \mathcal{C}(Y)$, the simple closed curves forming the vertices of α is denoted by γ_α. Simplices α, β in $\mathcal{C}(Y)$ are said to fill an essential subsurface Y of S if all non-trivial non-peripheral curves in Y have essential intersection with at least one of γ_α or γ_β. Fixing α, β in $\mathcal{C}(S)$, we can form a regular neighborhood of $\gamma_\alpha \cup \gamma_\beta$ and fill in all disks to obtain an essential subsurface Y. We say that Y is filled by α, β. If $X \subset Z$ is an essential subsurface, ${}_Z(X)$ will denote the relative boundary of X in Z, i.e. the set of all boundary components of X that are non-peripheral in Z. A pants decomposition of S is a disjoint collection of pairs of pants $P_1, \cdots, P_n$ embedded in S such that $S \setminus \bigcup_i P_i$ is a disjoint collection of non-homotopic annuli in S. A tube in an end $E \subset N$ is an embedded solid torus $R-$neighborhood of an unknotted geodesic.

5.1. Hierarchies. We recall the concept of hierarchies from [**MM00**].

Definition 5.1. *Fix an essential subsurface Y in S.*
Case 1: $\xi(Y) > 4$: A sequence of simplices $\{v_i\}_{i\in\mathcal{I}}$ in $\mathcal{C}(Y)$ for $\mathcal{I}$ a finite or infinite interval in $\mathbb{Z}$ is said to be tight if it satisfies the following:

- *For vertices w_i of v_i and w_j of v_j $(i \neq j)$, $d_{\mathcal{C}(Y)}(w_i, w_j) = |i - j|$,*
- *Let $F(v_{i-1}, v_{i+1})$ denote the subsurface of Y filled by the multicurves v_{i-1}, v_{i+1}. Then for $\{i-1, i, i+1\} \subset \mathcal{I}$, v_i equals ${}_Y F(v_{i-1}, v_{i+1})$.*

Case 2: $\xi(Y) = 4$: A tight sequence is the vertex sequence of a geodesic in $\mathcal{C}(Y)$.

Definition 5.2. *A tight geodesic g in $\mathcal{C}(Y)$ consists of a tight sequence $v_0, \cdots, v_n$, and two simplices in $\mathcal{C}(Y)$, $\mathbf{I} = \mathbf{I}(g)$ and $\mathbf{T} = \mathbf{T}(g)$, the initial and terminal markings respectively, such that v_0 is a sub-simplex of $\mathbf{I}$ and v_n is a sub-simplex of $\mathbf{T}$.*

n is called the length of g. v_i is called a simplex of g. Y is called the domain of g and is denoted as $Y = D(g)$. g is said to be supported in $D(g)$.

For a simplex v of $\mathcal{C}(W)$, **collar**(v) will denote a small tubular neighborhood of the union of the simple closed curves in v. If $\xi(W) \geq 4$, v is a simplex of $\mathcal{C}(W)$ and Y is a component of $W \setminus$ **collar**(v), then we shall call Y a component domain of (W, v). For a tight geodesic g having domain $D(g)$, $Y \subset S$ is called a component domain of g if Y is a component domain of $(D(g), v_j)$ for some simplex v_j of g.

An edge path $\rho : [0, n] \to \mathcal{P}(S)$ joining $\rho(0) = P_1$ to $\rho(n) = P_2$ is called a **hierarchy path** in the pants complex $\mathcal{P}(S)$ joining pants decompositions P_1 and P_2 if

(1) There exists a collection $\{Y\}$ of essential, non-annular subsurfaces of S, which we refer to as component domains of ρ, such that for each Y there exists a sub-interval $J_Y \subset [0, n]$ with $\partial Y \subset \rho(j)$ for all $j \in J_Y$.
(2) For any such component domain Y of ρ, there exists a tight geodesic g_Y supported in Y satisfying the following: for each $j \in J_Y$, there exists $\alpha \in g_Y$ with $\alpha \in \rho(j)$.

Definition 5.3. *A* **hierarchy path** *ρ in $\mathcal{P}(S)$ is a sequence $\{P_n\}_n$ of pants decompositions of S such that for any $P_i, P_j \in \{P_n\}_n$, $i \leq j$, the sequence $P_i, P_{i+1}, \cdots, P_{j-1}, P_j$ of pants decompositions is a hierarchy path joining P_i and P_j.*

The collection of tight geodesics g_Y supported in component domains of ρ will be referred to as the **hierarchy** *of tight geodesics associated to ρ.*

In [**Min10**], Minsky considered a finer construction called a *resolution*. The notion of a hierarchy path above differs from a resolution in the sense of Minsky by ignoring geodesics supported in curve complexes of annuli. Thus the resolution of a hierarchy path we shall define below continues to ignore annular domains.

Given a hierarchy H associated to a hierarchy path ρ, a **slice** is a set τ of pairs (h, v), with $h \in H$ and v a simplex of h, such that the following is satisfied:

- Any geodesic h occurs in at most one element (pair) of τ.
- There exists a distinguished pair (h_τ, v_τ) in τ, called the bottom pair of τ and h_τ is called the bottom geodesic.
- For every $(k, w) \in \tau$ not equal to the bottom pair, $D(k)$ is a component domain of $(D(h), v)$ for some $(h, v) \in \tau$.

DEFINITION 5.4. *Given a hierarchy path $\rho : I \to \mathcal{P}(S)$ and a hierarchy H associated to it, a* **resolution** *of H is a sequence of slices $\tau_i = \{(h_{i1}, v_{i1}), (h_{i2}, v_{i2}), \cdots, (h_{in_i}, v_{in_i})\}$, $i \in I$, such that the set of vertices of $\{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ coincides with the set of non-peripheral boundary curves of pairs of pants in $\rho(i) \in \mathcal{P}(S)$.*

We introduce the notion and structure of Minsky Blocks following [**Min10**]. These are the building blocks of the model in [**Min10**]. A 4-geodesic is a tight geodesic in H supported in a component domain of complexity 4. An edge e of such a 4-geodesic is called a 4-edge. Let g be the 4-geodesic containing e, $D(e)$ the domain $D(g)$. e^- and e^+ will denote the initial and terminal vertices of e and **collar** v will denote a small neighborhood of v in $D(e)$.

In [**Min10**], Minsky constructs a block $B(e)$ which we refer to as a **Minsky block** as follows:

$$B(e) = (D(e) \times [-1, 1]) \setminus (\mathbf{collar}(e^-) \times [-1, -1/2) \cup \mathbf{collar}(e^+) \times (1/2, 1]).$$

The horizontal boundary of $B(e)$, consisting of pairs of pants, is given by

$$_{\pm}B(e) \equiv (D(e) \setminus \mathbf{collar}(e^{\pm})) \times \{\pm 1\}.$$

The top (resp. bottom) horizontal boundaries of $B(e)$ are $(D(e) \setminus \mathbf{collar}(e^+)) \times \{1\}$ (resp. $(D(e) \setminus \mathbf{collar}(e^-)) \times \{-1\}$). The rest of the boundary, called the vertical boundary, is a union of annuli.

5.2. Split level surfaces. Let E be (a neighborhood of) a degenerate end of a hyperbolic 3-manifold N. Let $\mathcal{T}$ denote a collection of disjoint, uniformly separated tubes in E containing all Margulis tubes. In particular, there is a uniform lower bound $\epsilon_0 > 0$ on the injectivity radius at all points in E outside $\bigcup_{T \in \mathcal{T}} Int(T)$. In [**Min10**], Minsky constructs a model M for E from a model $M(0)$ for $E \setminus \bigcup_{T \in \mathcal{T}} Int(T)$. In [**BCM12**], it is shown that there exists a bi-Lipschitz homeomorphism $F : E \to M$. There is a unique *standard* hyperbolic pair of pants $(Q, \partial Q)$ such that each component of ∂Q

has length one. A copy of $(Q, \partial Q)$ in $(M(0), \partial M(0))$ is called *flat* if it is isometrically embedded.

Definition 5.5. *A* **split level surface** *associated to a pants decomposition* $\{Q_1, \cdots, Q_n\}$ *is an embedding* $f : \cup_i (Q_i, \partial Q_i) \to (M(0), \partial M(0))$ *such that*

(1) $f(Q_i, \partial Q_i)$ *is flat for all* i*. (We shrink each* Q_i *slightly so that the complement of* $\cup_i Q_i$ *in* S *is a union of small annular neighborhoods of the pants curves.)*
(2) f *extends to a topological embedding (which we also denote by* f*) of* S *into* M *such that* $f(S \setminus \bigcup_i Q_i) \subset F(\bigcup_{T \in \mathcal{T}} Int(T))$.

For an embedded split level surface S_i as above, the union of the collection of flat pairs of pants in its image is denoted as S_i^s. Hence $S_i \setminus S_i^s$ consists of annuli properly embedded in the tubes $\mathcal{T}$.

The set of equivalence classes of topological embeddings from S to M that agree with a split level surface f corresponding to a pants decomposition $\{Q_1, \cdots, Q_n\}$ on $Q_1 \cup \cdots \cup Q_n$ can be partially ordered. Denote the equivalence class of f by $[f]$. The partial order $\leq_E$ is given as follows. $f_1 \leq_E f_2$ if there exist $g_i \in [f_i]$, $i = 1, 2$, such that $g_2(S)$ lies in the unbounded component of $E \setminus g_1(S)$.

We shall say that a sequence $\{S_i\}$ of split level surfaces exits E if

(1) $i < j$ implies $S_i \leq_E S_j$.
(2) For all compacts $B \subset E$, $S_i \cap B = \emptyset$ for all sufficiently large i.

Definition 5.6. *A pair of split level surfaces* S_i *and* S_j $(i < j)$ *is* k**-separated** *if*

(1) *for all* $x \in S_i^s$, $d_M(x, S_j^s) \geq k$.
(2) *for all* $x \in S_j^s$, $d_M(x, S_i^s) \geq k$.

The following Definition is the take-off point for split geometry.

Definition 5.7. *A simple closed curve* v **splits** *a pair of split level surfaces* S_i *and* S_j $(i < j)$ *if* v *occurs as a pants curve of both* S_i *and* S_j.

Definition 5.8. *A simple closed curve* $v \subset S$ *is* l**-thin** *if the corresponding tube* T_v *has core curve of length at most* l*. A tube* $T \in \mathcal{T}$ *is* l*-thin if its core curve is* l*-thin. A tube* $T \in \mathcal{T}$ *is* l*-thick if it is not* l*-thin. We denote the collection of all* l*-thin tubes as* $\mathcal{T}_l$ *and* $M(0)$ *along with all* l*-thick tubes as* $M(l)$.

A pair of split level surfaces S_i *and* S_j $(i < j)$ *is said to be an* l**-thin pair** *if there exists an* l*-thin curve* v *that splits* S_i, S_j.

5.2.1. *The Minsky model, the bi-Lipschitz model theorem and consequences.* For $\theta, \omega > 0$, a neighborhood $N_\epsilon(\gamma)$ of a closed geodesic γ is called a (θ, ω)-thin tube if the length of γ is at most θ and the length of the shortest geodesic on $\partial N_\epsilon(\gamma)$ is at least ω.

THEOREM 5.9 ([**Min10**], [**BCM12**]). *Let N be the convex core of a simply or doubly degenerate hyperbolic 3-manifold without cusp corresponding to a representation of the fundamental group of a closed surface S. There exist*

(1) $L \geq 1$, $\theta, \omega, \epsilon, \epsilon_1 > 0$,
(2) *a collection $\mathcal{T}$ of (θ, ω)-thin tubes containing all Margulis tubes in N,*
(3) *a 3-manifold M,*
(4) *an L-bi-Lipschitz homeomorphism $F : N \to M$,*

such that if

(1) $M(0) = F(N \setminus \bigcup_{T \in \mathcal{T}} Int(T))$
(2) *$F(\mathcal{T})$ denotes the image of the collection $\mathcal{T}$ under F,*
(3) *and $\leq_E$ denotes the partial order on the collection of split level surfaces in an end E of M,*

then there exists a sequence S_i of split level surfaces associated to pants decompositions P_i exiting E such that

(1) *$S_i \leq_E S_j$ if $i \leq j$.*
(2) *The sequence $\{P_i\}$ gives a hierarchy path in $\mathcal{P}(S)$.*
(3) *If $P_i \cap P_j = \{Q_1, \cdots Q_l\}$ then $f_i(Q_k) = f_j(Q_k)$ for $k = 1 \cdots l$, where f_i, f_j are the embeddings defining the split level surfaces S_i, S_j respectively.*
(4) *For all i, $P_i \cap P_{i+1} = \{Q_{i,1}, \cdots Q_{i,l}\}$ consists of a collection of pairs of pants, such that $S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ has a single non-annular component of complexity 4. Further, there exists a Minsky block W_i and an isometric map G_i of W_i into $M(0)$ such that $f_i(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l})$ (resp. $f_{i+1}(S \setminus (Q_{i,1} \cup \cdots \cup Q_{i,l}))$) is contained in the bottom (resp. top) gluing boundary of W_i.*
(5) *For each flat pair of pants Q in a split level surface S_i there exists an isometric embedding of $Q \times [-\epsilon, \epsilon]$ into $M(0)$ such that the embedding restricted to $Q \times \{0\}$ agrees with f_i restricted to Q.*
(6) *For each $T \in \mathcal{T}$, there exists a split level surface S_i associated to pants decomposition P_i such that the core curve of T is isotopic to a non-peripheral boundary curve of P_i. The boundary $F(\partial T)$ of $F(T)$ with the induced metric d_T from $M(0)$ is a Euclidean torus equipped with a product metric $S^1 \times S^1_v$, where any circle of the form $S^1 \times \{t\} \subset S^1 \times S^1_v$ is a round circle of unit length and is called a horizontal circle; and any circle of the form $\{t\} \times S^1_v$ is a round circle of length l_v and is called a vertical circle.*
(7) *Let g be a tight geodesic other than the bottom geodesic in the hierarchy H associated to the hierarchy path $\{P_i\}$, let $D(g)$ be the support of g and let v be a boundary curve of $D(g)$. Let T_v be the tube in $\mathcal{T}$ such that the core curve of T_v is isotopic to v. If a vertical circle of $(F(\partial T_v), d_{T_v})$ has length l_v less than $n\epsilon_1$, then the length of g is less than n.*

5.3. Split surfaces and weak split geometry. We are now in a position to describe the general case of split geometry. We first extend split level surfaces to split surfaces by adjoining bi-Lipschitz annuli.

Definition 5.10. *An L-bi-Lipschitz* **split surface** *in $M(l)$ associated to a pants decomposition $\{Q_1, \cdots, Q_n\}$ of S and a collection $\{A_1, \cdots, A_m\}$ of (some of the) complementary annuli in S is an embedding $f : \cup_i Q_i \bigcup \cup_i A_i \to M(l)$ such that*

1. *the restriction $f : \cup_i (Q_i, \partial Q_i) \to (M(0), \partial M(0))$ is a split level surface,*
2. *the restriction $f : A_i \to M(l)$ is an L-bi-Lipschitz embedding,*
3. *f extends to an embedding (also denoted f) of S into M such that each annulus component of $f(S \setminus (\cup_i Q_i \bigcup \cup_i A_i))$ is properly contained in $F(\bigcup_{T \in \mathcal{T}_l} Int(T))$.*

Denote split surfaces by Σ_i. Let Σ_i^s denote the union of the collection of flat pairs of pants and bi-Lipschitz annuli in the image of the split surface Σ_i.

We recall one of the technical tools from [**Mj14a**].

Theorem 5.11. ([**Mj14a**, Theorem 4.8]) *Let N, M, $M(0)$, S, F be as in Theorem 5.9 above. Let E be an end of M. Fix a hyperbolic metric on S, l less than the Margulis constant, and let $M(l)$ be the union of $M(0)$ and $l-$thick Margulis tubes. Then there exist $L_1 \geq 1$, $\epsilon_1 > 0$, $n \in \mathbb{N}$, and a sequence Σ_i of L_1-bi-Lipschitz, ϵ_1-separated split surfaces exiting E such that for all i, either an l-thin curve splits the pair (Σ_i, Σ_{i+1}); or there exists an L_1-bi-Lipschitz embedding $G_i : (S \times [0,1], (\partial S) \times [0,1]) \to (M, \partial M)$ such that $\Sigma_i^s = G_i(S \times \{0\})$ and $\Sigma_{i+1}^s = G_i(S \times \{1\})$.*

Further, each l-thin curve in S splits at most n split level surfaces in the sequence $\{\Sigma_i\}$.

A crucial feature of Theorem 5.11 above is that each $l-$thin in S splits a uniformly bounded number of split level surfaces. This necessarily involves passing to a subsequence $\{\Sigma_i\}$ of the sequence S_i of split level surfaces given by Theorem 5.9 (hence the change in notation).

Pairs of split surfaces satisfying the first (resp. second) alternatives of Theorem 5.11 are called **l-thin pairs** (resp. **l-thick pairs**) of split surfaces. We shall drop l if it is understood. We now provide a weakened version of the promised notion of split geometry.

Definition 5.12. *A model end E identified with $S \times [0, \infty)$ satisfying the following conditions is said to have* **weak split geometry***:*

1. *There exists a sequence of split surfaces $S_i^s (\subset S \times \{i\})$ exiting E.*
2. *A collection $\mathcal{T}$ of Margulis tubes in (the corresponding end of N) with image $F(\mathcal{T})$ in E. The image is also referred to as a Margulis tube.*
3. *Each complementary annulus of S_i^s with core curve v is mapped properly into a Margulis tube $F(T)$ which is said to split S_i^s.*

(4) *Either there exists a Margulis tube splitting both S_i^s and S_{i+1}^s, or else both $S_i(= S_i^s)$ and $S_{i+1}(= S_{i+1}^s)$ have bounded geometry and bound a thick block B_i.*
(5) *$F(T) \cap S_i^s$ is either empty or consists of a pair of boundary components of S_i^s that are parallel in S_i.*
(6) *There is a uniform upper bound on the number of surfaces that any $F(T)$ splits.*

We shall sketch a proof of a special case of the following Theorem in the next subsection.

THEOREM 5.13 ([**Mj14a**]). *Any degenerate end of a hyperbolic 3-manifold is bi-Lipschitz homeomorphic to a Minsky model and hence to a model of weak split geometry.*

To proceed to define split geometry, we have to impose a quasiconvexity condition. But before we do this, we need to identify the analog of amalgamation components in weak split geometry (Definition 5.12). We do this now.

If $(\Sigma_i^s, \Sigma_{i+1}^s)$ is a thick pair of split surfaces in M, then the closure of the bounded component of $E \setminus (\Sigma_i^s \cup \Sigma_{i+1}^s)$ between Σ_i^s, Σ_{i+1}^s will be called a **thick block** as in Section 4.

DEFINITION 5.14. *Let $(\Sigma_i^s, \Sigma_{i+1}^s)$ be an l-thin pair of split surfaces in M and $F(\mathcal{T}_i)$ be the collection of l-thin Margulis tubes that split both Σ_i^s, Σ_{i+1}^s. The closure of the union of the bounded components of $M \setminus ((\Sigma_i^s \cup \Sigma_{i+1}^s) \bigcup_{F(T) \in F(\mathcal{T}_i)} F(T))$ between Σ_i^s, Σ_{i+1}^s will be called a* **split block***.*

The closure of any bounded component is called a **split component***.*

Each split component is allowed to contain Margulis tubes, called **hanging tubes** that do not go all the way across from the top to the bottom, i.e. they do not split both Σ_i^s, Σ_{i+1}^s.

Topologically, therefore, a split component $B^s \subset B = S \times I$ is a topological product $S^s \times I$ for some, necessarily connected S^s. However, the upper and lower boundaries of B^s need only be split subsurfaces of S^s to allow for hanging tubes starting or ending (but not both) within the split block.

Electrifying split components as in Section 4.3, we obtain a new electric metric called the **graph metric** d_G on E.

DEFINITION 5.15. *A model of weak split geometry is said to be of* **split geometry** *if the convex hull of each split component has uniformly bounded d_G-diameter.*

5.4. Cannon-Thurston maps for degenerate manifolds. Let M be a hyperbolic 3-manifold homotopy equivalent to a closed hyperbolic surface S. Once we establish that M has split geometry, the proof proceeds as in Section 4.3 by electrifying split components, constructing a hyperbolic ladder $\mathcal{L}_\lambda$ and finally recovering a hyperbolic geodesic from an electric one. We shall therefore dwell in this subsection on showing that any degenerate

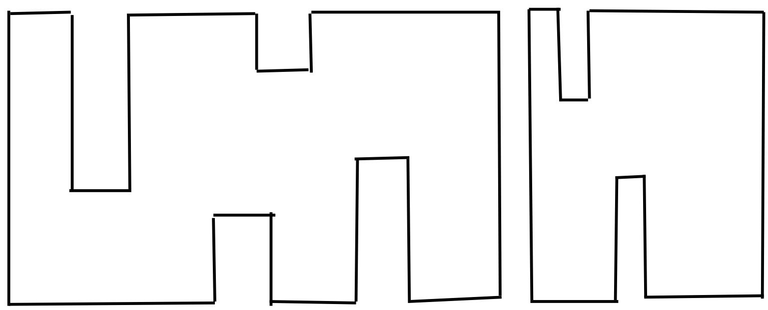

FIGURE 1. Split Block with hanging tubes

end has split geometry. We shall do this under two simplifying assumptions, directing the reader to [**Mj14a**] (especially the Introduction) for a more detailed road-map.

We borrow extensively from the hierarchy and model manifold terminology and technology summarized in Sections 5.1 and 5.2 above. The model manifold of [**Min10**] (Theorem 5.9) furnishes a resolution, or equivalently, a sequence $\{P_m\}$ of pants decompositions of S exiting E and hence a hierarchy path. Let τ_m denote simple multicurve on S constituting P_m. The P_m in turn furnish split level surfaces $\{S_m\}$ exiting E. Next, corresponding to the hierarchy path is a tight geodesic in $\mathcal{C}(S)$ consisting of the bottom geodesic $\{\eta_i\}$ of the hierarchy. We proceed to extract a subsequence of the resolution τ_m using the bottom geodesic $\{\eta_i\}$ under two key simplifying assumptions:

(1) For all i, the length of exactly one curve in η_i is sufficiently small, less than the Margulis constant in particular. Call it η_i for convenience.
(2) Let S_i correspond to the first occurrence of the vertex η_i in the resolution τ_m. Assume further that the S_i's are actually split surfaces and not just split level surfaces, i.e. they all have injectivity radius uniformly bounded below,

It follows that the Margulis tube η_i splits both S_i and S_{i+1} and that the tube T_i corresponding to η_i is trapped entirely between S_i and S_{i+1}. The product region B_i between S_i and S_{i+1} is therefore a split block for all i and T_i splits it. The model manifold thus obtained therefore satisfies the conditions of Definition 5.12. In a sense, this is a case of 'pure split geometry', where all blocks have a split geometry structure (no thick blocks). To prove that the model is indeed of split geometry, it remains to establish the quasiconvexity condition of Definition 5.15.

Let K be a split component and $\widetilde{K}$ an elevation to $\widetilde{E}$. Let v be a boundary short curve for the split component and let T_v be the Margulis tube corresponding to v abutting K. Denote the hyperbolic convex hull by $CH(\widetilde{K})$ and pass back to a quotient in M. A crucial observation that is needed here is the fact that any pleated surface has bounded d_G-diameter. This is because thin parts of pleated surfaces lie inside Margulis tubes that get electrified in the graph metric. It therefore suffices to show that any point in $CH(K)$ lies close to a pleated surface passing near the fixed tube T_v. This last condition follows from the Brock-Bromberg drilling theorem [**BB04**] and the fact that the convex core of a quasi-Fuchsian group is filled by pleated surfaces [**Fan99**]. This completes our sketch of a proof of the following main theorem of [**Mj14a**]:

THEOREM 5.16. *Let $\rho : \pi_1(S) \to PSL_2(\mathbb{C})$ be a simply or doubly degenerate (closed) surface Kleinian group. Then a Cannon-Thurston map exists.*

It follows that the limit set of $\rho(\pi_1(S))$ is a continuous image of S^1 and is therefore locally connected. As a first application of Theorem 5.16, we shall use the following Theorem of Anderson and Maskit [**AM96**] to prove that connected limit sets of Kleinian groups without parabolics are locally connected.

THEOREM 5.17 ([**AM96**]). *Let Γ be an analytically finite Kleinian group with connected limit set. Then the limit set $\Lambda(\Gamma)$ is locally connected if and only if every simply degenerate surface subgroup of Γ without accidental parabolics has locally connected limit set.*

Combining the remark after Theorem 5.16 with Theorem 5.17, we immediately have the following affirmative answer to Question 1.4.

THEOREM 5.18. *Let Γ be a finitely generated Kleinian group without parabolics and with a connected limit set Λ. Then Λ is locally connected.*

Further generalizations and applications of Theorem 5.16 appear in the following section.

6. Generalizations and applications: Kleinian groups

6.1. Finitely generated Kleinian groups. In [**Mj14b**], we show that the point preimages of the Cannon-Thurston map for a simply or doubly degenerate surface Kleinian group given by Theorem 5.16 corresponds to end-points of leaves of ending laminations. In particular, the ending lamination corresponding to a degenerate end can be recovered from the Cannon-Thurston map. This was extended further in [**DM16**] and [**Mj17b**] to obtain the following general version for finitely generated Kleinian groups.

THEOREM 6.1 ([**Mj17b**]). *Let G be a finitely generated Kleinian group. Let $i : \Gamma_G \to \mathbb{H}^3$ be the natural identification of a Cayley graph of G with the orbit of a point in $\mathbb{H}^3$. Then i extends continuously to a Cannon-Thurston map $\hat{i} : \widehat{\Gamma_G} \to \mathbb{D}^3$, where $\widehat{\Gamma_G}$ denotes the (relative) hyperbolic compactification of Γ_G.*

Let ∂i denote the restriction of $\hat{i}$ to the boundary $\partial\Gamma_G$ of Γ_G. Let E be a degenerate end of $N^h = \mathbb{H}^3/G$ and $\widetilde{E}$ a lift of E to $\widetilde{N^h}$ and let M_{gf} be an *augmented Scott core of N^h. Then the ending lamination $\mathcal{L}_E$ for the end E lifts to a lamination on $\widetilde{M_{gf}} \cap \widetilde{E}$. Each such lift $\mathcal{L}$ of the ending lamination of a degenerate end defines a relation $\mathcal{R}_{\mathcal{L}}$ on the (Gromov) boundary $\partial \widetilde{M_{gf}}$ (or equivalently, the relative hyperbolic boundary $\partial_r \Gamma_G$ of Γ_G), given by $a\mathcal{R}_{\mathcal{L}}b$ iff a, b are end-points of a leaf of $\mathcal{L}$. Let $\{\mathcal{R}_i\}$ be the entire collection of relations on $\partial \widetilde{M_{gf}}$ obtained this way (taking all ends E and all lifts $\mathcal{L}$). Let $\mathcal{R}$ be the transitive closure of the union $\bigcup_i \mathcal{R}_i$. Then $\partial i(a) = \partial i(b)$ iff $a\mathcal{R}b$.*

6.2. Cannon-Thurston maps and the ending lamination theorem. Theorems 5.16 and 6.1 in conjunction with the Ending Lamination Theorem [**Min10, BCM12**] establish the slogan:

Dynamics on the Limit Set determines Geometry in the Interior.

The model built in [**Min10, BCM12**] is of course necessary for the proofs of Theorems 5.16 and 6.1. We now combine the Ending Lamination Theorem with these two theorems to establish the above slogan. In the simplest case, the Ending Lamination Theorem says that for a simply or doubly degenerate surface Kleinian group $\Gamma (= \rho(\pi_1(S))$ without accidental parabolics, the geometry i.e. the isometry type of the manifold $M = \mathbf{H}^3/\Gamma$ is determined by its end-invariants. For a doubly degenerate group, the end-invariants are two ending laminations, one each for the two geometrically infinite ends of M. For a simply degenerate group, the end-invariants are an ending lamination corresponding to the geometrically infinite end of M and a conformal structure on S corresponding to the geometrically finite end of M. The ending lamination corresponding to a geometrically infinite end is independent of the hyperbolic structure on S and hence may be regarded as a purely topological piece of data associated to the end. Thus, in the context of geometrically infinite Kleinian groups, the Ending Lamination Theorem justifies the slogan

Topology implies Geometry.

It may thus be considered an analog of Mostow Rigidity for infinite covolume Kleinian groups.

The structure of Cannon-Thurston maps ([**Mj14b**], or more generally, Theorem 6.1) show that the point preimage data of Cannon-Thurston maps captures precisely the ending lamination. By the Ending Lamination Theorem, it follows that the Cannon-Thurston map determines the isometry type of M

(1) completely when M is doubly degenerate, and
(2) up to bi-Lipschitz homeomorphism (with a uniformly bounded constant) when M is simply degenerate.

Further, since topological conjugacies are compatible with Cannon-Thurston maps, it follows that a topological conjugacy of the $\Gamma-$action on limit sets comes from a bi-Lipschitz homeomorphism of quotient manifolds. Hence the Ending Lamination Theorem [**Min10, BCM12**] in conjunction with Theorem 5.16 and the structure theorem of [**Mj14b**] (or Theorem 6.1) show that we can recover the geometry of M from the action of Γ on the limit set Λ_Γ.

6.3. Primitive stable representations. In [**Min09**] Minsky introduced an open subset of the $PSL_2(\mathbb{C})$ character variety for a free group, properly containing the Schottky representations, on which the action of the outer automorphism group is properly discontinuous. He called these **primitive stable representations.** Let F_n be a free group of rank n. An element of F_n is primitive if it is an element of a free generating set. Let $\mathcal{P} = \cdots www \cdots$ be the set of bi-infinite words with w cyclically reduced primitive. A representation $\rho : F_n \to PSL_2(\mathbb{C})$ is primitive stable if all elements of $\mathcal{P}$ are mapped to uniform quasigeodesics in $\mathbf{H}^3$.

Minsky conjectured that primitive stable representations are characterized by the feature that every component of the ending lamination is blocking.

Using Theorem 6.1, Jeon and Kim [**JK10**], and Jeon, Kim, Ohshika and Lecuire [**JKOL14**] resolved this conjecture. We briefly sketch their argument for a degenerate free Kleinian group without parabolics.

Let $\{D_1, \cdots, D_n\} = \mathcal{D}$ be a finite set of essential disks on a handlebody H cutting H into a 3-ball. A free generating set of F_n is dual to $\mathcal{D}$. For a lamination $\mathcal{L}$, the **Whitehead graph** $Wh(\Lambda, \mathcal{D})$ is defined as follows. Cut ∂H along $\partial \mathcal{D}$ to obtain a sphere with $2n$ holes, labeled by $D_i^{\pm}$. The vertices of $Wh(\mathcal{L}, \mathcal{D})$ are the boundary circles of ∂H, with an edge whenever two circles are joined by an arc of $\mathcal{L} \setminus \mathcal{D}$. For the ending lamination $\mathcal{L}_E$ of a degenerate free group without parabolics, $Wh(\mathcal{L}_E, \Delta)$ is connected and has no cutpoints.

Let ρ_E be the associated representation, i.e. a representation of a free Kleinian group with ending lamination $\mathcal{L}_E$. If ρ_E is not primitive stable, then there exists a sequence of primitive cyclically reduced elements w_n such that $\rho_E(w_n^*)$ is not an $n-$ quasi-geodesic. After passing to a subsequence, w_n and hence w_n^* converges to a bi-infinite geodesic w_∞ in the Cayley graph with two distinct end points w_+, w_- in the Gromov boundary of F_n. The Cannon-Thurston map identifies w_+, w_-. Hence by Theorem 6.1 they are either the end points of a leaf of $\mathcal{L}_E$ or ideal end-points of a complementary ideal polygon of $\mathcal{L}_E$. It follows therefore that $Wh(w_\infty, \mathcal{D})$ is connected and has no cutpoints. Since the w_n's converge to w_∞, $Wh(w_n, \mathcal{D})$ is connected and has no cutpoints for large enough n. A Lemma due to Whitehead says that if $Wh(w_n, \mathcal{D})$ is connected and has no cutpoints, then w_n cannot be primitive, a contradiction.

6.4. Discreteness of commensurators. In [**LLR11**] and [**Mj11b**], Theorems 5.16 and 6.1 are used to prove that commensurators of finitely generated, infinite covolume, Zariski dense Kleinian groups are discrete. The basic fact that goes into the proof is that commensurators preserve the structure of point pre-images of Cannon-Thurston maps. The point preimage structure is known from Theorem 6.1.

6.5. Motions of limit sets. We discuss the following question in this section, which paraphrases the second part of [**Thu82**, Problem 14]. A detailed survey appears in [**Mj17a**].

QUESTION 6.2. *Let G_n be a sequence of Kleinian groups converging to a Kleinian group G. Does the corresponding dynamics of G_n on the Riemann sphere S^2 converge to the dynamics of G on S^2?*

To make Question 6.2 precise, we need to make sense of 'convergence' both for Kleinian groups and for their dynamics on S^2. There are three different notions of convergence for Kleinian groups.

DEFINITION 6.3. *Let $\rho_i : H \to PSL_2(\mathbb{C})$ be a sequence of Kleinian groups. We say that ρ_i converges to ρ_∞* **algebraically** *if for all $h \in H$, $\rho_i(h) \to \rho_\infty(h)$.*

Let $\rho_j : H \to PSL_2(\mathbb{C})$ be a sequence of discrete, faithful representations of a finitely generated, torsion-free, nonabelian group H. If $\{\rho_j(H)\}$ converges as a sequence of closed subsets of $PSL_2(\mathbb{C})$ in the Gromov-Hausdorff topology to a torsion-free, nonabelian Kleinian group Γ, Γ is called the geometric limit of the sequence.

$G_i(= \rho_i(H))$ converges **strongly** *to $G(= \rho_\infty(H))$ if the convergence is both geometric and algebraic.*

Question 6.2 then splits into the following three questions.

QUESTION 6.4.

(1) *If $G_n \to G$ geometrically, then do the corresponding limit sets converge in the Hausdorff topology on S^2?*
(2) *If $G_n \to G$ strongly then do the corresponding Cannon-Thurston maps converge uniformly?*
(3) *If $G_n \to G$ algebraically then do the corresponding Cannon-Thurston maps converge pointwise?*

We give the answers straight off and then proceed to elaborate.

ANSWERS 6.5.

(1) *The answer to Question 6.4 (1) is Yes.*
(2) *The answer to Question 6.4 (2) is Yes.*
(3) *The answer to Question 6.4 (3) is No, in general.*

The most general answer to Question 6.4 (1) is due to Evans [**Eva00**], [**Eva04**]:

THEOREM 6.6 ([**Eva00**], [**Eva04**]). *Let $\rho_n : H \to G_n$ be a sequence of weakly type-preserving isomorphisms from a geometrically finite group H to Kleinian groups G_n with limit sets Λ_n, such that ρ_n converges algebraically to $\rho_\infty : H \to G^a_\infty$ and geometrically to G^g_∞. Let Λ_a and Λ_g denote the limit sets of G^a_∞ and G^g_∞. Then $\Lambda_n \to \Lambda_g$ in the Hausdorff metric. Further, the sequence converges strongly if and only $\Lambda_n \to \Lambda_a$ in the Hausdorff metric.*

The answer to Question 6.4 (2) is due to the author and Series [**MS17**] in the case that $H = \pi_1(S)$ for a closed surface S of genus greater than one. This can be generalized to arbitrary finitely generated Kleinian groups as in [**Mj17a**]:

THEOREM 6.7. *Let H be a fixed group, ρ_n a sequence of strongly type-preserving isomorphisms such that $\rho_n(H) = \Gamma_n$ is a sequence of geometrically finite Kleinian groups converging strongly to a Kleinian group Γ. Let M_n and M_∞ be the corresponding hyperbolic manifolds. Let K be a fixed complex with fundamental group H.*

Consider embeddings $\phi_n : K \to M_n$, $n = 1, \cdots, \infty$ such that the maps ϕ_n are homotopic to each other by uniformly bounded homotopies (in the geometric limit). Then Cannon-Thurston maps for $\widetilde{\phi_n}$ exist and converge uniformly to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.

Finally we turn to Question 6.4 (3), which turns out to be the subtlest. In [**MS13**] we showed that the answer to Question 6.4 (3) is 'Yes' if the geometric limit is geometrically finite. We illustrate this with a concrete example due to Kerckhoff and Thurston [**KT90**]

THEOREM 6.8. *Fix a closed hyperbolic surface S and a simple closed geodesic σ on it. Let tw^i denote the automorphism of S given by an i-fold Dehn twist along σ. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, tw^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i-} denote the lower boundary component of the convex core of G_i, $i = 1, \cdots, \infty$ (including ∞). Let $\phi_i : S \to S_{i-}$ be such that if $0 \in \mathbf{H}^2 = \widetilde{S}$ denotes the origin of $\mathbf{H}^2$ then $\widetilde{\phi_i}(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbf{H}^3 = \widetilde{M_m}$. We also assume (using the fact that M_∞ is a geometric limit of M_m's) that S_{i-}'s converge geometrically to $S_{\infty-}$. Then the Cannon-Thurston maps for $\widetilde{\phi_i}$ converge pointwise, but not uniformly, on $\partial\mathbf{H}^2$ to the Cannon-Thurston map for $\widetilde{\phi_\infty}$.*

However, if the geometric limit is geometrically infinite, then the answer to Question 6.4 (3) may be negative. We illustrate this with certain examples of geometric limits constructed by Brock in [**Bro01**].

Fix a closed hyperbolic surface S and a separating simple closed geodesic σ on it, cutting S up into two pieces S_- and S_+. There is a natural map G defined on $\partial\widetilde{S}(= S^1)$ that identifies $p, q \in \partial\widetilde{S}$ if and only if p, q are end-points of a lift of σ to $\widetilde{S}$. Then $G(\partial\widetilde{S})$ is naturally identifiable with the limit set of a(ny) representation of $\pi_1(S)$ where σ corresponds to an accidental parabolic and the induced representations of $\pi_1(S_-)$ and $\pi_1(S_+)$ are quasi-Fuchsian with parabolic boundary curves (corresponding to σ). Thus $G(\partial\widetilde{S})$ is a tree of circles corresponding to limit sets of the quasi-Fuchsian representations of $\pi_1(S_-)$ and $\pi_1(S_+)$.

Let ϕ denote an automorphism of S such that $\phi|_{S_-}$ is the identity and $\phi|_{S_+} = \psi$ is a pseudo-Anosov of S_+ fixing the boundary. Let G_i be the quasi-Fuchsian group given by the simultaneous uniformization of $(S, \phi^i(S))$. Let G_∞ denote the geometric limit of the G_m's. Let S_{i0} denote the lower boundary component of the convex core of G_i, $i = 1, \cdots, \infty$ (including ∞). Let $\phi_i : S \to S_{i0}$ be such that if $0 \in \mathbf{H}^2 = \widetilde{S}$ denotes the origin of $\mathbf{H}^2$ then $\widetilde{\phi}_i(0)$ lies in a uniformly bounded neighborhood of $0 \in \mathbf{H}^3 = \widetilde{M_m}$. We also assume (using the fact that M_∞ is a geometric limit of M_m's) that S_{i0}'s converge geometrically to $S_{\infty 0}$.

Let Σ be a complete hyperbolic structure on S_+ such that σ is homotopic to a cusp on Σ. Let Λ consist of pairs (ξ_-, ξ) of endpoints (on $\mathbb{S}^1_\infty$ of stable leaves λ of the stable lamination of ψ acting on $\widetilde{\Sigma}$. Also let $\partial\widetilde{\mathcal{H}}$ denote the collection of basepoints of lifts (to $\widetilde{\Sigma}$) of the cusp in Σ corresponding to σ. Let

$$\Theta = \{\xi : \text{There exists } \xi_- \text{ such that } (\xi_-, \xi) \in \Lambda; \xi_- \in \partial\widetilde{\mathcal{H}}\}.$$

Identifying $\partial\widetilde{\Sigma}$ with one of the (topological) circles in the tree of circles $G(\partial\widetilde{S})$ above, define

$$\Xi_0 = G^{-1}(\Theta)$$

to be the preimage of Θ under G and Ξ to be the $\pi_1(S)-$orbit of Ξ_0. We then have the following characterization of points where Cannon-Thurston maps do not converge:

THEOREM 6.9 ([**MS17**]). *Let $\partial\phi_i$, $i = 1 \cdots, \infty$ denote the Cannon-Thurston maps for $\widetilde{\phi}_i$. Then*

(1) *$\partial\phi_i(\xi)$ does not converge to $\partial\phi_\infty(\xi)$ for $\xi \in \Xi$.*
(2) *$\partial\phi_i(\xi)$ converges to $\partial\phi_\infty(\xi)$ for $\xi \notin \Xi$.*

In [**MO17**], we identify the exact criteria that lead to the discontinuity phenomenon of Theorem 6.9.

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