Toward and after virtual specialization in 3-manifold topology

Yi Liu and Hongbin Sun

Abstract. This article surveys works in 3–manifold topology after the proof of the virtual Haken conjecture. It reviews the virtual specialization of 3–manifold groups, and some following development of methods, and applications in recent years.

1. Introduction

A topological space is said to *virtually* have a certain property if it admits a finite cover with that property. A compact orientable irreducible 3–manifold is called a *Haken manifold* if it contains a properly embedded two-sided incompressible subsurface. In the 1980s, as he proved his revolutionary hyperbolization theorem for aspherical atoroidal Haken manifolds, Thurston went on to ask whether every closed hyperbolic 3–manifold is virtually a Haken manifold. This amazing question survived Perelman's proof of the geometrization in 2003, and stood for another decade until Agol eventually resolved it. Besides the core argument, Agol's proof is built on Kahn and Markovic's surface subgroup theorem, Sageev's construction of cubulating groups, Haglund and Wise's special cube complex theory, Wise's quasiconvex hierarchy theory for special word hyperbolic groups, and Groves and Manning's malnormal filling theorems for word hyperbolic groups. In this survey, we summarize some important ideas toward the solution of the virtual Haken conjecture and its related conjectures, and then browse through some more recent advances on various topics in 3–manifold topology, which are inspired or motivated by the new theorems.

The organization of this survey is as follows. In Section 2, we provide an introductory review on the virtual specialization of 3–manifold groups.

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In Section 3, we summarize some significant techniques that have been developed or utilized during the past few years. In Sections 4–10, we survey applications of the new techniques to a list of topics, including separability of finitely generated subgroups, virtual homological torsion, virtual domination, twisted Alexander invariants, virtual representation volume, profinite completion, and quantitative aspects. While the topic sections are presented generally independently from each other, we sort them out so that Section 4 is about topological characterization of group theoretic properties, and Sections 5–8 about construction of individual finite covers, and Sections 9–10 about all finite covers being a whole collection.

Throughout the survey, manifolds and complexes are all assumed to be connected, unless otherwise declared. Basepoints are implicitly assumed in the notations of fundamental groups. Kneser–Milnor prime decomposition and prime summands are only considered for compact orientable 3–manifolds. Jaco–Shalen–Johanson (JSJ) torus decomposition, JSJ tori, and JSJ pieces are only considered for compact orientable irreducible (hence prime) 3–manifolds.

We adopt the following terminology in this survey. Caution that other authors may use the same terms referring to a different range of objects. A *mixed* 3–manifold is an orientable compact irreducible 3–manifold with at least one JSJ torus and at least one non-elementary atoroidal JSJ piece. It is allowed to have negative Euler characteristic in this survey, although many articles assume mixed 3-manifolds have zero Euler characteristic (empty or tori boundary). A *graph manifold* is an orientable compact irreducible 3–manifold whose JSJ pieces are all Seifert fibered. It is allowed to be a Seifert fibered space or a virtual Anosov torus bundle over a circle. An *orientable thickened Klein bottle* is a compact interval bundle over a Klein bottle whose total space is orientable. It is unique up to homeomorphism, and may occur as an elementary, Seifert fibered JSJ piece, which we do not consider to be geometric.

For background review, we take advantage of several excellent surveying works, including Agol [**Ago14**], Aschebrenner–Friedl–Wilton [**AscFW15a, AscFW15b**], Friedl–Vidussi [**FriV11b**], Wang [**Wan02**], Reid [**Rei18**]. In particular, one of our motivations is to provide a list of recent updates on virtual properties of 3–manifold groups since [**AscFW15a**]. The reader is furthermore referred to the classical references Hempel [**Hem76**] and Jaco [**Jac80**] for 3–manifold topology, and Thurston [**Thu79**] for the topology and geometry of hyperbolic 3–manifolds.

Although we have struggled to cover as many existing post-virtual-specialization works in 3–manifold topology as possible, we claim no completeness of our scope. The exposition also reflects the taste and the comprehension of the authors unavoidably.

2. From geometrization to virtual specialization

While the geometrization theorem successfully reduces the topological classification of 3–manifolds to hyperbolic 3–manifolds, it is far from easy to *really* understand the latter. Among the finite-volume ones, the unique hyperbolic 3–manifold of the smallest volume was only provably identified in the first decade of this century, (see [**GabMM10**]). As for the infinite-volume class, the study of the topology and geometry of hyperbolic 3–manifolds with finitely generated fundamental groups has led to the remarkable resolutions to Marden's tameness conjecture [**Ago04, CalG06**] and Thurston's ending lamination conjecture [**Min10, BrocCM12**].

Study on finite covers of hyperbolic 3–manifolds, along a different path, has led to fascinating combination of deep insights from topology, geometry, dynamics, and geometric group theory. The following Theorem 2.1 is a milestone in 3–manifold topology after geometrization, and is also a prominent achievement of a broad program in geometric group theory, known as virtual specialization:

Theorem 2.1. *The fundamental group of any finite-volume hyperbolic* 3*–manifold is virtually compact special.*

The cusped case of Theorem 2.1 (and part of the Haken closed case) is due to Wise [**Wis12b, Wis12a**]. The closed case in full generality is due to Agol [**Ago13**].

Before we explain the result in more detail, let us list four consequences of Theorem 2.1. They confirm four celebrated conjectures posed by Thurston [**Thu82**, Section 6, the questions 15–18]. Those conjectures have been a driving force in 3–manifold topology since the 1980s:

Theorem 2.2. *For any finite-volume hyperbolic* 3*–manifold M, the following statements hold true:*

(1) *All finitely generated subgroups of $\pi_1(M)$ are separable. In other words, $\pi_1(M)$ is LERF. (Separable and LERF will be defined in Section 4.)*
(2) *Some finite cover of M contains a properly embedded essential subsurface. In other words, M is virtually a Haken manifold.*
(3) *Some finite cover of M has non-vanishing rational first homology. In other words, M is virtually of positive first Betti number.*
(4) *Some finite cover of M is homeomorphic to a surface bundle over a circle. In other words, M is virtually fibered.*

The reader is referred to [**AscFW15a**, Corollary 4.2.3] for more information about the proof of Theorem 2.2. We only emphasize that the statements in Theorem 2.2 are not direct implications of virtual compact-specialness, (except the third one). The actual proof of the first statement uses the covering theorem [**Can96, Thu82**] and the tameness theorem [**Ago04, CalG06**]; the second relies at least on classical theorems in 3–manifold topology (plus virtual $b_1 > 0$); the fourth uses Agol's criterion for virtual fibering [**Ago08**].

The content of Theorem 2.1 is about group actions on CAT(0) cube complexes— their existence and well behavior. We unwrap its statement by briefly explaining some terminology.

A *cube complex* is a finite-dimensional locally finite cell complex whose n–cells are modeled on the standard Euclidean cube $[-1,1]^n \subset \mathbb{R}^n$ and whose characteristic maps restricted to faces are isometries onto lower-dimensional cubes. We furnish any connected cube complex with the canonical shortest-path metric using the standard Euclidean metric on the cubes, so the complex is said to be *non-positively curved* if its universal cover is CAT(0) as a geodesic metric space. In this case, deck transformation on the universal cover provides a typical example of a proper (isomorphic) group action on a CAT(0) cube complex, which is indeed free.

A *special* cube complex is a non-positively curved complex whose hyperplanes are positioned in a certain nice manner. Having a system of hyperplanes is perhaps one of the most significant features of cube complexes, which fundamentally distinguishes this subclass from general polyhedral complexes. Observe that there are exactly n *midcubes* in any n–cube, namely, $[-1,1]^k \times \{0\} \times [-1,1]^{n-k-1} \subset [-1,1]^n$, $k = 0, \cdots, n-1$ in the standard coordinates. Given any cube complex X, the disjoint union of midcubes of X altogether form a new cube complex $\mathcal{H}$, by gluing up according to the partial order of inclusion between their faces. An (immersed) *hyperplane* of X therefore refers the restriction of the obvious placing map $\mathcal{H} \to X$ to any connected component of $\mathcal{H}$. Formally, a special cube complex is defined as a non-positively curved cube complex whose hyperplanes avoid a list of pathologies, (termed *one-sided*, *self-intersecting*, *self-osculating*, and *inter-osculating*,) due to Haglund–Wise [**HglW08**]. (Haglund and Wise originally introduced the terms *A-special* and *C-special*. The former became the term *special* in its most commonly used sense in the literature.) For expository purpose, we offer an equivalent, globalized description, as follows. A non-positively curved cube complex is *special*, if and only if the following requirements are all satisfied: Every hyperplane is embedded; for every embedded hyperplane, its closed 1–neighborhood is an embedded cube subcomplex which is isomorphic to the product of a 1–cube with the hyperplane; and for any intersecting pair of embedded hyperplanes, the intersection of their closed 1–neighborhoods is an embedded cube subcomplex which is isomorphic to the product of a 2–cube with their intersection.

Definition 2.3. A finitely generated group G is said to be *special* if there exists a special cube complex X with $\pi_1(X) \cong G$. Moreover, G is said to be *compact special* if some such X is also compact.

Note that the special cube complex is not part of data, so Definition 2.3 is about a property of a finitely generated group itself. The following characterization is due to Haglund and Wise [**HglW08**, Theorem 1.1]. It gives one some idea how large the special group class is (and is not).

THEOREM 2.4. *A finitely generated group is special if and only if it is isomorphic to a finitely generated subgroup of a right-angled Artin group.*

There is a repeatedly used two-step strategy for *virtual specialization*, (that is, proving the virtual specialness of a suspected group): First, try to construct a proper action of the group in question on a CAT(0) cube complex, or to *cubulate* the group. After that, try to find a finite-index subgroup which acts freely and whose orbit space is a special cube complex, or to *virtually specialize* the cubulation. For *virtual compact specialization*, we also want the action to be cocompact in the first step. In general, both of the steps could be hopelessly difficult, but there are useful directions. The first problem can be reduced to the existence of a certain efficient collection of codimension–1 subgroups, using Sageev's construction [**Sag95**]. The second problem can be characterized as the separability of hyperplane subgroups and hyperplane double-cosets, according to Haglund–Wise's criterion [**HglW08**, Theorem 9.19].

For closed hyperbolic 3–manifolds, the proof of Theorem 2.1 exhibits a splendid, highly skillful example of the above two-step strategy. The first step is done by a Sageev-type construction due to Bergeron–Wise [**BerW12**, Theorem 1.5]. The participating codimension–1 subgroups are many quasi-Fuchsian surface subgroups, as produced by Kahn–Markovic's surface subgroup theorem [**KahM12a**]. The second step is exactly [**Ago13**, Theorem 1.1], the main theorem in that work of Agol. Agol's proof is inductive and relies on Wise's theory of quasi-convex virtual hierarchies for word-hyperbolic groups [**Wis12b, Wis12a**]. In fact, Wise's work also proves Theorem 2.1 for closed hyperbolic 3–manifolds which contain essentially embedded quasi-Fuchsian closed subsurfaces.

For cusped hyperbolic 3–manifolds, the first proof of Theorem 2.1 is due to Wise [**Wis12b**], (see also [**Wis12a**, Chapter 15]). His proof is an application of a generalized hierarchy theory that he develops in a relatively hyperbolic setting. For an alternative recent proof of the cusped case, which more closely follows the two-step strategy, see Groves–Manning [**GroM21**]. The first step there is done by the recent work Cooper–Futer [**CooF19**].

Virtual specialization for general 3–manifolds has also been studied, and there has been a clean and complete answer. We say that a compact 3–manifold is *non-positively curved* if its interior admits a complete Riemannian metric of non-positive sectional curvature. This class of 3–manifolds is known to consist of all aspherical virtually product 3–manifolds, hyperbolic 3–manifolds, mixed 3–manifolds, and some completely characterized non-geometric graph manifolds, which include all the boundary-nonempty ones, (see [**Liu13**, Section 2] and [**AscFW15a**, Section 4.7] for more information and guides to the literature).

THEOREM 2.5. *An aspherical compact* 3*–manifold has a virtually special fundamental group if and only if it is non-positively curved.*

For geometric 3–manifolds, Theorem 2.5 is not obvious only in the hyperbolic case, but that case follows from the works of Agol and Wise [**Ago13, Wis12b**]. For mixed 3–manifolds, Theorem 2.5 is due to Przytycki–Wise [**PrzW18**]. For non-geometric graph manifolds, Theorem 2.5 is due to Liu [**Liu13**], and also Przytycki–Wise [**PrzW14a**] assuming nonempty boundary. Most proofs of these results actually follow the aforementioned two-step strategy. Note that the characterization appears to reflect an underlying connection between two properties arising from very different contexts. However, the above case-by-case proof does not offer sufficient explanation.

To characterize virtual compact-specialness, observe that any orientable compact nonpositively curved 3–manifold is prime. Suppose moreover that the 3–manifold contains no essential Klein bottles. Then every JSJ piece under the JSJ torus decomposition is either atoroidal or Seifert fibered over an orientable hyperbolic base 2–orbifold. A Seifert fibered piece is said to be *internal* if it contains neither any boundary tori of the 3–manifold nor any JSJ tori adjacent to atoroidal pieces. Define the *charge* of an internal Seifert fibered piece to be the (unsigned) Euler number of the Seifert fibration relative to the boundary framing given by the adjacent-piece fibers, and here we consider charge as a rational number up to sign. Here the relative Euler number is defined to be the Euler number of the closed Seifert manifold obtained by Dehn filling of this piece along the slopes on its boundary. An orientable compact non-positively curved 3–manifolds without essential Klein bottles is said to be *chargeless* if its internal Seifert fibered pieces all have zero charge. Since being chargeless and the opposite are both preserved under passage to finite covers, we also speak of these properties for general compact nonpositively curved 3–manifolds by considering their orientable Klein-bottle-free finite covers.

THEOREM 2.6. *A compact non-positively curved* 3*–manifold with empty or tori boundary has a virtually compact special fundamental group if and only if it is chargeless.*

The geometric case of Theorem 2.6 follows again from [**Ago13, Wis12b**] and direct observation. The graph manifold case is due to Hagen–Przytycki [**HgePr15**]. The mixed 3–manifold case is due to Tidmore [**Tid18**].

3. Summary of methods

A series of important techniques have been invented toward the virtual specialization of 3–manifold groups. Some of them have been systematically developed over the past years, and some of them have been applied to other problems creatively. In this section, we provide a review of several methods which are frequently used in the surveyed works of subsequent sections.

To motivate our discussion, consider how to prove the virtual Haken conjecture (see Theorem 2.2 (2)). Given a non-Haken closed hyperbolic 3–manifold, one first produces a π_1–injectively immersed closed subsurface, (using Kahn–Markovic's surface subgroup theorem,) and then finds a finite cover to which the surface lifts to be embedded, (using separability of quasiconvex subgroups). This will complete the proof since the finite cover is Haken by definition. We see that such a proof is possible today basically because there are available (theoretical) constructions, for manageable immersed sub-objects and for interesting finite covers. The methods to be summarized below all belong to these two families.

3.1. Construction of subsurfaces. We review two kinds of constructions for producing π_1–injectively immersed closed subsurfaces in closed 3–manifolds. Both of them can be used to produce more general objects, such as certain subsurfaces with boundary, or 2–subcomplexes. There are also various extensions to serve particular technical purposes. To keep focused on the basic idea, we only briefly mention those related constructions, even though they may very often contain significant contribution to the whole subject. It should be emphasized that π_1–injectivity is crucial and will be insisted on. The point is that the output objects are supposed to represent subgroups of the fundamental group of the 3–manifold in question.

3.1.1. *Constructions in closed hyperbolic* 3*–manifolds.* Under extra hypotheses such as the Haken condition or arithmeticity, there are several known ways to construct large amount of π_1–injectively immersed subsurfaces. The advantage of the *good pants method* lies in its generality and maneuverability. This method was originally introduced by Kahn and Markovic [**KahM12a**] to prove Theorem 3.1, also known as the surface subgroup theorem. The idea was applied by the same authors, shortly after, to prove Ehrenpreis' conjecture about Riemann surfaces [**KahM15**]. For a more detailed introduction to the good pants method, see [**KahM14**].

THEOREM 3.1. *Every closed hyperbolic* 3*–manifold admits a* π_1*–injectively immersed closed subsurface of genus at least* 2*.*

The subsurface as asserted in Theorem 3.1 is obtained by gluing up a large finite collection of immersed pairs of pants along their common cuffs (boundary components). To ensure π_1–injectivity, geometric conditions must be imposed on both the shape of the pants and the rule of the gluing. All the immersed pants to be glued up are required to be intrinsically nearly isometric to a regular hyperbolic pair of pants with some large cuff length $R \gg 1$, and extrinsically nearly totally geodesic with small bending in the normal direction. Quantified suitably with some prescribed small error bound $\epsilon > 0$, such pants are called (R, ϵ)*–good pants.* Along any common cuff, two oriented pairs of good pants should only be glued up with matching orientation and small along-cuff bending, and slightly trickily, the seams (common perpendicular geodesic segments of two cuffs) should mismatch with a relative shift of distance ≈ 1 rightward, (viewed from either side measuring the other). Kahn and Markovic show that any closed immersed subsurface obtained this way is π_1–injective and quasi-Fuchsian. Using the fact that the frame flow for any closed hyperbolic 3–manifold is exponentially mixing (see [**Moo87**] and [**Pol92**]), Kahn and Markovic are able to produce a large finite collection of oriented good pants, which are somehow evenly distributed and orientation-type balanced. The technical conditions make sure that the collection of pants admits a gluing as above, so a subsurface as asserted in Theorem 3.1 can be constructed.

The effective use of frame flow dynamics brings about several unique qualities of the good pants method. Firstly, it allows fine shape control in constructing long geodesic segments, (see [**LiuM15**, Lemma 4.15] and [**Liu19**, Theorem 3.1]). Secondly, it works regardless of the shape or the homology of the closed hyperbolic 3–manifold in question. In fact, first homology classes all admit representative good curves, and second homology classes all admit representative good panted subsurfaces for some finite multiple, (see [**LiuM15**, Theorem 5.2 and Corollary 1.1]). Thirdly, the method tends to produce π_1–injectively immersed objects in a ubiquitous fashion. For example, limit sets of Kahn–Markovic surface subgroups can approximate any round circle on the sphere at infinity arbitrarily well, with respect to the Hausdorff topology, (see [**KahM12a**, Theorem 1.1]). We refer the readers to [**KahM15, LiuM15, Liu19**] for further developments of the good pants method regarding homological control, and [**Liu19, Sun15a, Sun15b, LiuS18**] for examples of object design in utilizing the method, and [**KahM12b**] for following study on the quantitative aspects. For quick reviews on the good pants method and more references, see the preliminary sections [**LiuS18**, Section 2], [**Liu19**, Section 2].

For cusped hyperbolic 3–manifolds, good pants constructions are not fully available yet. One major issue is that [**KahM12a**, Theorem 3.4] would fail to work if the injectivity radius became zero. A very recent generalization to the cusped case is given by Kahn–Wright [**KahW21**]. Another recent work Cooper–Futer [**CooF19**] provides a substitutional solution for certain interest of applications.

3.1.2. *Constructions in mixed 3–manifolds.* The virtual specialization of mixed 3–manifolds is obtained by Przytycki–Wise [**PrzW18**]. In the course of cubulation with Sageev's construction, Przytycki and Wise develop a set of tools to produce sufficiently many π_1–injectively immersed closed subsurfaces, which are all virtually embedded. Their constructions are extended by Derbez–Liu–Wang [**DerLW15**, Section 4] to a relative version.

Let us illustrate some central ingredients with a sample problem. Suppose that M is an orientable closed mixed 3–manifold, and $R \subset J$ is a properly embedded, essential and ∂–essential, connected subsurface of a JSJ piece $J \subset M$. In general, it is impossible to extend R to be an essential closed subsurface S of M. However, we may ask whether some finite cover R' of R admits some inclusion $R' \subset S'$, where S' is a virtually embedded closed surface immersed π_1–injectively in M, and where the inclusion lifts the immersion $R' \to M$. In other words, we wonder if R can be virtually complemented to make a closed, virtually essentially embedded subsurface. This question has a positive answer. If every JSJ piece adjacent to J is hyperbolic, it can be solved using [**PrzW14b**, Proposition 4.6], which invokes Wise's omnipotence theorem for cusped hyperbolic 3–manifolds, (see [**Wis12b**, Theorem 16.15]). If no graph submanifold of M contains J, (so J is hyperbolic,) it can be solved using [**PrzW14a**, Corollary 3.3]. In general, it follows from [**DerLW15**, Theorem 4.4], which invokes [**PrzW14b**, Proposition 4.6] and strengthens [**PrzW14a**, Corollary 3.3].

We see in the above sample that in order to construct π_1–injectively immersed subsurfaces in mixed 3–manifolds, one may start with some constructions in the JSJ pieces, (or more precisely, the hyperbolic pieces and the maximal graph-submanifold parts). Then one may proceed by pasting finite covers of the piecewise constructions together, as sometimes called *merging*. Extra care will be needed to keep the resulting subsurface virtually embedded, or maybe the otherwise, depending on what is being looked for. The subtlety here comes from the fact that non-geometric 3–manifolds do not have LERF fundamental groups, (see Section 4 for more on this topic).

We also mention that there has been extensive study on π_1–injectively immersed subsurfaces in graph manifolds, thanks to works prior to the virtual specialization, (see Buyalo–Svetlov [**BuyS04**] for an exposition).

3.2. Construction of finite covers. We collect some powerful techniques that can be used to build finite covers with specific properties. They all arise from the study of virtual specialization for 3–manifolds.

THEOREM 3.2. *Let M be a closed hyperbolic 3–manifold. Then for any geometrically finite subgroup H of $\pi_1(M)$, there exists a finite index subgroup G' of $\pi_1(M)$ which contains H, and moreover, some homomorphism $G' \to H$ fixes every element of H. In other words, geometrically finite subgroups of $\pi_1(M)$ are virtual retracts.*

This is a particular case of the virtual retract property for quasiconvex subgroups of virtually compact special word-hyperbolic groups. See [**HglW08**, Theorem 7.3 and the proof]. The virtual retract property implies that the inclusion of the subgroup virtually induces injective homomorphisms on homology groups, and any cohomology class of the subgroup can be virtually extended to the whole group. These nice properties on (co)homology groups are very useful in topological applications, see Sections 4, 5, 6.

THEOREM 3.3. *Let M be a closed hyperbolic 3–manifold. Then for any infinite-index geometrically finite subgroup H of $\pi_1(M)$, there exists a quotient homomorphism $\pi_1(M) \to Q$ of $\pi_1(M)$ onto a virtually compact special, non-elementary word-hyperbolic group Q, and moreover, H has finite image in Q.*

This is a particular case of the virtually compact-special filling theorem for quasiconvex subgroups of virtually compact special word-hyperbolic groups. See [**Ago13**, Theorem A.1], (and also the proof of [**Liu19**, Lemma 8.4] for more explanation). For applications, see Theorems 5.6.

THEOREM 3.4. *Let M be a non-positively curved compact 3–manifold with empty or incompressible tori boundary. Then there exists a finite cover M' of M, and for any nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$ which is not fibered, the pull-back class $\phi' \in H^1(M';\mathbb{Z})$ lies on the boundary of a fibered cone for M'. In other words, non-fibered cohomology classes of M are virtually quasi-fibered.*

See [**Ago08**, Theorem 5.1 and Corollary 2.3], and also Theorem 2.4. Here we recall some terminology for the reader's reference. For any orientable compact irreducible 3–manifold M with empty or incompressible tori boundary, there exists a (possibly empty) finite collection of mutually disjoint open linear cones in $H^1(M;\mathbb{R})$, and they are determined by the following property: A nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$ lies in one of the cones if and only if ϕ is homotopically represented by a bundle projection $M \to S^1$, (possibly having disconnected fiber when ϕ is non-primitive), [**Thu86**]. These cones are therefore called the *fibered cones* for M. It is known that each fibered cone is convex, finite polyhedral, and has only rationally coordinated faces. Nontrivial integral classes in the fibered cones are called *fibered classes*, and those on the point-set boundary of fibered cones are called *quasi-fibered classes*. In general, fibered classes are better cohomology classes than non-fibered class. For example, the norm minimizing surface dual to a fibered class is unique, and it is easy to compute the twisted Alexander polynomial of a fibered class. Sometimes, these nice properties on fibered classes can be extended to quasi-fibered classes, then Theorem 3.4 virtually realizes all non-zero cohomology classes as quasi-fibered classes with such nice properties. For applications, see Section 7.

THEOREM 3.5. *Let M be a compact orientable irreducible 3–manifold with empty or incompressible tori boundary. Suppose that $J'_1, \cdots, J'_s$ are finite covers of the JSJ pieces $J_1, \cdots, J_s$ of M, respectively. Then there exists a finite cover $\tilde{M}$ of M, and any JSJ piece $\tilde{J}$ of $\tilde{M}$ covers a JSJ piece J_i of M factoring through J'_i. Moreover, $\tilde{M}$ can be required to be regular over M and characteristic restricted over the JSJ tori.*

This result says that finite covers of a 3–manifold can be built by merging given finite covers of the JSJ pieces. See [**DerLW15**, Proposition 4.2], which relies on Wise's omnipotence theorem for cusped hyperbolic 3–manifolds, (see [**Wis12b**, Theorem 16.15]). Theorem 3.5 is in particular useful for constructing some finite cover of a non-geometric 3-manifold that contains some pre-chosen Seifert or hyperbolic pieces (actually their finite covers). For applications, see Sections 4 and 8.

3.3. Utility of the constructions. The methods summarized above have a wide spectrum of applications. These are surveyed by topics in the subsequent sections, with an emphasis on the contribution of the new techniques. Generally speaking, constructions developed during the virtual specialization of 3–manifold groups are particularly good at producing individual finite covers of 3–manifolds with certain desired properties under fairly general topological assumptions. We also observe that most of the constructions do not make explicit use of a cube complex, even though the proofs of their existence do. On the other hand, the constructions usually lose track of the covering degree and the deck transformation group. Furthermore, despite the success of the virtual specialization in studying the topology and geometry of (hyperbolic) 3-manifolds, several important topics in 3-manifold theory remain completely disconnected from the new techniques. These include various gauge theories on 3–manifolds and quantum invariants of 3-manifolds (especially the volume conjecture). It would be very exciting if one can apply virtual specialization techniques to these fields.

4. Subgroup separability

One of the most influential methods in 3–manifold topology is to study incompressible subsurfaces and their induced decomposition of 3–manifolds [**Hem76**]. This method has inspired geometric group theory in certain aspects, notably the Bass–Serre theory about group actions on trees [**Ser80**] and the Haglund–Wise theory about special cube complexes [**HglW08**]. At cost of passage to finite covers, one is often able to produce incompressible subsurfaces (and other analogous objects) from π_1–injectively immersed ones downstairs, and the important notion here is subgroup separability. In fact, subgroup separability is used for proving many, if not all, post-virtual-specialization results. On the other hand, by applying virtual specialization techniques, subgroup separability of 3–manifold groups can be understood in quite satisfactory details. In this section, we review separability of finitely generated subgroups, LERF groups, and summarize their recent developments in 3–manifold groups.

For any group G, a subgroup H is said to be *separable* in G if for every $g \in G$ with $g \notin H$, there exists a finite index subgroup $G' \leq G$ with $H \leq G'$ and $g \notin G'$. For example, the trivial subgroup $\{1\}$ being separable in G means exactly that G is *residually finite*. Subgroup separability is related to virtual embedding for the following topological interpretation: Let X be a Hausdorff topological space and $\widetilde{X} \to X$ be a regular covering with a deck transformation group G. Then a subgroup H is separable in G if and only if for every compact subset K of $\widetilde{X}/H$, there is a finite covering $X' \to X$ such that the projection $\widetilde{X}/H \to X$ factors through X' and maps K homeomorphically into X', (see Scott [**Sco78**, Lemma 1.4 and the proof]).

For example, suppose that G is the fundamental group of an aspherical 3–manifold M, and H a separable subgroup isomorphic to the fundamental group of an aspherical closed surface S. Then the topological interpretation implies that the inclusion $H \to G$ is induced by a map $S \to M$ which lifts to be an embedding $S \to M'$. This is because any lifted map $S \to \widetilde{M}/H$ is a homotopy equivalence and is homotopic to an embedding.

The above simple example is actually quite illuminating. We see that subgroup separability serves as a condition that helps promoting immersions virtually to be embeddings. In this way, a topological (or homotopy-theoretical) problem is turned into a group-theoretical one. As we explain below, the latter can be addressed in 3–manifold groups by virtual specialization techniques. (We also point out that 3–manifold topology is involved even in such a simple example.)

A group G is said to be *locally extended residually finite*, or *LERF*, if all finitely generated subgroups are separable in G. (Besides LERF, there is the term *extended residually finite*, or *ERF*, which requires all subgroups to be separable. Both LERF and ERF mean to strengthen the notion of residual finiteness.) This property is inherited by all subgroups. Polycyclic-by-finite groups [**Mal58**], free groups [**Hall49**], and finitely generated Fuchsian groups [**Sco78**] are all LERF. A finitely generated right-angled Artin group is LERF if and only if the defining graph contains no complete subgraphs which are four-vertex cycles or four-vertex strings [**MetR09**, Theorem 2].

The following Theorems 4.1, 4.2, and Remark 4.3 provide a complete solution to effective characterization of LERF and non-LERF 3–manifold groups.

THEOREM 4.1. *Let M be an orientable prime closed 3–manifold. Then $\pi_1(M)$ is LERF if and only if M supports one of Thurston's eight geometries.*

THEOREM 4.2. *Let M be an orientable prime compact 3-manifold with nonempty incompressible boundary. Hence M is irreducible. Adopt the JSJ torus decomposition of M, possibly trivial. Then $\pi_1(M)$ is LERF if and only if every JSJ torus is adjacent to at least one JSJ piece of negative Euler characteristic.*

REMARK 4.3. The assumptions in Theorems 4.1 and 4.2 are added to simplify statements. For any compact 3–manifold N, one may pass to the canonical orientable cover of degree at most 2, fill up the spheres on the boundary with 3–balls, and inspect the LERF property for the sphere-disc decomposition summands. Then $\pi_1(N)$ is LERF if and only if all the summands have LERF fundamental groups.

The geometric case for Theorem 4.1 is due to Scott [**Sco78**] for the six Seifert fibered geometries, and due to Agol [**Ago13**] for the hyperbolic geometry, and elementary for the Sol geometry (as polycyclic-by-finite groups are LERF). Although the main ingredient of Scott's proof in [**Sco78**] was 2–dimensional hyperbolic geometry, it can be interpreted as a special cube complex proof. The seminal idea is generalized in [**Ago13**] by using virtual specialization for hyperbolic 3–manifold groups in full power, (see Theorem 2.2 and the comments). The non-geometric case of Theorem 4.1 and Theorem 4.2 are due to Sun [**Sun19a**] and [**Sun20**] respectively, relying on

[**Ago13, Wis12b**]. The proof makes use of characterization of separability for finitely generated subgroups obtained in [**Liu17a**] and [**Sun20**] (see Theorem 4.6), along with construction of non-separable finitely generated surface subgroups or free subgroups. Former examples of 3–manifolds with non-LERF fundamental groups include the first such graph manifold due to Burns–Karrass–Solitar [**BurKS87**] (which is the mapping torus of a Dehn twist on the one-punctured torus), and all non-geometric graph manifolds due to Niblo–Wise [**NibW01**], and certain mixed 3–manifolds due to Liu [**Liu17a**], see [**AscFW15a**, Section 5.2 (H.11) and Section 7.2.1] for more references. The proofs in [**BurKS87, NibW01**] are algebraic, while the proofs in [**Liu17a, Sun19a, Sun20**] are topological.

Let us mention some clues to suggest why Theorems 4.1 and 4.2 are possible. Since the geometric case follows already from the virtual specialization, it is reasonable to infer that failure of subgroup separability comes from the gluing of the geometric pieces. In [**RubW98**], Rubinstein and Wang discovered certain horizontally immersed subsurfaces in certain closed graph manifolds. (Being horizontal means that the immersion is transverse to the Seifert fiberation in every Seifert fibered piece.) They showed that those subsurfaces cannot lift to be embedded in any finite cover. They also identified the obstruction in terms of certain intersection numbers on the JSJ tori. To extend their construction to mixed 3–manifolds, (which essentially prove Theorems 4.1 and 4.2), one would expect some generalized constructions of non-separable subgroups and some similar characterization of subgroup separability. The first part becomes possible by the methods described in Section 3.1.2. (The non-separable subgroups in [**Liu17a, Sun19a, Sun20**] are all obtained topologically by pasting virtual-fiber subsurfaces in the JSJ pieces in a certain designed way. In particular, they are either surface groups or free groups.) The second part is what we explain in the rest of this section.

We characterize separability of surface subgroups in 3–manifold groups, before addressing the general case. Suppose that $f\colon S \to M$ is a π_1–injective map of closed surface S to an orientable prime closed 3–manifold M. By homotopy, f can be arranged to be immersed, intersecting the JSJ tori transversely and minimizing the total number of their preimage components. Then the JSJ decomposition of M induces via f a decomposition of S along essential curves into essential subsurfaces, called the *JSJ curves* and the *JSJ subsurfaces*, respectively. Up to isotopy of S the decomposition depends only on the homotopy class of f.

Every JSJ subsurface of S is properly immersed in a JSJ piece of M via f. It is either vertically or horizontally immersed in a non-elementary Seifert fibered piece; it is either geometrically finitely or geometrically infinitely immersed in a non-elementary hyperbolic piece; otherwise, it is an annulus or a Möbius band properly immersed in an orientable thickened Klein bottle. Note that a JSJ subsurface is a virtual fiber of the carrying piece precisely in the Seifert-fibered-horizontal case, the hyperbolic-geometrically-infinite case, and thickened-Klein-bottle case. Therefore, we obtain a distinguished (possibly empty or disconnected) subsurface $\Phi(S) \subset S$ by taking all the virtual-fiber type JSJ subsurfaces and gluing along their common JSJ curves. We call $\Phi(S)$ the *almost fiber part* of S with respect to f.

With the above notations, Liu [**Liu17a**] introduces a homotopy invariant $s(f) \in H^1(\Phi(S); \mathbb{Q}^\times)$ for π_1–injective maps $f: S \to M$, called the *spirality character* for (S, f). This invariant can be explicitly expressed, in terms of some boundary covering degrees associated to the JSJ subsurfaces of $\Phi(S)$ and their carrying pieces. It behaves naturally under finite coverings $\kappa: S' \to S$, in the sense that $\kappa^*(s(f)) = s(f \circ \kappa)$ holds with respect to the natural identification $\kappa^{-1}(\Phi(S)) = \Phi(S')$ up to isotopy. We say that the surface map (S, f) is *aspiral* with respect to f if $s(f)$ lies in the subgroup $H^1(\Phi(S); \{\pm 1\})$, or equivalently, as a homomorphism of abelian groups, $s(f): H_1(\Phi(S); \mathbb{Z}) \to \mathbb{Q}^\times$ takes values in $\{\pm 1\}$. Being aspiral intuitively means that there is no obstruction to virtually embedding the almost fiber part, (up to homotopy relative to $\partial\Phi(S)$ and their carrying covering tori).

Theorem 4.4. *Let M be an orientable prime closed 3–manifold. For any π_1–injective map $f: S \to M$ of a closed surface S, the image of $\pi_1(S)$ in $\pi_1(M)$ is separable if and only if the surface map (S, f) is aspiral.*

Remark 4.5.

(1) If f is π_1–injective, Przytycki and Wise [**PrzW14b**] show that f is virtually homotopic to an embedding if and only if $\pi_1(S)$ is separable in $\pi_1(M)$. Moreover, S can be lifted to some finite cover of M as a leaf of a taut foliation, up to homotopy, [**Liu17a**, Theorem 1.1].

(2) If M is a closed graph manifold and f is an immersion of S as a horizontal subsurface, the spirality character coincides with the invariant s introduced in Rubinstein–Wang [**RubW98**]. Rubinstein and Wang used this invariant to discover the first example of π_1–injectively immersed subsurfaces which are not virtually embedded.

Theorem 4.4 is due to Liu [**Liu17a**]. The proof invokes Wise's omnipotence theorem for cusped hyperbolic 3–manifolds, (see [**Wis12b**, Theorem 16.15]).

The idea for proving Theorem 4.4 is to generalize Rubinstein–Wang [**RubW98**], taking virtual fibers of hyperbolic pieces as the analogue of horizontally immersed subsurfaces in Seifert-fibered pieces. As mentioned in 4.5 (2), the spirality character is the suitable generalization of the original obstruction discovered by Rubinstein and Wang. Once it is correctly formulated, its triviality will imply by definition that the almost fiber part is virtually embedded, and vice versa. The virtual embedding property of the whole surface can be obtained by the methods described in Theorems 3.2 and 3.5. Then the characterization of Przytycki–Wise [**PrzW14b**] (Remark 4.5 (1)) implies Theorem 4.4.

For finitely generated subgroups of general 3–manifold groups, Sun [**Sun20**] obtains the following Theorem 4.6, which generalizes Theorem 4.4. Let M be an orientable irreducible compact 3–manifold with empty or incompressible boundary. For any finitely generated subgroup H of $\pi_1(M)$ (fixing a basepoint), denote by M_H the covering space corresponding to H. There is a canonically induced decomposition of M_H from the JSJ decomposition of M. For M not supporting the Sol geometry, Sun introduces a generalized almost fiber surface $\Phi(H)$ and an (unsigned) generalized spirality character $s(H) \in H^1(\Phi(H); \mathbb{Q}^\times/\{\pm 1\})$. Therefore, H is said to be *aspiral* if $s(H)$ is trivial. For convenience, we define $\Phi(H) = \emptyset$ for M either a virtual Anosov torus bundle over a circle (the Sol–geometric case) or a product of a sphere with a circle (the reducible prime case). A toy model of Theorem 4.6 is the first non-separable subgroup of 3-manifold group constructed in [**BurKS87**], which is actually not a subgroup of a properly immersed π_1–injective subsurface.

THEOREM 4.6. *Let M be an orientable prime compact 3–manifold with empty or incompressible boundary. Then a finitely generated subgroup is separable in $\pi_1(M)$ if and only if it is aspiral.*

We also mention that based on similar ideas as [**Liu17a, Sun20**], Sun [**Sun19b**] shows that nontrivial geometrically finite amalgamations of hyperbolic 3–manifold groups are not LERF. This result is applied in the same paper to show that most arithmetic hyperbolic manifolds of dimension ≥ 4 have non-LERF fundamental groups.

It seems natural to wonder if the failure of double coset separability in 3–manifold group is also due to a topological reason, since double coset separability is also an important ingredient in the theory of cube complexes. We pose the following question for completeness of the theory.

QUESTION 4.7. Let M be an orientable prime compact 3–manifold with empty or incompressible boundary. For any finitely generated subgroup H, H' of $\pi_1(M)$ and any element $g \in \pi_1(M)$, is there a topological characterization for separability of the double coset HgH' in $\pi_1(M)$, namely, when HgH' is closed with respect to the profinite topology of $\pi_1(M)$?

5. Virtual homological torsion

In this section, we survey on homological torsion for finite covers of 3–manifolds. The results that we mention here and in the next section exhibit a typical strategy to employ virtual specialization techniques. A motivating problem is as follows: Given a hyperbolic integral homology sphere M (which has the same integral homology as a 3–sphere), does there exist a finite cover M' of M such that $H_1(M'; \mathbb{Z})$ contains torsion? The problem looks as difficult as the virtual positive first Betti number conjecture (Theorem 2.2 (3)). However, good pants constructions allow us to make effective use of hyperbolic geometry, and subgroup separability arguments allow us to obtain pretty strong positive solutions to the problem, (see Theorems 5.4 and 5.5). To retrieve the origin of the problem, we first review a conjecture on the growth of virtual homological torsion, which has been drawing increasing attention during the past years, (see Conjecture 5.3). Then we mention some attempts in 3–manifold topology using techniques from virtual specialization.

Unlike the fundamental group, the homology is related to the topology of a 3–manifold in a remote and loose manner. However, experiments and heuristics suggest that homology of finite covers may still contain a dominating amount of topological information, and that the asymptotic behavior may be somewhat predictable. Theorem 5.1 below, which is an easy consequence of virtual specialization, indicates that Tits' alternative for the fundamental group can be characterized through the boundedness of virtual first Betti numbers, (implied by the facts of [**AscFW15a**, Chapter 5, Flowchart 4 and (H.13)]). The next Theorem 5.2 describes the linear gradient of the growth of virtual first Betti numbers, which is a special case of the Kazhdan equality due to Lück [**Lüc94**], (see also [**Lüc02**, Theorem 4.1]).

By a *cofinal tower* of pointed covers $\{(X'_n, x'_n)\}_{n\in\mathbb{N}}$ of a pointed topological space $(X, *)$, we mean an ascending sequence of pointed covering projections

$$\cdots \longrightarrow (X'_n, x'_n) \longrightarrow \cdots \longrightarrow (X'_2, x'_2) \longrightarrow (X'_1, x'_1) \longrightarrow (X, *)$$

with the property $\bigcap_{n\in\mathbb{N}} \pi_1(X'_n, x'_n) = \{1\}$, as subgroups of $\pi_1(X', *)$. If all the covers X'_n are regular over X, choices of basepoints will no longer affect the intersection, so we simply omit mentioning the basepoints.

THEOREM 5.1. *Let M be an aspherical compact 3–manifold. Then either $\pi_1(M)$ is virtually solvable, or exclusively, there exists a cofinal tower of finite regular covers $\{M'_n\}_{n\in\mathbb{N}}$ of M with unbounded first Betti number $b_1(M'_n)$.*

THEOREM 5.2. *Let M be an aspherical compact 3–manifold with an infinite fundamental group. Then for any cofinal tower of finite regular covers $\{M'_n\}_{n\in\mathbb{N}}$ of M, the first Betti number grows at most linearly, and indeed,*

$$\lim_{n\to\infty} \frac{b_1(M'_n)}{[M'_n : M]} = -\chi(M).$$

Here $\chi(M)$ stands for the Euler characteristic of M, (which equals the first L^2–Betti number of M under the assumption).

It is therefore natural to ask about the existence and the growth of homological torsion in finite covers of 3–manifolds. The following conjecture is an average sample of its various versions, which are formulated by many authors including Bergeron–Venkatesh [**BerV13**, Conjecture 1.3], Lê [**Lê18**, Conjecture 1.3], and Lück [**Lüc02**, Question 13.73]:

CONJECTURE 5.3. Let M be an aspherical compact 3–manifold of Euler characteristic 0. Then for any cofinal tower of finite regular covers $\{M'_n\}_{n\in\mathbb{N}}$

of M, the growth of the logarithmic homological torsion size satisfies

$$\lim_{n\to\infty} \frac{\log |H_1(M'_n;\mathbb{Z})_{\mathrm{tors}}|}{[M'_n : M]} = \frac{v_3}{6\pi} \cdot \|M\|.$$

Here $\|M\|$ stands for the simplicial volume of M and $v_3 = \sum_{m=1}^{\infty} \sin(m\pi/6)/m^2 \approx 1.0149$ stands for the volume of a hyperbolic regular ideal tetrahedron.

The right-hand side of the formula in Conjecture 5.3 equals -1 times the L^2–torsion of M, (see [**Lüc02**, Theorem 4.5]). It is known to be an upper bound of the left hand-side with the limit replaced by the limit superior, due to Lê [**Lê18**]. In particular, the conjecture holds true for graph manifolds. For 3–manifolds of nonzero simplicial volume, Conjecture 5.3 remains widely open. Indeed, by the time the present survey is written, there have been no known examples of 3–manifolds with exponential growth of homological torsion size for even *some* cofinal towers of regular finite covers. On the other hand, for certain abelian towers or certain twisted homological torsions, analogous convergence results to Conjecture 5.3 have been obtained, see Lê [**Lê14**, Theorem 5] and Bergeron–Venkatesh [**BerV13**, Section 8.2], for example. We refer the reader to [**AscFW15a**, Section 7.5.1] for further references on this topic, and [**BrocD15**] and [**AbéBBGNS17**, Section 8] for some more recent discussion about virtual homological torsion growth.

For closed hyperbolic 3–manifolds, the good pants constructions and virtual specialization techniques are powerful tools for producing homologically interesting finite covers. Although the output covers are usually irregular, the following Theorems 5.4 and 5.5 do agree with our intuition that a finite-volume hyperbolic 3–manifold should be complicated enough to exhibit numerous virtual homological torsion of unlimited patterns. In particular, the theorems appear to vote up for Conjecture 5.3.

THEOREM 5.4. *Let M be a closed hyperbolic 3–manifold. Then for any finite abelian group A, there exists a finite (irregular) cover M' of M for which the homological torsion subgroup $H_1(M';\mathbb{Z})_{\mathrm{tors}}$ contains a direct summand isomorphic to A.*

THEOREM 5.5. *Let M be a closed hyperbolic 3–manifold. Fix an auxiliary basepoint. Then there exists a cofinal tower of finite (irregular) pointed covers $\{M'_n\}_{n\in\mathbb{N}}$ of M for which the homological torsion size grows exponentially, namely,*

$$\lim_{n\to\infty} \frac{\log |H_1(M'_n;\mathbb{Z})_{\mathrm{tors}}|}{[M'_n : M]} > 0.$$

Theorem 5.4 is due to Sun [**Sun15a**]. The proof develops the good pants constructions [**KahM12a**], and invokes virtual specialization of closed hyperbolic 3–manifolds [**Ago13, Wis12a**] and virtual retract property of quasiconvex subgroups [**HglW08**]. Theorem 5.5 is due to Liu [**Liu19**]. In addition to the above ingredients, the proof develops the constructions of

[**LiuM15**], and invokes the malnormal special quotient theorem [**AgoGM16, Wis12a**].

To prove Theorem 5.4, the key idea is to produce a quasiconvex π_1–injectively immersed 2–complex $f\colon X \to M$, designing so that A occurs as a direct summand of $H_1(X;\mathbb{Z})$. It will follow that X embeds into some finite cover M' and that $\pi_1(X)$ also embeds as a retract of $\pi_1(M')$. Then $H_1(X;\mathbb{Z})_{\mathrm{tors}}$ embeds into $H_1(M';\mathbb{Z})_{\mathrm{tors}}$ as a direct summand. For a special (but essential) case, assume that A is a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$. Then the 2–complex X constructed in [**Sun15a**] is homeomorphic to the quotient of a surface with two boundary components, such that on each boundary component the points are identified by a cyclic rotation of order n. The construction of X is basically as described in Section 3.1.1, using a large collection of good pants, but some technical control is added to make the desired homeomorphism type. Theorem 5.5 also starts with a similar idea by producing a π_1–injectively immersed closed nonorientable subsurface of odd Euler characteristic. Then deep consequences of the virtual specialization theorem can be applied to construct finite (irregular) covers of hyperbolic 3-manifolds, so that liftings of the odd Euler characteristic subsurface contribute enough homological torsion comparable with the covering degree.

For cusped hyperbolic 3–manifolds or mixed 3–manifolds of Euler characteristic 0, generalizations of Theorems 5.4 and 5.5 are expected true, but still unknown. The major difficulty lies in the good pants constructions, as they are not yet available for cusped hyperbolic 3–manifolds. One may hope that the new techniques in the recent work Kahn–Wright [**KahW21**] provide a key to unlock the generalizations.

We mention another recent result, Theorem 5.6, which reaches a partial extension of Theorem 5.4 to cusped hyperbolic 3–manifolds and mixed 3–manifolds of Euler characteristic 0.

Theorem 5.6. *Let M be an aspherical compact 3–manifold of Euler characteristic 0 and of nonzero simplicial volume. Then there exists a tower of finite covers $\{M'_n\}_{n\in\mathbb{N}}$ of M with unbounded homological torsion size $|H_1(M'_n;\mathbb{Z})_{\mathrm{tors}}|$. The finite covers can be taken to be regular if M fibers over a circle.*

Theorem 5.6 is due to Liu [**Liu20**, Corollary 1.4]. It also follows from Hadari [**Had20**] assuming nonempty boundary. It would be overwhelming to explain the proof in this survey. We only point out that the method for proving Theorem 5.6 is very different from those of Theorems 5.4 and 5.5. In particular, it does not employ good pants constructions, (except implicitly in the closed hyperbolic case applying virtual specialization). However, [**Liu20**] still relies on virtual specialization. In fact, it combines dynamics of (pseudo-Anosov) surface automorphisms with virtual special filling techniques (Theorem 3.3). The work [**Had20**] does not rely on virtual specialization.

QUESTION 5.7. Given any finite-volume hyperbolic 3–manifold, what are the possible growth types of the first Betti number for cofinal towers of regular finite covers, with respect to the covering degree?

QUESTION 5.8. As in Question 5.7, how about the possible growth types of the first mod p Betti number, for any prime p?

QUESTION 5.9. Given any finite-volume hyperbolic 3–manifold, does every finite abelian group embed into the first integral homology of some regular finite cover?

REMARK 5.10. For a weaker form of Question 5.9, we also ask if it holds for a generic finite abelian group, for example, $\bigoplus^k(\mathbb{Z}/m\mathbb{Z})$ where $k, m \in \mathbb{N}$ satisfy $p \nmid m$ for all $p \in S$ and where S is a finite set of primes that depend on the 3–manifold.

6. Virtual domination

In this section, we survey on construction of virtual dominations for oriented closed hyperbolic 3–manifolds onto other 3–manifolds. The strategy for the results of this section is very similar of Theorem 5.4.

For any integer $k \neq 0$ and any pair of closed oriented n–manifolds M, N, we say that M *k–dominates* N if there exists a map $f\colon M \to N$ of mapping degree k, namely, such that $f_*\colon H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$ satisfies $f_*[M] = k[N]$. We say that M *dominates* N if M k–dominates N for some integer $k \neq 0$. The general philosophy on (1–)domination between manifolds is that, if M (1–)dominates N, then the topology of M is more complicated than the topology of N. This philosophy can be made explicit and be rigorously proved for some invariants, e.g. rank of fundamental group, homology groups, simplicial volume, etc; while it is still mysterious for some other invariants, e.g. Heegaard genus and Heegaard Floer homology in dimension 3, etc.

For dominations between manifolds, results in dimension 2 and 3 are more fruitful. In dimension 2, all domination results can be deduced by considering the simplicial volume and doing simple topological constructions. If M k–dominates N, it is known that their simplicial volumes satisfy the comparison

$$\|M\| \geq |k| \cdot \|N\|,$$

which is an equality if M covers N of degree $|k|$. For a closed orientable surface S of genus $g \geq 1$, its simplicial volume equals $2\pi|\chi(X)|$.

For closed orientable 3–manifolds, its simplicial volume equals the sum of the simplicial volume of its prime summands, and for each prime summand, the simplicial volume equals the sum of the hyperbolic volumes of its hyperbolic pieces divided by v_3 (the volume of a hyperbolic regular ideal tetrahedron). By covering space argument, any ±1–domination map is all π_1–surjective. Since 3–manifold groups are Hopfian (any surjective endomorphism is an isomorphism) and the fundamental group almost determines a

3-manifold, the existence of ±1–domination provides a partial ordering for all oriented closed aspherical 3–manifolds. Moreover, since the geometrization theorem gives a (partial) classification of 3-manifolds, dominations between 3-manifolds can be investigated by considering dominations between geometric 3-manifolds (which have special structures), and it makes the domination problem in dimension 3 more accessible.

We refer the readers to the slightly early survey Wang [**Wan02**] for basic facts about nonzero degree maps between 3–manifolds.

Theorem 6.1. *For any closed oriented hyperbolic* 3*–manifold* M *and any closed oriented* 3*–manifold* N*, there exists a degree–*1 *map* $f\colon M' \to N$ *for some finite cover* M' *of* M*. In other words,* M *virtually* 1*–dominates* N*.*

Corollary 6.2. *Under the same assumption as Theorem 6.1, for any integer* $k \neq 0$*,* M *virtually* π_1*–surjectively* k*–dominates* N*.*

Theorem 6.1 is due to Liu–Sun [**LiuS18**], (see the same work for the proof of Corollary 6.2). The virtual (π_1–surjective) 2–domination, and hence $2k$–domination, were obtained earlier by Sun [**Sun15b**]. The work [**Sun15b**] relies on the good pants method of [**KahM12a, LiuM15**], and [**LiuS18**] relies on [**Liu19**] besides [**KahM12a, LiuM15, Sun15b**].

The construction of the virtual 1–(or 2–) domination map is roughly as the following. First take a cell structure of the target manifold N, and denote its 2–skeleton $N^{(2)}$. After replacing each 2-cell in $N^{(2)}$ by a higher genus surface to get a 2–complex Z, we construct a π_1–injective immersion $Z \looparrowright M$. By hyperbolicity of M, $\pi_1(M)$ is LERF and there exists a finite cover M' of M such that Z embeds into M'. Then a neighborhood of Z in M' may admits a pinching to a neighborhood of $N^{(2)}$ in N, and the virtual domination map is an extension of the pinching. In this construction, one actually need to control the geometry of $N^{(2)}$ carefully, so that the desired pinching from a neighborhood of Z in M' to a neighborhood of $N^{(2)}$ exists.

The degree 2 in the earlier construction is a technical effect of a $\mathbb{Z}/2\mathbb{Z}$–valued obstruction in the relative good pants method discovered in [**LiuM15**]. Roughly speaking, as one attempts to build a desired π_1–injectively immersed 2–complex in a closed hyperbolic 3–manifold using good pants, one starts by building some 1–complex of controlled shape using long geodesic segments. This can be done with a fundamental construction called the *connection principle*. Next, instead of attaching 2–cells to cycles of edges, one moves on to attach some subsurfaces built from good pants with boundary, making sure that the boundary good curves are freely homotopic to the cycles wanted. It is shown in [**LiuM15**] that a null-homologous good curve bounds a good panted subsurface precisely when a $\mathbb{Z}/2\mathbb{Z}$–valued obstruction vanishes. However, the original connection principle does not provide control of the obstruction to that accuracy. In [**Sun15b**], the difficulty is overcomed by passing to a double of the 1–complex before attaching the good panted subsurfaces, and this extra operation accounts for the degree 2 of the final output. Among other modifications, [**LiuS18**] improves the method of

[**Sun15b**] by invoking an enhanced version of the connection principle, as introduced by [**Liu19**]. With this new ingredient and more careful control of the obstruction, Theorem 6.1 is proved.

Similar as commented in the virtual homological torsion case (see Section 5), for oriented closed mixed 3–manifold M, the work in [**KahW21**] may lead to a virtual domination result like Theorem 6.1. However, for cusped hyperbolic 3–manifolds M and virtual proper dominations, it remains in doubt whether a similar statement would still hold, as there are apparently more topological constraints such as the number of boundary components.

Question 6.3. Does every oriented closed mixed 3–manifold virtually 1–dominate every closed oriented 3–manifold?

Question 6.4. Does every oriented cusped hyperbolic 3–manifold virtually properly dominate every topologically finite oriented 3–manifold with nonempty toral ends? If so, what are the possible mapping degrees?

We also mention the following question from [**Ago14**].

Question 6.5. Does every closed oriented hyperbolic 3–manifold virtually dominate any other oriented closed fibered 3–manifold by a fibered cover and a fiber-preserving map?

Given these virtual domination results in dimension 3, one may also wonder whether for arbitrary dimension n, all closed orientable hyperbolic n-manifolds virtually dominates all closed orientable n-manifolds. It is proved in [**Gai13**] that for any dimension n, there exists a particular closed orientable n-manifold that virtually dominates everything, but the manifolds in [**Gai13**] are not hyperbolic.

7. Twisted Alexander invariants

In this section, we collect some recent results in twisted Alexander polynomials, twisted Reidemeister torsion, and related invariants. These results represent another family of applications of virtual fibrations and other virtual techniques, which are quite different from those of Sections 5 and 6. The study of twisted Alexander invariants started at early stages of topology and remained active in the past few decades. It was known that twisted Alexander polynomials and twisted Reidemeister torsion yield lower bounds for the Thurston norm of the defining integral first cohomology class, and exhibit certain features when the cohomology class is fibered. Consequences of virtual specialization supply desired constructions of finite covers for improving those results, and indeed, they enable one to detect the Thurston norm and the fiberedness. These are the contents of Theorems 7.1, 7.3 and Corollary 7.2.

We recall some notations and relevant facts in order to make precise statements. For a comprehensive survey on twisted Alexander polynomials, see Friedl–Vidussi [**FriV11b**]. In this section, M is always an orientable compact 3–manifold with empty or tori boundary. Suppose that R is a Noetherian unique factorization domain (UFD), and k a positive integer, and t an indeterminant. For any representation $\rho\colon \pi_1(M) \to \mathrm{GL}(k, R)$ and any cohomology class $\phi \in H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(M), \mathbb{Z})$, the n–th (ρ, ϕ)*–twisted homology* $H^{\rho\otimes\phi}_*(M; R[t, t^{-1}])$ is a finitely generated $R[t, t^{-1}]$–module, which is well-defined up to isomorphism for each $n \in \mathbb{Z}$, and which vanishes unless $n \in \{0, 1, 2, 3\}$. The n–th (ρ, ϕ)*–twisted Alexander polynomial* $\Delta^{\rho\otimes\phi}_{M,n}$ of M is defined to be the order of $H^{\rho\otimes\phi}_*(M; R[t, t^{-1}])$, which is treated as an element of $R[t, t^{-1}]$ up to a monomial factor with invertible coefficient in R. (Recall that the *order* of any finitely generated module over a Noetherian UFD is any generator of the smallest principal ideal that contains the zeroth elementary ideal of that module, which is unique up to multiplication by a unit.) It is customary to denote $\Delta^{\rho\otimes\phi}_{M,1}$ as $\Delta^{\rho\otimes\phi}_{M}$, called particularly the (ρ, ϕ)*–twisted Alexander polynomial* of M. It is known that $\Delta^{\rho\otimes\phi}_{M} \neq 0$ implies $\Delta^{\rho\otimes\phi}_{M,n} \neq 0$ for all $n \in \mathbb{Z}$. In this case, the (ρ, ϕ)*–twisted Reidemeister torsion* $\tau^{\rho\otimes\phi}_{M}$ of M can be expressed explicitly by the formula

$$\tau^{\rho\otimes\phi}_{M} \doteq \frac{\prod_{n \text{ odd}} \Delta^{\rho\otimes\phi}_{M,n}}{\prod_{n \text{ even}} \Delta^{\rho\otimes\phi}_{M,n}},$$

where the dotted equal means an equality in the field of rational functions $R(t)$ up to a monomial with invertible coefficient in R. Note that the products are finite, since $\Delta^{\rho\otimes\phi}_{M,n} \doteq 1$ holds for $n \notin \{0, 1, 2, 3\}$. (The convention $\tau^{\rho\otimes\phi}_{M} = 0$ is often adopted for $\Delta_M = 0$.)

Theorem 7.1. *Let M be a compact orientable irreducible 3–manifold with empty or incompressible tori boundary. Then the following statements hold true:*

(1) *For any nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$ and any representation $\rho\colon \pi_1(M) \to \mathrm{GL}(k, R)$ of rank $k > 0$ over a Noetherian UFD R, the following comparison holds,*

$$x_M(\phi) \geq \frac{1}{k} \cdot \deg\left(\tau^{\rho\otimes\phi}_{M}\right),$$

unless $\Delta^{\rho\otimes\phi}_{M} = 0$.

(2) *There exists a unitary representation $\rho\colon \pi_1(M) \to U(k)$ of some finite dimension $k > 0$ and with the following property: For every nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$, the twisted Alexander polynomial $\Delta^{\rho\otimes\phi}$ does not vanish, and the following equality holds:*

$$x_M(\phi) = \frac{1}{k} \cdot \deg\left(\tau^{\rho\otimes\phi}_{M}\right).$$

Moreover, ρ can be taken to have finite image in $U(k)$.

Corollary 7.2. *Under the same assumption as Theorem 7.1, for any nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$, the following maximum is achieved and the equality holds true:*

$$x_M(\phi) = \max_{(\rho,V)} \left\{ \frac{\deg\left(\tau_M^{\rho\otimes\phi}\right)}{\dim(V)} \right\},$$

where (ρ, V) runs over all the (isomorphism classes of) finite-dimensional complex linear representations $\rho\colon \pi_1(M) \to \mathrm{GL}(V)$.

The part of Theorem 7.1 about degree bound is known without geometrization, due to Friedl–Kim [**FriK06**], (see also Friedl [**Fri14**]). The second part of Theorem 7.1 is due to Friedl and Vidussi [**FriV15**] when M is not a closed graph manifold. In fact, their proof also works for non-positively curved 3–manifolds. For the extension to graph manifolds, see Friedl–Nagel [**FriN15**].

The fibered case of Theorem 7.1 can be obtained by direct calculation with a suitable cell decomposition. For non-positively curved 3–manifolds, virtual specialization implies that any non-fibered class is virtually quasi-fibered, (see Theorem 3.4). As quasi-fibered classes can be virtually approached by rational fibered classes, the degree equality follows from a continuity argument. See also [**FriL19, Liu17b**] for generalized results on twisted L^2–Alexander torsions of 3–manifolds.

Theorem 7.3. *Let M be an orientable compact irreducible 3–manifold with empty or incompressible tori boundary. Then for any nontrivial cohomology class $\phi \in H^1(M;\mathbb{Z})$, the following statements hold true:*

(1) *If ϕ is fibered, for every representation $\rho\colon \pi_1(M) \to \mathrm{GL}(k, R)$ of rank $k > 0$ over a Noetherian UFD R, the twisted Alexander polynomial $\Delta_M^{\rho\otimes\phi}$ is monic. In this case, the following equality holds true:*

$$x_M(\phi) = \frac{1}{k} \cdot \deg\left(\tau_M^{\rho\otimes\phi}\right).$$

(2) *If ϕ is not fibered, there exists some finite quotient $\gamma\colon \pi_1(M) \to \Gamma$ and the twisted Alexander polynomial $\Delta_M^{\gamma\otimes\phi}$ vanishes.*

The fibered case of Theorem 7.3 is again known without geometrization, see [**FriK06, Fri14**] and the references thereof. The converse of the fibered case was first proved by Friedl and Vidussi [**FriV11a**] based on geometrization, but without virtual specialization. It was applied in the same work to resolve a well-known conjecture motivated by Taubes' work [**Tau94, Tau95**], which asserts that any closed 4–manifold of the form $S^1 \times M$ admits a symplectic structure (if and) only if M fibers over a circle. The non-fibered case of Theorem 7.3 is due to Friedl–Vidussi [**FriV13**]. It also implies the converse of the fibered case obviously. The proof for the non-fibered case carries out a plan as proposed earlier by the same authors

[**FriV08**]. It is done by employing new ingredients about subgroup separability, including results of [**Ago13, PrzW14b**].

We mention the following questions from [**Ago14**] and [**AscFW15a**, Question 7.5.5]:

Question 7.4. Does every finite-volume hyperbolic 3–manifold virtually fibers over a circle with an orientable invariant foliation for the pseudo-Anosov monodromy?

Question 7.5. Does every compact orientable non-positively curved 3–manifold with empty or incompressible tori boundary admit a finite cover whose Thurston norm ball has all faces being fibered?

8. Virtual representation volume

In this section, we survey on virtual representation volumes of closed oriented 3–manifolds. In view of the techniques, the recent results that we mention below employ ingredients of virtual specialization for 3–manifold groups in a similar way as with Theorems 4.4, 4.6, and the non-geometric cases of Theorem 2.5. On the other hand, they exhibit yet another application of the new methods to a topic of independent interest. For a rather quick introduction, we only recall some basic facts that are necessary for understanding the update. We refer the readers to Derbez–Liu–Sun–Wang [**DerLSW19**] for an exposition and a more complete background review.

Let G be a connected Lie group. Fix a maximal compact subgroup K of G, and a G–invariant volume form on the homogeneous space $X = G/K$. For any closed oriented smooth manifold M of dimension the same as X and any representation $\rho\colon \pi_1(M) \to G$, Goldman [**Gol82**] introduces a quantity $\mathrm{vol}_G(M, \rho) \in \mathbb{R}$, called the *volume* of (M, ρ). The *G–representation volume* of M is therefore defined as

$$\mathrm{V}(M, G) = \sup_{\rho} |\mathrm{vol}_G(M, \rho)|,$$

which lies in $[0, +\infty]$. One of the motivations for studying representation volumes lies in the *domination property*. That is, for any map $f\colon M \to N$ between closed oriented manifold of the same dimension as X, the following comparison holds true:

$$\mathrm{V}(M, G) \geq |\deg(f)| \cdot \mathrm{V}(N, G).$$

Note that the equality does not necessarily hold even if f is a finite covering.

For dimension 3, there are essentially two interesting representation volumes, coming from the identity components of the isometric transformation groups of the hyperbolic geometry $\mathbb{H}^3$ and the Seifert geometry $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. The former is called *hyperbolic representation volume*, corresponding to $G = \mathrm{PSL}(2, \mathbb{C})$, and the latter is called *Seifert (representation) volume*, corresponding to $G = \mathrm{Isom}(\widetilde{SL(2, \mathbb{R})}) \cong \widetilde{\mathrm{SL}}(2, \mathbb{R}) \times_{\mathbb{Z}} \mathbb{R}$. For any closed oriented 3–manifold M, we denote these two representation volumes by $\mathrm{HV}(M)$ and

$\mathrm{SV}(M)$, accordingly. It is known that $\mathrm{HV}(M)$ and $\mathrm{SV}(M)$ are both finite values. With the standard hyperbolic volume form on $\mathbb{H}^3$, $\mathrm{HV}(M)$ equals the hyperbolic volume of M when M is hyperbolic. With a suitably normalized volume form on $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\mathrm{SV}(M)$ equals $4\pi^2|\chi|^2/|e|$ when M is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–geometric, where $\chi \in \mathbb{Q}$ stands for the Euler characteristic of the base 2–orbifold and $e \in \mathbb{Q}^\times/\{\pm 1\}$ is the (unsigned) Euler number of the (unoriented) Seifert fibration.

It is hard to compute $\mathrm{HV}(M)$ and $\mathrm{SV}(M)$ in general if M fails to support the correct geometry. There are no known topological characterizations for their non-vanishing occasions either. However, we know significantly better about their behavior in finite covers.

THEOREM 8.1. *Let M be a closed oriented prime 3–manifold. Adopt the geometric decomposition of M, possibly trivial.*

1. *If M contains at least one hyperbolic geometric piece, then $\mathrm{HV}(M') > 0$ holds for some finite cover M' of M.*
2. *If M contains at least one $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–geometric piece, then $\mathrm{SV}(M') > 0$ holds for some finite cover M' of M.*
3. *If M contains at least one hyperbolic geometric piece, then $\mathrm{SV}(M') > 0$ holds for some finite cover M' of M.*

COROLLARY 8.2. *A closed oriented 3–manifold has virtually non-vanishing hyperbolic representation volume if and only if the simplicial volume is nonzero.*

COROLLARY 8.3. *A closed oriented 3–manifold has virtually non-vanishing Seifert volume if and only if no closed oriented 3–manifold admits a sequence of maps onto this 3–manifold with unbounded mapping degrees.*

See [**DerLW15**] (and also [**DerLSW17**, Remark of Theorem 1.7]) for proving Corollary 8.2, and [**DerLSW17**, Corollary 1.6] for Corollary 8.3. The first two statements in Theorem 8.1 are due to Derbez–Liu–Wang [**DerLW15**]; for graph manifolds, the second statement was originally proved by Derbez–Wang [**DerW12**]; the third statement is due to Derbez–Liu–Sun–Wang [**DerLSW17**].

To prove the first two statements of Theorem 8.1, one need to be able to virtually extend a representation of the given geometric piece. The actual construction in [**DerLW15**] uses a representation that factors through some Dehn filling of the piece and has positive volume. The virtual extension part relies on virtual specialization as it constructs a virtually embedded π_1–injectively immersed 3–complex by attaching immersed compact surfaces along their boundary to a (disjoint) cover of given piece. Besides, one need to be able to calculate the representation volume by summing up the contribution from the pieces. This is done with a formula called the *additivity principle*, as established in [**DerLW15**]. Observe that the third statement of Theorem 8.1 would follow from the domination property if we knew closed oriented mixed 3–manifold dominates another 3–manifold of positive Seifert volume, (see Question 6.3). The proof in [**DerLSW17**] takes a substitute approach, and invokes the virtual extension theorem for representations in [**DerLW15**] and the virtual domination theorem for closed oriented hyperbolic 3–manifolds in [**Sun15b**].

THEOREM 8.4. *For any closed oriented* 3*–manifold* M *of nonzero simplicial volume, the following set of values is unbounded:*

$$\left\{ \frac{\mathrm{SV}(M')}{[M':M]} \in [0,+\infty) \,\middle|\, M' \text{ any finite cover of } M \right\}.$$

Theorem 8.4 is again proved in Derbez–Liu–Sun–Wang [**DerLSW17**]. Replacing SV with HV, the similar value set is known to be bounded by $v_3\|M\|$, (see [**DerLSW17**, Remark of Theorem 1.7]; for the notation, see Conjecture 5.3).

Theorem 8.4 is clearly much stronger than Theorem 8.1 (3), as it claims a large amount of virtual Seifert volume compared to the covering degree. The efficiency comes from a crucial ingredient due to Gaifullin [**Gai13**]. In fact, the results of [**Gai13, Sun15b**] implies that given any closed oriented hyperbolic 3–manifold M and any closed oriented 3–manifold N, there is a degree–d cover of M that k–dominates N, and the ratio $d\|M\|/k(\|N\|+\epsilon)$ does not exceed some constant that depends only on M.

We mention the following questions from [**DerLSW17**, Section 6].

QUESTION 8.5. Given any closed mixed 3–manifold, what is the fastest linear growth rate of virtual hyperbolic representation volume, among all cofinal towers of regular finite covers?

QUESTION 8.6. Given any prime compact closed 3–manifold with positive simplicial volume, what are the possible growth types of Seifert volume, among all cofinal towers of regular finite covers?

QUESTION 8.7. Given any non-geometric closed graph 3–manifold, what is the fastest linear growth rate of virtual Seifert volume, among all cofinal towers of regular finite covers?

9. Profinite completion

The research on profinite completions of 3–manifold groups is a relatively new topic and represents certain trends of the study. This topic lies on the intersection of two different areas: profinite groups and 3-manifold topology. In this section, we collect some applications of virtual specialization techniques to profinite completion of 3–manifold groups. For a more detailed and comprehensive introduction to profinite properties of discrete groups, see the survey article [**Rei18**] of Reid.

Let Γ be a group. Denote by $\mathcal{C}(\Gamma)$ the set consisting of all the finite quotients of Γ (as isomorphism classes of finite groups), sometimes called the *genus* of Γ. Then one can study the situations when $\mathcal{C}(\Gamma)$ determines Γ up to isomorphism, which is an interesting problem from both theoretical and computational viewpoint. As all infinite simple groups have only trivial finite quotients, it is reasonable to restrict the domain to smaller classes of groups. Among finitely generated residually finite groups, the abelian ones are all determined by their finite quotients, (see [**Rei18**, Proposition 3.1]). It is known that any finitely generated 3–manifold group (or equivalently, the fundamental group of a compact 3–manifold) is residually finite [**Hem87**], and in fact, virtually residually p for all but finitely many primes p [**AscF13**]. However, even among finitely generated 3–manifold groups, there are non-isomorphic Sol–geometric pairs [**Fun13**], $\mathbb{H}^2\times\mathbb{E}^1$–geometric pairs [**Hem14**], and non-geometric graph manifold pairs [**Wlk18**] that have pairwise the same sets of finite quotients. The following open question is an instance of *profinite rigidity problems* for torsion-free lattices of $\mathrm{PSL}(2,\mathbb{C})$.

QUESTION 9.1. For any pair of finite-volume hyperbolic 3-manifolds M_1, M_2, does $\mathcal{C}(\pi_1(M_1)) = \mathcal{C}(\pi_1(M_2))$ imply $\pi_1(M_1)\cong\pi_1(M_2)$?

The *profinite completion* of a group Γ is defined as

$$\widehat{\Gamma} = \varprojlim_{N} \Gamma/N$$

where N runs over the inverse system of finite-index normal subgroups of Γ. This is a totally disconnected compact topological group with respect to the *profinite topology*, namely, the coarsest topology to keep all finite quotient homomorphisms continuous. The naturally induced homomorphism $\Gamma\to\widehat{\Gamma}$ has dense image. It is injective if and only if Γ is residually finite. For any pair of finitely generated residually finite groups Γ_1, Γ_2, it is known that $\mathcal{C}(\Gamma_1)=\mathcal{C}(\Gamma_2)$ holds if and only if there is a group isomorphism between $\widehat{\Gamma_1}$ and $\widehat{\Gamma_2}$. Moreover, the group isomorphism is necessarily continuous forward and backward, thanks to a deep result of Nikolov and Segal [**NikS07**]. In this sense, complete data for recovering the topological group $\widehat{\Gamma}$ is already stored in $\mathcal{C}(\Gamma)$, only $\widehat{\Gamma}$ is more organized as a mathematical object.

Hyperbolic once-punctured-torus bundles provide the first, and currently the only known family of infinitely many positive examples for Question 9.1:

THEOREM 9.2. *Let M be a hyperbolic once-punctured-torus bundle over a circle. If N is any compact 3–manifold with $\widehat{\pi_1(N)}\cong\widehat{\pi_1(M)}$, then N is homeomorphic to M.*

Theorem 9.2 is due to Bridson–Reid–Wilton [**BriRW17**]. Before, the first two authors proved the theorem for M being the figure-eight knot complement [**BriR20**]. Both of the proofs rely on virtual specialization and separability of (surface) subgroups. For any finitely generated subgroup $H<\pi_1(M)$ of a hyperbolic 3-manifold group, the LERFness of $\pi_1(M)$ implies that the image of H under the inclusion $\pi_1(M)\to\widehat{\pi_1(M)}$ is isomorphic to $\widehat{H}$. Another important fact here is that almost one-punctured-torus bundles have $b_1=1$. So there is essentially a unique homomorphism $\pi_1(N)\to\widehat{\mathbb{Z}}$

that factors through a surjective homomorphism $\widehat{\pi_1(N)} \to \widehat{\mathbb{Z}}$. These facts are used to proof that fiberedness can be determined through profinite completions in the $b_1 = 1$ case. This is also confirmed by [**BriRW17**] under some condition on the isomorphism between profinite completions, while in general by [**Jai20**]. See also [**BoiF20, Uek18**] for similar results on profinite rigidity and profinite detection of fiberedness among knot groups. More recently, it is known that $\mathrm{PGL}(2, \mathbb{Z}[\omega])$ and $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ are profinitely rigid among finitely generated residually finite groups, where ω stands for a primitive cubic root of unity, [**BriMRS20**].

Theorem 9.3. *Suppose that M, N are closed orientable 3-manifolds with a given isomorphism $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. Then the following statements hold true.*

(1) *In the prime decompositions of M and N, there are the same number of $S^2 \times S^1$ summands. There is a bijective correspondence between the irreducible prime summands determined by the following property: For each irreducible prime summand M_i of M and the corresponding N_i of N, $\widehat{\pi_1(M_i)}$ is conjugate to $\widehat{\pi_1(N_i)}$ under the given isomorphism.*
(2) *If M and N are both prime, then M supports one of Thurston's eight geometries if and only if N does, and they support the same geometry.*
(3) *If M and N are both prime and non-geometric, then the dual graphs of the JSJ decompositions of M and N are isomorphic to each other, while corresponding vertex groups have isomorphic profinite completions.*

Theorem 9.3 is due to Wilton and Zalesskii, see [**WltZ17**] for the second statement, and the more recent [**WltZ19**] for the first and the third statements. The proof for the hyperbolic case of the second statement relies heavily on device from virtual specialization [**Ago13, HglW08, Wis12b**]. For certain closed subgroups of profinite completions of hyperbolic virtually special groups, Wilton and Zalesskii actually prove a structure theorem [**WltZ17**, Theorem D]. In particular, they prove that the profinite completion of a closed hyperbolic 3-manifold group does not contain a subgroup isomorphic to $\widehat{\mathbb{Z}^2}$. Although closed hyperbolic 3-manifold groups do not contain a subgroup isomorphic to $\mathbb{Z}^2$, it requires substantial work to pass it to the profinite completion. The proof makes use of a malnormal hierarchy, besides other geometric group theoretic tools including profinite trees. For the first and third statements of Theorem 9.3, Wilton and Zalesskii used techniques on profinite trees to prove that profinite completions of 3–manifold groups do not have unexpected splittings, other than the ones induced by the prime and JSJ decompositions. We refer to [**Wlk18, WltZ19**] for some extensions of Theorem 9.3 to compact orientable 3–manifolds with nonempty tori boundary.

One important ingredient in most of the proofs of results on profinite completions is the goodness of 3-manifold groups. Among various different tools invoked to study profinite completions of 3–manifold groups, the cohomology theory is in particular very useful. An isomorphism between two profinite completions of 3-manifold groups induces isomorphisms between their closed subgroups, as well as isomorphisms on the cohomology of corresponding subgroups. In general, it is difficult to visualize the cohomology of these closed subgroups, while it is easier to visualize the cohomology of the corresponding subgroups of 3–manifold groups, since they have more explicit topological meanings. The following theorem establishes a relation between the cohomology of a 3-manifold group $\pi_1(M)$ and the cohomology of its profinite completion $\widehat{\pi_1(M)}$.

THEOREM 9.4. *For any compact 3–manifold M and any finite $\pi_1(M)$–module V, the natural embedding $\pi_1(M) \to \widehat{\pi_1(M)}$ induces an isomorphism of twisted group cohomology*

$$H^*(\widehat{\pi_1(M)}; V) \cong H^*(\pi_1(M); V).$$

In other words, $\pi_1(M)$ is good in the sense of Serre.

Theorem 9.4 is due to Wilton–Zalesskii [**WltZ10**] for graph manifolds, and due to Cavendish [**Cav12**] for hyperbolic and mixed 3–manifold, invoking virtual specialization, especially virtual fiberedness for hyperbolic 3–manifolds [**Ago13, Wis12b**]. The rest cases are derived by simple arguments.

10. Quantitative aspects

Most results that we have mentioned so far, especially Theorems 2.1 and 2.2 in Section 2, assert the existence of certain finite covers without giving any degree bound. If one attempted to follow the proofs in [**Ago08, Ago13, KahM12a, Sag95, Wis12b**], making all the constants explicit, and even if that actually worked out, it is not hard to imagine that the resulting bound would be terrible. By contrast, Dunfield and Thurston tested 10,986 samples in the Hodgson–Weeks census of small-volume closed hyperbolic 3-manifolds. According to their report, 42% of the samples have covers of degree at most 6 and b_1 at least 1, and the percentage goes up to 95% for degree at most 100, [**DunT03**]. So it seems that having a huge minimal degree for a b_1–positive cover (or a Haken cover) is a comparatively rare phenomenon among hyperbolic 3–manifolds of a bounded volume.

There have been a number of works concerning quantified answers to virtual problems. As the last topic of this survey, we collect miscellaneous results on quantitative aspects that are related to 3–manifold topology. They shed light on the vast land of finite covers and also offer fresh perspectives to existing theorems.

Quantitative residual finiteness has been considered for many familiar discrete groups. Given a residually finite group G, for any nontrivial element $g \in G$, define the quantities

$$k_G(g) = \min_{H \lhd G,\, g \notin H} [G : H], \qquad d_G(g) = \min_{H < G,\, g \notin H} [G : H].$$

Note d_G and k_G are both conjugacy invariant. Given any generating set S of G, for any natural number $n \in \mathbb{N}$, define the quantities

$$K_G^S(n) = \max_{g \in G,\, |g|_S \leq n} k_G(g), \qquad D_G^S(n) = \max_{g \in G,\, |g|_S \leq n} d_G(g).$$

Note $K_G^S(n) \geq D_G^S(n)$. For two functions $f, f' \colon \mathbb{N} \to \mathbb{N}$, we adopt the comparison notations $f \preceq f'$ if $f(n) \leq Cf'(Cn)$ holds for some constant $C > 0$ and for all $n \in \mathbb{N}$, and $f \simeq f'$ if $f \preceq f'$ and $f' \preceq f$ both hold. For any finitely generated group G, it is elementary to check $K_G^S \simeq K_G^{S'}$ and $D_G^S \simeq D_G^{S'}$ for any pair of finite generating sets S, S'. So we simply write K_G and D_G in this case when considering their growth type in the above sense.

Theorem 10.1.

(1) *For any finitely generated free group F_r, $K_{F_r}(n) \succeq n^{2/3}$ holds.*
(2) *$D_{F_r}^S(n) \leq \frac{n}{2} + 2$ holds for the standard generating set S.*
(3) *For any field finite extension L over $\mathbb{Q}$ and its ring of integers O_L, $K_{\mathrm{SL}(k,O_L)}(n) \preceq n^{k^2} - 1$ holds for $k \geq 2$, and $K_{\mathrm{SL}(k,O_L)}(n) \succeq n$ holds for $k \geq 3$.*
(4) *$D_{\mathrm{SL}(k,\mathbb{Z})}(n) \simeq n^{k-1}$ holds for $k \geq 3$.*
(5) *For any right-angled Artin group A_Γ with a defining finite simplicial graph Γ, $D_{A_\Gamma}^S(n) \leq n + 1$ holds for the standard generating set S.*
(6) *For any geometrically finite hyperbolic manifold M which admits a totally geodesic immersion into a compact, right-angled Coxeter orbifold of dimension 3 or 4, $d_{\pi_1(M)}(g) \preceq l_M(g)$ holds for all nontrivial $g \in \pi_1(M)$. Here $l_M(g)$ stands for the length of the geodesic representative of g.*
(7) *For any topologically finite surface S of negative Euler characteristic, $d_{\pi_1(S)}(g) \leq 32.3\, l_\Sigma(g)$ holds for some complete hyperbolic structure Σ on S and for all nontrivial $g \in G$.*

The statements in Theorem 10.1 are proved in Kassabov–Matucci [**KasM11**] for (1), Buskin [**Bus09**] for (2), Bou-Rabee [**Bou10**] for (3), Bou-Rabee–Hagen–Patel [**BouHP15**] for (4) and (5), Patel [**Pat16**] for (6), and Patel [**Pat14**] for (7).

In a topological flavor, it is interesting to consider the minimal covering degree for virtually embedding a homotopically nontrivial loop on a surface, up to homotopy, since subgroup separability is closely related with lifting immersed objects to be embedded in finite covers. For any topologically finite surface S of negative Euler characteristic, and for any homotopically nontrivial loop γ of S, denote by $\deg(\gamma)$ the smallest covering degree among all finite covers $S' \to S$ which admits an embedded homotopy lift of γ.

For all $n \in \mathbb{N}$, denote by $f_S(n)$ the maximum for $\deg(\gamma)$ as γ runs over all immersed loops of S with at most n self-intersections. Given any complete hyperbolic structure Σ on S, we also consider the metric analogue of $f_S(n)$. For all $L > 0$, denote by $f_\Sigma(L)$ the maximum for $\deg(\gamma)$ among all immersed closed geodesics γ of Σ of length at most L.

THEOREM 10.2. *Let S be a topologically finite surface of negative Euler characteristic.*

1. *For some complete hyperbolic structure Σ on S and for all $L > 0$, $f_\Sigma(L) \leq 16.2L$ holds.*
2. *For all $n \in \mathbb{N}$, $f_S(n) \geq n+1$ holds. Moreover, for any complete hyperbolic structure Σ on S, and for all sufficiently large $L > 0$, $C_1 L \leq f_\Sigma(L) \leq C_2 L$ holds if Σ has no cusps, and $f_\Sigma(L) \geq e^{L/(2+\epsilon)}$ holds if Σ has at least one cusp. Here the constants $C_1, C_2, \epsilon > 0$ depend only on Σ.*
3. *For all $n \in \mathbb{N}$, $C_1 n \leq f_S(n) \leq C_2 n$ holds, where $C_1, C_2 > 0$ depend only on S. Moreover, for any complete hyperbolic structure Σ on S with at least one cusp, and for all sufficiently large $L > 0$, $C_1 e^{L/2} \leq f_\Sigma(L) \leq C_2 L e^{L/2}$ holds, where $C_1, C_2 > 0$ depend only on Σ.*

Theorem 10.2 is due to Patel [**Pat14**] for (1), and Gaster [**Gas16**] for (2), and Aougab–Gaster–Patel–Sapir [**AouGPS17**] for (3).

The next theorem considers quantitative separation of group elements from a quasiconvex subgroup, in virtually special word hyperbolic groups.

THEOREM 10.3. *Let G be a word hyperbolic group having a compact special subgroup of index J, and S be a finite generating set of G. Then there exists a constant $P = P(G,S)$ with the following property:*

For any K–quasiconvex subgroup H of G and any element $g \notin H$ in G of word length $\leq n$, there exists a subgroup G' of G that contains H but not g, and satisfies

$$[G : G'] \leq \left(n e^{PK}\right)^{J!} P.$$

Here quasiconvexity and word length are considered with respect to S.

Theorem 10.3 is due to Hagen–Patel [**HgePa16**]. In fact, Hagen and Patel give a quantitative proof of a result of Haglund–Wise [**HglW08**] that any local isometry from a compact cube complex to a Salvetti complex has a virtual completion.

Quantitative virtual specialness has been studied for certain arithmetic groups. Authors in this direction are interested in the smallest index of *C-special* subgroups, namely, subgroups that can be embedded into a right-angled Coxeter group, (see [**HglW08**]). In general, a finitely generate group is virtually C-special if and only if it is virtually (A-)special. However, arithmetic groups often have a better chance to contain small-index C-special subgroups, as many reflections occur as their commensurators. For example, $\Gamma_6 = \mathrm{SO}_0(\langle 1,1,1,1,1,1,-1\rangle;\mathbb{Z})$ contains a C-special index–2 subgroup in its level–2 principal congruence subgroup [**EveRT12**]. It is also known that all Bianchi groups $\mathrm{PSL}(2, O_d)$ can be virtually embedded into Γ_6, where O_d stands for the ring of integers in the imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$, [**AgoLR01**].

THEOREM 10.4.

(1) *The Bianchi group* $\mathrm{PSL}(2, O_d)$ *contains a C-special subgroup* Δ_d *of index given as follows:*

$$[\mathrm{PSL}(2,O_d):\Delta_d] = \begin{cases} 48 & d \equiv 1,2 \bmod 4 \\ 120 & d \equiv 3 \bmod 8 \\ 72 & d \equiv 7 \bmod 8 \end{cases}$$

(2) *Let* $\mathcal{M} = \mathbb{H}^3/\Gamma$ *be an arithmetic hyperbolic* 3*–orbifold that is commensurable with* $\mathrm{SO}^+(q;\mathbb{Z})$*, where* q *is a bilinear form over* $\mathbb{Q}$ *of signature* $(3,1)$*. Then for any* $\epsilon > 0$*, there exists some constant* $K = K(q,\epsilon)$*, and* $\mathcal{M}$ *has a C-special finite cover of covering degree at most* $K \cdot \mathrm{vol}(\mathcal{M})^{\epsilon}$*.*

Theorem 10.4 is due to Chu [**Chu20**] for (1), and DeBlois–Miller–Patel [**DebMP20**] for (2). Their proofs explore ideas from the above mentioned works [**AgoLR01, EveRT12**].

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Beijing International Center for Mathematical Research, Peking University, No. 5 Yiheyuan Road, Haidian District, Beijing 100871, China
Email address: liuyi@bicmr.pku.edu.cn

Department of Mathematics, Rutgers University, Hill Center–Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA
Email address: hongbin.sun@rutgers.edu