New developments in Ricci flow with surgery

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Abstract. We give a survey of the developments surrounding Ricci flow in dimension 3 after Perelman's work and explain the connection with the partial resolution of the Generalized Smale Conjecture.

Contents

1. Introduction and brief overview

One of the most interesting results concerning the topology of 3-manifolds was the resolution of the Poincaré and Geometrization Conjectures:

Poincaré Conjecture. *Any simply connected, closed 3-manifold is diffeomorphic to a 3-sphere.*

Geometrization Conjecture. *Any closed, oriented, prime 3-manifold has a geometric decomposition, i.e. a set of pairwise disjoint, incompressible 2-tori whose complement consists of components that each admit a complete locally homogeneous metric of finite volume.*

Both conjectures were proven by Perelman [**Per02, Per03b, Per03a**] in 2002/03 using Ricci flow, following a program initiated by Hamilton. The Ricci flow equation, which was introduced by Hamilton [**Ham82**], is a weakly parabolic equation governing the evolution of a family of Riemannian metrics on a manifold. Under this equation, curvature terms evolve under a diffusion-reaction equation (for more details see Subsection 2.1). The general hope is that a Ricci flow "distributes curvature evenly" across the manifold, improving the metric towards a more "symmetric" metric, such as a metric of constant curvature or a locally homogeneous metric. Such a behavior was first made rigorous by Hamilton in [**Ham82**], where he showed that in dimension 3, starting from a metric of positive Ricci curvature the flow converges (modulo rescaling) to a metric of constant curvature. This allowed him to classify the topology of such manifolds and motivated further study of the Ricci flow equation in connection with topological questions.

The proof of the Poincaré and Geometrization Conjectures required the analysis of general Ricci flows, starting from metrics without a Ricci curvature bound. These flows may develop singularities, which need to be excised via a surgery construction at a discrete set of times. Perelman managed to gain sufficient structural understanding of these singularities and devised an appropriate surgery process (for more details see Subsection 2.2), producing a *Ricci flow with surgery*, which he eventually used to resolve both topological conjectures.

Interestingly, however, Perelman's proof relied only on a very cursory analytic and asymptotic characterization of the Ricci flow with surgery, leaving many basic properties of the flow uncharacterized. For example, the locally homogeneous metrics predicted by the Geometrization Conjecture were ultimately constructed via classical geometric-topological methods and the question whether those metrics arise as limits of the flow remained unanswered after Perelman's work.

After about 15 years, we finally have a more precise understanding of the geometric and analytic properties of Ricci flows (with surgery) in dimension 3. Some of this work has resulted in the partial resolution of the Generalized Smale Conjecture — another purely topological result proved using geometric-analytic techniques. The purpose of this survey is to describe these new developments since Perelman's work.

The focus of this survey will be primarily Ricci flow in dimension 3. Unless stated otherwise, all manifolds are assumed to be 3-dimensional and orientable. We will also mainly focus on work aimed at furthering our understanding of Perelman's Ricci flow with surgery and the resulting topological applications. We remark that there has been some other interesting work relating to Perelman's techniques in dimension 3, for example on Ricci flow with surgery in the non-compact [**BBM11**, **BBM15**] and orbifold cases [**DL09**, **KL14a**] — the latter reproving the Geometrization Conjecture for orbifolds. For further applications of Ricci flow with surgery to the study of hyperbolic manifolds see [**AST07**].See also [**Sim12**, **ST17**, **BCRW17**] for smoothing results, which partially rely on Perelman's singularity analysis.

2. Background

In the following, we will provide a very condensed introduction, or recapitulation, of Ricci flow and Ricci flow with surgery in dimension 3 up to and including Perelman's work. We will often state results in a very simplified form and use a language that is somewhat different from Perelman's, but more suitable to present further work. A more detailed survey of Perelman's work can be found in [**And04**, **Mor05**], for example.

2.1. Ricci flows and singularity formation. The Ricci flow equation describes the evolution of a smooth family of Riemannian metrics $(g(t))_{t\in[0,T)}$ on a manifold M^n, depending on a time-parameter $t \in [0,T)$:

$$\partial_t g(t) = -2\,\mathrm{Ric}(g(t)), \qquad g(0) = g_0. \tag{2.1}$$

If M is compact, then for any initial condition g_0 the initial-value problem (2.1) has a unique solution on a maximal time-interval of the form $[0,T)$, $T \leq \infty$. If $T < \infty$, then the flow develops a singularity, meaning that the curvature tensor blows up as $t \nearrow T$.

A very basic class of examples of Ricci flows arises if the initial metric g_0 is an Einstein metric with $\mathrm{Ric}(g_0) = \lambda g_0$. In this case, the Ricci flow takes the form

$$g(t) = (1 - 2\lambda t) g_0.$$

So depending on the sign of the Einstein constant λ, the flow shrinks, expands or is constant. If $\lambda > 0$, then the flow develops a singularity at time $T = \frac{1}{2\lambda}$. By taking Cartesian products, we can generate more examples, for example the *round shrinking cylinder* on $S^2 \times \mathbb{R}$, which takes the form $g(t) = (1-2\lambda t) g_{S^2} + g_{\mathbb{R}}$.

Let us now describe the behavior of flows starting from more arbitrary initial data. In dimension 2, this behavior is well understood, due to the work of Hamilton and Chow [**Ham88**, **Cho91**].

Theorem 2.1. *If $n = 2$ and M is compact, then $g(t)$ remains in the same conformal class,*

$$T = \begin{cases} \dfrac{\mathrm{area}(M, g_0)}{4\pi\chi(M)} & \text{if } \chi(M) > 0 \\ \infty & \text{if } \chi(M) \leq 0 \end{cases}$$

and $g(t)$ is asymptotic to the examples mentioned earlier. More specifically, depending on whether $\chi(M) > 0$, $= 0$ or < 0, the rescaled metrics $(T -$

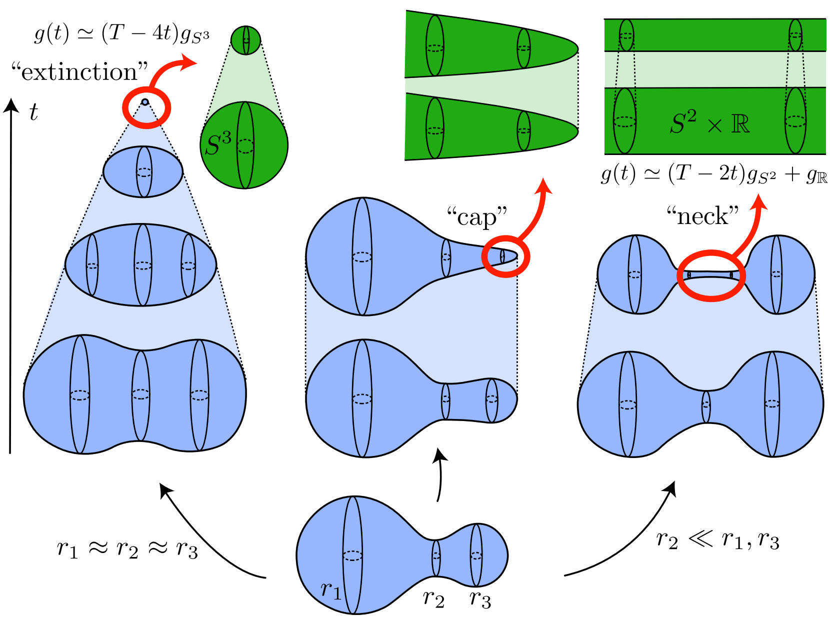

FIGURE 1. Different singularity formations in the rotationally symmetric case, depending on the choice of the radii r_1, r_2, r_3. The flows depicted on the top are the corresponding singularity models.

$t)^{-1}g(t)$, $g(t)$ or $t^{-1}g(t)$ smoothly converge to a metric of constant curvature with $K \equiv +1, 0$ or -1.

In dimension 3, the behavior of the flow — and its singularity formation — is far more complicated. To get an idea of possible behaviors, it is useful to consider the famous dumbbell example (see Figure 1), as analyzed in [**AK04**, **AIK15**]. In this example (M, g_0) is constructed by connecting two round spheres of radii r_1, r_3 by a certain type of rotationally symmetric neck of radius r_2. So $M \approx S^3$ and $g_0 = f^2(s)g_{S^2} + ds^2$ is a warped product away from two points. It turns out that any flow starting from a metric of this form must develop a singularity in finite time and its singularity formation depends on the choice of the radii r_1, r_2, r_3. More specifically, if all radii are comparable, then the diameter of the manifold converges to zero and, after rescaling, the flow (often[1]) becomes asymptotically round — similarly as in the case $\chi(M) > 0$ of Theorem 2.1. This case is called *extinction*. On the other hand, if $r_2 \ll r_1, r_3$, then the flow develops a *neck singularity*, which is modeled on a round shrinking cylinder on $S^2 \times \mathbb{R}$. Note that in this case, the singularity only occurs in a certain region of the manifold, while the metric

[1]For pedagogical reasons, we have omitted the discussion of an interesting non-generic case, called the *peanut solution*. In this case $r_1 = r_3$ and the diameter converges to zero in finite time. However, after rescaling, the metric looks like an expanding oval with a slight indentation in the center, where some sectional curvatures are negative.

converges to a smooth limiting metric everywhere else, as $t \nearrow T$. Lastly, there is also an intermediate case, which is expected to be non-generic. In this case the flow develops a singularity that is modeled on the Bryant soliton — a one-ended paraboloid-like singularity. This singularity is called a *cap singularity.*

Perelman found that for general (non-rotationally symmetric) initial metrics, the singularity formation is qualitatively similar to that of the previous example. The remainder of this Subsection is concerned with making this statement more precise. The concepts introduced in the following will be important later. We first define a class of Ricci flows that we expect to observe as singularity models.

DEFINITION 2.2 (κ-solution). An ancient Ricci flow $(M, (g(t))_{t \leq 0})$ is called *κ-solution* if

(1) The flow has uniformly bounded curvature and complete time-slices.
(2) The curvature operator is non-negative everywhere (i.e. the sectional curvature is non-negative in dimension 3).
(3) The scalar curvature is positive everywhere.
(4) The flow is *κ-noncollapsed at all scales*, meaning that whenever $|\mathrm{Rm}| < r^{-2}$ on a time-t ball of the form $B(x,t,r)$, then $\mathrm{vol}_t\, B(x,t,r) \geq \kappa r^3$.

The precise meaning of the properties (1)–(4) will not be of importance for this survey. The key point here is that a κ-solution is defined via a collection of properties that hold on any non-trivial blow-up limit of a 3-dimensional Ricci flow. By "blow-up limit" we mean a geometric limit of parabolically rescaled Ricci flows; here the limit is understood in the sense of Hamilton [**Ham95**], which is a generalization of Cheeger-Gromov convergence to Ricci flows. So in summary, we have (note that all our results from here on are in dimension $n = 3$ and all manifolds are assumed to be orientable):

THEOREM 2.3. *Suppose that $g(t)$ develops a singularity at some finite time T. Then any smooth blow-up limit of $g(t)$ is a κ-solution.*

Perelman subsequently classified κ-solutions in a qualitative way and showed that they essentially fall into one of three categories: spherical, cylindrical and one-ended. These categories correspond to the three different behaviors in the rotationally symmetric example above. In the following, we will make Perelman's classification more precise and we will restate Theorem 2.3 in a language that will be useful later. We first introduce the following notion of local geometric closeness.

DEFINITION 2.4 (Curvature scale and local ε-models). Let (M, g) be a Riemannian manifold and $x \in M$ a point. We define the *curvature scale of x* to be $\rho(x) = |\mathrm{Rm}|^{-1/2}(x)$. So ρ has the dimension of length.

A pointed Riemannian manifold $(\overline{M}, \overline{g}, \overline{x})$ is called *local ε-model at x* if there is a diffeomorphism onto its image

$$\psi : B(\overline{x}, \varepsilon^{-1}) \longrightarrow M$$

such that $\psi(\overline{x}) = x$ and

$$\big\| \rho^{-2}(x) \psi^* g - \overline{g} \big\|_{C^{[\varepsilon^{-1}]}(B(\overline{x}, \varepsilon^{-1}))} < \varepsilon.$$

The case in which the model space $(\overline{M}, \overline{g})$ is a round cylinder will be particularly important for us.

DEFINITION 2.5 (Centers of ε-neck). We say that a point x on a Riemannian manifold (M, g) is a *center of an ε-neck* if the pointed round cylinder $(S^2 \times \mathbb{R}, g_{S^2 \times \mathbb{R}}, \overline{x})$ (where $\overline{x}$ is an arbitrary point) is a local ε-model at x.

We can now state Perelman's qualitative classification of 3-dimensional κ-solutions.

THEOREM 2.6. *For any κ-solution $(M, (g(t))_{t \leq 0})$ one of the following is true:*

- *(a)* Cylindrical case: *$(M, (g(t))_{t \leq 0})$ is homothetic to a geometric quotient of the round shrinking cylinder on $S^2 \times \mathbb{R}$.*
- *(b)* One-ended case: *$M \approx \mathbb{R}^3$ and each time-slice $(M, g(t))$ is asymptotically cylindrical in the following sense: Given any $\varepsilon > 0$, there an embedded 3-disk $D \subset M$ such that all points of the complement $M \setminus D$ are centers of ε-necks.*
- *(c)* Spherical case: *M is diffeomorphic to a spherical space form and either*
  - *(c′) $(M, (g(t))_{t \leq 0})$ is homothetic to a quotient of the round shrinking sphere.*
  - *(c″) $M \approx S^3, \mathbb{RP}^3$ and as $t \searrow -\infty$ all non-trivial geometric limits are of the form (a) or (b).*

We refer to [**KL08**, Corollary 48.1] for a more detailed statement. A prominent example in case (b) is the Bryant soliton [**Bry05**]. Recently, Brendle [**Bre18**] showed that the Bryant soliton is the only κ-solution in case (b). For more details, see Section 3.

For future applications, we need to discuss a more quantitative version of Perelman's blow-up result Theorem 2.3. The following result states that a flow is close enough to a κ-solution whenever the curvature scale is below (i.e. the curvature is above) a certain *uniform* threshold r_2.

DEFINITION 2.7 (Canonical Neighborhood Assumption). A Riemannian manifold (M, g) satisfies the *ε-canonical neighborhood assumption at scales (of the interval)* (r_1, r_2) if all $x \in M$ of scale $\rho(x) \in (r_1, r_2)$ are locally ε-modeled on the pointed final time-slice $(\overline{M}, \overline{g}(0), \overline{x})$ of a κ-solution $(\overline{M}, (\overline{g}(t))_{t \leq 0})$.

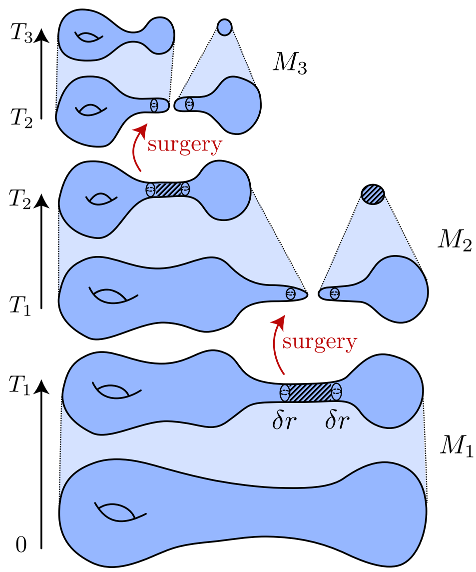

FIGURE 2. A schematic depiction of a Ricci flow with surgery. The almost-singular parts $M_{\text{almost-sing}}$, i.e. the parts that are discarded under each surgery construction, are hatched diagonally.

Note that we may often take $r_1 = 0$. Using this notion we have:

THEOREM 2.8 (Perelman [**Per02**]). *For any 3-dimensional Ricci flow* $(M, (g(t))_{t\in[0,T)})$, $T < \infty$ *and every* $\varepsilon > 0$, *there is a scale* $r_0 = r_0(\varepsilon, M, g(0), T) > 0$ *such that the flow satisfies the* ε*-canonical neighborhood assumption at scales* $(0, r_0)$.

So in summary, Perelman's characterization of the singularity formation in dimension 3 consists of two steps: First, Theorem 2.8 describes quantitatively where the flow is close to a κ-solution, and second, Theorem 2.6 characterizes κ-solution geometrically in a qualitative way.

2.2. Ricci flows with surgery. Based on Theorem 2.8, Perelman [**Per03b**] specified a surgery process, in which the manifold is cut open along small cross-sectional 2-spheres, the high curvature part of the manifold and extinct components are removed, and the resulting spherical boundary components are filled in with 3-disks endowed with a standard cap metric. This produces a new smooth metric, from which the Ricci flow can be restarted. The process may then be iterated to yield a sequence of Ricci flows of the form (see Figure 2)

(2.2)
$$(M_1, (g_1(t))_{t\in[0,T_1]}), \quad (M_2, (g_2(t))_{t\in[T_1,T_2]}), \quad (M_3, (g_3(t))_{t\in[T_2,T_3]}), \quad \ldots$$

where the time-slices $(M_{i+1}, g_{i+1}(T_i))$ arise from $(M_i, g_i(T_i))$ by a certain *surgery process*. The collection of these Ricci flows together with some additional structure, which characterizes the surgery process itself, is called a *Ricci flow with surgery* and will be denoted by $\mathbf{M}$ for the sake of this survey. If $t \in [T_{i-1}, T_i]$, then we will call $(M_i, g_i(t))$ a *time-t-slice*; note that for any $t \in \{T_1, T_2, \ldots\}$ there are two time-t-slices.

Let us provide some more details on Perelman's construction. We will disregard several fine points that are not of essence for this survey. For a more precise discussion see Perelman's original paper [**Per02**] or several variations of his construction [**KL08, MT07, BBB$^+$10, Bam07**].

We begin by describing the surgery process. Let for now (M, g) be a compact (3-dimensional) Riemannian manifold that satisfies the ε-canonical neighborhood assumption below some positive scale r for some small $\varepsilon > 0$, which we will later choose smaller than some universal constants. The manifold (M, g) will later be taken to be the final time-slice $(M_i, g_i(T_i))$ of a Ricci flow from (2.2). We will moreover assume later that the time T_i is chosen sufficiently close to a singular time, so that (M, g) contains some points of scale $\rho \ll r$.

Let $\delta > 0$ be another small constant (possibly much smaller than ε). Choose some pairwise disjoint embedded 2-spheres $\Sigma_1, \ldots, \Sigma_m \subset M$ that occur as central 2-spheres of δ-necks at a curvature scale of $\approx \delta r$. Call each component of the complement $M \setminus (\Sigma_1 \cup \ldots \cup \Sigma_m)$ either *non-singular* or *almost-singular*, depending on whether it contains a point of curvature scale $> 10\delta r$ or not. We can then write

$$M \setminus (\Sigma_1 \cup \ldots \cup \Sigma_m) = M_{\text{non-sing}} \mathbin{\dot\cup} M_{\text{almost-sing}}. \tag{2.3}$$

Assuming that ε and δ are sufficiently small, it can be shown that the components of $M_{\text{almost-sing}}$ are topologically controlled in the sense that their closures must be diffeomorphic to either a 3-disk, $S^2 \times [0, 1]$ or its $\mathbb{Z}_2$-quotient, a spherical space form, $S^2 \times S^1$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$. So these components can be discarded without losing too much information on the topology of M.

Next, attach 3-disks to the (spherical) boundary components of $M_{\text{non-sing}}$ and call the resulting manifold M'. Endow this manifold with a metric g' that equals g on $M_{\text{non-sing}}$ and is isometric to a fixed cap-like rotationally symmetric metric with positive sectional curvature on the added 3-disks, except in an interpolation region near the boundary. We say that (M', g') *was constructed from (M, g) via surgery at scale δr*. Each disk attached to $M_{\text{non-sing}}$ is called *a surgery*. A surgery is called *trivial* if the boundary 2-sphere Σ_j of this disk (in M') also bounds a 3-disk in M, i.e. if it didn't change the topology of M. We record:

LEMMA 2.9. *M is diffeomorphic to a connected sum of components of M' and of copies of $S^2 \times S^1$ and spherical space forms. If all surgeries are trivial, then M and M' are diffeomorphic.*

Given this surgery process, we can construct the Ricci flows (2.2) iteratively: Starting from an initial metric $g_i(T_{i-1})$ on M_i, consider the maximal flow $(g(t))_{t\in[T_i,T_i^*)}$. If $T_i^* = \infty$, then we are done and $(M_i, (g_i(t))_{t\in[T_i,\infty)})$ is the last element of (2.2). If $T_i^* < \infty$, so if the flow develops a singularity, then we choose $T_i < T_i^*$ close enough to T_i^* such that $(M_i, g_i(T_i))$ has a non-empty almost-singular-part $M_{\text{almost-sing}}$, carry out the surgery construction above to determine $(M_{i+1}, g_{i+1}(T_i))$ and repeat the process. Note that we also allow the possibility that $M_i = \emptyset$ for some i, in which case we say that the flow has become *extinct*.

Recall that the surgery construction above depends on a scale r below which the ε-canonical neighborhood assumption holds. This scale could potentially deteriorate (i.e. decrease) significantly with each surgery, possibly leading to an accumulation of surgery times. However, Perelman found that for sufficiently small δ, the scale r can be chosen independently of the number of previous surgeries. More specifically, he showed that there is a continuous positive function $r : [0,\infty) \to \mathbb{R}_+$ such that at each time t the ε-canonical neighborhood assumption holds below scale $r(t)$, except near regions where a surgery had recently been performed. This exception turned out to be inessential to the surgery process.

So in summary:

THEOREM 2.10 (Perelman [**Per03b**]). *Given any closed (3-dimensional) Riemannian manifold (M, g), there are continuous positive functions $r, \overline{\delta} : [0,\infty) \to \mathbb{R}_+$, such that if each surgery step at time T_i is performed at scale $\delta(T_i)r(T_i)$, where $\delta : [0,\infty) \to \mathbb{R}_+$ is continuous with $0 < \delta(t) \leq \overline{\delta}(t)$, then the surgery process can be continued indefinitely, producing a Ricci flow with surgery $\mathbf{M}$ such that:*

(a) The initial time-slice $(M_1, g_1(0))$ is isometric to (M, g).
(b) The surgery times $T_1, T_2, \ldots$ don't accumulate.
(c) At any time t the ε-canonical neighborhood assumption holds below scales $r(t)$, except in "certain parabolic regions after surgery caps".

By iterated application of Lemma 2.9, one can see that if $i < j$, then M_i is diffeomorphic to a connected sum of components of M_j and of copies of $S^2 \times S^1$ and spherical space forms. So if the Ricci flow with surgery $\mathbf{M}$ goes extinct in finite time, then the prime decomposition of the initial manifold $M = M_1$ consists only of copies of $S^2 \times S^1$ and spherical space forms.[2] If the initial manifold has finite fundamental group, then extinction in finite time is guaranteed [**Per03a, CM08**], which implies the Poincaré Conjecture:

[2]An interesting application of this fact is the case in which the initial metric has positive scalar curvature. In this case, finite-time extinction is guaranteed by a simple maximum principle argument, which implies that the minimum of the scalar curvature has to go to infinity in finite time. Therefore, any compact orientable 3-dimensional manifold only admits a metric of positive scalar curvature if it is a connected sum of copies of $S^2 \times S^1$ and spherical space forms. The reverse statement is also true.

THEOREM 2.11. *If $\pi_1(M)$ is finite, then* $\mathbf{M}$ *becomes extinct in finite time. Thus M is diffeomorphic to a spherical space form and if M is simply connected, then $M \approx S^3$.*

Lastly, let us introduce some more terminology, which we will use later. Recall that each initial time-slice $(M_{i+1}, g_{i+1}(T_i))$ is constructed from $(M_i, g_i(T_i))$ by attaching 3-disks to its non-singular part. From now on, we will denote this non-singular part (i.e. the set of points that survive the surgery) by $U_i^- \subset M_i$ and $U_i^+ \subset M_{i+1}$. As the metric is not altered on the non-singular part, the surgery construction induces a natural isometry

$$\phi_i : (U_i^-, g_i(T_i)) \longrightarrow (U_i^+, g_{i+1}(T_i)), \tag{2.4}$$

which we will call *transition map*. The points in the complements $M_i \setminus U_i^-$ and $M_{i+1} \setminus U_{i+1}^-$ are called *surgery points*. We record for future purpose:

PROPOSITION 2.12. *$\rho \leq C\delta(T_i)r(T_i)$ on surgery points $M_i \setminus U_i^-$ and $M_{i+1} \setminus U_{i+1}^-$ for some universal constant C.*

2.3. Long-time behavior. Let us now consider the case in which a Ricci flow with surgery does not become extinct in finite time. We briefly recapitulate Perelman's results on the long-time asymptotics of the flow and their connection with the resolution of the Geometrization Conjecture. We will discuss refinements of these results in Section 4.

THEOREM 2.13. *Let* $\mathbf{M}$ *be a Ricci flow with surgery that does not become extinct in finite time. Then there is a continuous positive function $w : [0, \infty) \to \mathbb{R}_+$ with $\lim_{t\to\infty} w(t) = 0$ and for large t any time-t-slice $(\mathbf{M}_t, g_t)$ has a decomposition of the form*

$$\mathbf{M}_t = \mathbf{M}_t^{\text{thick}} \cup \mathbf{M}_t^{\text{thin}},$$

where $\mathbf{M}_t^{\text{thick}}$ is open and $\mathbf{M}_t^{\text{thin}}$ is closed, such that the following holds:

(a) The components of $\mathbf{M}_t^{\text{thick}}$ are diffeomorphic to a finite set of finite-volume hyperbolic manifolds (with cusps) $H_1, \ldots, H_m$. Moreover the rescaled metric $(4t)^{-1}g_t$ restricted to these components is $w(t)$-close to the corresponding hyperbolic metrics on each H_j, where we truncate cusps at distance $w^{-1}(t)$ from a basepoint.

(b) The boundary tori of $\mathbf{M}_t^{\text{thin}}$ are incompressible in $\mathbf{M}_t$.

(c) $\mathbf{M}_t^{\text{thin}}$ is locally collapsed at different scales in the following sense: For any $x \in \mathbf{M}_t^{\text{thin}}$ there is a scale $0 < r_x \leq \sqrt{t}$ such that $\sec \geq -r_x^{-2}$ on the ball $B(x, r_x)$ and $\operatorname{vol} B(x, r_x) \leq w(t) r_x^3$.

We recall that a 3-dimensional hyperbolic metric g_{hyp} evolves by the Ricci flow as $g(t) = (1+4t)g_{\text{hyp}}$. This illustrates the rescaling in Assertion (a). A basic example for Assertion (c) would be a Cartesian product of a hyperbolic surface with S^1; in this example we can choose $r_x = \sqrt{t}$.

By analyzing the collapse of Assertion (c) further (and using some additional estimates coming from the Ricci flow) [**SY05**, **KL14b**, **MT14**,

PBB⁺09], it is possible to decompose $\mathbf{M}_t^{\text{thin}}$ into Seifert pieces, which correspond to different collapsing behaviors. The existence of this Seifert decomposition implies the existence of a geometric decomposition, therefore establishing the Geometrization Conjecture. Note, however, that a priori the Seifert decomposition arising from Assertion (c) may be quite different from the final geometric decomposition.

We refer to [**Sco83, Thu97, Hat00**] for excellent references concerning prime and geometric decompositions.

2.4. Open questions after Perelman's work. Perelman's work revolutionized the field of Ricci flow (with surgery) and provided sufficient understanding to conclude two important topological conjectures. However, from an analytic and geometric perspective his work left several natural questions open, some of which have recently been resolved. More specifically:

(1) Perelman's qualitative description of singularity models (κ-solution) helped to locate necks, the basis of the surgery construction, and to understand the topology of the almost-singular part. However, his work did not fully classify all κ-solutions. See Section 3 for more details. In addition, the higher order asymptotic behavior of the flow near a singularity is still largely unknown. Related to this, it is currently even unclear whether the diameter of the manifold remains uniformly bounded in time close to a singular time.
(2) Perelman's surgery procedure is a non-canonical construction since it requires the choice of several parameters, such as the surgery scale δr. Perelman recognized this disadvantage and conjectured that there is a more canonical flow, which produces a *unique* flow for any initial data. See Section 5 for more details.
(3) Theorem 2.13, characterizing the long-time behavior of the flow, provided enough information on the thin part in order to find a geometric decomposition using topological techniques. However, it left open whether and in what way the asymptotic behavior of the flow reflects this geometric decomposition. It also did not exclude the occurrence of surgeries for large t. See Section 4 for more details.

As we will see in Section 6, recent work on (2) has resulted in the resolution of the Generalized Smale Conjecture and the analysis of the space of metrics with positive scalar curvature, so our efforts in gaining a better geometric and analytic understanding of the flow have paid off from a topological perspective. It is expected that our insights will have even further topological consequences.

3. Classification of κ-solutions

In [**Per02**], Perelman showed that singularities in 3-dimensional Ricci flows are modeled on κ-solutions $(M, (g_t)_{t\leq 0})$ (see Subsection 2.1 for further details). He also gave a partial classification of κ-solutions (see Theorem 2.6).

Let us recall the cases in which this classification was only of qualitative nature:

(A) $M \approx \mathbb{R}^3$. In this case Perelman showed that the solution must be asymptotically cylindrical, in the sense that all points in the complement of a compact set are centers of necks. An example in this case is the Bryant soliton. κ-solutions of this topological type occur in "cap singularities", as illustrated in Figure 1.

(B) $M \approx S^3$. In this case, the metric is almost cylindrical away from two bounded sets, near which the metric converges to a κ-solution of type (A) as $t \searrow -\infty$. An example in this case was constructed by Perelman and is similar to the 2-dimensional King-Rosenau solution [**Kin93**, **Kin94**, **Ros95**]. κ-solutions of this topological type cannot occur as a singularity model of a single flow, but may occur as limits of certain degenerating sequences of flows.

(C) $M \approx \mathbb{R}P^3$. Same as case (B) after passing to the double cover.

Before proceeding, we need to consider an important subclass of Ricci flows, called *solitons*. Solitons are self-similar Ricci flows, whose time-slices are homothetic. In other words, the metric at any time t is given by $g(t) = a\phi^* g$ for some fixed metric g, where $a > 0$ and ϕ is a diffeomorphism. The Ricci flow equation then reduces to the following soliton equation for g

$$\mathrm{Ric} + \mathcal{L}_X g = \lambda g,$$

where X is a vector field and $\lambda \in \mathbb{R}$. We call a soliton *shrinking, steady or expanding*, if $\lambda > 0, = 0, < 0$, respectively. If $X = \nabla f$ is a gradient vector field, then $\mathcal{L}_X g = 2\nabla^2 f$ and we call the soliton *gradient*.The Bryant soliton is a rotationally symmetric steady gradient soliton on $\mathbb{R}^3$ that takes the following form in polar coordinates:

$$g = u^2(r) g_{S^2} + dr^2, \qquad u(r) \sim \sqrt{r} \quad \text{as} \quad r \to \infty.$$

Shrinking gradient solitons occur as models of Type I singularities (i.e. singularities where the curvature satisfies a bound of the form $|\mathrm{Rm}| < C(T-t)^{-1}$). In addition, shrinking and steady gradient solitons occur as certain blow-downs or limits of κ-solutions. This motivates the study of κ-solutions that are at the same time solitons. It can be seen easily that there are no κ-solutions that are also expanding solitons. Any (3-dimensional) κ-solution that is also a shrinking soliton must be a quotient of the round sphere or the round cylinder. In the case of steady solitons Brendle showed:

Theorem 3.1 (Brendle [**Bre13**]). *Any (3-dimensional) κ-solution that is also a steady soliton is homothetic to the Bryant soliton.*

This theorem has an interesting consequence, which follows via the work of Hamilton [**Ham93**]: Let $(M, (g_t)_{t \leq 0})$ be a general κ-solution of type (A), (B) or (C) (see the list above). Then there is a sequence of times $t_i \searrow -\infty$ such that any non-trivial pointed limit or blow-down limit of the metrics $g(t_i)$ is a soliton. By Theorem 3.1, this limit must be rotationally symmetric. So, in a certain sense, $(M,(g_t)_{t\le 0})$ is asymptotically rotationally symmetric for $t \searrow -\infty$. Recently, Brendle used this observation, combined with a clever stability analysis, to show:

THEOREM 3.2 (Brendle [**Bre18**]). *Any (3-dimensional) κ-solution of type (A) is homothetic to the Bryant soliton.*

In the same paper, Brendle remarks that the same technique also yields the following result. For an independent proof see also [**BK19a**].

THEOREM 3.3. *Any (3-dimensional) κ-solution of type (A), (B) or (C) is rotationally symmetric.*

An important consequence of Theorem 3.2 is:

COROLLARY 3.4 (Brendle). *If $(M,(g(t))_{t\in[0,T)})$ is a (3-dimensional) Ricci flow that develops a singularity at time $T<\infty$, then any blow-up limit is homothetic to the Bryant soliton or to a quotient of the round sphere or the round cylinder.*

It would still be desirable to classify κ-solutions in the cases (B) and (C). This may be complicated by the fact that these solutions are expected to be unstable. At the time of writing, partial progress was made by Angenent, Brendle, Daskalopoulos and Sesum in [**ABDS19**], where a full classification result in cases (B) and (C) was also announced.

4. Precise long-time asymptotics

Theorem 2.13 gives a relatively incomplete picture of the long-time asymptotics of a Ricci flow with surgery $\mathbf{M}$. For example:

(1) The decomposition arising from the collapse on $\mathbf{M}^{\text{thin}}_t$ may a priori be different from the geometric decomposition.
(2) The theorem does not identify any of Thurston's eight homogeneous geometries as asymptotic limits — except for the hyperbolic one.
(3) The theorem does not make any assertions on the occurrence of surgeries. A priori t could be a surgery time and $\mathbf{M}^{\text{thin}}_t$ could contain surgery caps or the thin-part of a surgery step.

Note that due to the finiteness of the prime decomposition and Lemma 2.9, all but a finite number of surgeries in $\mathbf{M}$ are trivial. It can furthermore be shown that any essential 2-sphere in the initial time-slice has to give rise to a surgery in finite time. Therefore, after some finite time, all surgeries must be trivial and all components must be prime.

Concerning Question (2), the convergence of the metric as $t\to\infty$ was further analyzed by Lott [**Lot07, Lot10**], under additional curvature and diameter assumptions.

THEOREM 4.1 (Lott). *Suppose that* $|\mathrm{Rm}_t|<Ct^{-1}$ *and* $\operatorname{diam}_t \mathbf{M}_t < C\sqrt{t}$ *for large t (this implies finiteness of surgeries). Then each component of $\mathbf{M}_t$ contains a single piece in its geometric decomposition and, depending on the* *type of this piece, the rescaled metric $t^{-1}g_t$ converges to a hyperbolic metric or collapses to a lower dimensional space. After passing to the universal cover, the metric converges to a homogeneous expanding soliton metric in the pointed Cheeger-Gromov sense. The type of this limit agrees with the geometric type of M, except for the case $SL(2,\mathbb{R})$, in which the limit is $\mathbb{H}^2 \times \mathbb{R}$.*

In [**LS14**], a similar characterization was achieved under an additional symmetry assumption.

Theorem 4.2 (Lott-Sesum). *Suppose that M contains a single piece in its geometric decomposition and suppose that (M, g) is either a warped product of a circle over a 2-dimensional space or it admits a local isometric torus action. Then the flow either goes extinct in finite time, or it is free of surgeries and the same conclusions as in Theorem 4.1 hold.*

In [**Bam18e, Bam18a, Bam18b, Bam18c, Bam18d**], the author showed finiteness of the surgeries in the general case and obtained an optimal asymptotic curvature bound (see also [**Bam17**] for a discussion of a more basic case):

Theorem 4.3 (Bamler). *Let $\mathbf{M}$ be a Ricci flow with surgery and suppose that the surgery function $\delta(t)$ is chosen sufficiently small. Then $\mathbf{M}$ has only finitely many surgeries. So there is a time $T \geq 0$ after which the flow is given by a classical Ricci flow $(M, (g(t))_{t\in[T,\infty)})$. Moreover we have $|\mathrm{Rm}_t| < Ct^{-1}$ for all $t \geq T$.*

Note that the smallness condition on $\delta(t)$ is qualitatively similar to the bound required to make the surgery construction possible (see Theorem 2.10). So this condition is not very restrictive. Also note that the asserted curvature bound has the optimal decay, as it is realized in the hyperbolic case.

Theorem 4.3 reduces the study of the long-time asymptotics of Ricci flows with surgery to the study of *classical* Ricci flows with $|\mathrm{Rm}_t| < Ct^{-1}$. So if M has trivial geometric decomposition, then it suffices to show that $\mathrm{diam}_t\, M < C\sqrt{t}$, which would allow us to apply Theorem 4.1 in order to obtain a complete characterization. In general, however, we have $\limsup_{t\to\infty} t^{-1/2}\,\mathrm{diam}_t\, M = \infty$.

The proof of Theorem 4.3 relies on a number of geometric, topological and analytic observations. The main goal is to obtain the estimate $|\mathrm{Rm}_t| < Ct^{-1}$ for large t; this bound immediately implies the finiteness of the surgeries. The key estimate is that a bound of this form holds on $\mathbf{M}_t^{\mathrm{thick}}$ and in regions of $\mathbf{M}_t^{\mathrm{thin}}$ where the metric is collapsed along incompressible fibers. A topological analysis of the collapse on $\mathbf{M}_t^{\mathrm{thin}}$ (compare with Assertion (c) of Theorem 2.13), yields that the desired curvature bound holds everywhere, except possibly on a finite set of pairwise disjoint embedded solid tori $\approx D^2 \times S^1$, where the metric is collapsed along compressible fibers. The main part of the proof is aimed at understanding the evolution

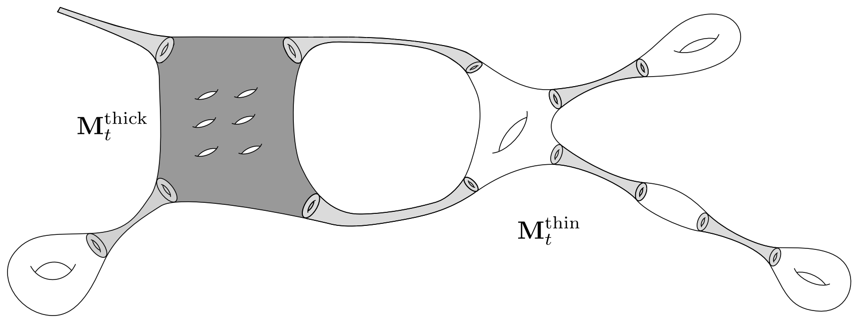

FIGURE 3. Schematic depiction of the metric $g(t)$ at large times t. The metric is collapsed along 2-tori in the shaded regions and along circles in the white regions. The 2-tori may not all belong to a geometric decomposition. The dark shaded region denotes the thick part $\mathbf{M}_t^{\text{thick}}$, which is nearly hyperbolic.

of these solid tori. It is shown that these solid tori have bounded diameter, from which one can deduce the desired curvature bound.

Theorem 4.3 implies that the collapse in $\mathbf{M}_t^{\text{thin}}$ occurs at scale $r_x = c\sqrt{t}$ and under 2-sided curvature bounds, as opposed to only lower curvature bounds (compare with Assertion (c) of Theorem 2.13). Therefore, the theory of Cheeger-Fukaya-Gromov [**CFG92**] is directly applicable to the rescaled metric $t^{-1}g_t$. This allows us, for example, to rule out collapses along spherical fibers. Combining this with arguments that were developed for the proof of Theorem 4.3, we arrive at the following characterization of the thin part (for a more detailed statement see [**Bam18e**, Theorem 1.4]).

THEOREM 4.4 (Bamler). *As $t \to \infty$, the rescaled metric $t^{-1}g_t$ on the thin part either collapses to a point or it collapses locally along incompressible 2-tori or circle fibers. There is a finite subset of these torus fibers that constitutes a geometric decomposition of the underlying manifold. Any 2-dimensional pointed Gromov-Hausdorff limit as $t \to \infty$ is a smooth orbifold of finite volume (possibly with cusps). Compare with Figure 3.*

A more precise characterization of the thin part would be desirable. For further questions and conjectures in this direction see [**Bam18e**]. We describe one particularly interesting behavior that we presently cannot exclude. Consider the case in which $M = \Sigma_2 \times S^1$, where Σ_2 denotes a surface of genus 2. Assume that the rescaled metric $t^{-1}g_t$ remains close to a Cartesian product of two metrics $g_t^{\Sigma_2}$, $g_t^{S^1}$, where $g_t^{\Sigma_2}$ is hyperbolic. Then it may a priori happen that $g_t^{\Sigma_2}$ degenerates and converges to two cusped hyperbolic metrics on a punctured torus. In this case the rescaled metric $t^{-1}g_t$ would collapse along 2-tori in a region that connects these two cusps. These torus fibers would give rise to a JSJ-decomposition whose pieces are diffeomorphic to a Cartesian product of a punctured torus with S^1; such a decomposition is geometric, however not minimal. It remains an interesting question whether such a behavior can occur.

5. Singular Ricci flows

Unfortunately, a Ricci flow with surgery, as described in Subsection 2.2, is not a canonical object, because its construction depends on a number of auxiliary parameters, such as:

- The surgery scale δr of the cross-sectional 2-spheres Σ_j along which the almost-singular part is excised (compare with (2.3)).
- The precise position and number of these 2-spheres.
- The standard cap metric that is placed on the 3-disks which are attached to the non-singular part during a surgery.
- The method used to interpolate between this metric and the metric on the nearby necks.

Different choices of these parameters may influence the future development of the flow significantly (as well as the space of future surgery parameters). Hence a Ricci flow with surgery is not *uniquely* determined by its initial metric.

This disadvantage was already recognized by Perelman in both of his ground breaking papers. Perelman conjectured that there should be another flow, in which surgeries are effectively carried out at an infinitesimal scale, or which in other words "flows through singularities". His conjecture was recently resolved by Lott, Kleiner and the author [**KL17, BK17b**]. In the following, we will first describe the construction of such a flow and then the unique and continuous dependence on its initial data.

5.1. Existence and Construction. In [**KL17**], Kleiner and Lott introduced a certain class of *singular Ricci flows*, which were a natural candidate for Perelman's conjecture (see also [**ACK12**] for earlier work in the rotationally symmetric case).

Theorem 5.1 (Kleiner, Lott). *Let (M, g) be a compact Riemannian 3-manifold. Then there is a singular Ricci flow $\mathcal{M}$ whose initial time-slice $(\mathcal{M}_0, g_0)$ is isometric to (M, g).*

A singular Ricci flow is described by a spacetime 4-manifold and time-slices are given by the level sets of a time-function (see Figure 4). It can be thought of as an analogue of other singular geometric flows, such as the Brakke flow or the level set flow for mean curvature flow, however, there are some important differences, which we will discuss later. Let us now give a more precise definition of a singular Ricci flow. To do this, we first need to define a broader class of flows, called *Ricci flow spacetimes*.

Definition 5.2 (Ricci flow spacetime). A *Ricci flow spacetime* is a tuple $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$, often abbreviated by $\mathcal{M}$, that consists of the following data:

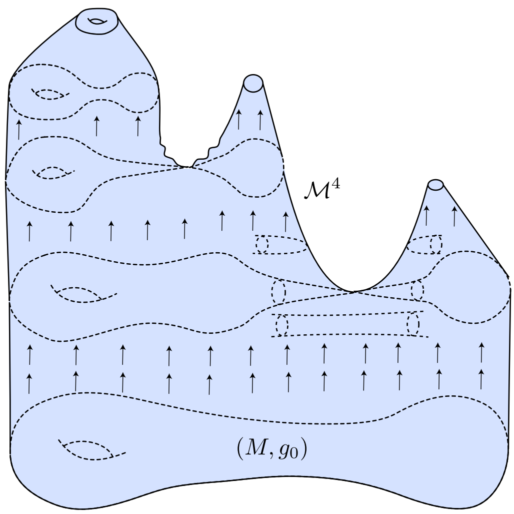

FIGURE 4. Illustration of a singular Ricci flow. The arrows indicate the time-vector field $\partial_{\mathfrak{t}}$.

(1) A smooth 4-manifold $\mathcal{M}$ with boundary, called the *spacetime manifold*.
(2) A smooth *time-function* $\mathfrak{t} : \mathcal{M} \to [0, \infty)$. Its level sets $\mathcal{M}_t := \mathfrak{t}^{-1}(t)$ are called *time-slices*. We require that the initial time-slice is the only boundary component of $\mathcal{M}$, i.e. $\partial\mathcal{M} = \mathcal{M}_0$.
(3) A smooth *time-vector field* $\partial_{\mathfrak{t}}$ on $\mathcal{M}$ such that $\partial_{\mathfrak{t}}\, \mathfrak{t} = 1$. Its maximal trajectories are called *worldlines*.
(4) A smooth inner product field g on $\ker(d\mathfrak{t}) \subset T\mathcal{M}$, which induces a Riemannian metric g_t on every time-slice $\mathcal{M}_t$ and which satisfies the Ricci flow equation

$$\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2\,\mathrm{Ric}(g).$$

Here $\mathrm{Ric}(g)$ denotes the tensor that restricts to the Ricci curvature of g_t on each time-slice $\mathcal{M}_t$.

A classical Ricci flow or a Ricci flow with surgery can be converted into a Ricci flow spacetime as follows: If $(M, (g(t))_{t\in[0,T)})$ is a classical Ricci flow, then we let $\mathcal{M} := M \times [0, T)$ and let $\mathfrak{t}$ and $\partial_{\mathfrak{t}}$ be the projection and standard vector field induced by the second factor. The metric g is defined to be the pullback of $g(t)$ under the projection $\mathcal{M}_t = M \times \{t\} \to M$ for any t. Note that the worldlines of $\mathcal{M}$ are of the form $t \mapsto (x, t + t_0)$.

Next, consider a Ricci flow with surgery $\mathbf{M}$ consisting of Ricci flows $(M_i, (g_i(t))_{t\in[T_{i-1},T_i]})$. As described in the last paragraph, we can convert each Ricci flow into a Ricci flow spacetime on $\mathcal{M}_i = M_i \times [T_{i-1}, T_i]$. Using the (isometric) transition maps $\phi_i : (U_i^-, g_i(T_i)) \to (U_i^+, g_{i+1}(T_i))$ (see (2.4)), we can glue these spacetimes together and construct the spacetime manifold:

$$\mathcal{M} := \big(M_1 \times [0, T_1] \cup_{\phi_1} M_2 \times [T_1, T_2] \cup_{\phi_2} M_3 \times [T_2, T_3] \cup_{\phi_3} \dots\big) \setminus \mathcal{S},$$

where

$$\mathcal{S} := \bigcup_i \big((M_i \setminus U_i^-) \cup (M_{i+1} \setminus U_i^+) \big) \times \{T_i\}$$

denotes the set of surgery points, i.e. the set of points that are removed or added during a surgery step. Then $\mathcal{M}$ is a manifold with boundary and the time functions $\mathfrak{t}$, time vector fields $\partial_\mathfrak{t}$ and inner products g on each $\mathcal{M}_i$ extend to smooth objects on all of $\mathcal{M}$. The resulting object $(\mathcal{M}, \mathfrak{t}, \partial_\mathfrak{t}, g)$ is called a *Ricci flow spacetime associated to* $\mathbf{M}$.

Note that the time-slices $(\mathcal{M}_t, g_t)$ are isometric to $(M_i, g_i(t))$ for any non-surgery time $t \in (T_{i-1}, T_i)$ and to $(U_i^-, g_i(T_i)) \cong (U_{i+1}^+, g_i(T_i))$ whenever $t = T_i$. So, by continuous extension, the associated Ricci flow spacetime $\mathcal{M}$ encodes the same information as the original Ricci flow with surgery $\mathbf{M}$.

Next, we introduce two important properties of Ricci flow spacetimes. The first is the canonical neighborhood assumption, whose definition is analogous to the classical definition.

Definition 5.3 (Canonical neighborhood assumption for Ricci flow spacetimes). A Ricci flow spacetime $\mathcal{M}$ is said to *satisfy the ε-canonical neighborhood assumption at scales* (r_1, r_2) if the same is true for each time-slice $(\mathcal{M}_t, g_t)$ (see Definition 2.7).

For the second property, note that given any Ricci flow spacetime, we can easily produce a new Ricci flow spacetime by removing a closed set of points. The following notion of completeness limits or excludes such constructions.

Definition 5.4 (Completeness). A Ricci flow spacetime $\mathcal{M}$ is said to be r_0-*complete* for some $r_0 \geq 0$ if the following holds: Consider a path $\gamma : [0, s_0) \to \mathcal{M}$ such that $\inf_{s\in[0,s_0)} \rho(\gamma(s)) > r_0$ and such that:

(1) The image $\gamma([0, s_0))$ lies in a single time-slice $\mathcal{M}_t$ and the length of γ, measured with respect to the metric g_t is finite or
(2) γ is a trajectory of $\partial_\mathfrak{t}$ or of $-\partial_\mathfrak{t}$.

Then the limit $\lim_{s\nearrow s_0} \gamma(s)$ exists.

We can now define the main object from Theorem 5.1.

Definition 5.5 (Singular Ricci flow). A *singular Ricci flow* is a Ricci flow spacetime $\mathcal{M}$ with the following two properties:

(1) It is 0-complete.
(2) For any $\varepsilon > 0$ there is a continuous positive function $r(t)$ such that $\mathcal{M}$ satisfies the ε-canonical neighborhood assumption below scale $r(t)$ at each time t.

We remark that in most cases property (2) can be simplified:

Lemma 5.6. *If the initial time-slice $(\mathcal{M}_0, g_0)$ is complete and has bounded curvature, then we only need to require assumption (2) for $\varepsilon = \varepsilon_{can}$, where $\varepsilon_{can} > 0$ is some universal constant.*

An interesting example of a singular Ricci flow is detailed in [**ACK12**]. In this example, $\mathcal{M}_0$ is a rotationally symmetric dumbbell as sketched in Subsection 2.1. The corresponding singular Ricci flow remains rotationally symmetric and develops a non-degenerate neckpinch with subsequent resolution at infinitesimal scale. By this we mean the following: There are times $0 < T_1 < T_2$ such that $\mathcal{M}$ restricted to the time-interval $[0, T_1)$ is given by a classical Ricci flow $g_1(t)$ on S^3 that becomes singular at time T_1, while $\mathcal{M}$ restricted to (T_1, T_2) is given by a classical Ricci flow $g_2(t)$ on two disjoint copies of S^3 that goes extinct at time T_2. As $t \nearrow T_1$, the diameter of the equatorial 2-sphere of S^3 measured with respect to the metric $g_1(t)$ converges to zero. On the complement of this 2-sphere, $g_1(t)$ converges to a smooth metric that is isometric to $(\mathcal{M}_{T_1}, g_{T_1})$. So the Gromov-Hausdorff limit of $(S^3, g_1(t))$ is isometric to $(\mathcal{M}_{T_1}, g_{T_1})$ after removing a single singular point. On the other hand, as $t \searrow T_1$, the metric $g_2(t)$ converges to a smooth metric on the complement of two points (one in each copy of S^3), which is also isometric to $(\mathcal{M}_{T_1}, g_{T_1})$. So $\mathcal{M}$ is diffeomorphic to a 4-disk with 3 punctures; one puncture for the non-degenerate neckpinch at time T_1 and two punctures for the extinction of the two copies of S^3. The time-function $\mathfrak{t}$ is equivalent to a Morse function on a 4-disk of indices 2, 2 and 3 — after removing the critical points.

Let us digest the definition of a singular Ricci flow a bit more. It is tempting to think of the time function $\mathfrak{t}$ as a Morse function, where critical points correspond to infinitesimal surgeries. However, there are two important differences: First, by definition $\mathfrak{t}$ cannot have critical points since $\partial_{\mathfrak{t}} \mathfrak{t} = 1$. In fact, a singular Ricci flow is a completely smooth object. The "singular points" of the flow are not part of $\mathcal{M}$, but can be obtained after a metric completion; however, such a construction does not seem to be very useful in practice. Second, it is currently unknown whether the set of singular times, i.e. the set of times whose time-slices are incomplete, is discrete. At present, it is only known that this set has Hausdorff dimension $\leq \frac{1}{2}$ [**KL18**].

Similar notions of singular flows have been developed for mean curvature flow, such as the level set flow and Brakke flow. However, these notions characterize the flow at singular points via barrier and weak integral conditions, respectively. This is possible, in part, because a mean curvature flow is an embedded object and its singular set has an analytic meaning. In contrast, the definition of a singular Ricci flow only characterizes the flow on its regular part. In lieu of a weak formulation of the Ricci flow equation on the singular set, we have to impose the canonical neighborhood assumption, which serves as an asymptotic characterization near the incomplete ends.

Let us finish this subsection by sketching the main idea of the construction of a singular Ricci flow in [**KL17**]. In brief terms, a singular Ricci flow arises as a subsequential limit of a sequence of Ricci flows with surgery, in which surgeries are performed at smaller and smaller scales. Therefore, a Ricci flow with surgery can be seen as an approximation of a singular Ricci flow. To make this more precise, let us first rephrase Theorem 2.10 in the language of Ricci flow spacetimes.

THEOREM 5.7. *Let* (M, g) *be a compact Riemannian manifold and assume that* $r, \delta : [0, \infty) \to \mathbb{R}_+$ *are chosen as in Theorem 2.10. Assume that* $\sup_t \delta(t) r(t) \leq r_0$. *Then there is a Ricci flow spacetime* $\mathcal{M}$ *such that for some universal constant* C*:*

(a) $(\mathcal{M}_0, g_0)$ *is isometric to* (M, g).
(b) $\mathcal{M}$ *is* Cr_0*-complete.*
(c) At any time t *the* ε*-canonical neighborhood assumption holds at scales* $(Cr_0, r(t))$.

Note we have introduced the lower bound Cr_0 in Assertion (c) in order to exclude points near surgery caps, which have scale $\leq C\delta(t) r(t)$.

Recall that the function $r(t)$ in Theorem 2.10 is independent of the choice of the surgery scale function $\delta(t)$, as long as $\delta(t) \leq \overline{\delta}(t)$. This allows us to construct a sequence of Ricci flow spacetimes for which completeness and the canonical neighborhood assumption hold at smaller and smaller scales. Passing to a subsequence and taking a limit produces a singular Ricci flow:

THEOREM 5.8 (Kleiner, Lott [**KL17**]). *Consider a sequence* $\delta_j : [0, \infty) \to \mathbb{R}_+$ *of continuous functions such that* $\delta_j \to 0$ *uniformly and denote by* $\mathcal{M}^j$ *the corresponding Ricci flow spacetimes from Theorem 5.7 for the same initial condition* (M, g). *Then, after passing to a subsequence, we have convergence* $\mathcal{M}^j \to \mathcal{M}^\infty$, *where the latter is a singular Ricci flow with initial time-slice isometric to* (M, g).

The convergence $\mathcal{M}^j \to \mathcal{M}^\infty$ is understood as follows: If all flows $\mathcal{M}^j$, $\mathcal{M}^\infty$ are non-singular, then the convergence is equivalent to Hamilton's convergence of Ricci flows [**Ham95**]. In the singular case write $\mathcal{M}^j = (\mathcal{M}^j, \mathfrak{t}^j, \partial^j_{\mathfrak{t}}, g^j)$ and view the tensor field g^j as a degenerate Riemannian metric on the 4-manifold $\mathcal{M}^j$, with a nullspace generated by $\partial^j_{\mathfrak{t}}$. So

$$\widetilde{g}^j := (d\mathfrak{t}^j)^2 + g^j$$

is a (possibly incomplete) *Riemannian* metric on $\mathcal{M}^j$. Using this new metric, the convergence $\mathcal{M}^j \to \mathcal{M}^\infty$ is equivalent to smooth Cheeger-Gromov convergence $(\mathcal{M}^j, \widetilde{g}^j) \to (\mathcal{M}^\infty, \widetilde{g}^\infty)$.

5.2. Uniqueness and continuity. Note that in Kleiner and Lott's construction a singular Ricci flow only arises as a *subsequential* limit of Ricci flows with surgery (see Theorem 5.8). A priori, there could be many subsequences, leading to different limiting singular Ricci flows. Recently, Kleiner and author showed that this is not the case:

THEOREM 5.9 (Bamler, Kleiner [**BK17b**]). *Any singular Ricci flow* $\mathcal{M}$ *is uniquely determined by its initial data* $(\mathcal{M}_0, g_0)$, *up to isometry.*

This result fully resolved Perelman's conjecture:

Corollary 5.10. *For every compact Riemannian 3-manifold* (M, g) *there is a* unique*, singular Ricci flow* $\mathcal{M}$ *with initial time-slice* $(\mathcal{M}_0, g_0) = (M, g)$.

Moreover, it showed:

Corollary 5.11. *In Theorem 5.8 we have convergence* $\mathcal{M}^j \to \mathcal{M}^\infty$ *without passing to a subsequence.*

We remark that Theorem 5.9 even holds in the non-compact case. Moreover, a slightly more technical wording of Theorem 5.9 even holds for singular initial data.

A byproduct of the work [**BK17b**] is the following *Stability Theorem*, which states that two singular Ricci flows are close if their initial data are sufficiently close. This theorem will be the cornerstone of the Continuity Theorem 5.13 below, which we will exploit in Section 6.

Theorem 5.12. *Let* $\mathcal{M}$ *be a singular Ricci flow with compact initial data and choose* $\varepsilon, r, T > 0$. *Then there is a* $\delta = \delta(\mathcal{M}, \varepsilon, r, T) > 0$ *such that the following holds. If* $\mathcal{M}'$ *is another singular Ricci flow whose initial time-slice* $(\mathcal{M}'_0, g'_0)$ *is* $(1+\delta)$*-bilipschitz to* g_0*, then* $\mathcal{M}$ *and* $\mathcal{M}'$ *are* ε*-close above scale* ε *and up to time* T*. By this we mean that there is a* $(1+\varepsilon)$*-bilipschitz embedding*

$$\phi : \mathcal{M} \supset \{\rho > \varepsilon, \mathfrak{t} < T\} \longrightarrow \mathcal{M}'$$

whose image contains the set $\{\rho > c\varepsilon, \mathfrak{t}' < T\} \subset \mathcal{M}'$*, for some universal constant* $c > 0$.

We remark that there is an even more general version of this theorem, called the *Strong Stability Theorem* [**BK17b**, Theorem 1.7], in which δ is independent of $\mathcal{M}$ and in which both flows $\mathcal{M}$, $\mathcal{M}'$ are only assumed to be δ-complete. So, in other words, even if the Ricci flow equation was violated below scale δ, then we still obtain stability.

Theorem 5.12 implies the following *Continuity Theorem*, which we will only state in a vague form; for more details see [**BK19b**, Section 4].

Theorem 5.13 (Bamler, Kleiner [**BK19b**]). *Let* M *be a manifold and for any metric* g_0 *on* M *consider the singular Ricci flow* $\mathcal{M}^{g_0}$ *with initial data* (M, g_0)*. Then* $\mathcal{M}^{g_0}$ *"depends continuously" on* g_0.

This continuity should be understood as follows. Let $\mathrm{Met}(M)$ be the space of smooth Riemannian metrics on M, equipped with the C^∞-topology. For each $g_0 \in \mathrm{Met}(M)$ write $\mathcal{M}^{g_0} = (\mathcal{M}^{g_0}, \mathfrak{t}^{g_0}, \partial_{\mathfrak{t}}^{g_0}, g^{g_0})$. Then we can find a topology on the disjoint union $\bigsqcup_{g_0 \in \mathrm{Met}(M)} \mathcal{M}^{g_0}$ such that the natural projection $\bigsqcup_{g_0 \in \mathrm{Met}(M)} \mathcal{M}^{g_0} \to \mathrm{Met}(M)$ is a topological submersion. Moreover, we can find a lamination structure on $\bigsqcup_{g_0 \in \mathrm{Met}(M)} \mathcal{M}^{g_0}$ compatible with this topology, whose leaves equal $\mathcal{M}^{g_0}$ and with respect to which the objects $\mathfrak{t}^{g_0}$, $\partial_{\mathfrak{t}}^{g_0}$, g^{g_0} vary continuously in the C^∞-topology.

Lastly, let us remark that uniqueness has been shown to be false for weak solutions to the mean curvature flow in dimensions $n \geq 3$; see [**Ilm98, Whi02**].

6. Topological applications

In the following, we will describe two topological applications that arise from our improved understanding of singular Ricci flows, more specifically from the Continuity Theorem 5.13. The first application concerns the classification of the homotopy types of the diffeomorphism groups of 3-manifolds. The second application concerns the contractibility of spaces of metrics with positive scalar curvature.

6.1. Topology of diffeomorphism groups and the Generalized Smale Conjecture. Let M be a manifold and denote by $\mathrm{Diff}(M)$ the group of self-diffeomorphisms equipped with the C^∞-topology. Pick a Riemannian metric g on M. We have a natural injection of the isometry group:

$$\mathrm{Isom}(M, g) \longrightarrow \mathrm{Diff}(M). \tag{6.1}$$

It will turn out that in many cases this map is a homotopy equivalence if g is chosen to be sufficiently symmetric.

The first result of this type was obtained by Smale [**Sma59**], who analyzed this question in dimension 2:

THEOREM 6.1 (Smale). *If (M, g) is isometric to the round 2-sphere, then (6.1) is a homotopy equivalence. Therefore $\mathrm{Diff}(S^2) \simeq O(3)$.*

In dimension 3 we have:

GENERALIZED SMALE CONJECTURE (GSC). *Suppose that (M, g) is closed and has constant curvature $K \equiv \pm 1$. Then (6.1) is a homotopy equivalence.*[3]

The case $M = S^3$ ($K \equiv 1$) is known as the Smale Conjecture and was resolved by Hatcher [**Hat83**]; see also earlier work by Cerf showing that $\mathrm{Diff}(S^3)$ has exactly two connected components [**Cer64d, Cer64a, Cer64b, Cer64c**]. In the $K \equiv 1$ case, the GSC was also resolved for lens spaces with the exception of $\mathbb{R}P^3$, as well as for prism spaces and quaternionic spherical space forms [**Iva82, Iva84, HKMR12**]. The GSC, however, remained open for $\mathbb{R}P^3$, as well as for the tetrahedral, octahedral and icosahedral families of spherical space forms. In the case $K \equiv -1$, the GSC was resolved by Hatcher and Ivanov [**Hat76, Iva76**] (in the Haken case) and Gabai [**Gab01**] (in the general case). Note that in the hyperbolic case $\mathrm{Isom}(M, g)$ is discrete due to Mostow rigidity. The proofs of all these results were purely topological and relied on Hatcher's result in the case $M = S^3$.

Using singular Ricci flow, Kleiner and the author [**BK17a, BK19b**] used singular Ricci flows to give a unified proof of the GSC in all cases.

[3] For simplicity, we have omitted the $K \equiv 0$ case. See [**Iva79**] for more details.

THEOREM 6.2 (Bamler, Kleiner). *The Generalized Smale Conjecture is true.*

The proof of Theorem 6.2, as detailed in [**BK19b**], is independent of Hatcher's work on the Smale Conjecture and therefore it gives us an alternative proof in the S^3-case. The same techniques also give a new proof of the following theorems [**BK19b**], which have previously been obtained by Hatcher [**Hat81**].

THEOREM 6.3. $\mathrm{Diff}(S^2 \times S^1)$ *is homotopy equivalent to* $O(2) \times O(3) \times \Omega O(3)$*, where* $\Omega O(3)$ *denotes the loop space of* $O(3)$*.*

THEOREM 6.4. $\mathrm{Diff}(\mathbb{R}P^3 \# \mathbb{R}P^3)$ *is homotopy equivalent to* $O(1) \times O(3)$*.*

Before continuing, let us point out the difference between the two papers [**BK17a, BK19b**], which both address Theorem 6.2. The paper [**BK17a**] contains a relatively short partial proof (about 30 pages). However, this proof fails in the $\mathbb{R}P^3$-case and it still relies on Hatcher's resolution of the Smale Conjecture. By contrast, the paper [**BK19b**] is much longer and more technical, but can handle the full GSC and does not rely on Hatcher's result.

We will now convey some ideas of the proof of Theorem 6.2 and, in particular, show how the study of diffeomorphism groups relates to the continuity of singular Ricci flows, as stated in Theorem 5.13. For this purpose, we will mainly focus the shorter proof in [**BK17a**], which is more instructional. Toward the end of this subsection, we will briefly describe the problems that arise in the general case and sketch how they are resolved.

The basis of the proof Theorem 6.2 in both approaches is the following elementary lemma, which reduces the GSC to the study of the space of constant curvature metrics. Denote by $\mathrm{Met}(M)$ the space of Riemannian metrics on M and by $\mathrm{Met}_{K\equiv\pm1}(M)$ the subset of metrics of constant curvature $K \equiv \pm 1$ (both equipped with the C^∞-topology). Then:

LEMMA 6.5. *Suppose that* (M, g) *is closed and has constant curvature* $K \equiv \pm 1$*. Then (6.1) is a homotopy equivalence if and only if* $\mathrm{Met}_{K\equiv\pm1}(M)$ *is contractible.*

PROOF. This follows via a long-exact sequence argument applied to the fiber bundle $\mathrm{Diff}(M) \to \mathrm{Met}_{K\equiv\pm1}(M)$, $\phi \mapsto \phi^* g$, whose fibers are homeomorphic to $\mathrm{Isom}(M, g)$. □

So it remains to show that $\mathrm{Met}_{K\equiv\pm1}(M)$ is contractible. In order to illustrate the proof of this fact, we will first sketch a proof in dimension 2 (i.e. of Theorem 6.1) using Ricci flow. This proof is different from Smale's original proof.

PROOF OF THEOREM 6.1. We need to show that $\mathrm{Met}_{K\equiv1}(S^2)$ is contractible. To see this, consider the map

$$\mathrm{Met}(S^2) \longrightarrow \mathrm{Met}_{K\equiv1}(S^2), \tag{6.2}$$

sending each g_0 to the limit of $(T-t)^{-1}g(t)$ as $t \nearrow T$, where $(g(t))_{t\in[0,T)}$ is a Ricci flow with initial condition $g(0)=g_0$. This limit exists and has $K \equiv 1$ by Theorem 2.1. The map (6.2) is a continuous retraction. Therefore, since $\mathrm{Met}(S^2)$ is contractible, so is $\mathrm{Met}_{K\equiv 1}(S^2)$. □

Let us now try to replicate this proof in dimension 3. Our goal will be to define a continuous retraction of the form

$$\mathrm{Met}(M) \longrightarrow \mathrm{Met}_{K\equiv\pm 1}(M) \tag{6.3}$$

Let $g_0 \in \mathrm{Met}(M)$ and consider the solution of the *singular* Ricci flow $\mathcal{M}^{g_0}$ with initial time-slice (M, g_0) (see Corollary 5.10); recall that the *classical* Ricci flow starting from g_0 will not be of much use in dimension 3, as it develops a non-round singularity in finite time.

We now need to extract a constant curvature metric on M from $\mathcal{M}$. This step will be more difficult, since the topology of the time-slices depends on time. To explain how we get around this issue, we will have to introduce some basic terminology on the geometry of Ricci flow spacetimes. Recall that a *worldline* in a Ricci spacetime $\mathcal{M}$ is a maximal trajectory of the time-vector field $\partial_\mathfrak{t}$; so a worldline corresponds to a non-moving point. We will now call a point $x \in \mathcal{M}$ *good* or *bad* depending on whether the worldline through x intersects the initial time-slice $\mathcal{M}_0$ or not. In the rotationally symmetric example of a non-degenerate neckpinch in Subsection 2.1, each time-slice $\mathcal{M}_t$ contains exactly two bad points for $t \in (T_1, T_2)$ (one per S^3) and none for $t \in (0, T_1)$. For any time t we can define a map $\phi_t : \mathcal{M}_t^{\mathrm{good}} \to \mathcal{M}_0$ mapping each good point x to the intersection of the worldline through x with the initial time-slice $\mathcal{M}_0$. If $\mathcal{M} = M \times [0,T)$ is given by a classical Ricci flow on a manifold M, then this map is nothing else than the projection $M \times \{t\} \to M \times \{0\}$. In the general case, it can be seen easily that ϕ_t is a smooth diffeomorphism onto its image.

The following is true (see Figure 5):

THEOREM 6.6. *Let $\mathcal{M}$ be a singular Ricci flow. Then*

(a) The set of bad points in any time-slice $\mathcal{M}_t$ is discrete.
(b) Suppose that M is a spherical space form other than S^3, $\mathbb{R}P^3$. Then there are two times $0 < T_1 < T_2$ with the following property: For any time $t \in (T_1, T_2)$ there is a unique component $\mathcal{C}_t \subset \mathcal{M}_t$ that is diffeomorphic to M. The flow restricted to $\cup_{t\in(T_1,T_2)}\mathcal{C}_t$ is given by a classical Ricci flow that converges modulo rescaling to a round metric. Consequently, the image

$$U := \phi_t(\mathcal{C}_t \cap \mathcal{M}_t^{\mathrm{good}}) \subset M$$

does not depend on $t \in (T_1, T_2)$ and we have smooth convergence

$$(T_2 - t)^{-1}(\phi_t)_* g_t \longrightarrow \overline{g} \qquad \text{on} \qquad U \tag{6.4}$$

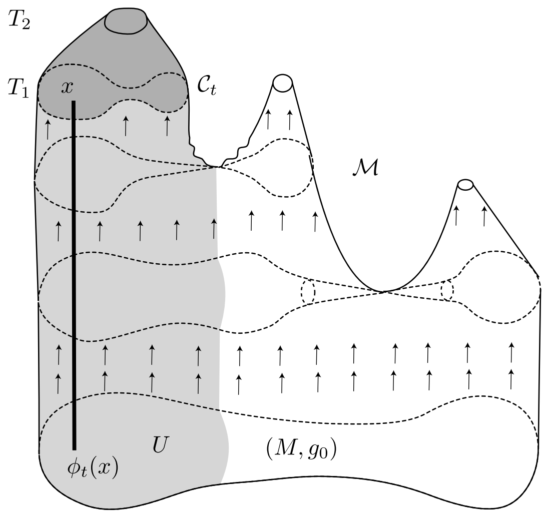

FIGURE 5. Illustration of Theorem 6.6. The bold vertical line through x is a worldline. The union $\bigcup_{t \in (T_1, T_2)} \mathcal{C}_t$ is shaded dark.

as $t \nearrow T_2$. The Riemannian manifold $(U, \overline{g})$ is isometric to a punctured round sphere where the punctures correspond to the bad points of $\mathcal{C}_t$.

(c) *If M is a hyperbolic manifold, then we have a similar characterization, where $T_2 = \infty$ and we have convergence $t^{-1}(\phi_t)_* g_t \to \overline{g}$ instead of (6.4).*

Note that we had to exclude the cases $M \approx S^3, \mathbb{R}P^3$ in Theorem 6.6(b) due to the possibility of a singularity modeled on a $\mathbb{Z}_2$-quotient of the peanut solution (see the footnote in Subsection 2.1) — a solution that goes extinct in finite time, but does not converge to a round sphere modulo rescaling.

Let us now assume that $M \not\approx S^3, \mathbb{R}P^3$. We define the set of *partially defined metrics* $\mathrm{PartMet}_{K \equiv \pm 1}(M)$ to be the set of pairs $(U, \overline{g})$, where $U \subset M$ is open and $\overline{g}$ is a metric of constant curvature $K \equiv 1$ such that $(U, \overline{g})$ is isometric to a punctured constant curvature metric on M. We equip $\mathrm{PartMet}_{K \equiv \pm 1}(M)$ with the C^∞_{loc}-topology. Theorem 6.6 induces a canonical map

$$\mathrm{Met}(M) \longrightarrow \mathrm{PartMet}_{K \equiv \pm 1}(M), \qquad g_0 \longmapsto (U, \overline{g}). \tag{6.5}$$

Using the continuous dependence of singular Ricci flows on their initial data (see Theorem 5.13) or using the Stability Theorem (Theorem 5.12) one can easily show that this map is continuous.

In the last step in the proof of Theorem 6.2 an obstruction theory argument and Hatcher's resolution of the Smale Conjecture are used to construct a map

$$\mathrm{PartMet}_{K \equiv \pm 1}(M) \longrightarrow \mathrm{Met}_{K \equiv \pm 1}(M), \qquad (U, \overline{g}) \longmapsto \widetilde{g}, \tag{6.6}$$

where $\widetilde{g}$ is essentially an extension of $\overline{g}$ onto the complement $M \setminus U$. Composing the maps (6.5) and (6.6) yields the desired retraction (6.3), finishing the proof of Theorem 6.2 if $M \not\approx S^3, \mathbb{R}P^3$.

Finally, let us briefly digest the argument above and discuss the difficulties that have to be overcome in the general case in [**BK19b**]. As mentioned above, in the cases $M \approx S^3, \mathbb{R}P^3$ our argument breaks down, because the flow may develop non-round limits. Moreover, the obstruction theory argument used to construct (6.6) relies on Hatcher's resolution of the Smale Conjecture. Lastly, in the case $M \approx S^3$ we face the further complication that we cannot uniquely characterize the component $\mathcal{C}_t \subset \mathcal{M}_t$ in Theorem 6.6(b) by its topology, because for almost all times t all components of $\mathcal{M}_t$ are spheres.

So in order to prove Theorem 6.2 for all spherical space forms and without relying on Hatcher's resolution of the Smale Conjecture, we have to revise our construction of (6.3) in such a way that every component that is separated by the flow is treated equally and fully until its extinction time. This is achieved in [**BK19b**] using a new topological notion called a *partial homotopy at time* T — among other things. A partial homotopy can be viewed as a hybrid between a continuous family of singular Ricci flows (for large T) and a homotopy in $\mathrm{Met}(M)$ (for $T = 0$). Partial homotopies can be modified via certain moves, which if applied correctly, allow us to reduce the parameter T by a small and uniform amount. By an induction argument, this implies the existence of partial homotopies at time 0 starting from a family of singular Ricci flows. These partial homotopies in turn imply the existence of the necessary homotopies in $\mathrm{Met}(M)$, allowing the construction of the map (6.3) via an obstruction theory argument.

6.2. Contractibility of the space of metrics with positive scalar curvature. Recall that M is assumed to be a compact, orientable 3-manifold and denote again by $\mathrm{Met}(M)$ the space of Riemannian metrics on M equipped with the C^∞-topology. Let $\mathrm{Met}_{PSC}(M) \subset \mathrm{Met}(M)$ the subspace of metrics with positive scalar curvature (PSC). In [**BK19b**], Kleiner and the author showed:

THEOREM 6.7 (Bamler, Kleiner). $\mathrm{Met}_{PSC}(M)$ *is either contractible or empty.*

This theorem was inspired by the work of Marques [**Mar12**], who showed that $\mathrm{Met}_{PSC}(M)$ is path connected — using Ricci flow with surgery. The analogous statement in dimension 2 — the contractibility of $\mathrm{Met}_{PSC}(S^2)$ — can be proven using the uniformization theorem, or by Ricci flow. Starting with the famous paper of Hitchin [**Hit74**], there has been a long history of results based on index theory, which show that $\mathrm{Met}_{PSC}(M)$ has non-trivial topology when M is high dimensional; we refer the reader to the survey [**Ros07**] for details. Theorem 6.7 provides the first examples of manifolds of dimension ≥ 3 for which the homotopy type of $\mathrm{Met}_{PSC}(M)$ is completely understood.

The proof of Theorem 6.7 relies on two basic facts: first, the PSC-condition is preserved by the (singular) Ricci flow and second, every singular Ricci flow on a manifold admitting a PSC-metric eventually acquires positive scalar curvature near its extinction time. Let us provide some more details. Our goal is to show that $\pi_k(\mathrm{Met}_{PSC}(M)) = 1$ for any $k \geq 0$. Since $\mathrm{Met}(M)$ is contractible, this is equivalent to showing that $\pi_{k+1}(\mathrm{Met}(M), \mathrm{Met}_{PSC}(M)) = 1$ for any $k \geq 0$. For this purpose, consider a continuous map $\alpha : D^{k+1} \to \mathrm{Met}(M)$ with $\alpha(\partial D^{k+1}) \subset \mathrm{Met}_{PSC}(M)$. This map can also be viewed as a family of metrics $(g^s = \alpha(s))_{s \in D^{k+1}}$ on M, where g^s is PSC for all $s \in \partial D^{k+1}$. To understand the main strategy of the proof let us first make the (very restrictive) simplifying assumption that $M \approx S^3$ and that every metric g^s can be evolved into a non-singular Ricci flow of the form $(g^s_t)_{t \in [0,T^s)}$ that converges to the round metric modulo rescaling, which of course is a PSC-metric. By reparameterizing the time parameter and rescaling the metric appropriately, the family of Ricci flows $(g^s_t)_{s \in D^{k+1}, t \in [0,T^s)}$ can be converted into a homotopy of the form $\beta : D^{k+1} \times [0,1] \to \mathrm{Met}(M)$ with the following properties:

(1) $\beta(\cdot, 0) = \alpha$.
(2) $\beta(s,t) \in \mathrm{Met}_{PSC}(M)$ for all $s \in \partial D^{k+1}$ and $t \in [0,1]$, because $\beta(s,0) = \alpha(s) \in \mathrm{Met}_{PSC}(M)$ and the PSC-condition is preserved by the flow.
(3) $\beta(s,1) \in \mathrm{Met}_{PSC}(M)$ for all $s \in D^{k+1}$, because the flow eventually acquires positive scalar curvature.

In other words, β is a nullhomotopy of α within $(\mathrm{Met}(M), \mathrm{Met}_{PSC}(M))$, showing that $\pi_k(\mathrm{Met}(M), \mathrm{Met}_{PSC}(M)) = 1$, as desired.

Lastly, let us briefly discuss the general case, in which the simplifying assumption is violated. In this case, Theorem 5.13 implies that the family of metrics $(g^s)_{s \in D^{k+1}}$ can be evolved into a continuous family of singular Ricci flows $(\mathcal{M}^s)_{s \in D^{k+1}}$ with initial data (M, g^s). These flows now need to be converted into a nullhomotopy of the form $\beta : D^{k+1} \times [0,1] \to \mathrm{Met}(M)$ that still satisfies Properties (1)–(3). During this conversion process we are confronted with similar problems as described in the end of Subsection 6.1, which can be overcome with similar techniques, such as the use of partial homotopies.

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