Removeability of a codimension four singular set for solutions of a Yang Mills Higgs equation with small energy

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ABSTRACT. The Yang-Mills-Higgs equations, while they arose in theoretical physics, have become a central object of study in geometric analysis. The moduli spaces of solutions are used to study problems in algebraic geometry as well as topology. To understand these moduli spaces, it is natural to ask what sort of objects are limits in a weak sense of sequences of solutions. In many examples, it can be shown that subsequences of solutions converge to a solution off a set of Hausdorff codimension 4. In four dimensions, these point singularities are removable. The question arises of when this singular set of dimension $n - 4$ is removable in higher dimensions. In a classic paper, Tao and Tian [11] prove an important step in the removability theorem. What they show is that, in general, independent of the equation, if the case the curvature of a connection defined on a set of Hausdorff codimension 4 is sufficiently small in a certain Morrey space, that there exists a smaller ball and a gauge in which the connection is $d + A$, where A is coclosed and bounded by norms on the curvature. One can now treat the Yang-Mills-Higgs equations as a standard elliptic system and obtain further regularity. This choice of Morrey space is natural, because monotonicity of solutions provides estimates exactly in this space. However, since the solution is only defined off a set of Hausdorff codimension 4, it is not known that the monotonicity theorem is true for limiting singular solutions. Only by adding the assumption of stationary with respect to diffeomorphisms in the entire domain can one apply the theorem to limiting singular connections. Unfortunately, we cannot fill in this gap. We provide a new proof of the Tao-Tian result. Our proof uses the equation and a maximum principle for a differential inequality derived from the equation via averaging. In this manner, we can avoid the geometric construction of a good gauge in a weak setting via averaging exponential gauges, as we first obtain further estimates on the curvature from the

inequality. Our main result Corollary 4 is that, if the curvature is sufficiently small in a Morrey space, and the solution is defined and smooth off a set of Hausdorff codimension 4 with the covariant derivative of the Higgs field in the same Morrey space, then the solution is smoothly gauge equivalent to one which extends smoothly over the singular set. In the case that the solution has small energy, and is a critical point of the integral, or even just critical with respect to diffeomorphisms in the entire domain, not just off the singular set, the smallness of the curvature in the Morrey space in a smaller domain can be verified and leads to Theorem 11 as a corollary of Corollary 4. There may be other ways to obtain the bounds on the curvature in the Morrey space; hence we state the main result in terms of small curvature in the Morrey space. The results apply to most of the coupled Yang-Mills-Higgs equations, and we do not go into the details in this paper. We do assume that the Higgs field is bounded, and explain why this follows from L^2 bounds on it. Many interesting equations, such as the Kapustin-Witten equations and the equations for a complex flat connection do not come with such a bound on sequences. Moreover, the singular set of a renormalized limit is Hausdorff codimension 2, and we sadly have nothing to say about these singularities.

We develop a new method for proving regularity for small energy stationary solutions of coupled gauge field equations. Our results duplicate those of [**10**] for the pure Yang Mills equations, but our proof is simpler, and obtains bounded curvature without the use of Coulomb gauges. It relies instead on the Weitzenblock formulae, and an improved Kato inequality. Our results also extend and simplify those of [**3**].

1. Introduction

We consider stationary solutions of the Euler Lagrange equations for the integral

$$(1.1) \qquad \mathcal{A}(D_A, \Phi) = \int_\Omega \left(|F_A|^2 + |D_A\Phi|^2 + Q(\Phi) \right) (dx)^n,$$

for $\Omega \subset \mathbb{R}^n$ $n \geq 4$, and D_A is a unitary connection on the product bundle $V \times \Omega$, Φ is a section of the product bundle $V \times \Omega$, and where $Q(\bullet)$ is a real valued equivariant function from sections of the product bundle $V \times \Omega$. The integrand on the right hand side of (1.1) is gauge invariant, but since the result is local, we do not need any of the topological considerations in its formulation. A typical term in $Q(\Phi)$ can be $|(|\Phi|^2 - \lambda^2)|^2$ if $V = \mathfrak{su}(N)$, but we impose only three conditions on Q, namely

$$(1.2\mathrm{a}) \qquad\qquad\qquad Q(\Phi) \geq 0$$

$$(1.2\mathrm{b}) \qquad\qquad\qquad (\Phi, Q_\Phi(\Phi)) \geq -K_1^2.$$

Here $Q_\Phi(\Phi)$ is the directional derivative of the function Q, in the direction ϕ, and, hence, $Q_\Phi(\Phi)$ is a section of V^*. By the usual duality argument, we can identify it with a section of V.

(1.2c) $Q(\Phi)$ is equivariant with respect to the group action.

REMARK 1. Our sign conventions are those of analysis. Hence, $\triangle = \operatorname{div} d = \sum_{i=1}^n \frac{\partial^2}{\partial^2 x^i}$, and D_A^* is the covariant derivative divergence, that is the formal adjoint of D_A on p-forms with sign $(-1)^p$.

First, we state the Euler–Equations for the energy functional

(1.3a) $$2(D_A)^* F_A = [\Phi, D_A \Phi]$$

(1.3b) $$2(D_A)^* D_A \Phi = Q_\Phi(\Phi).$$

As in the previous work on this subject, monotonicity theorems and change of scale are key ingredients. We generally only change scale by blow-up, from $|x - x_0| \le r$ to $|y| = \left(\frac{|x-x_0|}{r} \right) R$, for $0 < r < R$. Since, we are in a local trivialization, $D_A = d + A$ is a local covariant exterior derivative with fixed scaling given by

(1.4) $$A dx = \tilde{A} dy = (\tilde{A})(\frac{R}{r}) dx,$$

which implies $\tilde{A} = (\frac{r}{R})A$. Consistency of the first two terms in the integrand on the right hand side of (1.1) demands that $\Phi(x) = \tilde{\Phi}(y)(\frac{r}{R})$, that is that Φ scales like a 1-form. This requires $Q(\Phi)$ to scale like a 4-form. That is

(1.5) $$\tilde{Q}_R(\tilde{\Phi}) = \left(\frac{R}{r} \right)^4 (Q_r)(\Phi) = \left(\frac{R}{r} \right)^4 Q\left(\left(\frac{r}{R} \right) \tilde{\Phi} \right).$$

This looks formidable, however, we are saved by (1.2) and Theorem 1, which provides bounds on terms depending on $|\Phi|$, and hence bounds on terms depending on Q.

Singular solutions of (1.3) may arise as limits of smooth solutions. The following theorem indicates that, if there is a uniform bound in L^2 on the sequence of Higgs Fields Φ_j, there will also be a bound on the maximum of $|\Phi_j|$, and hence on the maximum of $|\Phi|$ in the singular limit.

THEOREM 1. *Let (D_A, Φ) be a smooth solution of the field equations (1.1) in the domain $\Omega \subset \mathcal{R}^n$. Let, $\Phi \bullet Q_\Phi(\Phi)\Phi \ge -(K_1)$, where $\bullet$ denotes inner product, and let $\int_\Omega |\Phi|^2 (dx)^n \le (K_2)^2$. Then $|\Phi|$ is bounded in the interior of Ω and for all $x_0 \in int(\Omega)$, $0 < d < dist(x_0, \partial\Omega)$*

(1.6) $$|\Phi(x_0)|^2 \le ((d)^{-n}) C_1(n)) \int_{|x-x_0| \le d} |\Phi|^2 (dx)^n + C_2(n) K_1^2 d^2.$$

PROOF. We will make use of the following identity:

$$(1.7) \qquad (\frac{1}{2})\triangle|\Phi|^2 = (\Phi) \bullet (D_A^* D_A \Phi) + |D_A \Phi|^2.$$

The equation (1.7) follows from the facts that Φ is locally a bundle section and the compatibility of the connection with the inner product on sections.

Here $\triangle$ is the co-ordinate laplacian on the base. We use the the field equation (1.3b) to replace the first term on the right hand side of (1.7), and obtain

$$(1.8) \qquad \triangle|\Phi|^2 = (\Phi) \bullet (Q_\Phi(\Phi)) + 2|D_A \Phi|^2.$$

Using the lower bound $\Phi \bullet Q_\Phi(\Phi) \geq -(K_1)^2$ in (1.8), we obtain:

$$(1.9) \qquad \triangle|\Phi|^2 \geq -K_1^2.$$

Thus, the function $|\Phi|^2 + (x - x_0)^2 K_1^2$ is non-negative smooth subharmonic function. Now apply the sub-mean value property for subharmonic functions, the triangle inequality for integrals, the triangle inequality for sums, and take the sup of $(x - x_0)^2 K_1^2$ inside the integral over $B_d(x_0) \subset\subset \Omega$. $\qquad \square$

We can use these estimates in several ways, depending on the actual structure of $Q(\Phi)$. The terms that we have in mind are terms like $(|\Phi|^2 - a^2)^p$, or $|[\Phi, \Phi]|^p$, for $p \geq 1$, where the simple cases occur for $p = 2$.

We just gave a description of the functional and motivated the bound that we assume on the maximum of Φ in subsequent chapters. In Section 2 we outline a proof of the monotonicity theorem. This motivates the assumption we place on the Morrey Space norm of the curvature F_A in our final theorem. We emphasis that for its validity, it requires the critical point (D_A, Φ) to be stationary with respect to diffeomorphisms that are the identity near the boundary of the domain. Section 3 contains the necessary improvement to the usual Kato inequality $|d|v|| \leq |\nabla v|$ where $v = (F_A, D_A \Phi)$, as well as a reminder of the Weitzenbock formulae for the same quantity. In this section, as in Section 1, our computations hold only on the set in which the connection is smooth. Estimates for smooth solutions are contained in Section 4. Section 4 is not essential but the estimates for smooth solutions with curvature small in a Morrey Space are easier to obtain and provide a warm-up for the singular case.

The core of the paper is in Section 5. Here we use the properties of Morrey Spaces to get regularity for singular solutions of a differential inequality (on functions f) with a coefficient small in a borderline Morrey space. The Morrey Spaces $X^m(\Omega_l)$ (with m an integer, and with $\Omega_l = [-l, l]$) that we use in the statement of Theorem 9 just below are defined in Appendix C. The theorem that we prove is Theorem 9:

THEOREM 9. *Let $u \geq 0$ and $f > 0$ be smooth functions on $\Omega_4 - \mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n - 4$ Hausdorff dimension. If $f \in X^2(\Omega_4)$,*

$0 < k < 2$, *and if*

$$(1.10) \qquad -\triangle f + \alpha \left(\frac{|df|^2}{f} \right) - uf \leq Qf,$$

then there exist constants $\eta_k > 0$, *and* $\kappa_k > 0$, *such that if* $\|u\|_{X^2(\Omega_4)} < \eta_k$, *then* $f \in X^k(\Omega_1)$. *Moreover*

$$(1.11) \qquad \|f\|_{X^k(\Omega_1)} \leq \kappa_k \|f\|_{X^2(\Omega_4)}.$$

Section 6 is a straightforward application of Theorem 9 and the improved Kato inequality of Section 3.

The idea of using differential inequalities to get estimates of the norms of curvature and covariant derivative of the Higgs field, and to thus to remove high codimension singular sets is due to Tom Otway [**5**].

We have Theorem 10:

THEOREM 10. *Let* (D_A, Φ) *be solutions to a Yang-Mills-Higgs system in* $\Omega_4 - \mathcal{S}$, *where* $\mathcal{S}$ *is a closed set of finite* $n-4$ *dimensional Hausdorff measure. Let* $\upsilon = (F_A, D_A\Phi)$. *Assume that* $\upsilon \in X^2(\Omega_4)$, *and that* $\Phi \in L^\infty(\Omega_4)$.

$$(1.12) \qquad Q_1 = \sup_{[-4,4]^n} \left(2|\Phi| + |Q_{\Phi,\Phi}(\Phi)| \right)$$

$$(1.13) \qquad Q_2^2 = \sup_{[-4,4]^n} \left(\frac{|Q_\Phi(\Phi)|^2}{Q_1} \right).$$

If $F_A \in X^2(\Omega_4)$ *is sufficiently small, then* $|\upsilon| \in L^\infty(\Omega_1)$. *We also have the explicit bound*

$$(1.14) \qquad \|\upsilon\|_{L^{2n}(\Omega_1)} \leq C(Q_1)(\|\upsilon\|_{X^2(\Omega_4)} + Q_2^2)$$

We state the main regularity theorem as a corollary of Theorem 10. It applies the Coulomb gauge construction of Appendix C. We have there:

COROLLARY 4. *Assume* $(D_{\tilde{A}}, \Phi)$ *satisfies a Yang-Mills-Higgs system in* $\Omega - \mathcal{S}$, *where* $\mathcal{S}$ *is a closed set of finite* $n-4$ *dimensional Hausdorff measure. Let* $\upsilon = (F_{\tilde{A}}, D_{\tilde{A}}\Phi)$

$$(1.15) \qquad Q_1 = \sup_\Omega \left(|Q(\Phi)|^2 + |Q_{\Phi,\Phi}(\Phi)| \right).$$

$$(1.16) \qquad Q_2^2 = \frac{\left(\sup_\Omega |Q_\Phi(\Phi)|^2 \right)}{Q_1}.$$

Suppose $\upsilon \in X^2(\Omega)$, *and* $Q_1\delta^2$ *as well as* $Q_2\delta^2$ *(scales like the two form* υ*) are bounded by a fixed constant. In addition suppose that* $F_{\tilde{A}} \in X^2(\Omega)$ *has small enough* $X^2(\Omega)$ *norm (independent of the other constants). If* $\Omega_{y,\delta} \subset \Omega$, *then* $\delta^2\upsilon \in L^\infty(\Omega_{y,\delta})$ *is bounded above by a constant, and* $(D_{\tilde{A}}, \Phi)$ *are smoothly gauge equivalent on* $\Omega_{y,\delta} - \mathcal{S}$ *to a smooth exterior covariant differential (corresponding to a smooth connection), and a smooth Higgs Field on* $\Omega_{y,\delta}$, *both of which extend smoothly across the singular set* $\mathcal{S} \cap \Omega_{y,\delta}$.

The Appendices are important. In Appendix A we outline the necessary background on Morrey Spaces. We refer the reader to the book by Adams [1], and give outlines of applications that are not in this reference, such as the invertibility of the Laplacian on cubes with Dirichlet boundary value. We introduce notation such as the space $X^k = M^{[\frac{n}{k}, \frac{4}{k}]}$ for the Morrey space which scales like $\frac{n}{k}$ and has integral power $\frac{4}{k}$. This simplifies our exposition. Appendix B contains the maximum principle that we use in Section 5, which seemed to fit better in an appendix than in the body of the paper. Appendix C is the proof of the existence of a Coulomb gauge for singular connections. This is not in the standard literature because we allow a singular set of Hausdorff codimension three (We only need Hausdorff codimension four for our application). Tao and Tian [11] prove this with much weaker conditions but their proof is much more elaborate. Since the point of this paper is to simplify their proof, we include it.

2. Monotonicity formulae

In this section we use the condition that $\mathcal{A}(D_A, \Phi)$ is stationary in the sense of Price [4], and Zhang [9] at (D_A, Φ) with respect to smooth diffeomorphisms, that are the identity near the domain boundary, to derive a monotonicity formula. We give an outline, since the method is the same as used by many authors, cf. [4], and Lemma 2.1 of [11].

THEOREM 2. *If (D_A, Φ) is a stationary point of $\mathcal{A}(D_A, \Phi)$ with respect to smooth diffeomorphisms of Ω which equal the identity in a neighborhood of $\partial\Omega$, then for all smooth vector fields $\boldsymbol{\nu}$ with compact support in Ω, we have*

$$(2.1) \quad \int_\Omega [\partial_k \boldsymbol{\nu}^k |F_A|^2 + |D_A \Phi|^2 + Q(\Phi) - \partial_j \boldsymbol{\nu}^k F_{ki} F_{ji} \\ -2\partial_j \boldsymbol{\nu}^k (D_{A,j}\Phi)(D_{A,k}\Phi)](dx)^n = 0$$

PROOF. This formula can be straightforwardly derived for smooth solutions F_A, Φ via Noether's formula. But, by direct calculation (compare formula (2.4) page 265 of [11]), it is also true for $|F_A|^2, |\Phi|^2, Q(\Phi) \in L^1(\Omega)$ Thus, if the singular points of $|F_A|^2, |D_A\Phi|^2, Q(\Phi) \in L^1(\Omega)$, occur on a set of measure zero, they do not affect formula (2.1)   □

COROLLARY 1. *If (2.1) holds, and if $\mathcal{A}(D_A, \Phi)$ is finite, then, for $\{|x - x_0| \le R\} \subset \Omega$, we have*

$$(2.2)$$
$$\int_{|x-x_0| \le R} [(n-4)|F_A|^2 + (n-2)|D_A\Phi|^2 + nQ(\Phi)] (dx)^n - $$
$$R \int_{|x-x_0|=R} [|F_A|^2 + |D_A\Phi|^2 + Q(\Phi) - 4|F_{A,r}|^2 - 2|D_{A,r}\Phi|^2] (dx)^{n-1} = 0.$$

Here, $F_{A,r} = i_{\frac{\partial}{\partial r}}(F_A)$ is the radial part of F_A, and $D_{A,r} = D_A(\frac{\partial}{\partial r})$ is the radial part of D_A.

PROOF. With no loss of generality we assume that $x_0 = 0$, since our calculations are translation invariant. Let $\boldsymbol{\nu} = \boldsymbol{\nu}_\epsilon$ be a smooth vector field, dependent on a small positive parameter ϵ, where $\boldsymbol{\nu}$ is defined by

$$(2.3) \qquad \begin{cases} \boldsymbol{\nu} = r\frac{\partial}{\partial r} & \text{if} \quad |x| \leq R - \epsilon \\ \boldsymbol{\nu} = 0 & \text{if} \quad |x| \geq R \\ \boldsymbol{\nu} = \eta(\frac{R-|x|}{\epsilon})r\frac{\partial}{\partial r} & \text{otherwise}, \end{cases}$$

Here η is a smooth function, satisfying

$$(2.4) \qquad \begin{cases} \eta(t) = 0 & \text{for} \quad t \leq 0 \\ \eta(t) = 1 & \text{for} \quad t \geq 1 \end{cases}$$

with $\phi' \geq 0$. Note that:

$$(2.5) \qquad \frac{\partial}{\partial x^k}\boldsymbol{\nu}^i = \eta(\frac{R-|x|}{\epsilon})\delta_k^i - (\frac{1}{\epsilon})\eta'(\frac{R-|x|}{\epsilon})\frac{x^i x^k}{|x|}.$$

Using this $\boldsymbol{\nu}$ in (2.1), we obtain

(2.6)
$$\int_{|x| \leq R} \eta(\frac{R-|x|}{\epsilon})[(n-4)|F_A|^2 + (n-2)|D_A\Phi|^2 + nQ(\Phi)]\,(dx)^n -$$
$$\int_{R-\epsilon \leq |x|=\rho \leq R} (\frac{1}{\epsilon})\eta'(\frac{R-|x|}{\epsilon})[|F_A|^2 + |D_A\Phi|^2 + Q(\Phi) - 4|F_A|^2 - 2|D_{A,r}\Phi|^2]$$
$$\times \rho\,(da)_{n-1}d\rho.$$

where $(da)_{n-1} = \rho^{n-1}(d\Theta)$ is the area element on $|x| = \rho$.

Let $t = R - \rho$, and note that $\lim_{\epsilon \downarrow 0}(\frac{1}{\epsilon})(\eta'(\frac{t}{\epsilon})) \to \delta$, in the sense of distributions, where δ is the delta distribution. Thus, letting $\epsilon \downarrow 0$ gives our result. $\qquad \square$

From Corollary 1, by ignoring the radial parts of F_A, and D_A which appear with a negative sign, and by replacing both n and $n-2$ by $n-4$, we derive a differential inequality on

$$\mathcal{E}(R) = \int_{|x| \leq R} |F_A|^2 + |D_A\Phi|^2 + Q(\Phi)\,(dx)^n.$$

$$(2.7) \qquad (n-4)\mathcal{E}(R) \leq R\mathcal{E}'(R).$$

Integrating (2.7) gives the monotonicity formula

$$(2.8) \qquad \mathcal{E}(r) \leq (\frac{r}{R})^{n-4}\mathcal{E}(R).$$

THEOREM 3. *If (D_A, Φ) is a stationary point of the functional $\mathcal{A}(D_A, \Phi)$, with respect to smooth diffeomorphisms of its domain, then, if $\{|x - x_0| \leq R\} \subset \Omega$ for $r \leq R$, we have*

$$(2.9) \qquad \int_{|x-x_0|\leq r} |F_A|^2 + |D_A\Phi|^2 + Q(\Phi)\,(dx)^n$$
$$\leq \left(\frac{r}{R}\right)^{n-4} \int_{|x-x_0|\leq R} |F_A|^2 + |D_A\Phi|^2 + Q(\Phi)\,(dx)^n.$$

REMARK 2. In the case where we have a Riemannian metric, instead of a Euclidean metric, this formula is easily seen to be valid with an error term.

In order to prove our final regularity theorem Theorem 11 (which is a Corollary of Lemma 3 and of Corollary 4, [all in Section 6], we need the scaling estimates below:

We take a small ball $|x - x_0| \leq r$ to a ball $|y| \leq 4$. In this case, as in Section 1, we may assume that we have a bound on Φ, and therefore on $Q(\Phi)$, and its derivatives with respect to Φ. That is:

$$(2.10) \qquad\qquad |\Phi| \leq h$$
$$(2.11) \qquad\qquad Q(\Phi) \leq h_0 = max_{|\Phi|\leq h}Q(\Phi)$$
$$(2.12) \qquad\qquad |Q_\Phi(\Phi)| \leq h_1 = max_{|Q_\Phi(\Phi)|\leq h}|Q_\Phi(\Phi)|.$$

Under rescaling, from $r \to R$, denoting the rescaled terms by $\tilde{\Phi}$, $\tilde{Q}$ etc., We obtain

$$(2.13) \qquad\qquad |\tilde{\Phi}| = \left(\frac{r}{R}\right)|\Phi| \leq \left(\frac{r}{R}\right)h$$
$$\tilde{Q} = \left(\frac{r}{R}\right)^4 Q \leq h_0$$
$$|\tilde{Q}_{\tilde{\Phi}}| = \left(\frac{r}{R}\right)^3 |Q_\Phi| \leq \left(\frac{r}{R}\right)^3 h_1.$$

If Φ is more regular, then analogous bounds hold for the higher derivatives of $\tilde{Q}$ with respect to Φ. This becomes important for the regularity theory.

We have a rescaled monotonicity type estimate.

THEOREM 4. *If*

$$(2.14) \qquad \int_{|x-x_0|\leq r} [|F_A|^2 + |D_A\Phi|^2 + Q(\Phi)]\,(dx)^n \leq C(r)^{n-4}$$

is rescaled to $y = (\frac{R}{r})(x - x_0)$, we have

$$(2.15) \qquad \int_{|x-x_0|\leq R} [|F_{\tilde{A}}|^2 + |D_{\tilde{A}}\tilde{\Phi}|^2 + \tilde{Q}(\tilde{\Phi})]\,(dy)^n \leq C(R)^{(n-4)}.$$

Moreover, the rescaled variables satisfy (2.13).

REMARK 3. Under blowup, since we assume a bound on Φ, not only do $F_{\tilde{A},r} \to 0$ and $D_{r,\tilde{A}}\Phi \to 0$, but $\tilde{\Phi}$ and $\tilde{Q} \to 0$. Thus, the theory of blow-ups is the same as for pure Yang-Mills. This is somewhat disappointing.

There is a direct application of the Monotonicity Theorem 3 in the proof of Lemma 3 and thus in the proof of Theorem 11, which are in of Section 6.

3. Improved Kato inequalities

Let $\nabla_A = \{\nabla_{i,A}\}$ (where the i refers to local co-ordinates x^i on the base) denote a local covariant derivative in a bundle (where this notation is used to make a clear distinction between the full covariant derivative on the bundle and the exterior covariant differential), and ν is a C^1 section, with $\nabla\nu$ continuous, the pointwise inequality

$$(3.1) \qquad |d|\nu||^2 \leq \beta |\nabla\nu|^2, \text{ for } \quad \beta = 1$$

is well known. However, if ν satisfies some elliptic equations, often the constant β can be improved. This is particularly useful in removing singularities.

Such "Refined Kato inequalities" have been used before. In particular see (23') page 219 [6] and the estimate of the second fundamental form in [9]. A very general approach to Refined Kato inequalities, not including our results in this section is contained in the very lovely paper of Calderon et alia [3]. Their results are based on deep knowledge of group representation theory. In keeping the expository nature of this Survey volume, we prove our results in this section be elementary computation.

In the following, we will choose a specific orthonormal frame at $x \in \Omega \subset \mathcal{R}^n$. We write the inequalities in such a way as to make the extension to a Riemannian manifold as a base space clear.

Our two examples are $\nu = F_A$, and $\nu = D_A\Phi$. Here, F_A, and Φ are assumed smooth in the domain $\Omega \subset \mathcal{R}^n$. In fact, we prove a generalization of the usual improved Kato inequalities, for arbitrary one and two forms, with error terms. Because a general two form $\boldsymbol{F}$ (unlike the curvature F_A does not satisfy either the Bianchi identity or the first Field equation, we expect the error terms to involve $D_A\boldsymbol{F}$ and $D_A^*\boldsymbol{F}$). Similarly, because a general one form $\boldsymbol{\theta}$ does not satisfy the second Field equation and is not in the kernel of D_A^*, we expect the error terms to involve $D_A\boldsymbol{\theta}$, and $D_A^*\boldsymbol{\theta}$).

We note that the constants are different for the one form and the two form inequalities.

In this section we make use of the connection ∇, locally associated with a 1-form A, in the usual way, and denoted by ∇^A. We also make use of the associated local covariant derivative ∇_A, and the associated local covariant exterior differential denoted by D_A. For a less cluttered notation, suppress the subscript A on the connection and the local covariant derivative, which will cause no confusion because we are working in a local trivialization, so that the connection is the local covariant derivative.

THEOREM 5. *Let ∇ be an arbitrary metric compatible connection on $\Omega \times V$, where $\Omega \subset \mathcal{R}^n$. If $\boldsymbol{F}\colon \Omega \to V \otimes T^*(\Omega) \otimes T^*(\Omega)$ is an arbitrary smooth vector valued 2-form then,*

$$(3.2) \qquad \left(\frac{n}{n-1}\right)\left(|d|\boldsymbol{F}||^2\right) \leq |\nabla\boldsymbol{F}|^2 + |D_A\boldsymbol{F}|^2 + |D_A^*\boldsymbol{F}|^2,$$

and if $\boldsymbol{\theta}\colon \Omega \to V \otimes T^*(\Omega)$ is an arbitrary smooth vector valued one form,

$$(3.3) \qquad \left(\frac{n+1}{n}\right)\left(|d|\boldsymbol{\theta}||^2\right) \leq |\nabla\boldsymbol{\theta}|^2 + |D_A\boldsymbol{\theta}|^2 + |D_A^*\boldsymbol{\theta}|^2.$$

PROOF. First, we prove (3.2). At any arbitrary point p in the fiber V, we choose a local exponential gauge, so that $A(p) = 0$. Choose an adapted orthonormal frame (inducing local adapted orthonormal coordinates), such that $d|\boldsymbol{F}| = d_1|\boldsymbol{F}| = \frac{\partial}{\partial_1}(|\boldsymbol{F}|)dx^1$. Note, that choosing such an orthonormal frame, still preserves the exponential gauge centered at p, because all we are doing is rotating the base coordinates, by a constant rotation at p. Then using the standard Kato inequality, we have:

$$(3.4) \qquad d|\boldsymbol{F}| = d_1|\boldsymbol{F}| \leq |\nabla_1\boldsymbol{F}|.$$

Note, that in our adapted orthonormal coordinates, at the arbitrary point p in the fiber–that is the center of our exponential coordinates, we have $\nabla_1\boldsymbol{F}$ has the coordinate representation $\sum_{k,l}\frac{\partial}{\partial x^1}(F_{k,l})$, because $A(p) = 0$. The idea of the proof is to make use of this formula for the coordinates of $\nabla_1\boldsymbol{F}$, in an expression resulting from replacing terms in the coordinate representation of $D_A\boldsymbol{F}$ at p by terms in the coordinate representation of $D_A^*\boldsymbol{F}$ at p. Then, we use that fact that p is arbitrary. First, we express $D_A(\boldsymbol{F})$ at p in components, with respect to our adapted coordinates. We note that at p, we have $D_A(\boldsymbol{F}) = d(\boldsymbol{F})$.

Consider the individual components of $D_A\boldsymbol{F}(p) = dF(p)$. We can compute these explicitly in our local coordinates. In particular, consider those component terms of the form $\left(\frac{\partial}{\partial x^1}\boldsymbol{F}_{ij}\atop{i\neq j\neq1}\right)$. We have (for i, j fixed):

$$(3.5) \qquad \left(\frac{\partial}{\partial x^1}\boldsymbol{F}_{ij}\atop{i\neq j\neq1}\right) = \frac{\partial F_{1,j}}{\partial x^i} \pm \frac{\partial F_{1,i}}{\partial x^j} \pm (d\boldsymbol{F})_{1,i,j}.$$

Here, $(d\boldsymbol{F})_{1,i,j}$ is the $dx^1 \wedge dx^i \wedge dx^j$ component of $d\boldsymbol{F}$.

For each fixed pair (i,j), on the left hand side of equation (3.5), there are three terms on the right hand side of equation (3.5).

Note that the $\pm$ parity of the terms on the right hand side of equation (3.5) is immaterial to our proof.

Consider the individual components of $D_A^* \boldsymbol{F}(p) = d^* \boldsymbol{F}(p)$. In particular, those component terms of the form $\left(\frac{\partial}{\partial x^1} \boldsymbol{F}_{1,l} \right)_{l \neq 1}$, are given by:

$$(3.6) \qquad \left(\frac{\partial}{\partial x^1} \boldsymbol{F}_{1,l} \right)_{l \neq 1} = \sum_{\substack{s \neq 1 \\ s \neq l}} \left[\pm \left(\frac{\partial}{\partial x^s} \boldsymbol{F}_{s,l} \right) \right] \pm (d^* \boldsymbol{F})_l$$

Here, $(d^* \boldsymbol{F})_l$ is the dx^l component of $d^* * \boldsymbol{F}$, where l is the fixed l on the left hand side of equation (3.6).

Note that for each fixed l on the left hand side of equation (3.6), there are $n-2$ terms in the sum on the right hand side of (3.6). This is because s takes the $n-2$ integer values $\{s = 2, \ldots, n\} - \{l\}$. Thus, the right hand side of equation (3.6) has $n-1$ terms.

Note that the $\pm$ parity of the terms on the right hand side of equation (3.6) is immaterial to our proof. We have

$$(3.7) \qquad |\nabla_1 \boldsymbol{F}(p)|^2 = \sum_{s \neq t} \left| \frac{\partial}{\partial x^1} \boldsymbol{F}_{st} \right|^2 .$$

Replacing each term on the right hand side of equation (3.7) by terms not involving $\frac{\partial}{\partial x^1}$, by using either equation (3.5) or equation (3.6), we obtain an expression for $|\nabla_1 \boldsymbol{F}|^2$, in which $\frac{\partial}{\partial x^1}$ does not appear.

Each such replacement has either 3 or $n-1$ terms. Note, that, since $n \geq 4$, we have $3 \leq n-1$.

Now, we use $\left(\sum_{i=1}^{n-1} a_i \right)^2 \leq (n-1) \left(\sum_{i=1}^{n-1} a_i^2 \right)$.

Thus, we have:

$$(3.8)$$
$$|\nabla_1 \boldsymbol{F}(p)|^2 \leq (n-1) \left(\sum_{i=2}^{n} |\nabla_i \boldsymbol{F}(p)|^2 \right) + (n-1)|D_A^* \boldsymbol{F}|^2 + (n-1)|D_A \boldsymbol{F}|^2.$$

Adding $(n-1)|\nabla_1 F(p)|^2$ to both sides of equation (3.8) we obtain

$$(3.9)$$
$$(n)|\nabla_1 \boldsymbol{F}(p)|^2 \leq (n-1) \left(\sum_{i=1}^{n} |\nabla_i \boldsymbol{F}(p)|^2 \right) + (n-1)|D_A^* \boldsymbol{F}|^2 + (n-1)|D_A \boldsymbol{F}|^2.$$

Dividing both sides of inequality (3.9) by $n-1$, we obtain

$$(3.10) \qquad \left(\frac{n}{n-1} \right) |\nabla_1 \boldsymbol{F}(p)|^2 \leq \sum_{i=1}^{n} |\nabla_i \boldsymbol{F}(p)|^2 + |D_A^* \boldsymbol{F}|^2 + |D_A \boldsymbol{F}|^2.$$

Now, we use inequality (3.4), in combination with inequality (3.10), and the fact that p is arbitrary, to obtain:

$$(3.11) \quad \left(\frac{n}{n-1}\right)|\nabla_1|\boldsymbol{F}(p)||^2 \leq \left(\sum_{i=1}^{n}|\nabla_i\boldsymbol{F}(p)|^2\right) + |D_A^*\boldsymbol{F}|^2 + |D_A\boldsymbol{F}|^2.$$

However, it follows from formulae (2.4) page 193, formula (2.12) page 194, and formula (2.13) page 194 of [2], that the inequality (3.11) is gauge invariant. Since p is arbitrary, inequality holds in any gauge and at any point in our local trivialization. Note that inequality (3.11) is inequality (3.2), and this completes the proof of inequality (3.2).

Now, in a similar way, we prove inequality (3.3). First, we prove (3.2). At any arbitrary point p in the fiber V, we choose a local exponential gauge, centered at p so that $A(p) = 0$. Choose an adapted orthonormal frame (inducing local adapted orthonormal coordinates), such that $d|\boldsymbol{\theta}(p)| = d_1|\boldsymbol{\theta}(p)| = \frac{\partial}{\partial_1}(|\boldsymbol{\theta(p)}|)dx^1$. Note, that choosing such an orthonormal frame, still preserves the exponential gauge centered at p, because all we are doing is rotating the base coordinates, by a constant rotation at p.

Then using the standard Kato inequality, we have:

$$(3.12) \qquad\qquad d|\boldsymbol{\theta}(p)| = d_1|\boldsymbol{\theta}(p)| \leq |\nabla_1\boldsymbol{\theta}(p)|.$$

Note that in our adapted orthonormal coordinates at the arbitrary point p in the fiber (that is the center of our exponential coordinates), we have $\nabla_1\boldsymbol{\theta}(p)$ has the coordinate representation $\sum_{k,l}\frac{\partial}{\partial x^1}(F_{k,l})$ because $A(p) = 0$. The idea of the proof is to make use of this formula for the coordinates of $\nabla_1\boldsymbol{\theta}$, in an expression resulting from replacing terms in the coordinate representation of $D_A\boldsymbol{\theta}(p)$ at p by terms in the coordinate representation of $D_A^*\boldsymbol{\theta}(p)$ at p. Then, we use that fact that p is arbitrary. First we express $D_A(\boldsymbol{\theta})$ at p in components, with respect to our adapted coordinates. We note that at p we have $D_A(\boldsymbol{\theta}) = d(\boldsymbol{\theta})$.

Consider the individual components of $D_A\boldsymbol{\theta}(p) = d\boldsymbol{\theta}(p)$. We can compute these explicitly in our local coordinates. In particular, we consider coefficient terms that are the coefficients of $dx^1 \wedge dx^l$. These terms satisfy

$$(3.13) \qquad\qquad \left(\frac{\partial}{\partial x^1}\boldsymbol{\theta}_{l\neq 1}(p)\right) = \pm\frac{\partial\boldsymbol{\theta_1}}{\partial x^{l\neq 1}}(p) \pm (d\boldsymbol{\theta})_{1,l}.$$

Here, $(d\boldsymbol{\theta})_{1,l}$ is the coefficient of $d\boldsymbol{\theta}$ corresponding to $dx^1 \wedge dx^l$, where l is the fixed l on the left hand side of equation (3.13). There are two terms on the right hand side of equation (3.13). Note that, since $n \geq 4$, we have $2 \leq n$.

We also consider the individual components of $D_A^*(p)\boldsymbol{\theta} = d^*\boldsymbol{\theta}(p)$. We have

$$(3.14) \qquad\qquad \left(\frac{\partial}{\partial x^1}\boldsymbol{\theta}_1(p)\right) = \pm\sum_{k=2}^{n}\left(\frac{\partial}{\partial x^k}\boldsymbol{\theta}_k(p)\right) + d^*(\boldsymbol{\theta}(p)).$$

Note that $d^*(\boldsymbol{\theta}(p))$ is a zero form, and so it has no indices. There are n terms on the right hand side of equation (3.14).

We have:

$$(3.15) \quad |\nabla_1\boldsymbol{\theta}(p)|^2 = |d_1\boldsymbol{\theta}(p)|^2 = \left|\left(\frac{\partial}{\partial x^1}\boldsymbol{\theta}_1(p)\right)\right|^2 + \sum_{l=2}^{n}\left|\frac{\partial}{\partial x^1}|\boldsymbol{\theta}_{l\neq 1}(p))\right|^2.$$

Replacing the first term on the right hand side of equation (3.15) using (3.14), and replacing the second term of the right hand side of equation (3.15) using (3.13), and using $\left(\sum_{i=2}^{n}a_i\right)^2 \leq (n)\sum_{i=2}^{n}a_i^2$, we obtain

$$(3.16) \quad |\nabla_1\boldsymbol{\theta}(p)|^2 \leq n\left(\sum_{i=2}^{n}|\nabla_i\boldsymbol{\theta}(p)|^2\right) + n|D_A^*\boldsymbol{\theta}(p)|^2 + n|D_A\boldsymbol{\theta}(p)|^2.$$

Adding $(n)|\nabla_1\boldsymbol{\theta}(p)|^2$ to both sides of inequality (3.16) we obtain:

$$(3.17) \quad (n+1)|\nabla_1\boldsymbol{\theta}(p)|^2 \leq (n)|\nabla\boldsymbol{\theta}(p)|^2 + n|D_A^*\boldsymbol{\theta}(p)|^2 + n|D_A\boldsymbol{\theta}(p)|^2$$

Dividing inequality (3.17) by $n+1$ on both sides we obtain:

$$(3.18) \quad |\nabla_1\boldsymbol{\theta}(p)|^2 \leq \left(\frac{n}{n+1}\right)|\nabla\boldsymbol{\theta}(p)|^2 + |D_A^*\boldsymbol{\theta}(p)|^2 + |D_A^*\boldsymbol{\theta}(p)|^2.$$

Finally, we use the standard Kato inequalities (3.12) together with inequality (3.18) to obtain

$$(3.19) \quad |\nabla|\boldsymbol{\theta}(p)||^2 \leq \left(\frac{n}{n+1}\right)\left(|\nabla\boldsymbol{\theta}(p)|^2 + |D_A^*\boldsymbol{\theta}(p)|^2 + |D_A^*\boldsymbol{\theta}(p)|^2\right)$$

at p in our exponential gauge centered at p. However, it follows from formulae (2.4) page 193, formula (2.12) page 194, and formula (2.13) page 194 of [2], that the inequality (3.19) is gauge invariant. Since p is arbitrary, inequality holds in any gauge and at any point in our local trivialization.

Since inequality (3.19) is inequality (3.2), we have proved inequality (3.2). This completes the proof of the theorem. $\square$

These estimates are used in conjunction with the standard Weitzenbock formulae. We remind the readers of these identities. First, we have

THEOREM 6.

$$(3.20) \qquad \nabla_A^*\nabla_A V = D_A^*D_A V + D_A D_A^* V + S(F_A)V.$$

Here, $\nabla_A^*\nabla_A$ is the "rough Laplacian", D_A is the exterior covariant differential, D_A^* is the exterior covariant codifferential, and $S(F_A)$ is a bundle curvature term. There is no base curvature term, as we are assuming (locally) that the base manifold is an open domain of $\mathcal{R}^n$.

PROOF. A good reference for this is section 3 of [2]. $\square$

THEOREM 7. *Let (D_A, Φ) be a smooth solution of the field equations* (1.3) *in Ω. Then*

$$(3.21a) \qquad \left(-\frac{1}{2}\right) \triangle |F_A|^2 + |\nabla_A F_A|^2 \leq |F_A|^3 + |D_A\Phi|^2 |F_A| + |\Phi|^2 |F_A|^2$$

$$(3.21b) \qquad \left(-\frac{1}{2}\right) \triangle |D_A\Phi|^2 + |\nabla_A D_A \Phi|$$

$$\leq 2|F_A||D_A\Phi|^2 + |\Phi|^2|D_A\Phi|^2 + |Q_{\Phi,\Phi}(\Phi)||D_A\Phi|^2.$$

PROOF. We only give a sketch of the important points: To prove (3.21a), we have:

$$(3.22) \qquad D_A D_A^* F_A + D_A^* D_A F_A =$$
$$\left(\frac{1}{2}\right) D_A[\Phi, D_A\Phi] = \left(\frac{1}{2}\right)[D_A\Phi, D_A\Phi]$$
$$+ \left(\frac{1}{2}\right)[\Phi, [F_A, \Phi]].$$

In equation (3.21a) D_A is the exterior covariant derivative, and D_A^* is the exterior covariant codifferential.

Recall, that the inner product $<,>$ of p-forms and q-forms with coefficients that are sections of an associated bundle is defined by taking the inner product of the section valued coefficients and producting with inner product of the form parts. Thus the Hodge star operator can be considered as acting on the form part alone. Thus we have (using that the inner product on sections is a metric compatible with the connection ∇_A)

$$(3.23) \qquad d < F_A, F_A >= 2 < F_A, \nabla_A F_A >$$

(3.24)
$$d * (< F_A, \nabla_A F_A >) = 2 < \nabla_A F_A, \nabla_A F_A > + 2 < F_A, \nabla_A^* \nabla_A F_A > .$$

Thus

$$(3.25) \quad d^* d(< F_A, F_A >) = [2 < \nabla_A F_A, \nabla_A F_A > + 2 < F_A, \nabla_A^* \nabla_A F_A >].$$

Now apply Theorem 6 to the last term of (3.25), with $V = F_A$, noting that $D_A F_A = 0$ by Bianchi's identity. Then apply (3.22) to the result. The first term of the right hand side of (3.25) accounts for the second term of the left-hand side of (3.21a).

Similarly we prove (3.21b), by using the identity

$$(3.26) \qquad \nabla_A^* \nabla_A(D_A\Phi) = (D_A D_A^* + D_A^* D_A)(D_A\Phi) + [F_A, D_A\Phi]$$

and the field equations (1.3). $\qquad\qquad\qquad\qquad\qquad\qquad\square$

COROLLARY 2. *Let $\nu = (F_A, D_A\Phi)$. Then*

$$(3.27) \quad -\triangle(|\nu|^2) + \left(1 + \frac{1}{n}\right)|d(|\nu|)|^2 \leq 3|F_A||\nu|^2 + Q_1|\nu|^2 + 4|Q_\Phi(\Phi)|^2.$$

Here $Q_1 = 2max\left(|\Phi|^2 + |Q_{\Phi,\Phi}(\Phi)|^2\right)$.

PROOF. If we add equation (3.21) and equation (3.21b), we get:

$$
(3.28) \quad -\triangle(|\nu|^2) + |\nabla_A \nu|^2 \le |F_A|^3 + 3|F_A||D_A\Phi|^2 + |\Phi|^2 \left(|F_A|^2 + |D_A\Phi|^2\right)
$$

$$
(3.29) \quad + |Q_{\Phi,\Phi}(\Phi)||D_A\Phi|^2 \le 3|F_A||\nu|^2 + \left(|\Phi|^2 + |Q_{\Phi,\Phi}(\Phi)||\nu|^2\right).
$$

From Theorem 5, we get:

$$
(3.30) \quad \left(1 + \frac{1}{n}\right)|d(|\nu|)|^2 \le |\nabla_A \nu|^2 + |D_A^* F_A|^2 + |D_A^* D_A \Phi|^2 + |D_A D_A \Phi|^2
$$

$$
(3.31) \quad \le |\nabla_A \nu|^2 + |\Phi|^2|\nu|^2 + |Q_\Phi(\Phi)|^2.
$$

Putting this inequalities together gives the result. □

4. Bounds on the solutions of an elliptic inequality: the smooth case

In this section we prove that a smooth function f which satisfies an elliptic inequality in Ω is bounded in the interior of Ω in terms of its $X^2(\Omega)$ norm. This result, which is weaker than the result of Section 5, can be used to prove that the limit of smooth solutions is smooth on the complement of a set of finite $n-4$ Hausdorff dimension. Also, it is a warmup for Section 5.

We prove this result for $\Omega_1 \subset \Omega_4$, where $\Omega_l = [-l, l]^n$. By the results on scaling and monotonicity in Section 1, Appendix A and Section 2, it can be modified for arbitrary domains.

We use the notation of Appendix A, where $X^k = M^{\frac{n}{k}, \frac{4}{k}}$ for the Morrey Space with integration power $\frac{n}{k}$ and scaling power $\frac{4}{k}$. The formulae are particularly simple in this notation.

THEOREM 8. *Let $u > 0$, and $f \ge 0$ be smooth functions in Ω_4 with $f \in X^2(\Omega_4)$.*

$$
(4.1) \quad -\triangle f + \alpha \left(\frac{|df|^2}{f}\right) - uf \le Q_1 f.
$$

Then there exist η_k and K_k, depending on $\alpha > 0$ and $0 < k \le 1$, such that if $\|u\|_{X^2(\Omega_4)} \le \eta_k$,

$$
(4.2) \quad \|f\|_{X^k(\Omega_1)} \le K_k\|f\|_{X^2(\Omega_4)}.
$$

Here η_k, and K_k depend on the norm of the inversion of $\triangle$ on $X^{k'+2}$ and the norm of $X^{k'} \subset X_2^{2+k'}$, where $k \le k' \le 1$.

PROOF. Note, that this problem is not linear in f, but it scales linearly in f. This allows us to fix the dependence in the conclusion as linear in $\|f\|_{X^2(\Omega_4)}$.

In fact, Theorem 8 is true without the condition $\alpha > 0$. However, we prove it with this condition as a warm up for the proof of Theorem 9 of Section 5.

Recall that the norm that we use for $X_\gamma^k(\Omega_l)$ denotes the cutoff of the odd extension. For $\gamma > 0$, this imposes Dirichlet boundary conditions, where the condition $\gamma > 0$ is necessary.

The first Lemma is a straightforward computation.

LEMMA 1. *Suppose $u, f > 0$ are smooth, and*

$$(4.3) \qquad -\triangle f + \alpha \left(\frac{|df|^2}{f} \right) \leq (u + Q_1)f.$$

Then

$$(4.4) \qquad \left\| \frac{|df|}{f^{1/2}} \right\|_{X^2(\Omega_3)}^2 \leq C_1 \left(\frac{1}{\alpha} \right) \|f\|_{X^2(\Omega_4)}^2 \left(1 + Q_1 + \|u\|_{X^2(\Omega_4)} \right).$$

PROOF. Let

$$(4.5) \qquad \Psi = \begin{cases} 1, & \text{for} \quad t \leq 1 \\ 0, & \text{for} \quad t \geq 2 \end{cases}$$

be a smooth cutoff function, and for arbitrary $y \in \Omega_3$, let $\Psi_r(x) = \Psi(\frac{|x-y|}{r})$, $r < 1$. Multiply equation (4.3), by $\Psi_r(x)$, integrate and move the term $\int \triangle \Psi_r(x) f(x) \, (dx)^n = \int \Psi_r(x) \triangle f(x) \, (dx)^n$ to the right hand side. This gives,

$$(4.6) \qquad \alpha \int \Psi_r(x) \frac{|df(x)|^2}{f(x)} \, (dx)^n \leq \int \left[|\triangle \Psi_r| + \Psi_r(x)(u + Q_1) \right] f \, (dx)^n$$

Now

$$(4.7) \qquad \alpha \int_{|x-y| \leq r} \left[\frac{|df(x)|^2}{f(x)} \right] (dx)^n \leq \alpha \int \Psi_r(x) \left[\frac{|df(x)|^2}{f(x)} \right] (dx)^n$$

$$(4.8) \qquad |\triangle \Phi_r(x)| \leq C_2 r^{-2}$$

$$(4.9) \qquad \int |\triangle \Psi_r(x)| f(x) \, (dx)^n)$$

$$\leq C_2 r^{-2} \left(\int_{|x-y| \leq 2r} f(x)^2 \, (dx)^n \right)^{\frac{1}{2}} \left(\int_{|x-y| \leq 2r} 1 \, (dx)^n \right)^{\frac{1}{2}}$$

$$\leq C_3 r^{n-4} \|f\|_{X^2(\Omega_4)}$$

and

$$(4.10) \qquad \int \triangle \Psi_r(u + Q_1) f \, (dx)^n \leq (2r)^{n-4} \|u + Q_1\|_{X^2(\Omega_4)} \|f\|_{X^2(\Omega_4)}.$$

Putting this all together gives:

$$(4.11) \quad \alpha \int_{|x-y| \leq r} \frac{|df|^2}{f} \leq C_3(r^{n-4} + (2r)^{n-4} \|u + Q_1\|_{X^2(\Omega_4)})(\|f\|_{X^2(\Omega_4)}).$$

We can take the constant $C_1 = C_3(1 + 2^{n-4})$ by using the triangle inequality, dividing both sides by αr^{n-4} and using the definition of the X^2 norm. $\quad \square$

The next step in the proof of theorem 8 is to bound $\|f\|_{X^k(\Omega_2)}$. Choose another smooth test function,

$$(4.12) \qquad \hat{\psi}(x) = \begin{cases} 0 & \text{for} \quad x \in \mathcal{R}^n - \Omega_3 \\ 1. & \text{for} \quad x \in \Omega_2. \end{cases}$$

According to theorem 14, in Appendix A, if u is suffcently small, we can solve

$$(4.13) \qquad -\triangle\hat{\phi} - u\hat{\phi} = Q_1\hat{\psi}f - 2d\hat{\psi}df - [\triangle(\hat{\psi})]f$$

for $\hat{\phi} \in X_2^3(\Omega_3)$, if we can get an estimate of the right hand side of equation (4.13) in $X^3(\Omega_3)$. Since, $f \in X^2(\Omega_3) \subset X^3(\Omega_3)$, the first and third terms are fine. But, $df = \left(\dfrac{df}{f^{\frac{1}{2}}}\right)(f^{\frac{1}{2}})$. According to lemma (1), $\dfrac{df}{f^{\frac{1}{2}}}$ can be estimated in $X^2(\Omega_3)$. Equation (A.18) of Appendix A shows that $\|f^{\frac{1}{2}}\|_{X^1(\Omega_3)} \le \|f\|_{X^2(\Omega_3)}^{\frac{1}{2}}$. Moreover $X^2(\Omega_3) \otimes X^1(\Omega_3) \hookrightarrow X^3(\Omega_3)$ by multiplication. Hence the right hand side can be estimated in $X^3(\Omega_3)$ by

$$(4.14) \qquad (c + max|\triangle\hat{\psi}|)\|f\|_{X^2(\Omega_3)} + C_4\|f\|_{X^3(\Omega_4)}.$$

Hence,

$$(4.15) \qquad \|\hat{\phi}\|_{X^1(\Omega_3)} \le C_5\|\hat{\phi}\|_{X_2^3(\Omega_3)} \le C_6\|f\|_{X^2(\Omega_4)}$$

where we have used the norm of the Morrey-Sobolev embedding $X_2^3(\Omega_3) \hookrightarrow X^1(\Omega_3)$. We omit the dependence on u, because the norm involved is small by assumption.

Now let $g = \hat{\psi}f - \hat{\phi}$.

$$(4.16) \qquad -\triangle g - ug \le 0.$$

According to Theorem 17 of Appendix B, if $\|u\|_{X^2(\Omega_3)}$ is suffcently small, then $g \le 0$, and $\hat{\psi}f \le \hat{\phi}$. Then inequality (4.15) immediately transfers to:

$$(4.17) \qquad \|\hat{\psi}f\|_{X^1(\Omega_3)} \le C_7\|f\|_{X^2(\Omega_4)}.$$

Now we take a second cutoff function

$$(4.18) \qquad \bar{\psi} = \begin{cases} 0 & \text{if} \quad x \in \mathcal{R}^n - \Omega_2 \\ 1 & \text{if} \quad x \in \Omega_1 \end{cases}.$$

Note that $d(\hat{\psi}\bar{\psi}) = \triangle(\hat{\psi}\bar{\psi}) = 0$, and $\hat{\psi}\bar{\psi} = \bar{\psi}$. Now

$$(4.19) \qquad -\triangle(\bar{\psi}\hat{\phi}) - u\bar{\psi}\hat{\phi} = c\bar{\psi}f - 2d\bar{\psi}d\hat{\phi} - (\triangle\bar{\psi})\hat{\phi}.$$

These identities hold on the open dense nonsingular set. We have $\bar{\psi}f \in X^1(\Omega_2)$, and $\triangle(\bar{\psi})\hat{\phi} \in X^1(\Omega_2)$, but $2d\bar{\psi}d\hat{\phi}$ is only estimated in $X_1^3(\Omega_2) \subset X^2(\Omega_2)$. We cannot invert $-\triangle - u$ on X^2, no matter how small $\|u\|_{X^2(\Omega_2)}$ is. However $X^2(\Omega_2) \subset X^{2+k}(\Omega_2)$, and for $\|u\|_{X^2(\Omega_2)} \le \eta_k$, we can invert $-\triangle - u$

on $X^{2+k}(\Omega_2)$. Now we have an estimate on $\bar{\psi}\hat{\phi} \in X_2^{2+k}(\Omega_2) \subset X^2(\Omega_2)$, which transfers to $f \in X^k(\Omega_1)$. $\qquad\square$

5. Bounds on the solution of an elliptic inequality: the singular case

This section is similar to Section 4, except that we allow singular sets in Ω_4.

THEOREM 9. *Let $u \geq 0$, and $f > 0$, be smooth functions on $\Omega_4 - \mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n - 4$ Hausdorff dimension. If $f \in X^2(\Omega_4)$, $0 < k < 2$ and*

$$(5.1) \qquad -\triangle f + \alpha \left(\frac{|df|^2}{f} \right) - uf \leq Q_1 f$$

then there exist constants $\eta_k > 0$, and $\kappa_k > 0$, such that if $\|u\|_{X^2(\Omega_4)} < \eta_k$, then $f \in X^k(\Omega_1)$. Moreover

$$(5.2) \qquad \|f\|_{X^k(\Omega_1)} \leq \kappa_k \|f\|_{X^2(\Omega_4)}.$$

PROOF. We now modify the proof of Section 4 to account for the singular Set $\mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n-4$ Hausdorff dimension. We observe that inequality (5.2), though not linear, scales linearly. So we may assume that $\|f\|_{X^2(\Omega_4)} = 1$, and get bounds on $\|f\|_{X^k(\Omega_1)}$ as a constant.

In the body of this proof, we need the following lemma:

LEMMA 2. *Suppose $u \geq 0$, $f > 0$, with $u, f \in X^2(\Omega_4)$ smooth off of a closed set $\mathcal{S}$ of finite $n - 4$ Hausdorff dimension. In addition $\alpha > 0$, and $x \in \Omega_4 - \mathcal{S}$, we have,*

$$(5.3) \qquad -\triangle f + \alpha \left(\frac{|df|^2}{f} \right) \leq (u + Q_1)f,$$

then $\frac{|df|}{f^{\frac{1}{2}}} \in X^2(\Omega_3)$, and

$$(5.4) \qquad \left\| \frac{|df|}{f^{\frac{1}{2}}} \right\|_{X^2(\Omega_3)}^2 \leq C_\alpha \|f\|_{X^2(\Omega_4)} \left(1 + \|u\|_{X_{\Omega_4}^2} \right).$$

We prove Lemma 2.

PROOF. Suppose that the test function $\mu \in C_0^\infty(\Omega_4)$. We will show that

$$(5.5) \quad \int (-\triangle \mu) f\, (dx)^n + \alpha \int \mu \left(\frac{|df|^2}{f} \right) (dx)^n \leq \int (u + Q_1)\mu f\, (dx)^n.$$

Given equation (5.5), the proof is exactly the same as that of Lemma 1, if we set $\Psi_R = \mu$. Let $S \colon \Omega_4 \to \mathcal{R}^+$ be a regularized distance function (11)

to the set $\mathcal{S}$. See Definition 11 of Appendix B. Let $\Psi\colon [0,\infty) \to [0,\infty)$ be a C^∞ function such that Ψ has bounded derivative and

$$(5.6) \qquad \Psi(t) = \begin{cases} 1, & \text{if} \quad t > 2 \\ 0, & \text{if} \quad t \le 1. \end{cases}$$

We define:

$$(5.7) \qquad \beta_\epsilon = 1 - \Psi\left(\frac{s(x)}{\epsilon}\right) = \begin{cases} 1, & \text{if} \quad s(x) > 2\epsilon \\ 0, & \text{if} \quad s(x) \le \epsilon. \end{cases}$$

Note by the usual computation, and by the definition of the regularized distance function, we have:

$$(5.8) \qquad |d\beta_\epsilon| \le K\epsilon^{-1}$$

$$(5.9) \qquad |\triangle\beta_\epsilon| \le K\epsilon^{-2}.$$

Here the size of K is inconsequential for the proof except that it is independent of ϵ. Then, equation (5.5) is true if we replace μ by $\beta_\epsilon\mu \in C_0^\infty(\Omega_4 - \mathcal{S})$. Now we compute:

$$(5.10) \qquad \triangle(\beta_\epsilon\mu) = \triangle(\mu)\beta_\epsilon + 2\,grad(\mu) \bullet grad(\beta_\epsilon) + \mu\triangle(\beta_\epsilon)$$

and

$$(5.11) \qquad |2\,grad(\mu) \bullet grad(\beta_\epsilon) + \mu\triangle(\beta_\epsilon)| \le K(\mu)\epsilon^{-2}$$

where $K(\mu)$ does depend on μ. This then implies:

$$(5.12) \qquad \int_{\Omega_4} \beta_\epsilon\left((\triangle\mu)f + \mu\frac{|df|^2}{f^{\frac{1}{2}}}\right)(dx)^n \le$$

$$\int_{\Omega_4} \beta_\epsilon(u + Q_1)f\,(dx)^n + K(\mu)\epsilon^{-2}\int_{|s(x)|\le 2\epsilon} f\,(dx)^n.$$

But,

$$(5.13) \qquad \int_{|s(x)|\le 2\epsilon} f\,(dx)^n \le \left(\int_{|s(x)|\le 2\epsilon} |f|^2\,(dx)^n\right)^{\frac{1}{2}}\left(\int_{|s(x)|\le 2\epsilon} 1^2\,(dx)^n\right)^{\frac{1}{2}}$$

$$\le \left(K(\mathcal{S})\epsilon^4\right)^{\frac{1}{2}}\left(\int_{|s(x)|\le 2\epsilon} |f|^2\,(dx)^n\right)^{\frac{1}{2}}.$$

Here the volume of $\{X \mid s(x) < 2\epsilon\}$ has been estimated in Lemma 4 of Appendix B. However, since $\lim_{\epsilon\downarrow 0}\left(\int_{|s(x)|\le 2\epsilon} |f|^2\,(dx)^n\right) = 0$, the ϵ^2 and the ϵ^{-2} cancel, and we have our result. Note that the constants that depend on $\mathcal{S}$ and μ disappear in the limit, as they are multiplied by a term that goes to zero. This finishes the proof of the lemma. $\qquad\square$

In the remaining part of the proof of Theorem 9, we again use $\alpha > 0$, and assume without loss of generality that $\alpha < \frac{1}{2}$. Multiply equation (5.1) by $f^{-\alpha}$, and use the fact that in $\Omega_3 - \mathcal{S}$, we have:

$$(5.14) \quad f^{-\alpha}\left(-\triangle f + \alpha\left(\frac{|df|^2}{f}\right)\right) = d^*(f^{-(\alpha)}df) = -\left(\frac{1}{1-\alpha}\right)(\triangle(f^{1-\alpha})).$$

Define $\overline{f} = f^{1-\alpha}$, and note that:

$$(5.15) \qquad\qquad \|\overline{f}\|_{X^{2(1-\alpha)}(\Omega)} \sim \|f\|_{X^2(\Omega)}$$

from equation (A.18) of Appendix A. Now, on $\Omega_3 - \mathcal{S}$, we have:

$$(5.16) \qquad\qquad -\triangle\overline{f} - u\overline{f} \le Q_1\overline{f},$$

for $\overline{f} \in X^{2(1-\alpha)}(\Omega_3)$. Now we proceed with the proof of Theorem 9.

In the proof of Theorem 8, we also needed an estimate on:

$$(5.17) \qquad d\overline{f} = (1-\alpha)\left(\frac{df}{f^{\frac{1}{2}}}\right)\left(f^{-\alpha+\frac{1}{2}}\right) \in X^{3-2\alpha}(\Omega_3).$$

This follows from the multiplication law and from $\frac{df}{f^{\frac{1}{2}}} \in X^2(\Omega_3)$, as well as $f^{\frac{1}{2}-\alpha} \in X^{2(\frac{1}{2}-\alpha)}(\Omega_3)$. Here assuming that $\|f\|_{X^2(\Omega_4)} = 1$ is invaluable, as we need not carry these powers around. In carrying out the proof, we obtain that

$$(5.18) \qquad\qquad -\triangle\overline{\phi} - u\overline{\phi} = Q_1\hat{\Psi}\overline{f} - 2d\hat{\Psi}d\overline{f} - \hat{\Psi}\triangle(\overline{f})$$

can be solved for $\overline{\phi} \in X_2^{3-2\alpha}(\Omega_3)$. We have that

$$(5.19) \qquad\qquad -\triangle(\overline{g}) - u\overline{g} \le 0,$$

for $\overline{g} = \hat{\Psi}f - \overline{\phi}$. Here $\overline{g} \in X^{2(1-\alpha)}(\Omega_3)$, so the hypotheses of Theorem 18 of Appendix B are satisfied with $\overline{g}^{1+\gamma} = \overline{g}^{\frac{1}{1-\alpha}} \in X^2(\Omega_3) \subset L^2(\Omega_3)$. It follows that $\overline{g} \le 0$, and that $\hat{\Psi}\hat{f} \in X^{1-2\alpha}(\Omega_3)$. In the next step, transferring estimates similarly to the proof of Theorem 8, we get $\overline{\Psi\phi} \in X^{\bar{k}}(\Omega_2)$ for arbitrary $\bar{k}$ with $\|u\|_{X^2(\Omega_2)} \le \eta_{\hat{k}}$. But $\overline{f} \in X^{\bar{k}}(\Omega_1)$ is equivalent to $f \in X^{\bar{k}(1-\alpha)}(\Omega_1)$. This completes the proof of Theorem 9. $\qquad\square$

The following is a Corollary of the above proof of Theorem 9:

PROPOSITION 1. *If $u \le \lambda f$, and the hypothesis of Theorem 9 of Appendix B are satisfied, then we have a bound on $f(x)$ for $x \in \Omega_1$.*

PROOF. We follow the proof of Theorem 9, until, with $\overline{f} \le \overline{\phi} \in \Omega_2$, we have $\overline{\phi} \in X_2^{3-2}(\Omega_3)$. We have:

$$(5.20) \quad -\triangle((\overline{\Psi})(\overline{\phi})) = u((\overline{\Psi})(\overline{\phi})) + Q_1(\overline{\Psi})(\overline{f}) - 2(d\overline{\phi})(d\overline{\Psi}) - (\triangle(\overline{\Psi}))(\overline{\phi}).$$

Now, all the terms on the right are in $X^{2-2\alpha}(\Omega_2)$, except for the term $(u\overline{\Psi})(\overline{\phi})$. Here $\overline{\phi} \in X^{1-2\alpha}(\Omega_2)$ and $u \le \lambda f \le \lambda(\overline{f})^{\frac{1}{1-\alpha}} \in X^{\frac{1-2\alpha}{1-\alpha}}(\Omega_2) \subset X^1(\Omega_2)$. By multiplication, we have $u(\overline{\Psi})(\overline{\phi}) \in X^{2-2\alpha}(\Omega_2)$. Hence $(\overline{\Psi})(\overline{\phi}) \in$

$X_2^{2-2\alpha}(\Omega_2) \subset L^\infty(\Omega_2)$, and $\overline{f}(x) \leq \overline{\phi}(x) \leq K$ in Ω_1. Keeping track of the explicit dependence of powers of $\|f\|_{X^2}(\Omega_4)$ in the final estimate is not straightforward. $\qquad\square$

6. Application to Yang-Mills-Higgs

The simpler results of Section 4 apply directly to getting estimates on smooth solutions, and we do not go into that here.

First, we directly apply the results of Section 5 to solutions of a Yang-Mills-Higgs system in a cube $\Omega_4 = [-4, 4]^n$. An immediate corollary, using Appendix C, shows when solutions (with Hausdorff codimension 4 singular set), extend to smooth solutions in the interior of $[-1, 1]^n$. Later, we show how this applies to the Yang-Mills-Higgs equations in arbitrary domains, and discuss how to verify the hypotheses.

THEOREM 10. *Let (D_A, Φ) be solutions to a Yang-Mills-Higgs system in $\Omega_4 - \mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n-4$ dimensional Hausdorff measure. Let $v = (F_A, D_A\Phi)$. Assume that $v \in X^2(\Omega_4)$, and that $\Phi \in L^\infty(\Omega_4)$.*

$$(6.1) \qquad Q_1 = \sup_{[-4,4]^n} (2|\Phi| + |Q_{\Phi,\Phi}(\Phi)|)$$

$$(6.2) \qquad Q_2^2 = \sup_{[-4,4]^n} \left(\frac{|Q_\Phi(\Phi)|^2}{Q_1} \right).$$

If $F_A \in X^2(\Omega_4)$ is sufficiently small, then $|v| \in L^\infty(\Omega_1)$. We also have the explicit bound

$$(6.3) \qquad \|v\|_{L^{2n}(\Omega_1)} \leq C(Q_1)(\|v\|_{X^2(\Omega_4)} + Q_2^2).$$

PROOF. Using the improved Kato formula, and the Weitzenbock formulae, we have, from Corollary (2) of Theorem 7 see (3.27).

$$(6.4) \qquad -\triangle(|v|^2) + (1 + \frac{1}{n})|d|v||^2 \leq 3|F_A||v|^2 + Q_1(|v|^2 + Q_2^2).$$

Let $u = 3|F_A|$, and $f^2 = |F_A|^2 + Q_2^2$. Then

$$(6.5) \qquad -(\frac{1}{2})\triangle(f^2) + (1 + \frac{1}{n})(|df|^2 \leq uf^2 + Q_1 f^2.$$

Divide by f and use the fact that

$$(6.6) \qquad f^{-1}d^*(fdf) + \frac{|df|^2}{f} = \triangle f$$

to get

$$(6.7) \qquad -\triangle f + \left(\frac{1}{n}\right)\left(\frac{|df|^2}{f}\right) - uf \leq Q_1 f.$$

If $u = 3|F_A| \in X^2(\Omega_4)$ is sufficiently small, we can apply Proposition 1 to get a bound on $f \in L^\infty(\Omega_1)$.

To obtain the explicit bound

$$(6.8) \qquad \|f\|_{L^{2n}(\Omega_1)} \leq C(Q_1)\|f\|_{X^2(\Omega_4)}$$

we apply Theorem 9 with $\frac{4}{k} = 2n$. This gives:

$$(6.9) \qquad \|v\|_{L^{2n}(\Omega_1)} \leq \|v\|_{X^k(\Omega_1)} \leq C(Q_1)\left(\|v\|_{X^2(\Omega_4)} + Q_2\right). \qquad \square$$

Now, we have a regularity result in small balls of Ω_1. This regularity theorem is proved by combining Theorem 10 and Corollary 5 from Appendix C.

COROLLARY 3. *If $(D_{\tilde{A}}, \Phi)$ satisfy the hypothesis of Theorem 10, with $\|F_{\tilde{A}}\|_{X^2(\Omega_4)}$ sufficently small, then for each point $y \in \Omega_1$ there is a neighborhood $B_y(\delta) \subset \Omega_1$, with δ chosen such that (6.14) below holds, such that, in $B_y(\delta) - \mathcal{S}$, $D_{\tilde{A}}$ is smoothly gauge equivalent to an exterior covariant derivative $d + A$ (corresponding to a connection ∇_A). If $v = (F_A, D_A\Phi)$, we have:*

$$(6.10) \qquad \|v\|_{L^{2n}(\Omega_1)} \leq C(Q_1)\left(\|v\|_{X^2(\Omega_4)} + Q_2\right)$$

$$(6.11) \qquad d^*A = 0$$

$$(6.12) \qquad \delta^{-1}\|A\|_{L^{2n}(B_y(\delta))} \leq C\|F_A\|_{L^{2n}(B_y(\delta))} \leq C\|F_A\|_{L^{2n}(\Omega_1)}$$

In the new gauge

$$(6.13) \qquad A \text{ and } \Phi \text{ are smooth on } B_y(\delta).$$

PROOF. Condition 6.10 is the conclusion of Theorem 10. In Theorem 19 Appendix C the local trivialization in which the Coulomb condition of 6.11 holds is constructed. Let $\epsilon > 0$. The size of the ball $B_y(\delta)$ is fixed so that

$$(6.14) \qquad \|F_{\tilde{A}}\|_{L^{2n}(B_y(\delta))} \leq \|F_{\tilde{A}}\|_{L^{2n}(\Omega_1)} \leq \delta^{-3}\epsilon.$$

If we rescale to a unit ball we have the L^{2n} norm of F bounded by ϵ and we can apply theorem 19 of Appendix C. Then, equation 6.11 is valid. The inequality 6.12 is the rescaled version of the estimate in Corollary 5 Appendix C.

Now collect the information that we have from Appendix C, and the Euler-Lagrange equations to get the following identities and equations:

$$(6.15) \qquad d^*A = 0$$

$$(6.16) \qquad dA + \left(\frac{1}{2}\right)[A, A] = F_A$$

$$(6.17) \qquad d^*F_A + [A, F_A] = [\Phi, [d + A, \Phi]]$$

$$(6.18) \qquad D\Phi = [d + A, \Phi]$$

$$(6.19) \qquad D^*(d\Phi) + [A, d\Phi] = Q_\Phi(\Phi).$$

Rearrange the above equations so that:

$$(6.20) \qquad \triangle A = (d^*d + dd^*)A = L_1(A, dA, \Phi, d\Phi)$$

and

$$(6.21) \qquad \triangle \Phi = d^*d\Phi = L_2(A, dA, \Phi, d\Phi).$$

Here the powers of $\{A, dA, \Phi, d\Phi\}$ in the expressions L_1 and L_2 are under control and Q is a smooth function of Φ, which – to begin with– is in L^∞. Standard bootstrap arguments then yield smoothness in the interior. $\qquad \square$

To see how Theorem 10 applies in a general domain, assume $(D_{\tilde{A}}, \Phi)$ satisfies a Yang-Mills-Higgs system in $\Omega - \mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n-4$ dimensional Hausdorff measure. For a fixed $y \in \mathcal{R}^n$, let $\Omega_{y,\delta} = \{x \colon x - y \in [-\delta, +\delta]^n\ \}$. Then, Theorem 10 translates into:

COROLLARY 4. *Assume $(D_{\tilde{A}}, \Phi)$ satisfies a Yang-Mills-Higgs system in $\Omega - \mathcal{S}$, where $\mathcal{S}$ is a closed set of finite $n-4$ dimensional Hausdorff measure. Let $\upsilon = (F_A, D_A\Phi)$.*

$$(6.22) \qquad Q_1 = \sup_\Omega \left(|Q(\Phi)|^2 + \ |Q_{\Phi,\Phi}(\Phi)| \right)$$

$$(6.23) \qquad Q_2^2 = \left(\sup_\Omega |Q_\Phi(\Phi)|^2 \right) Q_1$$

Suppose $\upsilon \in X^2(\Omega)$, and $Q_1\delta^2$ as well as $Q_2\delta^2$ (scales like the two form υ) are bounded by a fixed constant. In addition suppose that $F_{\tilde{A}} \in X^2(\Omega)$ has small enough $X^2(\Omega)$ norm (independent of the other constants). If $\Omega_{y,\delta} \subset \Omega$, then $\delta^2\upsilon \in L^\infty(\Omega_{y,\delta})$ is bounded above by a constant, and $(D_{\tilde{A}}, \Phi)$ and $(D_{\tilde{A}}, \Phi)$ are smoothly gauge equivalent on $\Omega_{y,\delta} - \mathcal{S}$ to a smooth exterior covariant differential (corresponding to a smooth connection), and a smooth Higgs Field on $\Omega_{y,\delta}$, both of which extend smoothly across the singular set $\mathcal{S} \cap \Omega_{y,\delta}$.

PROOF. In rescaling to $\tilde{x} = \frac{x-y}{\delta}$, the constants rescale as already described. Note that X^2, and the (X^2 norm) are invariant, when applied to a geometric quantity that scales like a two form. Some examples of such a quantity are $F_{\tilde{A}}$ and $D_{\tilde{A}}\Phi$. Thus Corollary 4 is a restatement of Theorem 10 for cubes of arbitrary size. $\qquad \square$

We indicated in Section 1 how a bound on the maximum of norm Φ can be obtained. This leads to bounds on the terms Q_1 and Q_2.

We emphasize that there are important examples where $\Phi \in L^2(\Omega)$ cannot be bounded. Our theory only applies when this bound is available.

The same can be said for the bound on $\upsilon = (F_A, D_A\Phi) \in X^2(\Omega)$. Bounds on $\upsilon = (F_A, D_A\Phi) \in X^2(\Omega)$ are available when it is a stationary solution of a Yang-Mills-Higgs system.

In many cases, limits of smooth solutions approach smooth solutions (in an appropriate topology) in $\Omega - \mathcal{S}$ (where $\mathcal{S}$ is a closed set of finite $n-4$

dimensional Hausdorff measure), but it is not clear that these limits are stable with respect to perturbation by smooth diffeomorphisms unless they fix the singular set $\mathcal{S}$.

We only have the following:

LEMMA 3. *Suppose that (D_A, Φ) is a smooth solution of a Yang-Mills-Higgs system, with $Q > 0$ on $\Omega - \mathcal{S}$, where $\mathcal{S}$ is a closed set of zero n dimensional Lebesque measure. Suppose, in addition that (D_A, Φ) is a stationary critical point of the Yang-Mills-Higgs Functional $\mathcal{A}(D_A, \Phi)$, with respect to all smooth diffeomorphisms that fix the boundary of Ω. Let $\Omega_{y,\delta} = \{x \mid |x - y| < \delta\}$, assume that $dist(\Omega_{y,\delta}, \mathcal{R}^n - \Omega) \geq R \geq \delta$ as well as*

$$(6.24) \qquad \int_\Omega |F_A|^2 + |D_A \Phi|^2 + Q(\Phi)\,(dx)^n \leq S^2.$$

Then $(F_A, D_A\Phi) \in X^2(\Omega_{y,\delta})$, and

$$(6.25) \qquad \|(F_A, D_A\Phi)\|_{X^2(\Omega_{y,\delta})} \leq R^{\frac{-(n-4)}{2}} S.$$

PROOF. By the Monotonicity Theorem 3 for $x \in \Omega_{y,\delta}$ we have:

$$(6.26) \qquad \rho^{-(n-4)} \int_{B_x(\rho)} |F_A|^2 + |D_A\Phi|^2 + Q(\Phi)\,(dx)^n$$

$$(6.27) \qquad \leq R^{-(n-4)} \int_{B_x(R)} |F_A|^2 + |D_A\Phi|^2 + Q(\Phi)\,(dx)^n \leq R^{-(n-4)}S^2.$$

If we use the norm $\| \bullet \|''_{X^2(\Omega_\delta)}$ defined in equation (A.14) of Appendix A, we obtain the required estimate. $\qquad \square$

Our final regularity theorem is a corollary of Lemma 3, and of Corollary 4.

THEOREM 11. *Assume the hypothesis of Lemma 3 (with A replaced by $\tilde{A}$ and the hypothesis of Corollary 4 except for the small $X^2(\Omega)$ norm condition on $F_{\tilde{A}}$ in Corollary 4. Suppose $\Omega_0 \in \Omega$ with $dist(\Omega_0, \mathcal{R}^n) = R$ where $R - S < \epsilon$. If ϵ is positive and sufficently small and if S^2 in the hypothesis of Lemma 3 is sufficently small, then $(D_{\tilde{A}}, \Phi)$ are smoothly gauge equivalent to a smooth exterior differential (corresponding to a smooth connection) and a smooth Higgs Field on $\Omega_0 - \mathcal{S}$, and both extend smoothly across the Singular Set $\mathcal{S} \cap \Omega_0$.*

PROOF. Using Lemma 3, we obtain, when S^2 is sufficently small that we can apply Theorem 10 in its equivalent form of Corollary 4. $\qquad \square$

Appendix A. Morrey Spaces

The function spaces which arise naturally from the monotonicity formulae are Morrey Spaces. We outline a few key properties of these spaces. We follow the discussion and notation of [**11**], which is useful for geometers.

DEFINITION 1. Let $p \geq q > 1$. The Morrey Space $M^{p,q}$ is the space of measurable functions on $\mathcal{R}^n$, with finite $M^{p,q}$ norm, where the Morrey norm $M^{p,q}$ is defined by

$$(A.1) \qquad \|f\|_{M^{p,q}} = max_{\substack{y \in \mathcal{R}^n \\ r>0}} (r^{n(\frac{1}{p}-\frac{1}{q})}) \left(\int_{|x-y| \leq r} |f|^q (dx)^n \right)^{\frac{1}{q}}$$

Here p is the scaling power, q is the power of integration, and $L^p \subset M^{p,q}$. These are the same spaces as defined by Adams [1], and there denoted by $L^{q,\lambda}$. In our notation, we have $L^{q,\lambda} = M^{p,q}$, where $\lambda = \frac{nq}{p}$. We use the spaces $M^{\frac{n}{k}, \frac{4}{k}} = L^{\frac{4}{k}, 4} = X^k$.

Similarly, If Ω is an open domain of $\mathcal{R}^n$, we define a Morrey Space of functions on Ω.

DEFINITION 2. Let Ω be an open domain of $\mathcal{R}^n$. Let $p \geq q > 1$. The Morrey Space $M^{p,q}(\Omega)$ is the space of measurable functions on $\mathcal{R}^n$, with finite $M^{p,q}(\Omega)$ norm, where the Morrey norm $M^{p,q}(\Omega)$ is defined by

$$(A.2) \qquad \|f\|_{M^{p,q}(\Omega)} = max_{\substack{x_0 \in \Omega \\ r>0}} (r^{n(\frac{1}{p}-\frac{1}{q})}) \left(\int_{B_{x_0}(r) \subset \Omega} |f|^q (dx)^n \right)^{\frac{1}{q}}$$

Here p is the scaling power, q is the power of integration, and $L^p(\Omega) \subset M^{p,q}(\Omega)$. These are the same spaces as defined by Adams [1], (localized to Ω) and there denoted by $L^{q,\lambda}(\Omega)$. In our notation, we have $L^{q,\lambda}(\Omega) = M^{p,q}\Omega$, where $\lambda = \frac{nq}{p}$. We use the spaces $M^{\frac{n}{k}, \frac{4}{k}}(\Omega) = L^{\frac{4}{k}, 4}(\Omega) = X^k(\Omega)$.

The Morrey–Sobolev spaces are spaces of functions $M_\alpha^{p,q}$, with α derivatives in $M^{p,q}$, and mutis mutandis the spaces $M_\alpha^{p,q}(\Omega)$ when Ω is an open domain of $\mathcal{R}^n$. For ease of notation, we drop the Ω in this case. The results and proofs of the next two Theorems 12 and 13 also hold in this case.

Our two basic facts are:

THEOREM 12. *The map $(f,g) \rightarrow fg$, (where $f \in M^{p,q}$, and $g \in M_{q'}^{p',q'}$) has the property that*

$$(A.3) \quad M^{p,q} \bigotimes M^{p',q'} \rightarrow M^{p'',q''} \quad for \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{p''} \quad and \quad \frac{1}{q} + \frac{1}{q'} = \frac{1}{q''}$$

This specializes to $X^k \bigotimes X^{k'} \rightarrow X^{k+k'}$.

PROOF. The proof is a simple application of Holder's inequality. $\square$

THEOREM 13.

$$(A.4)$$

$$M_\alpha^{p,q} \subset M^{p',q'} \quad for \ n > \alpha p, \ where \ q \geq 1 \quad and \quad \frac{1}{p'} = \frac{1}{p} - \frac{\alpha}{n} \quad and \quad \frac{p}{p'} = \frac{q}{q'}$$

$$(A.5) \qquad \qquad X_\alpha^k \subset X^{k-\alpha} \quad for \quad \alpha < k$$

and

$$(A.6) \qquad \qquad M_\alpha^{p,q} \subset C^\gamma \quad for \quad \alpha p > k$$

PROOF. This is on page 43 of Adams [**1**].   □

So far, our function spaces are defined on $\mathcal{R}^n$ or on Ω, with Ω an open domain of $\mathcal{R}^n$. We now define Morrey Spaces whose elements are defined on Closed cubes. Because we have choice of such a closed cubical domain, it suffices to fix a closed cubical domain Ω_l in $\mathcal{R}^n$. Following Tao–Tian [**10**], we fix $\Omega_l = [-l, l]^n$. We use two extensions for f defined on Ω_l.

In order to invert the Laplacian in Morrey Spaces, we extend Ω_l to a tessellation (parquet) of $\mathcal{R}^n$, and use the method of "reflection in a parquet".

We use the odd extension to impose Dirichlet boundary conditions on a closed cube and the even extension to impose Neumann conditions. We are mostly working with the Dirichlet problem.

DEFINITION 3 (Odd Extension).

$$(A.7) \qquad\qquad \hat{f}(x) = f(x) \quad , \text{ for } \quad x \in \Omega_l$$

$$(A.8) \qquad \hat{f}(lke_j + x) = -f(lke_j - x) \quad , \text{ for } \quad x \in \Omega_l \quad , k \text{ odd.}$$

and extend f to all other closed cubes in the parquet by odd reflection in the faces of the closed cubes where is already defined.

DEFINITION 4 (Even Extension).

$$(A.9) \qquad\qquad \bar{f}(x) = u(x) \quad , \text{ for } \quad x \in \Omega_l$$

$$(A.10) \qquad \bar{f}(lke_j + x) = \bar{f}(lke_j - x) \quad , \text{ for } \quad x \in \Omega_l \quad , k \text{ odd.}$$

and extend f to all other cubes in the parquet by even reflection in the faces of the closed cubes where is already defined.

We also fix a smooth cutoff function Ψ, with support in $[-2, 2]$, that is one on $[-1, 1]$. Let $\Psi_l(x) = \Phi(\frac{x}{l})$.

DEFINITION 5 (Morrey Norm Extension).

$$(A.11) \qquad\qquad \|f\|_{M^{p,q}(\Omega_l)} = \|\Psi_l \hat{f}\|_{M^{p,q}} = \|\Psi \bar{f}\|_{M^{p,q}}.$$

DEFINITION 6 (Morrey–Sobolev Norm Extension).

$$(A.12) \qquad\qquad \|f\|_{M^{p,q}_\alpha(\Omega_l)} == \|\Psi_l \overline{f}\|_{M^{p,q}_l}.$$

We note that Definition 6 is only useful for Dirichlet boundary conditions. We will use it in Theorem 14 to invert elliptic operators on Morrey Spaces with a closed cubical domain

The even extension is useful for Neumann boundary value problems, but we only need the odd extension for $u \in X^2(\Omega_l)$.

We only use estimates on $\Omega_l = [-l, +l]^n$, for $1 \leq l \leq 4$, and by dilation arguments, the constants for $l = 1$ differ from the constants for $1 \leq l \leq 4$ by fixed constants in the scale.

Some remarks about equivalent norms are in order: We recall Definition 5. We may also use:

DEFINITION 7.

$$(A.13) \qquad \|f\|'_{M^{p,q}(\Omega_l)} = \max_{\substack{r \leq l \\ x \in \Omega_l}} (r)^{\frac{n}{p} - \frac{n}{q}} \left(\int_{|x-y| \leq r} |\hat{f}(y)|^q \, (dy)^n \right)^{\frac{1}{q}}$$

and

DEFINITION 8.

$$(A.14) \qquad \|f\|''_{M^{p,q}(\Omega_l)} = \max_{\substack{r \leq 1 \\ x,y \in \Omega_l}} (r)^{\frac{n}{p} - \frac{n}{q}} \left(\int_{\substack{x \in \Omega_l \\ |x-y| \leq r}} |f(y)|^q \, (dy)^n \right)^{\frac{1}{q}}.$$

We now use the norm in Definition 8, to define Morrey Spaces on the closed cubical domains Ω_l.

DEFINITION 9. The space of functions f defined on Ω_l such that $\|f\|''_{M^{p,q}(\Omega_l)}$ is finite is the Morrey Space $M^{p,q}(\Omega_l)$.

Note that by equivalence of the norms of Definition 7, Definition 8 and Definition 5 we could equivalently have required finiteness of these norms in Definition 9.

Similarly, we use the norm in Definition 8 to define Morrey-Sobolev Spaces on closed cubical domains Ω_l.

We remind the reader again that these definitions impose Dirichlet boundary conditions on the function f. This does, of course, not influence the definition of Morrey spaces, but it influences the definition when $\alpha > 0$ in the Morrey-Sobolev spaces.

DEFINITION 10. The spaces $M^{p,q}_\alpha(\Omega_l)$ are the set of functions on Ω_l with α derivatives in $M^{p,q}(\Omega_l)$.

There are combinatorial constants involved in the comparison estimates that show the above equivalences of norms. But, each of the following are easily proved to be valid in any of these norms:

Restriction:

$$(A.15) \qquad M^{p,q}(\Omega_l) \hookrightarrow M^{p,q}(\Omega_{l'}) \quad , \text{where} \quad l' \leq l$$

Subspaces:

$$(A.16) \qquad M^{p,q}(\Omega_l) \hookrightarrow M^{p',q'}(\Omega_l) \quad , \text{where} \quad p' \leq p, \quad q' \leq q$$

Multiplication:

(A.17)
$$M^{p',q'}(\Omega_l) \otimes M^{p'',q''}(\Omega_l) \hookrightarrow M^{p,q}(\Omega_{l'}), \quad \text{where} \quad \frac{1}{p'} + \frac{1}{p''} \leq \frac{1}{p}, \quad \frac{1}{q'} + \frac{1}{q''} \leq \frac{1}{q}.$$

Power Law (for ' and " norms):

$$(A.18) \qquad \|f^\alpha\|_{M^{p,q}(\Omega_l)} \leq \left(\|f\|_{M^{\alpha p, \alpha q}(\Omega_l)} \right)^\alpha \quad , \text{where} \quad \alpha > 0$$

Invertibility of $\triangle$ and $\triangle - cu$ on Morrey Spaces

First, we have an invertibility result for the Laplace operator on Morrey Spaces defined on Ω, with Ω a closed cubical domain.

THEOREM 14. *Let Ω be a closed cubical domain. Then, $\triangle \colon M_2^{p,q}(\Omega) \to M^{p,q}(\Omega)$ is invertible.*

PROOF. We prove this for $\Omega = [-1, 1]$. The proof for arbitrary Ω_l is obtained by scaling. First, we solve $\triangle f = g$ in Ω, with Dirichlet boundary conditions. Then, $\triangle \hat{f} = \hat{g}$ in $\mathcal{R}^n$. Choose

$$(A.19) \qquad \psi_3(x) = \begin{cases} 0 & \text{for,} \quad x \notin \Omega_4 \\ 1 & \text{for,} \quad x \in \Omega_3 \end{cases}.$$

Then

$$(A.20) \qquad \triangle(\psi_3 \hat{f}) = \psi_3 \hat{g} + (\triangle(\psi_3))\hat{f} + 2(d\psi_3) \bullet (d\hat{f}).$$

Now let

$$(A.21) \qquad \psi_3 \hat{f} = f_1 + f_2$$

where

$$(A.22) \qquad \triangle f_1 = \psi_3 \hat{g} \in M^{p,q}$$

$$(A.23) \qquad \triangle f_2 = 2d\psi_3 \bullet d\hat{f} + (\triangle\psi_3)\hat{f} \in L_1^q.$$

Here $f_1 \in M_2^{p,q}$ by the invertibility of $\triangle$ on $M^{p,q}$, (see theorem 8.1 of Adams [1]). We have f_2 in an appropriate Sobolev space on $\mathcal{R}^n$. But $\triangle f_2 = 0$ in Ω_3. Thus by elliptic regularity $f_2 \mid \Omega_2 \in C^\infty(\Omega_2)$. Hence,

$$(A.24) \qquad \|f\|_{M_2^{p,q}(\Omega)} = \|\psi_3 \hat{f}\|_{M_2^{p,q}} \leq \|\psi_3 f_1\|_{M_2^{p,q}} + \|\psi_3 f_2\|_{M_2^{p,q}}$$

$$(A.25) \qquad \leq C_1 \|f_1\|_{M_2^{p,q}} + C_2 \|f_2 \mid \Omega_2\|_{C^\infty(\Omega_2)}. \qquad \square$$

We are next interested in the properties of $\triangle - u \colon M_2^{p,q}(\Omega) \to M^{p,q}(\Omega)$, with Ω a closed cubical domain. Again, to define the operator, we note that $\triangle - \bar{u}$, where $\bar{u}$ is the even extension of u to $\mathcal{R}^n$, has the desired properties. Consider

$$(A.26) \qquad (\triangle - \bar{u})\hat{f} = \hat{g}.$$

We need only to estimate the norm of $\psi\bar{u}\hat{f}$ in M_q^p in terms of $\psi\hat{f}$ in $M_{q,2}^p$.

THEOREM 15. *Let $2 < k < 4$. Let Ω be a closed cubical domain. If $u \in M^{\frac{n}{2},2}(\Omega) = X^2(\Omega)$ is sufficently small then*

$$(A.27) \qquad \triangle - u \colon X_2^k(\Omega) \to X^k(\Omega)$$

is invertible.

PROOF. This follows from $X_2^k \hookrightarrow X^{k-2}$, and $X^{k-2} \bigotimes X^2 \hookrightarrow X^k$. See Theorem 12 which holds for Morrey Spaces on closed cubical domains by inheritance from Theorem 12 in $\mathcal{R}^n$

Note that our spaces assume Dirichlet boundary data. $\qquad \square$

Appendix B. Eigenvalues and the maximum principle

The goal of this appendix is to prove a maximum principle for $-\triangle - u$, when $u \in X^2(\Omega_l)$ is small on $\Omega_l = [-l, l]^n$, $1 \leq l \leq 4$. Since the constants change by fixed amounts, without loss of generality, we can assume $\Omega = [-1, 1]^n$.

THEOREM 16. *There exists a constant λ, (depending on the norm of $\triangle^{-1}$ on $X^2(\Omega)$ and on the constants in the Morrey-Sobolev embedding $X_2^3(\Omega) \subseteq X^1(\Omega)$ such that*

$$(B.1) \qquad \lambda \int_\Omega u\phi^2 \, (dx)^n \leq \|u\|_{X^2(\Omega)} \int_\Omega |d|\phi||^2 \, (dx)^n$$

for $\phi \in L_{1,0}^2(\Omega)$, where $\phi \in L_{1,0}^2(\Omega)$ denotes the set of functions in $\phi \in L_1^2(\Omega)$ with vanishing Dirichlet boundary value on $\partial\Omega$.

PROOF. It is sufficient to prove this for ϕ smooth, since since smooth functions are dense in $L_{1,0}^2(\Omega)$. Fix such a ϕ_0. Then, choose ρ so that

$$(B.2) \qquad \int_\Omega |d\phi_0|^2 \, (dx)^n \leq \rho \int_\Omega u\phi_0^2 \, (dx)^n$$

$$\leq \rho \int_\Omega u_c\phi_0^2 \, (dx)^n + \rho \int_\Omega (u - u_c)(max\phi_0^2) \, (dx)^n$$

Here

$$(B.3) \qquad u_c = \begin{cases} u, & \text{if } u \leq c \\ c & \text{if } u \geq c \end{cases}$$

and $\lim_{c\to\infty} \int_\Omega (u - u_c) \, (dx)^n = 0$. Note that

$$(B.4) \qquad \|u_c\|_{X^2} \leq \|u\|_{X^2}$$

Minimize $\int_\Omega |d\phi|^2 \, (dx)^n$ subject to the constraint $\int_\Omega u_c\phi^2 \, (dx)^n = 1$, for $\phi \in L_{1,0}^2(\Omega)$. Since $L_{1,0}^2(\Omega) \subset L^2(\Omega)$ is compact and $u_c \leq c$ we get an eigenvalue ρ_c and an eigenfunction ϕ_c in $L_{2,0}^p$ for all p, such that

$$(B.5a) \qquad -\triangle\phi_c - \rho_c u_c \phi_c = 0$$

$$(B.5b) \qquad \rho_c \int_\Omega u_c\phi^2 \, (dx)^n \leq \int_\Omega |d\phi|^2 \, (dx)^n$$

for all $\phi \in L_{1,0}^2(\Omega)$ But from (B.5a), we see that

$$(B.6) \qquad \|\triangle\phi_c\|_{X^3(\Omega)} \leq \rho_c\|u_c\phi_c\|_{X^3(\Omega)} \leq \rho_c\|u_c\|_{X^2(\Omega)}\|\phi_c\|_{X^1(\Omega)}.$$

However

$$(B.7)$$
$$\|\phi_c\|_{X^1(\Omega)} \leq c_1\|\phi_c\|_{X_2^3(\Omega)} \leq c_1c_2\|\triangle\phi_c\|_{X^3(\Omega)} \leq c_1c_2\rho_c\|u_c\|_{X^2(\Omega)}\|\phi_c\|_{X^1(\Omega)}.$$

Hence

$$(B.8) \qquad 1 \leq c_1c_2\|u\|_{X^2(\Omega)}\rho_c.$$

Use inequality (B.4), inequality (B.5b) and inequality (B.8) to get:

$$\text{(B.9)} \qquad \int_\Omega u\phi_0^2(dx)^n \le c_1 c_2 \|u\|_{X^2(\Omega)} \rho_c \int_\Omega u_c \phi_0^2(dx)^n +$$

$$\int_\Omega (u - u_c)\phi_0^2(dx)^n \le c_1 c_2 \|u\|_{X^2(\Omega)} \int_\Omega |d\phi_0|^2(dx)^n +$$

$$\int_{\Omega)} (u - u_c)(dx)^n \max_{x \in \Omega} \phi_0^2.$$

Since $\lim_{c \to \infty} \int_\Omega (u - u_c)(dx)^n = 0$, we get the result with $\lambda = \frac{1}{c_1 c_2}$. $\square$

The proof of the smooth theorem is immediate from Theorem 16.

THEOREM 17. *If (u, g) are smooth, there exists a constant η depending on the norm of $\triangle^{-1}$ and on a Morrey-Sobolev embedding constant, such that if $\|u\|_{X^2(\Omega)} < \eta$, $g = 0$ on $\partial\Omega$ and*

$$\text{(B.10)} \qquad -\triangle g - ug \le 0,$$

then $g \le 0$

PROOF. Let

$$\text{(B.11)} \qquad g_+(x) = \begin{cases} 0 & \text{if} \quad g(x) \le 0 \\ g(x) & \text{if} \quad g(x) \ge 0 \end{cases}.$$

Then from (B.10)

$$\text{(B.12)} \qquad \int_\Omega \left(|dg_+|^2 - ug_+^2\right)(dx)^n = \int_\Omega (-\triangle g - ug)g_+(dx)^n \le 0.$$

But,

$$\text{(B.13)} \qquad \int_\Omega ug_+^2(dx)^n \ge \int_\Omega |dg_+|^2(dx)^n \ge \left(\frac{\lambda}{\|u\|_{X^2(\Omega)}}\right) \int_\Omega ug_+^2(dx)^n$$

from Theorem 16. If $\lambda > \|u\|_{X^2(\Omega)}$, then

$$\text{(B.14)} \qquad \int_\Omega ug_+^2(dx)^n = \int_\Omega |dg_+|^2(dx)^n = 0.$$

Hence $\lambda = \eta$ of Theorem 16. $\square$

After this warm-up, we only need a few additional ideas to handle the singular case.

DEFINITION 11. If $\mathcal{S} \subset \Omega$ is a closed singular set, a regularized distance function to $\mathcal{S}$ is a map $s: \Omega \to \mathcal{R}^+$, such that $s(x) = 0$, for $x \in \mathcal{S}$, is smooth and $s: \Omega - \mathcal{S} \to \mathcal{R}$, and

$$\text{(B.15)} \qquad c^{-1}(dist(x, \mathcal{S})) \le s(x) \le c(dist(x, \mathcal{S})).$$

Furthermore the k-th derivative of s satisfies

$$\text{(B.16)} \qquad |(\frac{d}{dx_i})^k s(x)| \le C_k(s(x))^{-k+1} C_k(dist(x, \mathcal{S})^{-k+1})$$

on $\Omega - \mathcal{S}$.

The existence of this regularized distance function is a theorem of Stein [6] (Theorem 2 page 171). The following lemma follows from the definition of Hausdorff measure and a counting argument.

LEMMA 4. *If $\mathcal{S} \subset \Omega$ is a closed set of finite k-dimensional Hausdorff measure, and $s \colon \Omega \to \mathcal{R}$ is a regularized distance function to $\mathcal{S}$, then*

$$(B.17) \qquad \int_{s(x) \leq r} 1 \leq \bar{K} r^{n-k}.$$

THEOREM 18. *Suppose $\mathcal{S} \subset \Omega$ is a closed set of finite $n - 4$ dimensional Hausdorff measure. Let $g^{2\beta} = g^{1+\gamma} \in L^2(\Omega)$ (which defines β) for $\gamma > 0$, where $u, g \in C^\infty(\Omega - \mathcal{S})$ and $g = 0$ on $\partial\Omega - \mathcal{S}$. If $u \in X^2(\Omega)$ is sufficiently small (depending on $\gamma > 0$) and*

$$(B.18) \qquad -\triangle g - ug \leq 0,$$

then $g \leq 0$.

PROOF. Let Ψ be a smooth cutoff function with

$$(B.19) \qquad \Psi(t) = \begin{cases} 0 & \text{for} \quad t \leq 1 \\ 1 & \text{for} \quad t \geq 1 \end{cases}.$$

Assume s is a regularized distance function, and define $\Psi_R(x) = \Psi(\frac{s(x)}{R})$, for $R > 0$. Let

$$(B.20) \qquad g_\epsilon = \begin{cases} g - \epsilon & \text{for} \quad g \geq \epsilon \\ 0 & \text{for} \quad g \leq \epsilon \end{cases}.$$

Choose $\epsilon > 0$ such that ϵ is a regular value of g on $\Omega - \mathcal{S}$. We will take $\epsilon \to 0$, so that

$$(B.21) \qquad g_0 = \begin{cases} g & \text{for} \quad g \geq 0 \\ 0 & \text{for} \quad g \leq 0 \end{cases}.$$

We wish to prove $g_0 = 0$. Now, we have on Ω

$$(B.22) \qquad -\Psi_R^2(g_\epsilon^\gamma \triangle g - ug_\epsilon^\gamma g) \leq 0$$

$$(B.23) \qquad ug_\epsilon^\gamma g = ug_\epsilon^{2\beta} + \epsilon ug_\epsilon^\gamma.$$

We also have:

(B.24)

$$\Psi_R^2 g_\epsilon^\gamma \triangle g$$

$$= \text{div}(\Psi_R^2 g_\epsilon^\gamma dg) - (\frac{1}{2\beta})(d\Psi_R^2)g_\epsilon^{2\beta}) - \left(\frac{\gamma}{\beta^2}\right)\Psi_R^2 |dg_\epsilon^\beta|^2 + (\frac{1}{2\beta})(\triangle \Psi_R^2)g_\epsilon^{2\beta}.$$

Continue, to get:

$$(\text{B.25}) \quad \Psi_R^2 |dg_\epsilon|^2 = |d(\Psi_R g_\epsilon^\beta) - (d\Psi_R)g_\epsilon^\beta|^2 \geq \left(\frac{1}{2}|d(\Psi_R g_\epsilon^\beta)|^2 - |d\Psi_R|^2 g_\epsilon^{2\beta}\right).$$

Putting (B.22) to (B.25) together, we obtain:

$$(\text{B.26}) \quad (\frac{1}{2})\left(\frac{\gamma}{\beta^2}\right)|d(\Psi_R g_\epsilon^\beta)|^2 \leq u g_\epsilon^{2\beta}\Psi_R^2 + \left(\frac{\gamma}{\beta^2}\right)|d\Psi_R|^2 g_\epsilon^{2\beta}$$
$$+ (\frac{1}{2\beta})(\triangle\Psi_R^2)g_\epsilon^{2\beta} + \text{div}\left(\Psi_R^2 g_\epsilon^\gamma dg - (\frac{1}{2\beta})d(\psi_R^2)g_\epsilon^{2\beta}\right).$$

If we let $g_{R,\epsilon} = \Psi_R g_\epsilon$, and integrate (B.22), we get

$$(\text{B.27})$$
$$(\frac{1}{2})(\frac{\gamma}{\beta^2})\int_\Omega |dg_{R,\epsilon}|^2 \, (dx)^n \leq$$
$$(\text{B.28})$$
$$\int_\Omega u|g_{R,\epsilon}|^2 \, (dx)^n + \int_\Omega u g_\epsilon^\gamma \Psi_R^2 \, (dx)^n + C_\beta \int_\Omega (|\triangle(\Psi_R^2)| + |d\Psi_R|^2)g_\epsilon^{2\beta} \, (dx)^n.$$

There is no contribution from the divergence term, as $(\Psi_R^2)g_\epsilon^2 dg + \frac{\gamma}{\beta^2}\Psi_R[d(\Psi_R)]g_\epsilon^{2\beta}$ vanishes on $\partial\Omega \cup g^{-1}(\epsilon) \cup \{s(x) \leq R\}$. Because $g^{-1}(\epsilon)$ is a smooth submanifold, one can verify the computation at $g^{-1}(\epsilon)$ by a one dimensional argument. Now from the definition of Ψ_R and $\mathcal{S}$

$$(\text{B.29}) \quad |\triangle\Psi_R^2| + |d\Psi_R|^2$$
$$\leq (\frac{1}{R})\|d\Psi^2\|_{L^\infty}|d^2 s| + (\frac{1}{R^2}) + \left(\|d^2\Psi^2\|_{L^\infty} + \|d\Psi\|_{L^\infty}^2 |ds|^2\right)$$
$$\leq (\frac{\hat{K}}{R^2}).$$

Apply this inequality (B.29), inequality (B.27), and Theorem 16 to get

$$(\text{B.30}) \quad (\frac{1}{2})(\frac{\gamma}{\beta^2})\int_\Omega |dg_{R,\epsilon}|^2 \, (dx)^n \leq \lambda^{-1}\|u\|_{X^2(\Omega)}\int_\Omega |dg_{R,\epsilon}|^2 \, (dx)^n$$
$$+ \epsilon\|u\|_{L^2(\Omega)}\int_\Omega \|g_\epsilon^\gamma\|_{L^2(\Omega)} + (C(\beta))\hat{K}\bar{K}\left(\int_{s(x)\leq 2R}|g_\epsilon|^{4\beta}\,(dx)^n\right)^{\frac{1}{2}}.$$

Since $\lim_{2R\downarrow 0}\int_{s(x)\leq 2R}|g_\epsilon^{4\beta}|(dx)^n = 0$, if $\|u\|_{X^2(\Omega)} < (\frac{\gamma}{2\beta})\lambda$, then $g_0 = \lim_{\substack{R\to 0 \\ \epsilon\to 0}} g_{R,\epsilon} = 0$. $\qquad\square$

Appendix C. Coulomb gauges

In order to get further regularity beyond an estimate on $F_A = dA + \frac{1}{2}[A, A]$, it is necessary to construct a Coulomb gauge, i.e. a local trivialization in which $D_A = d + A$, and $d^*A = 0$, and to control the norm of A by a norm of F_A. Tian and Tao [**11**] do this in a very weak setting but their proof is very difficult. In our situation we can assume a bound on F in L^p for p

large, and their proof simplifies enormously. We include it here for complete-
ness. In the following $B_y(\delta) := \{x : |x - y| \leq \delta\}$, and $B := \{x : |x| \leq 1\}$.

THEOREM 19. *Let $\nabla_{\hat{A}}$ be a connection that is smooth on $\Omega - \mathcal{S}$, where
$\mathcal{S}$ is a closed singular set with finite $n - 3$ dimensional Hausdorff measure.
Suppose $F_{\hat{A}} \in L^p(\Omega)$ for $p > n$. Then for each $\epsilon > 0$, every point $y \in \Omega$ is
the center of a ball $B_y(\delta) \subset \Omega$, such that*

$$(C.1) \qquad \int_{B_y(\delta)} |F_{\hat{A}}|^p \, (dx)^n \leq \epsilon^p \delta^{-2p-n}.$$

Note that δ depends on ϵ.

*For such an $\epsilon > 0$, there exists a local trivialization in which the con-
nection ∇_A induces a local exterior covariant differential $D_A = d + A$ such
that A satisfies $d^*A = 0$ and*

$$(C.2) \quad \|A\|_{L^p(B_y(\delta))} \leq \delta_A C(p, n) \|F_A\|_{L^p(B_y(\delta))} = \delta_A C(p, n) \|F_{\hat{A}}\|_{L^p(B_y(\delta))}.$$

PROOF. The first statement is clear. Moreover, for interior domains of
Ω there is a uniform covering of such a domain by such balls. Choose a
coordinate system for such a ball, $(\tilde{x} := \frac{x-y}{\delta})$ that transforms this ball into
the unit ball. In this rescaled system

$$(C.3) \qquad \int_B |F_{\hat{A}}|^p \, (d\tilde{x})^n \leq \epsilon^p$$

and the conclusion is that there exists a trivialization at the scale of B such
that:

$$(C.4) \qquad \|A\|_{L^p(B)} \leq C(p, n) \|F_A\|_{L^p(B)} \leq C(p, n)\epsilon.$$

In the rescaled system 0 is not in $\mathcal{S}$. Parallel translate the fiber at 0 along
every ray in B until each ray intersects $\mathcal{S}$. This provides a smooth trivializa-
tion of the bundle over $B - \mathcal{S}'$, where $\mathcal{S}' := \{\lambda y \mid y \in \mathcal{S}, \lambda > 0\}$. Then, $\mathcal{S}'$
is a closed set of finite $n - 2$ dimensional Hausdorff measure. In this gauge
(trivialization) $x^k A^k = 0$ (for less cluttered notation we drop the subscript
A on F_A) and

$$(C.5) \qquad \frac{\partial}{\partial r}(rA_j) = A_j + x^k F_{k,j} + x^k \frac{\partial}{\partial x^j}(A_k) = F_{k,j}.$$

Integrating equation (C.5) we get

$$(C.6) \qquad rA_j = \int_0^r \rho F_{\rho,j} \, d\rho$$

where $\rho F_{\rho,j} = x^k F_{k,j}$. Then

$$(C.7) \quad |A|^p r^p \leq \left(\int_0^r |F| \rho \, d\rho \right)^p \leq \left(\int_0^r (|F| \rho^\alpha)^p \, d\rho \right) \left(\int_0^r \left(\rho^{1-\alpha} \right)^{\frac{p}{p-1}} \right)^p$$

$$\leq \left(\frac{r}{\beta} \right)^{p-1} \int_0^r |F|^p \rho^{\alpha p} \, d\rho.$$

We set $\alpha = \frac{n-1}{p}$, and compute:

$$(C.8) \qquad \beta\left(\frac{(1-\alpha p)}{p-1} + 1\right) = \frac{2p - \alpha p - 1}{p-1} = \frac{2p - n}{p-1} > 1.$$

Integrating $|A|^p r^p$ again in r and also in the spherical angle, gives:

$$(C.9) \qquad \int_0^1 \int_{S^1} r^{n-1}|A|^p \, d\theta dr \le \int_0^1 r^{p-1} \, dr \left(\int_{S^1} \int_0^1 \rho^{n-1}|F|^p \, d\rho d\theta\right).$$

Hence

$$(C.10) \qquad \|A\|_{L^p(B)} \le \|F\|_{L^p(B)} \le \epsilon.$$

Now consider the equation for $g = e^u$:

$$(C.11) \qquad d^*(g^{-1}dg + g^{-1}Ag) = s,$$

which is a smooth map from $\{u, A\}$ (with $u \in L_1^p(B)) \subset C^0(B)$ and $A \in L^p(B)$) to $s \in L_{-1}^p(B)$. Since at $u = 0$ the linearization is $\triangle u + A$, and $\triangle \colon L_0^p(B) \to L_{-1}^p(B)$ is invertible, equation (C.11) is solvable for small A for u near 0 in $L_{1,0}^p(B)$. Now, in the new gauge, $\tilde{A} = g^{-1}dg + g^{-1}Ag$, and $F_{\tilde{A}} = g^{-1}F_A g$. We have:

$$(C.12) \qquad d^*\tilde{A} = 0$$

$$(C.13) \qquad d\tilde{A} + (\frac{1}{2})[\tilde{A}, \tilde{A}] = F_{\tilde{A}}.$$

and,

$$(C.14) \quad \|\tilde{A}\|_{L^p(B)} \le \|dg\|_{L^p(B)} + \|A\|_{L^p(B)}$$
$$\le (C_p + 1)\|A\|_{L^p(B)} \le (C_p + 1)\|F_A\|_{L^p(B)} \le (C_p + 1)\|\tilde{F}_A\|_{L^p(B)}$$

Here C_p is roughly the norm of the inversion of $\triangle \colon L_{1,0}^p(B) \to L_{-1}^p(B)$. $\quad\square$

This construction gives a gauge transformation between smooth connections on $B - \mathcal{S}$. Because the singular set $\mathcal{S}$ is of Hausdorff codimension at least two, the gauge transformation, which is a map to a compact group, must extend over $B - \mathcal{S}'$. By the same argument, the singularities can only be removed in one way.

COROLLARY 5. *On $B(\frac{1}{2})$, we have $\tilde{A} \in L_1^p(B(\frac{1}{2}))$.*

PROOF. Since

$$(C.15) \qquad \|\tilde{A}\|_{L^p(B)} \le (C_p + 1)\|\tilde{F}_A\|_{L^p(B)},$$

and since $\tilde{A}$ solves the elliptic system:

$$(C.16) \qquad d^*\tilde{A} = 0$$

$$(C.17) \qquad d\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = \tilde{F},$$

standard interior elliptic regularity gives an estimate of $\|\tilde{A}\|_{L_1^p(B(\frac{1}{2}))}$ in terms of $\|F_{\tilde{A}}\|_{L^p(B)}$. $\quad\square$

Note that we do not set the radial $x^i A_i$ equal to zero on the boundary. We only use an interior regularity estimate.

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