Counting curves in quintic Calabi-Yau threefolds and Landau-Ginzburg models

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Dedicated to Professor Shing-Tung Yau on the occasion of his seventieth birthday

Abstract. In this note, we survey the developments that led to the invention of Mixed-Spin-P field theory (MSP theory) which interpolates Gromov-Witten theory of quintic Calabi-Yau threefolds and Landau-Ginzburg theory of the corresponding quintic polynomials.

Contents

1. Introduction

We look at the Fermat quintic polynomial in five variables

$$G(x_1, \dots, x_5) = x_1^5 + \cdots + x_5^5,$$

and the Fermat quintic threefold defined by

$$Q := \{[x_1, \dots, x_5] \in \mathbb{P}^4 \mid G(x_1, \dots, x_5) = 0\}.$$

The variety Q is a smooth projective Calabi-Yau threefold of $h^{1,1}(Q) = 1$ and $h^{2,1}(Q) = 101$, where the latter is the dimension of the moduli space of Calabi-Yau threefolds that are deformations of Q.

The mirror family $\check{Q}_\psi$ of the Fermat quintic Q is obtained by taking a crepant resolution of each member of the quotient family

$$\{y_1^5 + \cdots + y_5^5 - 5\psi y_1 \cdots y_5 = 0\}/\Gamma \subset \mathbb{P}^4/\Gamma.$$

where Γ is the quotient of $\{(a_i) \in (\mu_5)^5 : a_1 \cdots a_5 = 1\}$ by $\{(a, \ldots, a) : a \in \mu_5\}$. It acts on $\mathbb{P}^4$ via scaling the five homogeneous coordinates of $\mathbb{P}^4$.

The complex moduli $\mathcal{M}$ of $\check{Q}_\psi$ is 1-dimensional; its affine part is given by letting the variable ψ shown above to be in the weighted projective line $\mathbb{P}[5,1]$, after gluing $\mathbb{C}_z = \operatorname{Spec}\mathbb{C}[z]$ and $[\mathbb{C}_\psi/\mu_5] = [\operatorname{Spec}\mathbb{C}[\psi]/\mu_5]$ via $\mathbb{C}_z^* \to \mathbb{C}_\psi^*/\mu_5$, $z \mapsto (5\psi)^{-5}$.

The moduli $\mathcal{M}$ has three special points: the maximally unipotent monodromy (MUM) point at $\psi = \infty$ ($z = 0$); the conifold point at $\psi = 1$ ($z = 5^{-5}$) and the orbifold point at $\psi = 0$ ($z = \infty$).

Mirror symmetry conjecture for Fermat quintic threefold Q is the beginning of the subject now called the Mirror Symmetry. It is illustrated in the following Figure 1.

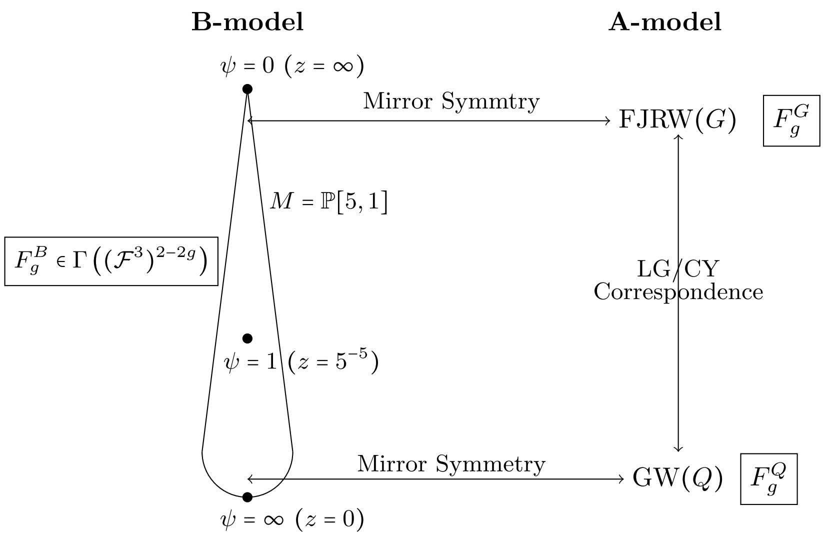

FIGURE 1. Mirror symmetry for quintic Calabi-Yau threefolds. The right hand side is the LG/CY correspondence; the left hand side is the B-model theory.

The lower-right corner is the Gromov-Witten (GW) theory of Q, with its generating function of the genus g GW invariants F_g^Q, mathematically a virtual count of genus g algebraic curves in Q.

The upper-right corner is the Fan-Jarvis-Ruan-Witten (FJRW) theory of the Fermat quintic polynomial W, with its generating function of the genus g FJRW invariants F_g^G, mathematically a virtual count of solutions to the Witten's equation associated to G.

The left hand side two corners are B-model theory on the mirror quintic $\check{Q}$, representing the genus g free energy F_g^B. Genus zero B-model on $\check{Q}$ is defined in terms of classical variation of Hodge structures and period integrals. The periods $\int_\gamma \Omega$ of the holomorphic 3-form Ω on $\check{Q}$ satisfy the Picard-Fuchs equation and can be expressed in terms of explicit hypergeometric series $I_k(z)$ (resp. $\omega_k(\psi)$) near $z = 0$ (resp. $\psi = 0$) at the lower-left (resp. upper-left) corner. Bershadsky-Cecotti-Ooguri-Vafa (BCOV) developed a physical theory of all genus B-model [**BCOV**]. Genus one BCOV theory is defined in terms of analytic torsion. A mathematical theory of higher genus B-model has been developed by Costello and S. Li [**CoLi1, CoLi2**], though the two corners remain to be further developed.

Heuristically, this theory is along the following line. Let $\mathcal{F}^3 \longrightarrow \mathcal{M}$ be the holomorphic line bundle on $\mathcal{M}$ such that $\mathcal{F}^3|_\psi = H^{3,0}(\check{Q}_\psi, \mathbb{C})$. Following the Kodaira-Spencer theory in [**BCOV**], F_g^B (where $g > 1$) is a non-holomorphic section of the line bundle $(\mathcal{F}^3)^{\otimes(2-2g)}$, which after taking certain holomorphic limit and expanding in specifically chosen local holomorphic coordinate q (resp. t) around $\psi = \infty$ (resp. $\psi = 0$) at the lower-left (resp. upper-left) corner of the figure, give rise to a power series, which mirror conjecture predicts that it should be equal to $F_g^Q(q)$ (resp. $F_g^G(t)$). As the B-model theory often is calculated explicitly by string theorists at low genera, the mirror symmetry conjecture often gives the precise expression of the GW generating function $F_g^Q(q)$ up to some genus.

We now recount the milestone mirror symmetry conjectures on quintic Calabi-Yau threefold since early 90's.

Genus zero. In [**CdGP**], Candelas-de la Ossa-Green-Parkes derived their genus zero mirror formula for $F_0^Q(q)$ by relating it to $\{I_k(z) : k = 0,1,2,3\}$, where $q(z) = I_1(z)/I_0(z)$ is known as the mirror map. Their derivation was proved a few years later (in 1996-7) by Givental [**Giv1**] and Lian-Liu-Yau [**LLY1**]. They proved the genus zero mirror theorems for complete intersections in projective spaces, and later extended their theory to smooth complete intersections in projective toric manifolds [**Giv2, LLY2, LLY3**].

For FJRW theory, Chiodo-Ruan proved a genus-zero mirror formula for $F_0^G(t)$ in terms of $\{\omega_k(\psi) : k = 1,2,3,4\}$, and established a genus-zero LG/CY correspondence for quintic Calabi-Yau threefolds [**ChRu**]. The LG/CY correspondence involves change of variables (the GW mirror map $q = q(z)$ and the FJRW mirror map $t = t(\psi)$) and analytic continuation of the hypergeometric series (or more Hodge theoretically, parallel transport with respect to the Gauss-Manin connection).

Genus one. Shortly after [**CdGP**], Bershadsky-Cecotti-Ooguri-Vafa (BCOV) derived the genus-one and genus-two mirror formulae for the quintic 3-fold [**BCOV**]. The BCOV genus-one mirror formula for $F_1^Q(q)$ was proved by A. Zinger fifteen years later in [**Zin**], via genus one reduced Gromov-Witten theory developed by Li-Zinger [**LiZi**], using a result of Vakil-Zinger [**VaZi**]. Zinger proved a mirror formula for a smooth Calabi-Yau hypersurface in a projective space of any dimension. Zinger's result was extended to smooth Calabi-Yau complete intersections in projective spaces by A. Popa [**Pop**]. There is an alternative proof based on quasimap theory developed by Ciocan-Fontanine, Kim, and Maulik [**CFKM**]: the main results in [**Zin, Pop**] also follow from (i) Kim and Lho's mirror theorem for genus-one quasimap invariants of smooth Calabi-Yau complete intersections [**KiLh**], and (ii) Ciocan-Fontanine and Kim's wall-crossing formula relating all-genus GW invariants and quasimap invariants of these targets [**CFK**]. Recently, Chang-Guo-Li-Zhou provided another proof of the BCOV genus-one mirror formula for F_1^Q [**CGLZ**] via torus localization in Mixed-Spin-P (MSP) theory developed in [**CLLL**].

The mathematical treatment of the genus-one BCOV theory was done by Fang-Lu-Yoshikawa in [**FLY**], where they defined what is nowadays called the BCOV invariant for compact Calabi-Yau threefolds, and confirmed the explicit formula of the BCOV invariant of the mirror quintic $\check{Q}_\psi$ predicted in [**BCOV**]. Recently, D. Eriksson, G. Freixas i Montplet, and C. Mourougane introduced and studied the BCOV invariant of Calabi-Yau manifolds of arbitrary dimension [**EFM**].

For FJRW theory, Guo-Ross proved a genus-one mirror formula for F_1^G via torus localization in MSP theory [**GuR1**], and proved a genus-one LG/CY correspondence for quintic Calabi-Yau threefolds [**GuR2**]. Indeed, the MSP theory is defined for $G = x_1^r + \cdots + x_r^r$ for any positive integer $r > 1$. It is expected that genus-one mirror formula for $G = x_1^r + \cdots + x_r^r$ and LG/CY correspondence for Calabi-Yau hypersurfaces in $\mathbb{P}^{r-1}$ can be proved via torus localization in MSP theory for any $r > 1$.

Genus $g \geq 2$. For all genus GW invariants F_g^X for any Calabi-Yau threefold X, the conceptual breakthrough came after BCOV developed their Kodaira-Spencer theory. Based on it, Yamaguchi-Yau derived their polynomiality statement and their functional equation (a version of Holomorphic Anomaly Equation) for F_g^Q [**YYau**], known as Yamaguchi-Yau polynomiality conjecture and Yamaguchi-Yau functional equation conjecture. The Kodaira-Spencer theory allows BCOV to develop their Feynman rule for quintic Q, a rule that can effectively derive F_g^Q after knowing all $F_{<g}^Q$, and knowing $3g-2$ more constraints of F_g^Q.

Built on [**BCOV, YYau**], Huang-Klemm-Quackenbuch (HKQ) argued how to determine the 3g-3 constants of F_g^Q, for $g \leq 51$ [**HKQ**]. The constant term of F_g^Q (where $g > 1$) is the genus g degree zero GW invariant of Q, which is known. The boundary conditions at the orbifold point $\psi = 0$ impose $\lceil \frac{3}{5}(g-1) \rceil$ constraints on the $3g-3$ unknowns, whereas the "gap condition" at the conifold point $\psi = 1$ imposes $2g-2$ constraints on the $3g-3$ unknowns.

To determine the remaining $\lfloor\frac{2}{5}(g-1)\rfloor$ unknowns, HKQ used Gopakumar-Vafa conjecture to express GW invariants in terms Gopakumar-Vafa (GV) invariants, also known as BPS invariants, which they fix up to genus $g=51$ by the classical Castelnuovos' bound.

Recently, mathematical advancements along this direction are accelerating. In [**GJR1, GJR2**], Guo-Janda-Ruan made some penetrating discovery on high genus GW invariants of quintic Calabi-Yau threefold; later Chen-Janda-Ruan [**CJR**] developed the theory of logarithm gauged linear sigma model (log GLSM), which represents a big step toward mathematical theory of GW invariants of Calabi-Yau threefold.

Around the same time, inspired by Witten's vision, Chang-Li-Li-Liu developed their MSP theory [**CLLL**]. This theory and its more recent development, the N-Mixed-Spin-P (NMSP) theory by Chang-Guo-Li-Li [**CGLL**], allows Chang-Guo-Li to prove the Yamaguchi-Yau polynomiality conjecture, the Yamaguchi-Yau functional equation conjecture [**YYau**], the BCOV's Feynman rule for quintic Q [**CGL1, CGL2**], and verifying BCOV genus two formula on quintic. We expect this theory will lead to further understanding of all genus GW invariants of the complete intersection Calabi-Yau threefolds in the product of weighted projective spaces.

In this note, we will survey the developments that led to the invention of Mixed-Spin-P field theory (MSP theory).

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2. Gromov-Witten (GW) theory

Gromov-Witten theory of the quintic threefold Q can be viewed as a mathematical theory of A-model topological strings on Q.

2.1. Moduli of curves.

Let g,ℓ be non-negative integers. A genus g, ℓ pointed prestable curve is $(C, z_1,\dots,z_\ell)$, where C is a genus g connected projective curve with at most nodal singularities; $z_1,\dots,z_\ell$ are distinct smooth points on C. A prestable curve is stable if its automorphism group (as a pointed curve) is finite.

The moduli $\mathfrak{M}_{g,\ell}$ of genus g, ℓ pointed prestable curves is a smooth Artin stack of dimension $3g-3+\ell$. It contains the moduli $\overline{\mathcal{M}}_{g,\ell}$ of genus g, ℓ pointed stable curves as an open substack. The moduli $\overline{\mathcal{M}}_{g,\ell}$ is non-empty if and only if $2g-2+\ell>0$. When non-empty, it is a smooth Deligne-Mumford stack of dimension $3g-3+\ell$.

2.2. Moduli of stable maps. Gromov's compactness theorem says that a sequence of smooth closed pseudo-holomorphic curves in a symplectic manifold of a fixed genus and uniformly bounded area converges to a pseudo-holomorphic cusp-curve (which is a curve with nodal singularities) [**Gro**]. In algebraic geometry this led to the notion of stable maps by Kontsevich [**Kon2**].

A genus g, ℓ pointed, degree d prestable map to Q is a morphism

$$u : (C, z_\bullet) \longrightarrow Q \quad z_\bullet = (z_1, \cdots, z_\ell),$$

where $(C, z_\bullet)$ is a genus g, ℓ pointed prestable curve, and $d = \deg u^*\mathcal{O}_{\mathbb{P}^4}(1)$. An isomorphism between two prestable maps $((C, z_\bullet), u)$ and $((C', z'_\bullet), u')$ is an isomorphism $(C, z_\bullet) \to (C', z'_\bullet)$ that commutes with the u and u'. An automorphism of an object is an isomorphism from the object to itself. A prestable map is *stable* if its group of automorphisms is finite.

Let $\overline{\mathcal{M}}_{g,\ell}(Q, d)$ be the moduli of genus g, ℓ pointed, degree d stable maps to Q. It is a proper, Deligne-Mumford stack with a projective coarse moduli space. When $d = 0$, it is $\overline{\mathcal{M}}_{g,\ell} \times Q$.

2.3. Perfect obstruction theory and virtual fundamental class. The need to construct virtual fundamental class was called for setting up an algebro-geometric construction of the GW invariants of projective manifolds [**KoMa**]. In [**LiTi, BeFa**], Li-Tian and Behrend-Fantechi fulfilled this task by constructing virtual fundamental classes of moduli spaces that have (relative) perfect obstruction theories.

Usually, a moduli stack in algebraic geometry has its tautological obstruction theory. Simply said, associated to each object ξ in the moduli stack we have vector spaces T^{-1}_ξ, T^0_ξ, $T^1_\xi, \cdots$, where T^{-1}_ξ is the space of infinitesimal automorphisms of ξ, T^0_ξ is the space of infinitesimal deformations of ξ, and T^1_ξ is the space of the obstructions to deforming ξ, etc. The moduli stack is said to have a perfect obstruction theory if for all ξ in the moduli stack,

$$T^{-1}_\xi = T^{i>1}_\xi = 0. \tag{1}$$

The meaning of $T^{-1}_\xi = 0$ is that it makes the moduli stack a DM stack at ξ. The (non-)vanishing of the higher obstruction class $T^{i>1}_\xi = 0$ remains a mystery. However, when all higher obstruction vanishes, T^1_ξ glue to an obstruction sheaf of the moduli stack. In this case we say that the moduli stack is virtually smooth.

We now look at the moduli space $\overline{\mathcal{M}}_{g,\ell}(Q, d)$. It comes with a forgetful map

$$\pi_{\mathcal{Q}/\mathfrak{M}} : \mathcal{Q} := \overline{\mathcal{M}}_{g,\ell}(Q, d) \longrightarrow \mathfrak{M}_{g,\ell}, \tag{2}$$

forgetting the u in $((C, z_\bullet), u)$. The map $\pi_{\mathcal{Q}/\mathfrak{M}}$ is *virtually smooth*: at $\xi = ((C, z_\bullet), u)$ in $\mathcal{Q}$, the relative tangent and obstruction spaces are

$$H^0(C, u^*T_Q) \quad \text{and} \quad H^1(C, u^*T_Q),$$

representing the infinitesimal deformations of the map u, and the obstructions to deforming the map u, with $(C, z_\bullet)$ fixed.

Since $H^{i>1}(C, u^*T_Q) = 0$ by dimension reason, the map $\pi_{\mathcal{Q}/\mathfrak{M}}$ is virtually smooth. By Riemann-Roch, we see that the relative virtual dimension of the map $\pi_{\mathcal{Q}/\mathfrak{M}}$ is $3-3g$. Since $\mathfrak{M}_{g,\ell}$ is smooth and $\mathcal{Q}$ is a DM stack, $\pi_{\mathcal{Q}/\mathfrak{M}}$ comes with a relative perfect obstruction theory. Applying the virtual cycle construction mentioned to the pair $\overline{\mathcal{M}}_{g,\ell}(Q,d) \to \mathfrak{M}_{g,\ell}$, we obtain a virtual cycle

$$[\overline{\mathcal{M}}_{g,\ell}(Q,d)]^{\mathrm{virt}} \in A_\ell(\overline{\mathcal{M}}_{g,\ell}(Q,d);\mathbb{Q}),$$

of degree

$$\mathrm{vir.}\dim \overline{\mathcal{M}}_{g,\ell}(Q,d) = (3-3g) + \dim \mathfrak{M}_{g,\ell} = \ell.$$

For instance, for $d=0$, $[\overline{\mathcal{M}}_{0,\ell}(Q,0)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{0,\ell}] \times [Q]$, and

$$[\overline{\mathcal{M}}_{g,\ell}(Q,0)]^{\mathrm{virt}} = c_{3g}(\mathbb{E}^\vee \boxtimes T_Q) \cap \big([\overline{\mathcal{M}}_{g,\ell}] \times [Q]\big) \in A_\ell(\overline{\mathcal{M}}_{g,\ell} \times Q;\mathbb{Q}), \quad g>0,$$

where $\mathbb{E}^\vee$ is the dual of the Hodge bundle[1] $\mathbb{E} = \mathbb{E}_{\overline{\mathcal{M}}_{g,\ell}}$ over $\overline{\mathcal{M}}_{g,\ell}$, and T_Q is the tangent bundle of Q.

2.4. Gromov-Witten invariants. Given a pair of non-negative integers $(g,d) \neq (0,0)$ and $(1,0)$, the genus g degree d GW invariant of Q is

$$N_{g,d} := \int_{[\overline{\mathcal{M}}_{g,0}(Q,d)]^{\mathrm{virt}}} 1 \in \mathbb{Q}. \tag{3}$$

The genus g GW potential of Q is

$$F_g^Q(q) = \begin{cases} \dfrac{5}{6}(\log q)^3 + \displaystyle\sum_{d=1}^{\infty} N_{0,d}q^d, & g=0; \\ -\dfrac{25}{12}\log q + \displaystyle\sum_{d=1}^{\infty} N_{1,d}q^d, & g=1; \\ \displaystyle\sum_{d=0}^{\infty} N_{g,d}q^d, & g \geq 2. \end{cases} \tag{4}$$

For instance, when $d=0$ and $g \geq 2$, Faber-Pandharipande in [**FaPa**] calculated

$$\begin{aligned} N_{g,0} =& \frac{(-1)^g}{2} \int_{[\overline{\mathcal{M}}_{g,0}]} \lambda_{g-1}^3 \int_{[Q]} c_3(T_Q) \\ =& (-1)^g \frac{|B_{2g}||B_{2g-2}|}{4g(2g-2)(2g-2)!}\chi(Q) = -200(-1)^g \frac{|B_{2g}||B_{2g-2}|}{4g(2g-2)(2g-2)!}. \end{aligned}$$

In [**MaPa**], Maulik-Pandharipande provide an algorithm of evaluating $N_{g,d}$ based on the degeneration formula proved by the first author [**Li2**].

[1] It is a rank g vector bundle on $\mathfrak{M}_{g,\ell}$ whose fiber at $(C, z_\bullet)$ is $H^0(C, \omega_C)$.

3. Chang-Li theory of stable maps with fields

String-theorists have been viewing the GW invariants of the quintics a field theory on Riemann surfaces. For mathematicians, it is a theory on the moduli of stable maps to Q. This paradox was finally resolved after the discovery of the theory of stable maps with fields, after the work of Guffin-Sharpe in [**GuSh**].

3.1. The LG model $(K_{\mathbb{P}^4}, \widehat{W})$. Let $\mathbb{C}^*$ act on $\mathbb{C}^6 = \mathbb{C}^5 \times \mathbb{C}$ by weights $(1^5, -5)$, via $s \cdot (x, p) = (sx, s^{-5}p)$, $s \in \mathbb{C}^*$. Then

$$\left((\mathbb{C}^5 - \{0\}) \times \mathbb{C}\right) / \mathbb{C}^* = K_{\mathbb{P}^4},$$

the canonical line bundle over $\mathbb{P}^4$. The function

$$W := p \cdot G(x) = p \cdot (x_1^5 + \cdots + x_5^5)$$

on $(\mathbb{C}^5 - \{0\}) \times \mathbb{C}$ is invariant under the $\mathbb{C}^*$-action, thus descends to a regular

$$\widehat{W} : K_{\mathbb{P}^4} \longrightarrow \mathbb{C}. \tag{5}$$

The pair $(K_{\mathbb{P}^4}, \widehat{W})$ is an LG model, with $\widehat{W}$ plays the role of superpotential.

The differential of the superpotential will play a significant role later. It is

$$d\widehat{W} = 5 \sum_{i=1}^{5} p x_i^4 dx_i + \Big(\sum_{i=1}^{5} x_i^5\Big) dp. \tag{6}$$

Its critical locus is the Fermat quintic threefold embedded in $\mathbb{P}^4$ via the 0-section:

$$\mathrm{Crit}(\widehat{W}) = \{[x_\bullet, p] \in K_{\mathbb{P}^4} \mid p = G(x) = 0\} = Q \ \subset \ K_{\mathbb{P}^4}. \tag{7}$$

3.2. Moduli of stable maps with fields. The Chang-Li theory of stable maps with p-fields [**ChLi1**] is a mathematical theory of A-model topological strings on the LG model $(K_{\mathbb{P}^4}, \widehat{W})$. It generalizes the genus zero Guffin-Sharpe-Witten model [**GuSh, Wit2**] to all genus cases.

Let g, ℓ, d be non-negative integers such that either $d > 0$ or $2g-2+\ell > 0$. A genus g, ℓ pointed, degree d stable maps (to $\mathbb{P}^4$) with a p-field is a triple

$$((C, z_\bullet), u, \rho), \tag{8}$$

where $u : (C, z_\bullet) \to \mathbb{P}^4$ is a genus g ℓ pointed degree d stable map;

$$\rho \in H^0(C, u^*\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \omega_C), \tag{9}$$

called a p-field. Isomorphisms between two objects in (8) are defined in obvious way.

We let

$$\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p$$

be the moduli of stable maps to $\mathbb{P}^4$ with fields, of the given topological data. When $g = 0$, it is $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)$ as the p-fields all vanish in this case. When

$g > 0$, there are always stable maps $(C, z_\bullet, u)$ with non-trivial p-fields. Thus $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p$ is never proper.

We rewrite it to make it a field theory over algebraic curves. A map $u : C \to \mathbb{P}^4$ is given by five sections $\varphi_1, \dots, \varphi_5$ of $u^*\mathcal{O}_{\mathbb{P}^4}(1)$ with no common zeros:

$$u(z) = [\varphi_1(z), \dots, \varphi_5(z)].$$

Replacing $u^*\mathcal{O}_{\mathbb{P}^4}(1)$ by an $\mathcal{L} \in \mathrm{Pic}_d(C)$, replacing u by $(\varphi_1(z), \dots, \varphi_5(z))$, we arrive at

$$\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p = \left\{ [(C, z_\bullet), \mathcal{L}, \varphi, \rho] \text{ stable} \;\middle|\; \begin{array}{l} (C, z_\bullet) \in \mathfrak{M}_{g,\ell},\ \mathcal{L} \in \mathrm{Pic}_d(C), \\ \varphi \in H^0(C, \mathcal{L}^{\oplus 5}),\ \rho \in H^0(\mathcal{L}^{\otimes(-5)} \otimes \omega_C) \end{array} \right\}.$$

Here $\xi = [(C, z_\bullet), \mathcal{L}, \varphi, \rho]$ is stable if φ has no common zero, and $\mathrm{Aut}(\xi)$ is finite.

The six-tuple $\varphi_1, \dots, \varphi_5, \rho$ are called fields in the physics literature. In this form, the objects are pointed curves with fields. It becomes a linear theory. Subsequently, we will view $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p$ as the moduli space of pointed curves with fields.

3.3. Perfect obstruction theory. Let $\mathfrak{D}_{g,\ell}$ be the moduli space of pairs $((C, z_\bullet), \mathcal{L})$ of genus g ℓ pointed prestable curves with $\mathcal{L} \in \mathrm{Pic}(C)$. It is a smooth Artin stack. By sending a stable

$$\xi = [(C, z_\bullet), \mathcal{L}, \varphi, \rho] \in \overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p \tag{10}$$

to $((C, z_\bullet), \mathcal{L})$ in $\mathfrak{D}_{g,\ell}$, and to $(C, z_\bullet)$ in $\mathfrak{M}_{g,\ell}$, we get forgetful morphisms

$$\mathcal{P} = \overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p \xrightarrow{\pi_{\mathcal{P}/\mathfrak{D}}} \mathfrak{D} = \mathfrak{D}_{g,\ell} \xrightarrow{\pi_{\mathfrak{D}/\mathfrak{M}}} \mathfrak{M} = \mathfrak{M}_{g,\ell}.$$

We now see that $\pi_{\mathcal{P}/\mathfrak{D}}$ is virtually smooth. Indeed, the relative tangent space of $\pi_{\mathcal{P}/\mathfrak{D}}$ at ξ is the space of infinitesimal deformations of the fields (φ, ρ) with $((C, z_\bullet), \mathcal{L})$ fixed, which is

$$T_{\mathcal{P}/\mathfrak{D},\xi} = H^0(C, \mathcal{L}^{\oplus 5}) \oplus H^0(C, \mathcal{L}^{\otimes(-5)} \otimes \omega_C);$$

its relative obstruction space to deforming the fields with $((C, z_\bullet), \mathcal{L})$ fixed is

$$Ob_{\mathcal{P}/\mathfrak{D},\xi} = H^1(C, \mathcal{L}^{\oplus 5}) \oplus H^1(C, \mathcal{L}^{\otimes(-5)} \otimes \omega_C),$$

By dimension reason, all higher relative obstructions vanish.

This gives a perfect relative obstruction theory of $\pi_{\mathcal{P}/\mathfrak{D}}$. Thus the virtual cycle construction gives the virtual cycle

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, d)^p]^{\mathrm{virt}} \in A_\ell(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, d)^p, \mathbb{Q}).$$

The dimension of the cycle is the virtual dimension of $\mathcal{P}$, which by a direct calculation is ℓ.

As $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4, d)^p$ is not proper for $g > 0$, an alternative construction that gives a compactly supported virtual cycle is called for.

3.4. Cosection localized virtual cycle. Let $\mathbb{E}_{\mathfrak{D}} = \pi^*_{\mathfrak{D}/\mathfrak{M}}\mathbb{E}_{\mathfrak{M}}$ be the Hodge bundle over $\mathfrak{D} = \mathfrak{D}_{g,\ell}$, c.f. at the end of subsection 2.3. Its total space

$$\pi_{\mathbb{E}/\mathfrak{D}} : \mathbb{E}_{\mathfrak{D}} \longrightarrow \mathfrak{D}$$

is a smooth stack over $\mathfrak{D}$, of relative dimension g.

The relative tangent space of $\pi_{\mathbb{E}/\mathfrak{D}}$ at $((C, z_\bullet), \theta)$ is $H^0(C, \omega_C)$, and the relative obstruction space is $H^1(C, \omega_C) \equiv \mathbb{C}$. We define a $\mathfrak{D}$-morphism $\mathbf{w}$,

$$\mathcal{P} = \overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p \xrightarrow{\ \mathbf{w}\ } \mathbb{E}_{\mathfrak{D}}, \tag{11}$$

via the assignment

$$[(C, z_\bullet), \mathcal{L}, \varphi, \rho] \longrightarrow [(C, z_\bullet), \rho \textstyle\sum_{i=1}^5 \varphi_i^5].$$

Notice that $\mathbf{w}$ is well-defined because under the scaling of $\mathcal{L}$, the term $\rho\sum_{i=1}^5 \varphi_i^5$ has total weight zero.

This morphism induces a homomorphism between the two obstruction sheaves,

$$\sigma_{\mathcal{P}/\mathfrak{D}} : Ob_{\mathcal{P}/\mathfrak{D}} \longrightarrow \mathbf{w}^* Ob_{\mathbb{E}_{\mathfrak{D}}/\mathfrak{D}} \cong \mathcal{O}_{\mathcal{P}},$$

which we call the LG-cosection. At ξ (see (10)), $\sigma_{\mathcal{P}/\mathfrak{D}}|_\xi$ sends

$$\begin{aligned} Ob_{\mathcal{P}/\mathfrak{D},\xi} \ni (\dot\varphi_1, \ldots, \dot\varphi_5, \dot\rho) &\longmapsto 5\sum_{i=1}^5 \rho\varphi_i^4\dot\varphi_i + \left(\sum_{i=1}^5 \varphi_i^5\right)\dot\rho \\ &= d\widehat{W}\Big|_{\substack{x_i \mapsto \varphi_i,\, dx_i \mapsto \dot\varphi_i,\\ p\mapsto \rho,\, dp \mapsto \dot\rho}} \end{aligned} \tag{12}$$

(cf. (6).) It is easy to argue that the cosection of the relative obstruction sheaf factors through a cosection of the absolute obstruction sheaf $Ob_{\mathcal{P}}$, defined via

$$\mathcal{T}_{\mathfrak{D}} \longrightarrow Ob_{\mathcal{P}/\mathfrak{D}} \longrightarrow Ob_{\mathcal{P}} \longrightarrow 0.$$

This allows one to apply the theory of cosection localized virtual cycle constructed by Kiem-Li [**KiLi**]:

THEOREM 3.1 (Cosection localized virtual cycle). *Suppose a moduli stack has a perfect obstruction theory whose obstruction sheaf has a cosection σ. Then the cosection localized virtual cycle is a cycle which lies in $Deg(\sigma) = (\sigma = 0)$, and is rationally equivalent to its ordinary virtual cycle of the perfect obstruction theory.*

Applying this theorem, Chang-Li obtained a cosection localized virtual cycle [**ChLi1**]:

$$[\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p]^{\mathrm{virt}}_\sigma \in A_\ell(Deg(\sigma_{\mathcal{P}/\mathfrak{D}})).$$

By (7), one sees that

$$D\sigma_{\mathcal{P}/\mathfrak{D}} = (\sigma_{\mathcal{P}/\mathfrak{D}}(\xi) = 0) = \left\{\text{locus where } \rho = \sum_{i=1}^5 \varphi_i^5 = 0\right\} = \overline{\mathcal{M}}_{g,\ell}(Q, d).$$

embedded in $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4, d)^p$ via zero p-fields.

In [**ChLi1**], Chang and Li proved

THEOREM 3.2 (Chang-Li). *The cosection localized virtual cycle of* $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4,d)^p$*, up to a sign, is identical to the virtual cycle of* $\overline{\mathcal{M}}_{g,\ell}(Q,d)$*:*

$$[\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^4,d)^p]^{\text{virt}}_\sigma = (-1)^{5d+1-g}[\overline{\mathcal{M}}_{g,\ell}(Q,d)]^{\text{virt}} \in A_\ell(\overline{\mathcal{M}}_{g,\ell}(Q,d),\mathbb{Q}). \tag{13}$$

Defines the p-field invariants $N^p_{g,d}$ to be the degree of $[\overline{\mathcal{M}}_{g,0}(\mathbb{P}^4,d)^p]^{\text{virt}}$, the theorem says that

$$N^p_{g,d} = (-1)^{5d+1-g}N_{g,d}.$$

4. Fan-Jarvis-Ruan-Witten (FJRW) theory

Fan-Jarvis-Ruan-Witten (FJRW) invariant is a generalization of Witten's top Chern class. after constructing a virtual cycle via the perturbed Witten's equation and Kuranishi structures [**FJR1, FJR2**].

There were other algebraic construction of Witten's top Chern class, etc., notably by Polishuke-Vaintrob [**PoVa**] and Chiodo [**Chi**]. In [**ChLiL**], Chang-Li-Li reconstruct FJRW invariants using the LG model parallel to the theory of stable maps with fields. This provides the technical tool to mathematically investigate the LG/CY correspondence.

4.1. The LG model $([\mathbb{C}^5/\mu_5],\widehat{G})$**.** The LG model is essentially identical to the LG model described in Section 3.1.

Let $\mathbb{C}^*$ act on $\mathbb{C}^6 = \mathbb{C}^5\times\mathbb{C}$ by weights $(1^5,-5)$. Then

$$[(\mathbb{C}^5\times(\mathbb{C}-0))/\mathbb{C}^*] = [(\mathbb{C}^5\times\{1\})/\mu_5] = [\mathbb{C}^5/\mu_5]$$

is a 5-dimensional Calabi-Yau orbifold. The function

$$W = p\cdot G(x) = p\cdot(x_1^5+\cdots+x_5^5)$$

on $\mathbb{C}^5\times(\mathbb{C}-0)$ is invariant under the $\mathbb{C}^*$-action. Its restriction to $\mathbb{C}^5\times\{1\} = \mathbb{C}^5$ is $G(x)$, which is invariant under the diagonal μ_5-action on $\mathbb{C}^5$, thus descends to a regular

$$\widehat{G}: [\mathbb{C}^5/\mu_5] \longrightarrow \mathbb{C}. \tag{14}$$

The pair $([\mathbb{C}^5/\mu_5],\widehat{G})$ is our LG model, with $\widehat{G}$ plays the role of superpotential.

The differential of the superpotential will be important later. It is

$$d\widehat{G} = 5\sum_{i=1}^5 x_i^4 dx_i \tag{15}$$

Its critical locus is the orbifold origin in $[\mathbb{C}^5/\mu_5]$,

$$\text{Crit}(\widehat{G}) = \{[x_\bullet]\in[\mathbb{C}^5/\mu_5] \mid x_1^4=\cdots=x_5^4=0\} \subset [\mathbb{C}^5/\mu_5].$$

4.2. Moduli of stable 5-spin curves. We follow the presentation of [**AbVi, AGV**] on twisted curves. A genus g, ℓ pointed twisted prestable curve is a connected proper one-dimensional DM stack $\mathcal{C}$ together with ℓ disjoint zero-dimensional integral closed substacks $\mathfrak{z}_1, \ldots, \mathfrak{z}_n \subset \mathcal{C}$, such that

(i) $\mathcal{C}$ is étale locally a nodal curve;
(ii) formally locally near a node, $\mathcal{C}$ is isomorphic to the quotient stack

$$[\mathrm{Spec}\,(\mathbb{C}[x,y]/(xy))/\mu_r], \quad \text{where } \zeta\cdot(x,y)=(\zeta x, \zeta^{-1}y),\ \zeta\in\mu_r;$$

(iii) each marking $\mathfrak{z}_i \subset \mathcal{C}$ is contained in the smooth locus of $\mathcal{C}$;
(iv) $\mathcal{C}$ is a scheme away from the markings and the singular points of $\mathcal{C}$; the coarse moduli space C of $\mathcal{C}$ is a nodal curve of arithmetic genus g.

Let $\pi : \mathcal{C} \to C$ be the coarse moduli morphism; let $z_i = \pi(\mathfrak{z}_i)$. The resulting $(C, z_\bullet)$ is a genus g, ℓ pointed prestable curve. We say $(\mathcal{C}, \mathfrak{z}_\bullet)$ is stable if $(C, z_\bullet)$ is stable.

Let $\mathfrak{M}^{\mathrm{tw}}_{g,\ell}$ be the moduli of genus g, ℓ pointed twisted prestable curves. It is a smooth Artin stack of dimension $3g-3+\ell$. The coarse moduli morphism $\mathfrak{M}^{\mathrm{tw}}_{g,\ell} \to \mathfrak{M}_{g,\ell}$ is the morphism sending $(\mathcal{C}, \mathfrak{z}_\bullet)$ to its coarse moduli space $(C, z_\bullet)$.

We introduce the notion of 5-spin curves. A genus g, ℓ pointed stable 5-spin curve is a triple $((\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell), \mathcal{L}, \rho)$ such that

(v) $(\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell)$ is a genus g, ℓ pointed twisted stable curve,
(vi) $\mathcal{L}$ is a representable line bundle on $\mathcal{C}$; $\rho : \mathcal{L}^{\otimes 5} \to \omega^{\log}_{\mathcal{C}}$ is an isomorphism.

Here the log dualizing sheaves of $(\mathcal{C}, \mathfrak{z}_\bullet)$ is $\omega^{\log}_{\mathcal{C}} = \omega_{\mathcal{C}}(\mathfrak{z}_1 + \cdots + \mathfrak{z}_\ell)$. Comparing with its coarse moduli $\pi : \mathcal{C} \to C$,

$$\omega^{\log}_{\mathcal{C}} = \pi^*\omega^{\log}_{C}, \quad \omega^{\log}_{C} = \omega_C(z_1 + \cdots + z_\ell). \tag{16}$$

Let $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho)$ be a 5-spin curve. Let $\mathfrak{z}_j$ be its marking. By the representability assumption, the group homomorphism $\mathrm{Aut}(\mathfrak{z}_j) \to \mathrm{Aut}(\mathcal{L}|_{\mathfrak{z}_j}) \cong \mathbb{C}^*$ is injective. Further, (iv) implies that $\mathrm{Aut}(\mathfrak{z}_j)$ acts trivially on $\mathcal{L}^{\otimes 5}|_{\mathfrak{z}_j}$. Therefore, $\mathrm{Aut}(\mathfrak{z}_j)$ is either trivial or isomorphic to μ_5.

Definition 4.1. *Let $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho)$ be a 5-spin curve. We say its marking $\mathfrak{z}_i$ is narrow if* $\mathrm{Aut}(\mathfrak{z}_j)$ *act non-trivially on $\mathcal{L}|_{\mathfrak{z}_j}$. We say $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho)$ is narrow if all its markings are narrow.*

Let $\mathfrak{z}_i$ be its narrow marking. Then $T\mathcal{C}|_{\mathfrak{z}_j}$ is a non-trivial representation of $\mathrm{Aut}(\mathfrak{z}_j)$. Hence there is a unique $m_j \in \{1,2,3,4\}$ so that as $\mathrm{Aut}(\mathfrak{z}_j)$-representations.[2]

$$\mathcal{L}|_{\mathfrak{z}_j} \cong (T\mathcal{C}|_{\mathfrak{z}_j})^{\otimes m_j} \tag{17}$$

[2]This is consistent with [**CLLL**, p. 323], that for $\mathfrak{z}_i$ a stacky point, $\mathcal{O}_{\mathcal{C}}(\mathfrak{z}_i)|_{\mathfrak{z}_i} \cong \mathrm{T}\mathcal{C}|_{\mathfrak{z}_i}$ has $m_i = 1$.

This way, to every marking $\mathfrak{z}_j$ we associate a unique integer $m_j \in [0,4]$, so that (17) holds. (When $\mathfrak{z}_j$ is a scheme point, we let $m_j = 0$.) Thus we associate to $(\mathcal{C}, \mathfrak{z}_\bullet)$ its type ℓ,

(18)
$$\gamma = (\gamma_1, \ldots, \gamma_\ell) = (\zeta^{m_1}, \ldots, \zeta^{m_\ell}) \in (\mu_5)^\ell, \quad \zeta = e^{2\pi\sqrt{-1}/5}. \quad m_j \in [0,4] \cap \mathbb{Z}.$$

Given a γ as above, we say a 5-spin curve $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho)$ is γ-marked if the m_i in γ are that associated with its i-th marking.

Let $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$ be the moduli space of stable genus g γ-marked 5-spin curves. It is a proper smooth DM stack of dimension $3g-3+\ell$, and is an open and closed substack of $\overline{\mathcal{M}}_{g,\ell}^{1/5}$, where the later is the moduli space of stable genus g ℓ pointed 5-spin curves. It is direct to see that $\overline{\mathcal{M}}_{g,\gamma}^{1/5}$ is non-empty if and only if $2g-2+\ell > 0$ and

$$\frac{2g-2+\ell-\sum_{j=1}^{\ell} m_j}{5} \in \mathbb{Z}.$$

4.3. Moduli of stable 5-spin curves with fields. We fix an ℓ-pointed type γ. A genus g γ-marked 5-spin curve with five fields is a quadruple

$$\xi = ((\mathcal{C}, \mathfrak{z}_1, \ldots, \mathfrak{z}_\ell), \mathcal{L}, \rho, \varphi), \tag{19}$$

where $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho)$ is a genus g γ-marked 5-spin curve, and its five fields φ:

$$\varphi = (\varphi_1, \ldots, \varphi_5) \in H^0(\mathcal{C}, \mathcal{L}^{\oplus 5}),$$

We say ξ is stable if $\mathrm{Aut}(\xi)$ is finite. One checks that it is stable if and only if the 5-spin curve is stable.

Let

$$\overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi}$$

be the moduli space of genus g γ-marked stable 5-spin curve with five fields. When $g > 0$, it is non-proper because there are points with $0 \neq H^0(C, \mathcal{L}^{\oplus 5})$.

4.4. Witten's top Chern class, its cosection localized construction. We use the LG model $([\mathbb{C}^5/\mu_5], \widehat{G})$ to reconstruct the FJRW invariants for narrow γ. Let γ be narrow. Let

$$\pi_{\Phi_\gamma/\mathcal{M}_\gamma} : \Phi_\gamma := \overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi} \longrightarrow \mathcal{M}_\gamma := \overline{\mathcal{M}}_{g,\gamma}^{1/5}$$

be the forgetful morphism (forgetting φ). Let

$$\xi = ((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \rho, \varphi) \in \Phi_\gamma = \overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi} \tag{20}$$

be any closed point. The relative tangent space of $\pi_{\Phi_\gamma/\mathcal{M}_\gamma}$ at ξ and the relative obstruction space to deforming ξ, are respectively

$$\mathrm{T}_{\Phi_\gamma/\mathcal{M}_\gamma,\xi} = H^0(C, \mathcal{L}^{\oplus 5}) \quad \text{and} \quad Ob_{\Phi_\gamma/\mathcal{M}_\gamma,\xi} = H^1(C, \mathcal{L}^{\oplus 5}).$$

Because $\mathcal{M}_\gamma$ is smooth, and because $\mathrm{Aut}(\xi)$ is finite when $\xi \in \Phi_\gamma$, we see that $\pi_{\Phi_\gamma/\mathcal{M}_\gamma}$ is virtually smooth, and admits a relative perfect obstruction theory, of relative virtual dimension

$$\chi(\mathcal{L}^{\oplus 5}) = \sum_{i=1}^{5}(\deg\mathcal{L} + 1 - g - \sum_{j=1}^{\ell}\frac{m_j}{5}) = 3 - 3g + \ell - \sum_{j=1}^{\ell} m_j.$$

Combined with $\dim\mathcal{M}_\gamma = 3g - 3 + \ell$, we see that the virtual dimension of Φ_γ is

$$d_\gamma := (3g - 3 + \ell) + (3 - 3g + \ell - \sum_{j=1}^{\ell} m_j) = \sum_{j=1}^{\ell}(2 - m_j). \tag{21}$$

Like before, we define a morphism

$$\Phi_\gamma = \overline{\mathcal{M}}_{g,\gamma}^{1/5,5\varphi} \xrightarrow{\ \mathbf{w}\ } \mathbb{E}_{\mathcal{M}_\gamma}, \tag{22}$$

where $\mathbb{E}_{\mathcal{M}_\gamma}$, $\mathbb{E}_{\mathcal{M}_\gamma}|_{(\mathcal{C},\mathfrak{z}_\bullet,..)} = H^0(C,\omega_C)$, is the Hodge bundle of $\mathcal{M}_\gamma = \overline{\mathcal{M}}_{g,\gamma}^{1/5}$. Let $\xi = (\mathcal{C},\mathfrak{z}_\bullet,..) \in \Phi_\gamma$, let $\pi : \mathcal{C} \to C$ be the coarse moduli morphism, and let $z_j = \pi(\mathfrak{z}_j)$ be the image markings. We get

$$\textstyle\sum_{i=1}^{5}\varphi_i^5 \in H^0(\mathcal{C},\mathcal{L}^{\otimes 5}) \ \xrightarrow[\cong]{\ \rho\ }\ H^0(\mathcal{C},\omega_{\mathcal{C}}^{\log}) = H^0(C,\omega_C^{\log}).$$

On the other hand since γ is narrow, $H^0(\mathcal{L}|_{\mathfrak{z}_i}) = 0$ since $\mathcal{L}|_{\mathfrak{z}_i}$ is a non-trivial $\mathrm{Aut}(\mathfrak{z}_i)$-module. Thus

$$0 = \sum(\varphi_i)^5|_{\mathfrak{z}_i} \in H^0(\omega_{\mathcal{C}}^{\log}|_{\mathfrak{z}_i}) = H^0(\omega_{\mathcal{C}}^{\log}|_{z_i}).$$

Consequently, $\sum\varphi_i^5$ lifts to

$$(\sum\varphi_i^5)^{\mathrm{lift}} \in H^0(C,\omega_C^{\log}(-z_1 - \cdots - z_\ell)) = H^0(C,\omega_C) = \mathbb{E}_{\mathcal{M}_\gamma}|_{(\mathcal{C},\mathfrak{z}_\bullet,..)}. \tag{23}$$

Its family version defines the morphism $\mathbf{w}$.

The morphism $\mathbf{w}$ induces a homomorphism of relative obstruction sheaves

$$\sigma_{\Phi_\gamma/\mathcal{M}_\gamma} : Ob_{\Phi_\gamma/\mathcal{M}_\gamma} \longrightarrow \mathbf{w}^* Ob_{\mathbb{E}_{\mathcal{M}_\gamma}/\mathcal{M}_\gamma} \cong \mathcal{O}_{\Phi_\gamma}.$$

In explicit form, it is

$$\sigma_{\Phi_\gamma/\mathcal{M}}|_\xi : Ob_{\Phi/\mathcal{M},\xi} = H^1(C,\mathcal{L})^{\oplus 5} \longrightarrow H^1(C,\omega_C) = \mathbb{C},$$

which sends

$$H^1(C,\mathcal{L})^{\oplus 5} \ni (\dot\varphi_1,\ldots,\dot\varphi_5) \longmapsto 5\sum_{i=1}^{5}(\varphi_i^4\dot\varphi_i)^{\mathrm{lift}}. \tag{24}$$

Indeed, it is (15) after substitutions $x_i \mapsto \varphi_i$ and $dx_i \mapsto \dot\varphi_i$; the superscript "lift" is as in (23).

The homomorphism $\sigma_{\Phi_\gamma/\mathcal{M}}$ gives a cosection of the relative obstruction sheaf $Ob_{\Phi_\gamma/\mathcal{M}_\gamma}$. Using $\mathbf{w}$, one argues that it factors through the obstruction sheaf of Φ_γ:

$$\sigma : Ob_{\Phi_\gamma} \longrightarrow \mathcal{O}_{\Phi_\gamma}.$$

As the (reduced part of) degeneracy locus of σ is

$$Deg(\sigma)_{\rm red} = \{\xi \mid \sigma(\xi) = 0\}_{\rm red} = \big\{\varphi_1^4 = \cdots = \varphi_4^4 = 0\big\}_{\rm red} = \overline{\mathcal{M}}^{1/5}_{g,\gamma},$$

embedded in Φ_γ via "$\varphi = 0$"-section.

Applying Kiem-Li's cosection localized virtual cycle construction ([**KiLi**]), we obtain the cosection localized virtual cycle

$$[\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma}]^{\rm virt}_\sigma \in A_{d_\gamma}(\overline{\mathcal{M}}^{1/5}_{g,\gamma}, \mathbb{Q}), \tag{25}$$

where d_γ is given in (21).

THEOREM 4.2 (Chang-Li-Li [**ChLiL**]). *For narrow γ, the numerical invariants derived from the cycle $[\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma}]^{\rm virt}_\sigma$ is identical to the FJRW-invariants of the quintic G.*

4.5. FJRW invariants. Let $\gamma_\ell := (\zeta^2, \ldots, \zeta^2)$, ℓ-tuple of ζ^2. In this case $d_{\gamma_\ell} = 0$, and $[\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma_\ell}]^{\rm virt}_\sigma$ are zero cycles. We define the primary FJRW invariants to be

(26)
$$\theta_{g,\ell} := \begin{cases} \deg[\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma_\ell}]^{\rm virt}_\sigma, & \text{when } 2g-2+\ell>0 \text{ and } 2g-2-\ell \in 5\mathbb{Z};\\ 0, & \text{otherwise.}\end{cases}$$

It is direct to see that these are the primary FJRW invariants. Let γ be narrow so that some $\gamma_i = \zeta$, say $\gamma_\ell = \zeta$. We let $\gamma' = (\gamma_1, \cdots, \gamma_{\ell-1})$. Then an easy argument shows that forgetting the ℓ-th defines a morphism

$$\phi_{\gamma,\gamma'} : \overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma} \longrightarrow \overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma'}$$

such that $\phi_{\gamma,\gamma'}$ has relative dimension one, and

$$[\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma}]^{\rm virt}_\sigma = \phi^*_{\gamma,\gamma'}([\overline{\mathcal{M}}^{1/5,5\varphi}_{g,\gamma'}]^{\rm virt}_\sigma).$$

After eliminating all the $m_i = 1$ from γ, we will have $m_j \geq 2$ left. Then by $d_\gamma = \sum(2-m_j)$ (see (21)), the invariant is non-trivial only when all $m_j = 2$. This confirms that the primary FJRW invariants are all γ_ℓ-marked ones.

We define the genus g FJRW invariants of G generating function be

$$F^G_g(t) = \sum_{\ell=0}^{\infty} \theta_{g,\ell} t^\ell. \tag{27}$$

5. Witten's vision

The MSP theory originated from the attempt to realize the vision of Witten. In [**Wit2**] he postulated that *"there a continuous family of conformal field theories (parameterized by the real line) interpolating from Landau-Ginzberg to Calabi-Yau"* ([p. 179]), and ([p. 193])

> *the C-Y/L-G correspondence arises upon examining this relation (family) in the presence of a particular common superpotential.*

The superpotential is $W = p \cdot G(s_i)$. Over the positive part of the real line, *"W restricts and descends to a holomorphic function $\widehat{W}$"* and the theory *"reduces to the Calabi-Yau hypersurface $G = 0$"* in the projective space. Over the negative part of the real line, *"W restricts and descends to a holomorphic function $G(x)$"* on $\mathbb{C}^5/\mu_5$, which is *"the superpotential of the familiar Landau-Ginzberg orbifold."* ([p. 193]) He stated that the only *"singularity is at the origin."* He suggested that these two theories are *"equivalent to each other on dense open subsets."* ([p. 192])

Witten's vision led to the journey to search for a "master theory" that interpolates between the GW of quintic $Q \subset \mathbb{P}^4$ and FJRW of the Fermat quintic $G(x)$.

5.1. The LG models and their quantizations. In Section 3.1 and 4.1, we have introduced the LG models $(K_{\mathbb{P}^4}, \widehat{W})$ and $([\mathbb{C}^5/\mu_5], \widehat{G})$. They are the restriction of the LG model on the Artin stack

$$\mathcal{W} = \text{descent of } p \cdot G(x) : [(\mathbb{C}^5 \times \mathbb{C})/\mathbb{C}^*] \longrightarrow \mathbb{C},$$

where $\mathbb{C}^*$ acts via the same weights $(1^5, -5)$.

It is evident that the cosection localized virtual cycles of *the moduli of stable maps with fields*, and of *the moduli of stable 5-spin curves with five fields* are respectively the quantizations of the LG models $\widehat{W}$ and $\widehat{G}$.

As we have seen, the quantization is via the process of associating to the weight 1 variable x_i the line bundle $\mathcal{L}$; associating to the weight -5 variable p the line bundle $\mathcal{L}^{\otimes(-5)}$ tensored with ω_C.[3] Thus for the LG model

$$\widehat{W} : [((\mathbb{C}^5 - 0) \times \mathbb{C})/\mathbb{C}^*] = K_{\mathbb{P}^4} \longrightarrow \mathbb{C},$$

we substitute the symbol x_i and p by $\varphi_i \in H^0(\mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{\otimes(-5)} \otimes \omega_C)$; substitute the GIT stability condition $(x_1, \cdots, x_5) \neq 0$ by $(\varphi_1, \cdots, \varphi_5)$ is nowhere vanishing, which makes it a morphism $u = [\varphi_1, \cdots, \varphi_5]$ to $\mathbb{P}^4$; add the stability condition $\mathrm{Aut}(\xi)$ finite to make the moduli of stable maps with fields a DM stack. In the end, the LG function $\widehat{W}$ is used to induce a cosection of the obstruction sheaf, allowing us to obtain a compactly supported virtual cycles that gives an alternative construction of the GW invariants of Q.

[3] Tensoring with ω_C is the twisting by mass in Super-String theories. (We learned the twisting by mass from [**GuSh**].)

Similarly, for the LG model

$$\widehat{G} : [(\mathbb{C}^5 \times (\mathbb{C} - 0))/\mathbb{C}^*] = [\mathbb{C}^5/\mu_5] \longrightarrow \mathbb{C},$$

we substitute the symbol x_i and p by $\varphi_i \in H^0(\mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{\otimes(-5)} \otimes \omega_{\mathcal{C}}^{\log})$; substitute the GIT stability condition $p \neq 0$ by ρ is nowhere vanishing, which makes it an isomorphism $\rho : \mathcal{L}^{\otimes 5} \cong \omega_{\mathcal{C}}^{\log}$; add the stability condition $\mathrm{Aut}(\xi)$ finite to make the moduli of 5-spin curves with fields a DM stack. In the end, we use the LG function $\widehat{G}$ to induce a cosection of the obstruction sheaf, to obtain a compactly supported virtual cycles that reconstructs us the FJRW invariants of G for narrow γ.

5.2. The master space LG model. We now introduce the master space $\mathbf{M}$ that interpolates the two GIT quotients $K_{\mathbb{P}^4}$ and $[\mathbb{C}^5/\mu_5]$.

We introduce a new $\mathbb{C}^*$-equivariant space. Let $\mathbb{C}^*$ act on $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1$ by weight $(1^5, -5, 1)$. We choose the semistable locus of this $\mathbb{C}^*$-action to be

$$(\mathbb{C}^5 \times \mathbb{C} \times \mathbb{CP}^1)_{\mathrm{ss}} = \{(x, p, [u, v]) \mid (x, u) \neq (0, 0), (p, v) \neq (0, 0)\}. \tag{28}$$

We introduce the master space as the GIT quotient

$$\mathbf{M} := [\mathbb{C}^5 \times \mathbb{C} \times \mathbb{CP}^1 /\!/ \mathbb{C}^*] = [(\mathbb{C}^5 \times \mathbb{C} \times \mathbb{CP}^1)_{\mathrm{ss}}/\mathbb{C}^*].$$

The function $p \cdot G$ on $\mathbb{C}^5 \times \mathbb{C}$ pulls-back to a $\mathbb{C}^*$-invariant function on $\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1$, thus descends to a regular

$$\mathbf{W} : \mathbf{M} \longrightarrow \mathbb{C}.$$

It is this pair we will be working with.

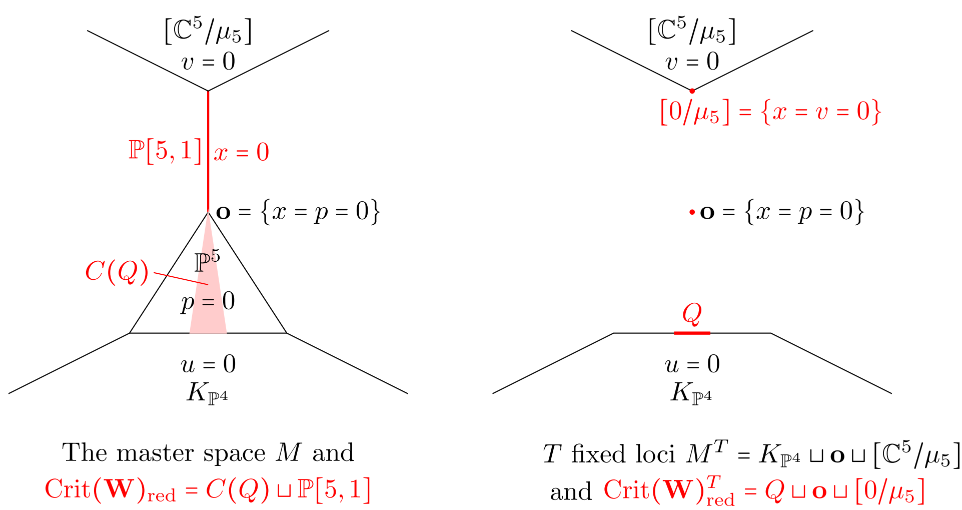


FIGURE 2. This is the geometric illustration of $\mathrm{LG}(\mathbf{M}, \mathbf{W})$ and its T-fixed part. Objects are placed according to their $\mathbb{P}^1$ coordinates $[u, 1]$.

The pair $(\mathbf{M}, \mathbf{W})$ is a $\mathbb{C}^*$-equivariant pair. Let $T = \mathbb{C}^*$ and let it act on $\mathbf{M}$ by

$$t \cdot [x, p, [u, v]] = [x, p, [tu, v]].$$

Its T-fixed locus is

$$\mathbf{M}^T = K_{\mathbb{P}^4} \times [0,1] \sqcup \mathbf{o} \sqcup [\mathbb{C}^5/\mu_5] \times [1,0], \quad \mathbf{o} = [0,0,[1,1]].$$

It is a disjoint union of three subvarieties of $\mathbf{M}$.

We denote the LG model $\mathbf{W} : \mathbf{M} \to \mathbb{C}$ by $\mathrm{LG}(\mathbf{M}, \mathbf{W})$. Then we have

$$\mathrm{LG}(\mathbf{M}, \mathbf{W})^T = \mathrm{LG}(K_{\mathbb{P}^4}, \widetilde{W}) \sqcup \mathrm{LG}(\mathbf{o}, 0) \sqcup \mathrm{LG}([\mathbb{C}^5/\mu_5], W). \tag{29}$$

Heuristically speaking, it says that the T-equivariant theory $\mathrm{LG}(\mathbf{M}, \mathbf{W})$ interpolates between $\mathrm{LG}(K_{\mathbb{P}^4}, \widehat{W})$ and $\mathrm{LG}([\mathbb{C}^5/\mu_5], \widehat{G})$, with extra contributions coming from $\mathrm{LG}(\mathbf{o}, 0)$.

For the cosection localized virtual cycle, it is important to know $\mathrm{Crit}(\mathbf{W})$ and $\mathrm{Crit}(\mathbf{W})^T$. We calculate

$$d\mathbf{W} = 5 \sum_{i=1}^5 p x_i^4 dx_i + \Big(\sum_{i=1}^5 x_i^5 \Big) dp.$$

Its critical locus is

$$\mathrm{Crit}(\mathbf{W})_{\mathrm{red}} = \Big\{ \sum_{i=1}^5 x_i^5 = p = 0 \Big\} \cup \Big\{ x_1 = \cdots = x_5 = 0 \Big\} = C(Q) \cup \mathbb{P}[5,1],$$

where

$$\Big\{ \sum_{i=1}^5 x_i^5 = p = 0 \Big\} = \{(x_1, \cdots, x_5, u) \mid \sum_{i=1}^5 x_i^5 = 0\}/\mathbb{C}^* = C(Q)$$

is the cone over $Q \subset \mathbb{P}^4$;

$$\{x = 0\} = \{(0, p, [1, v]) \mid p, v \in \mathbb{C}\}/\mathbb{C}^* \cong \mathbb{P}[5,1].$$

Combined with the expression of $\mathbf{M}^T$, we conclude

$$\mathrm{Crit}(\mathbf{W})^T_{\mathrm{red}} = Q \sqcup \mathbf{o} \sqcup [\zeta/\mu_5],$$

where $Q = \big\{ u = p = \sum_{i=1}^5 x_i^5 = 0 \big\}$, and $\zeta = (0, 1, [1, 0])$ is an orbifold point in $\mathbf{M}$.

Taking the T-equivariant LG model $\mathbf{W} : \mathbf{M} \to \mathbb{C}$, and following the quantization recipes specified in the previous subsection, we arrive at the notion of Mixed-Spin-P (MSP) fields.

6. Mixed-Spin-P (MSP) fields as the quantization

The MSP theory is a mathematical theory of A-model topological strings on the T-equivariant Landau-Ginzburg model $(\mathbf{M}, \mathbf{W})$. It provides an interpolation between the GW theory of the Calabi-Yau Q and the FJRW theory of the Landau-Ginzburg model $([\mathbb{C}^5/\mu_5], \widehat{G})$. In this section, we review the theory of Mixed-Spin-P (MSP) fields developed by Chang-Li-Li-Liu [**CLLL**].

6.1. Moduli of MSP fields. We let $\mu_5 \le \mathbb{C}^*$ be the subgroup of 5-th roots of unity. We let

$$\mu_5^{\mathrm{br}} = \mu_5 \cup \{(1,\rho),(1,\varphi)\}, \quad \text{and} \quad \mu_5^{\mathrm{na}} = \mu_5^{\mathrm{br}} - \{1\}.$$

For $\gamma \in \mu_5$, we let $\langle\gamma\rangle \le \mathbb{C}^*$ be the subgroup generated by γ; for the two exceptional elements $(1,\rho)$ and $(1,\varphi)$, we agree that $\langle(1,\rho)\rangle = \langle(1,\varphi)\rangle = \langle 1\rangle$.

Let g and $\ell \ge 0$ be two non-negative integers; let d_0, d_∞ be two rational numbers, and let $\gamma = (\gamma_1,\ldots,\gamma_\ell)$, where $\gamma_j \in \mu_5^{\mathrm{na}} = \{\zeta,\zeta^2,\zeta^3,\zeta^4,(1,\rho),(1,\varphi)\}$.

Definition 6.1 (Prestable MSP fields). *A genus g degree $\mathbf{d} = (d_0, d_\infty)$ γ-marked prestable MSP field is a 7-tuple*

$$\xi = ((\mathcal{C},\mathfrak{z}_\bullet),\mathcal{L},\mathcal{N},\varphi,\rho,\mu,\nu),$$

where

MSP-1 *(curve) $(\mathcal{C},\mathfrak{z}_\bullet) = (\mathcal{C},\mathfrak{z}_1,\ldots,\mathfrak{z}_\ell)$ is a genus g, ℓ pointed twisted curve;*

MSP-2 *(line bundles) $\mathcal{L}$, $\mathcal{N}$ are representable line bundles on $\mathcal{C}$ such that*
- *(degrees) $\deg(\mathcal{L}\otimes\mathcal{N}) = d_0$, $\deg(\mathcal{N}) = d_\infty$;*
- *(monodromies) if $\gamma_j = \zeta^{m_j}$ where $m_j \in \mathbb{Z}\cap[1,4]$ then $\mathfrak{z}_j = \mathcal{B}\mu_5$ and $\mathcal{L}|_{\mathfrak{z}_j} \cong (T\mathcal{C}|_{\mathfrak{z}_j})^{\otimes m_j}$ as $\mathrm{Aut}(\mathfrak{z}_j)$-modules; if $\gamma_j \in \{(1,\rho),(1,\varphi)\}$ then $\mathfrak{z}_j$ is a scheme point and we define $m_j = 0$;*

MSP-3 *(fields) $\varphi \in H^0(\mathcal{L}^{\oplus 5})$, $\rho \in H^0(\mathcal{L}^{\otimes(-5)}\otimes\omega_{\mathcal{C}}^{\log})$, $\mu \in H^0(\mathcal{L}\otimes\mathcal{N})$, and $\nu \in H^0(\mathcal{N})$;*

MSP-4 *(constraints) letting*

$$\Sigma^{\mathcal{C}}_{(1,\rho)} = \{\mathfrak{z}_j \mid \gamma_j = (1,\rho)\}, \quad \Sigma^{\mathcal{C}}_{(1,\varphi)} = \{\mathfrak{z}_j \mid \gamma_j = (1,\varphi)\},$$

then for $\mathfrak{z}_j \in \Sigma^{\mathcal{C}}_{(1,\rho)}$ (resp. $\Sigma^{\mathcal{C}}_{(1,\varphi)}$) we have $\rho(\mathfrak{z}_j) = 0$ (resp. $\varphi(\mathfrak{z}_j) = 0$).

Note that MSP-3 and MSP-4 can be combined into

MSP-3' $\varphi\in H^0(\mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus 5})$, $\rho\in H^0(\mathcal{L}^{\otimes(-5)}\otimes\omega_{\mathcal{C}}^{\log}(-\Sigma^{\mathcal{C}}_{(1,\rho)}))$, $\mu \in H^0(\mathcal{L}\otimes\mathcal{N})$, and $\nu \in H^0(\mathcal{N})$.

An *isomorphism*

$$((\mathcal{C},\mathfrak{z}_\bullet),\mathcal{L},\mathcal{N},\varphi,\rho,\mu,\nu) \longrightarrow ((\mathcal{C}',\mathfrak{z}'_\bullet),\mathcal{L}',\mathcal{N}',\varphi',\rho',\mu',\nu')$$

between two MSP fields is a triple (a,b,c), where $a : (\mathcal{C},\mathfrak{z}_\bullet) \cong (\mathcal{C}',\mathfrak{z}'_\bullet)$, $b : a^*\mathcal{L} \cong \mathcal{L}'$ and $c : a^*\mathcal{N} \cong \mathcal{N}'$ are isomorphisms that induce the obvious isomorphisms of the data in the two MSP fields. As usual, an isomorphism from ξ to itself is an automorphism of ξ.

Definition 6.2 (Stability). *A prestable MSP field ξ is* stable *if (φ,μ), (ρ,ν), and (μ,ν) are nowhere zero, and $\mathrm{Aut}(\xi)$ is finite.*

Lemma 6.3. *Let ξ be a γ-marked stable MSP field. Then $\mathcal{L}\otimes\mathcal{N}|_{\mathfrak{z}_j}$ is the trivial $\mathrm{Aut}(\mathfrak{z}_j)$-module for all marked points $\mathfrak{z}_j$. Therefore, $\mathcal{L}\otimes\mathcal{N}$ descends to a line bundle on the coarse moduli C of $\mathcal{C}$, and $d_0 = \deg(\mathcal{L}\otimes\mathcal{N}) \in \mathbb{Z}$.*

PROOF. Suppose $\mathrm{Aut}(\mathfrak{z}_j)$ is non-trivial. By MSP-2, $\mathrm{Aut}(\mathfrak{z}_j)$ acts non-trivially on $\mathcal{L}|_{\mathfrak{z}_j}$, so $\varphi(\mathfrak{z}_j) = 0$, which forces $\mu(\mathfrak{z}_j) \neq 0$ since ξ is stable, implying $\mathcal{L} \otimes \mathcal{N}|_{\mathfrak{z}_j} \cong \mathbb{C}$. □

Let $\mathcal{W}_{g,\gamma,\mathbf{d}}$ be the moduli of genus g, γ-marked, degree $\mathbf{d} = (d_0, d_\infty)$ stable MSP fields. It is a DM stack, locally of finite type. We endow it with the T action

$$t \cdot [(\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu] = [(\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \mathcal{N}, \varphi, \rho, t\mu, \nu]. \tag{30}$$

6.2. Cosection localized virtual cycle. Let $\mathfrak{D}'' = \mathfrak{D}_{g,\gamma,\mathbf{d}}$ be the moduli space of triples $((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \mathcal{N})$ satisfying MSP-1 and MSP-2. It is a smooth Artin stack of dimension $5g - 5 + \ell$.

Let

$$\pi_{\mathcal{W}/\mathfrak{D}''} : \mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}} \longrightarrow \mathfrak{D}'' = \mathfrak{D}_{g,\gamma,\mathbf{d}}$$

be the forgetful morphism, forgetting φ, ρ, μ and ν. The relative tangent space of $\pi_{\mathcal{W}/\mathfrak{D}''}$ at $\xi = ((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \mathcal{N}, \dots)$ is

$$\begin{aligned} \mathrm{T}_{\mathcal{W}/\mathfrak{D}'',\xi} = & H^0(\mathcal{C}, \mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus 5}) \oplus H^0(\mathcal{C}, \mathcal{L}^{\otimes(-5)} \otimes \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})) \\ & \oplus H^0(\mathcal{C}, \mathcal{L} \otimes \mathcal{N}) \oplus H^0(\mathcal{C}, \mathcal{N}); \end{aligned}$$

its relative obstruction space to deforming ξ is

$$\begin{aligned} Ob_{\mathcal{W}/\mathfrak{D}'',\xi} = & H^1(\mathcal{C}, \mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)})^{\oplus 5}) \oplus H^1(\mathcal{C}, \mathcal{L}^{\otimes(-5)} \otimes \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})) \\ & \oplus H^1(\mathcal{C}, \mathcal{L} \otimes \mathcal{N}) \oplus H^1(\mathcal{C}, \mathcal{N}). \end{aligned}$$

Since all higher obstructions vanish, $\pi_{\mathcal{W}/\mathfrak{D}''}$ is virtually smooth, and gives a relative perfect obstruction theory.

We calculate the virtual dimension of $\mathcal{W}_{g,\gamma,\mathbf{d}}$. First by Riemann-Roch, the relative virtual dimension of $\pi_{\mathcal{W}/\mathfrak{D}''}$ is

$$\begin{aligned} & 5\big(\deg \mathcal{L}(-\Sigma^{\mathcal{C}}_{(1,\varphi)}) + 1 - g - \sum_{j=1}^{\ell} \frac{m_j}{5}\big) \\ & \quad + \big(\deg \mathcal{L}^{\otimes(-5)} \otimes \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)}) + 1 - g\big) \\ & \quad + \big(\deg \mathcal{L} \otimes \mathcal{N} + 1 - g\big) + \big(\deg \mathcal{N} + 1 - g - \sum_{m_j \neq 0} (1 - \frac{m_j}{5})\big) \\ & = d_0 + d_\infty + 6 - 6g - 4\big(\ell_\varphi + \frac{1}{5} \sum_{j=1}^{\ell} m_j\big) \end{aligned}$$

where $\ell_\varphi = \sharp\{j \mid \gamma_j = (1, \varphi)\}$. The virtual dimension of $\mathcal{W}_{g,\gamma,\mathbf{d}}$ then is

$$d_{g,\gamma,\mathbf{d}} = d_0 + d_\infty + 1 - g + \ell - 4\big(\ell_\varphi + \frac{1}{5} \sum_{j=1}^{\ell} m_j\big). \tag{31}$$

As before we let $\mathbb{E}_{\mathfrak{D}''}$ be the Hodge bundle over $\mathfrak{D}''$. We form a $\mathfrak{D}''$-morphism

$$\mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}} \xrightarrow{\ \mathbf{w}\ } \mathbb{E}_{\mathfrak{D}''}.$$

It sends $\xi \in \mathcal{W}$ to

$$\mathbf{w}(\xi) = \big((\mathcal{C}, \mathfrak{z}_\bullet), \mathcal{L}, \mathcal{N}, \rho \cdot \sum_{i=1}^{5} \varphi_i^5\big).$$

We claim that $\rho \cdot \sum_{i=1}^{5} \varphi_i^5$ belongs to $H^0(C, \omega_C)$, which is $\mathbb{E}_{\mathfrak{D}''}|_{(\mathcal{C},\mathfrak{z}_\bullet),\mathcal{L},\mathcal{N})}$. Indeed, by abuse of notation we denote by $\Sigma^{\mathcal{C}}_{(1,\varphi)}$ (resp. $\Sigma^{\mathcal{C}}_{(1,\rho)}$) the divisor of all $z_i \in \Sigma^{\mathcal{C}}_{(1,\varphi)}$ (resp $z_i \in \Sigma^{\mathcal{C}}_{(1,\rho)}$), and denote by $\Sigma^{\mathcal{C}}_o$ the divisor of all $\mathfrak{z}_i$ which are orbifold points $\mathcal{B}\mu_5$. Then

$$\begin{aligned} &\Sigma^{\mathcal{C}} = \Sigma^{\mathcal{C}}_{(1,\rho)} + \Sigma^{\mathcal{C}}_{(1,\varphi)} + \Sigma^{\mathcal{C}}_o \quad \text{and} \\ &H^0(\omega_C) = H^0\left(\omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)} - \Sigma^{\mathcal{C}}_{(1,\varphi)} - 5\Sigma^{\mathcal{C}}_o)\right). \end{aligned}$$

Like in 5-spin curve case, we have

$$\begin{aligned} &\sum_{i=1}^{5} \varphi_i^5 \in H^0\left(\mathcal{C}, \mathcal{L}^{\otimes 5}(-5\Sigma^{\mathcal{C}}_{(1,\varphi)} - 5\Sigma^{\mathcal{C}}_o)\right) \quad \text{and} \\ &\rho \in H^0\left(C, \mathcal{L}^{\otimes(-5)} \otimes \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)})\right), \end{aligned}$$

where we also use the condition (MSP-3'). Therefore,

$$\rho \cdot \sum_{i=1}^{5} \varphi_i^5 \in H^0\left(\mathcal{C}, \omega^{\log}_{\mathcal{C}}(-\Sigma^{\mathcal{C}}_{(1,\rho)} - 5\Sigma^{\mathcal{C}}_{(1,\varphi)} - 5\Sigma^{\mathcal{C}}_o)\right) \subset H^0(C, \omega_C).$$

This proves that $\mathbf{w}$ is well-defined.

Like before, the morphism $\mathbf{w}$ induces a relative cosection

$$\sigma_{\mathcal{W}/\mathfrak{D}''} : Ob_{\mathcal{W}/\mathfrak{D}''} \longrightarrow \mathbf{w}^* Ob_{\mathbb{E}_{\mathfrak{D}''}/\mathfrak{D}''} \cong \mathcal{O}_{\mathcal{W}}.$$

The cosection takes the same form as (12) and (24), and factors through a cosection of absolute obstruction sheaf

$$\sigma : Ob_{\mathcal{W}} \longrightarrow \mathcal{O}_{\mathcal{W}}.$$

It is shown in [**CLLL**] that $Deg(\sigma)$ is proper. Applying Kiem-Li's work of cosection localized virtual cycle [**KiLi**], we obtain a properly supported virtual cycle.

THEOREM 6.4 (Chang-Li-Li-Liu [**CLLL**]). *The moduli of stable Mixed-Spin-P fields is a T-equivariant DM stack, locally of finite type. It has a T-equivariant virtual cycle*

$$[\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{virt}}_\sigma \in A^T_{d_{g,\gamma,\mathbf{d}}}(Deg; \mathbb{Q}),$$

supported on a proper substack $Deg \subset \mathcal{W}_{g,\gamma,\mathbf{d}}$.

7. Equivariant cohomology

In preparation for the discussion in the final section (Section 8), we give a brief review on equivariant cohomology. In this section $T = \mathbb{C}^*$.

Given any T space Y, the projection $p_{Y,T} : Y \to [Y/T]$ induces a ring homomorphism

$$p_{Y,T}^* : H_T^*(Y;\mathbb{Q}) = H^*([Y/T];\mathbb{Q}) \longrightarrow H^*(Y;\mathbb{Q}).$$

If $\phi^T \in H_T^*(Y;\mathbb{Q})$ and $\phi = p_{Y,T}^*\phi^T \in H^*(Y;\mathbb{Q})$, we say ϕ is the non-equivariant limit of ϕ^T, and say ϕ^T is a T-equivariant lift of ϕ.

EXAMPLE 7.1. If $Y = \bullet$ (a point), then

$$p_{\bullet,T}^* : H_T^*(\bullet;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}] \longrightarrow H^*(\bullet;\mathbb{Q}) = \mathbb{Q}$$

is a surjective ring homomorphism which sends a polynomial $f(\mathfrak{t}) \in \mathbb{Q}[\mathfrak{t}]$ to $f(0) \in \mathbb{Q}$. In other words, it is given by evaluation at zero.

EXAMPLE 7.2. Let $T = \mathbb{C}^*$ act on $\mathbb{P}^5$ by

$$t \cdot [\varphi_1, \ldots, \varphi_5, \nu] = [\varphi_1, \ldots, \varphi_5, t\nu]$$

where $t \in T$ and $\varphi_1, \ldots, \varphi_5, \nu$ are homogenous coordinates on $\mathbb{P}^5$. Then

$$p_{\mathbb{P}^5,T}^* : H_T^*(\mathbb{P}^5;\mathbb{Q}) = \mathbb{Q}[H,\mathfrak{t}]/\langle H^5(H+\mathfrak{t})\rangle \longrightarrow H^*(\mathbb{P}^5;\mathbb{Q}) = \mathbb{Q}[H]/\langle H^6\rangle$$

is a surjective ring homomorphism given by $H \mapsto H$, $\mathfrak{t} \mapsto 0$.

If $h : Y \to Z$ is a T-equivariant map, then we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ \downarrow{\scriptstyle p_{Y,T}} & & \downarrow{\scriptstyle p_{Z,T}} \\ [Y/T] & \xrightarrow{h_T} & [Z/T] \end{array}$$

which induces the following commutative diagram

$$(32) \qquad \begin{array}{ccc} H^*(Y;\mathbb{Q}) & \xleftarrow{h^*} & H^*(Z;\mathbb{Q}) \\ \uparrow{\scriptstyle p_{Y,T}^*} & & \uparrow{\scriptstyle p_{Z,T}^*} \\ H_T^*(Y;\mathbb{Q}) & \xleftarrow{h_T^*} & H_T^*(Z;\mathbb{Q}) \end{array}$$

In the above diagram, all the arrows are homomorphisms of $\mathbb{Q}$-algebras. The homomorphism h_T^* (resp. h^*) defines a structure of $H_T^*(Z;\mathbb{Q})$-module (resp. $H^*(Z;\mathbb{Q})$-module) on $H_T^*(Y;\mathbb{Q})$ (resp. $H^*(Y;\mathbb{Q})$). In particular, let $Z = \bullet$, we see that $H_T^*(Y;\mathbb{Q})$ is a module over $H_T^*(\bullet;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}]$, and

$$p_{Y,T}^*(f(\mathfrak{t})\phi) = f(0)p_{Y,T}^*\phi$$

for all $\phi \in H_T^*(Y;\mathbb{Q})$ and $f(\mathfrak{t}) \in \mathbb{Q}[\mathfrak{t}]$.

Suppose that Y is a proper smooth algebraic variety or a proper smooth DM stack which is of pure dimension r and is equipped with a T-action.

There is a T-equivariant fundamental class $[Y]^T \in A_r^T(Y;\mathbb{Q})$, and there is a $\mathbb{Q}[\mathfrak{t}]$-linear map

$$\int_{[Y]^T} : H_T^*(Y;\mathbb{Q}) \longrightarrow H_T^*(\bullet;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}] \tag{33}$$

sending $H_T^k(Y;\mathbb{Q})$ to $H_T^{k-2r}(\bullet;\mathbb{Q})$, where

$$H_T^\ell(\bullet;\mathbb{Q}) = \begin{cases} \mathbb{Q}\mathfrak{t}^{\ell/2}, & \text{if } \ell \in 2\mathbb{Z}_{\geq 0}, \\ 0, & \text{otherwise.} \end{cases}$$

There is a fundamental class $[Y] \in A_r(Y;\mathbb{Q})$, and there is a $\mathbb{Q}$-linear map

$$\int_{[Y]} : H^*(Y;\mathbb{Q}) \longrightarrow H^*(\bullet;\mathbb{Q}) = H^0(\bullet;\mathbb{Q}) = \mathbb{Q} \tag{34}$$

sending $H^k(Y;\mathbb{Q})$ to $H^{k-2r}(\bullet;\mathbb{Q})$. The maps (33) and (34) fit in the following commutative diagram

$$\begin{array}{ccc} H_T^*(Y;\mathbb{Q}) & \xrightarrow{\int_{[Y]^T}} & \mathbb{Q}[\mathfrak{t}] \\ \downarrow{\scriptstyle p_{Y,T}^*} & & \downarrow{\scriptstyle p_{\bullet,T}^*} \\ H^*(Y;\mathbb{Q}) & \xrightarrow{\int_{[Y]}} & \mathbb{Q} \end{array} \tag{35}$$

EXAMPLE 7.3. Let T act on $\mathbb{P}^5$ as in Example 7.2. Then

$$\int_{[\mathbb{P}^5]^T} H^k = \begin{cases} (-\mathfrak{t})^{k-5}, & k \geq 5, \\ 0, & k < 5, \end{cases} \quad \text{and} \quad \int_{[\mathbb{P}^5]} H^k = \delta_{k,5}.$$

Suppose that T acts on a non-proper, possibly singular DM stack $\mathcal{W}$ equipped a T-equivariant perfect obstruction theory of virtual dimension r, and that there is a T-equivariant cosection σ such that the degeneracy locus $Deg\sigma$ is a proper substack of $\mathcal{W}$. There is a T-equivariant cosection localized virtual cycle

$$[\mathcal{W}]_\sigma^{\mathrm{virt},T} \in A_r^T(Deg;\mathbb{Q}),$$

and there is a $\mathbb{Q}[\mathfrak{t}]$-linear map

$$\int_{[\mathcal{W}]_\sigma^{\mathrm{virt},T}} : H_T^*(Y;\mathbb{Q}) \longrightarrow H_T^*(\bullet;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}] \tag{36}$$

sending $H_T^k(\mathcal{W};\mathbb{Q})$ to $H_T^{k-2r}(\bullet;\mathbb{Q})$. There is a cosection localized virtual cycle

$$[\mathcal{W}]_\sigma^{\mathrm{virt}} \in A_r^T(Deg;\mathbb{Q}),$$

and there is a $\mathbb{Q}$-linear map

$$\int_{[\mathcal{W}]_\sigma^{\mathrm{virt}}} : H^*(\mathcal{W};\mathbb{Q}) \longrightarrow H^*(\bullet;\mathbb{Q}) = H^0(\bullet;\mathbb{Q}) = \mathbb{Q} \tag{37}$$

sending $H^k(\mathcal{W};\mathbb{Q})$ to $H^{k-2r}(\bullet;\mathbb{Q})$. The maps (36) and (37) fit in the following commutative diagram

$$\begin{array}{ccc} H^*_T(\mathcal{W};\mathbb{Q}) & \xrightarrow{\int_{[\mathcal{W}]^{\mathrm{vir},T}_\sigma}} & \mathbb{Q}[\mathfrak{t}] \\ \Big\downarrow{\scriptstyle p^*_{Y,T}} & & \Big\downarrow{\scriptstyle p^*_{\bullet,T}} \\ H^*(\mathcal{W};\mathbb{Q}) & \xrightarrow{\int_{[\mathcal{W}]^{\mathrm{virt}}_\sigma}} & \mathbb{Q} \end{array} \tag{38}$$

8. Toward a mathematical theory of LG/CY correspondence

8.1. MSP invariants. Using the universal family

$$\pi : \Sigma^{\mathcal{C}} \subset \mathcal{C} \to \mathcal{W} = \mathcal{W}_{g,\gamma,\mathbf{d}} \quad \text{with} \quad (\mathcal{L}, \mathcal{N}, \varphi, \rho, \mu, \nu)$$

we define the evaluation maps (associated to the marked sections $\Sigma_i^{\mathcal{C}}$):

$$\mathrm{ev}_i : \mathcal{W} \to X := \mathbb{P}^5 \cup \mu_5.$$

In case $\langle\gamma_i\rangle \neq 1$, define ev_i to be the constant map to $\gamma_i \in \mu_5$; in case $\gamma_i = (1,\varphi)$, define $\mathrm{ev}_i(\gamma_i) = 1 \in \mu_5$. In case $\gamma_i = (1,\rho)$, let $s_i : \mathcal{W} \to \mathcal{C}$ be the i-th marked section of the universal curve,[4] by (MSP-4) of Definition 6.1 we have $s_i^*\rho = 0$, thus $s_i^*\nu$ is nowhere zero and $s_i^*\mathcal{N} \cong \mathcal{O}_{\mathcal{W}}$. Thus, $s_i^*(\varphi,\mu)$ is a nowhere zero section of $s_i^*\mathcal{L}^{\oplus 6}$, defining the evaluation morphism

$$\mathrm{ev}_i = [s_i^*\varphi_1, \cdots, s_i^*\varphi_5, s_i^*\mu] : \mathcal{W} \to \mathbb{P}^5 \tag{39}$$

such that $\mathrm{ev}_i^*\mathcal{O}_{\mathbb{P}^5}(1) = s_i^*\mathcal{L}$.

Let $T = \mathbb{C}^*$ act on $\mathbb{P}^5$ by

$$t \cdot [\varphi_1, \ldots, \varphi_5, \mu] = [\varphi_1, \ldots, \varphi_5, t\mu],$$

and let T act trivially on μ_5. It makes ev_i T-equivariant.

We introduce the MSP state space. As a vector space over $\mathbb{Q}$ (resp. module over $H^*(BT;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}]$), The MSP state space is the cohomology with rational coefficient

$$\mathcal{H}^{\mathrm{MSP}} = H^*(X;\mathbb{Q}), \quad X = \mathbb{P}^5 \cup \mu_5.$$

Parallelly, the T-equivariant MSP state space is the T-equivariant cohomology

$$\mathcal{H}^{\mathrm{MSP},T} = H^*_T(X;\mathbb{Q}).$$

In terms of generators of $\mathbb{Q}[\mathfrak{t}] = H^*_T(pt;\mathbb{Q})$-modules,

$$H^*_T(\mathbb{P}^5;\mathbb{Q}) = \mathbb{Q}[H,\mathfrak{t}]/\langle H^5(H+\mathfrak{t})\rangle = \mathbb{Q}[\mathfrak{t}]\mathbf{1}_\rho \oplus \bigoplus_{i=1}^{5} \mathbb{Q}[\mathfrak{t}]H^i$$

and

$$H^*_T(\mu_5;\mathbb{Q}) = \mathbb{Q}[\mathfrak{t}]\mathbf{1}_\varphi \oplus \bigoplus_{m=1}^{4} \mathbb{Q}[\mathfrak{t}]\mathbf{1}_{\frac{m}{5}}$$

[4] As $\gamma_i = (1,\rho)$, the i-th marking is a scheme marking.

as graded $\mathbb{Q}[\mathfrak{t}]$-modules, where the degrees are given by (cf. the formula (31) of the virtual dimension):

(40)

$$\deg \mathbf{1}_\rho = 0, \quad \deg H = \deg \mathfrak{t} = 2, \quad \deg \mathbf{1}_\varphi = 8, \quad \text{and} \quad \deg \mathbf{1}_{\frac{m}{5}} = \frac{8}{5}m.$$

The (non-equivariant) MSP state space is a graded vector space over $\mathbb{Q}$, obtained by replacing $\mathbb{Q}[\mathfrak{t}]$ by $\mathbb{Q}$ in the above formula.

We formulate the gravitational descendants. Given

$$a_1, \dots, a_\ell \in \mathbb{Z}_{\geq 0}, \quad \phi_1, \dots, \phi_\ell \in \mathcal{H}^{\mathrm{MSP}} = H^*(X; \mathbb{Q}),$$

we define the MSP-invariants

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP}}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{virt}}_\sigma} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k \in \mathbb{Q}, \tag{41}$$

where

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{virt}}_\sigma = \sum_{\gamma \in (\mu_5^{\mathrm{na}})^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{virt}}_\sigma.$$

Similarly for $\phi_i^T \in \mathcal{H}^{\mathrm{MSP},T}$, define T-equivariant genus g MSP-invariants:

(42)

$$\langle \tau_{a_1}\phi_1^T \cdots \tau_{a_\ell}\phi_\ell^T \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} := \int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{virt},T}_\sigma} \prod_{k=1}^{\ell} (\psi_k^T)^{a_k} \mathrm{ev}_k^* \phi_k \in H^*(BT; \mathbb{Q}) = \mathbb{Q}[\mathfrak{t}],$$

$$[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{virt},T}_\sigma = \sum_{\gamma \in (\mu_5^{\mathrm{na}})^\ell} [\mathcal{W}_{g,\gamma,\mathbf{d}}]^{\mathrm{virt},T}_\sigma.$$

In (42), $\psi_k^T \in H^2(\mathcal{W}_{g,\ell,\mathbf{d}}; \mathbb{Q})$ is some natural T-equivariant lift of $\psi_k \in H^2(\mathcal{W}_{g,\ell,\mathbf{d}}; \mathbb{Q})$ in (41).

Note that the map

$$p_{X,T}^* : \mathcal{H}^{\mathrm{MSP},T} = H_T^*(X; \mathbb{Q}) \longrightarrow \mathcal{H}^{\mathrm{MSP}} = H^*(X; \mathbb{Q}) = H_T^*(X; \mathbb{Q})\Big|_{\mathfrak{t}=0}$$

is surjective. Given $\phi_1, \dots, \phi_\ell \in \mathcal{H}^{\mathrm{MSP}}$ there exist $\phi_1^T, \dots, \phi_\ell^T \in \mathcal{H}^{\mathrm{MSP},T}$ such that $p_{X,T}^* \phi_i^T = \phi_i$, i.e., ϕ_i^T is a T-equivariant lift of ϕ_i. Then

$$\begin{aligned} \langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP}}_{g,\ell,\mathbf{d}} &= p_{\bullet,T}^* \langle \tau_{a_1}\phi_1^T \cdots \tau_{a_\ell}\phi_\ell^T \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} \\ &= \langle \tau_{a_1}\phi_1^T \cdots \tau_{a_\ell}\phi_\ell^T \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}}\Big|_{\mathfrak{t}=0}. \end{aligned}$$

8.2. Torus localization. The T-fixed substack $\mathcal{W}^T_{g,\ell,\mathbf{d}} \subset \mathcal{W}_{g,\ell,\mathbf{d}}$ is a disjoint union of connected components:

$$\mathcal{W}^T_{g,\ell,\mathbf{d}} = \bigcup_{\Gamma \in G_{g,\ell,\mathbf{d}}} F_\Gamma \tag{43}$$

where $G_{g,\ell,\mathbf{d}}$ is a finite set of decorated graphs.

Let $i_\Gamma : F_\Gamma \hookrightarrow \mathcal{W}_{g,\ell,\mathbf{d}}$ be the inclusion. By torus localization of cosection localized virtual cycles proved by Chang-Kiem-Li [**CKL**] and the irregular vanishing proved by Chang-Li [**ChLi3**],

(44)
$$\int_{[\mathcal{W}_{g,\ell,\mathbf{d}}]^{\mathrm{virt}}_\sigma} \prod_{k=1}^{\ell} \psi_k^{a_k} \mathrm{ev}_k^* \phi_k = p^*_{\bullet,T}\left(\sum_{\Gamma \in G^{\mathrm{reg}}_{g,\ell,\mathbf{d}}} \int_{[F_\Gamma]^{\mathrm{virt},T}_{\sigma_\Gamma}} \frac{i_\Gamma^* \prod_{k=1}^{\ell} (\psi_k^T)^{a_k} \mathrm{ev}_k^* \phi_k^T}{e_T(N_\Gamma^{\mathrm{virt}})} \right),$$

where $G^{\mathrm{reg}}_{g,\ell,\mathbf{d}} \subset G_{g,\ell,\mathbf{d}}$ is the subset of regular graphs; $[F_\Gamma]^{\mathrm{virt},T}_{\sigma_\Gamma}$ is the cosection localized virtual cycle of F_Γ; $e_T(N_\Gamma^{\mathrm{virt}})$ is the T-equivariant Euler class of the virtual normal bundle N_Γ^{virt} of F_Γ in $\mathcal{W}_{g,\ell,\mathbf{d}}$; $\phi_i^T \in \mathcal{H}^{\mathrm{MSP},T}$ is a T-equivariant lift of $\phi_i \in \mathcal{H}^{\mathrm{MSP}}$; the map $p^*_{\bullet,T} : \mathbb{Q}[\mathfrak{t}] \to \mathbb{Q}$ is given by evaluation at zero: $f(\mathfrak{t}) \mapsto f(0)$.

Note that as the virtual dimension of $\mathcal{W}_{g,\ell,\mathbf{d}}$ is known, the identity (44) gives a infinitely many vanishing relations. To get a hold of these relations, we notice that the right hand side of (44) can be expressed in terms of the invariants of the following three theories:

(0) $\mathrm{LG}(K_{\mathbb{P}^4}, \widehat{W}) = \mathrm{GW}(Q)$ = GW theory of a quintic threefold,
(1) $\mathrm{LG}(\mathbf{o}, 0) = \mathrm{GW}(\text{point})$ = GW theory of a point, determined by Witten's conjecture [**Wit1**] first proved Kontsevich [**Kon1**], and
(∞) $\mathrm{LG}([\mathbb{C}^5/\mu_5], \widehat{G}) = \mathrm{FJRW}(G)$ = FJRW theory of the quintic polynomial G.

Suppose that $\phi_j \in \mathcal{H}^{\mathrm{MSP},T}$ is homogeneous of degree $2b_j$. Then

$$\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,\mathbf{d}} \in \mathbb{Q}\mathfrak{t}^{\sum_{j=1}^{\ell}(a_j+b_j-1)+g-1-d_0-d_\infty} \cap \mathbb{Q}[\mathfrak{t}] \tag{45}$$

which is zero unless

$$\sum_{j=1}^{\ell}(a_j+b_j-1)+g-1-d_0-d_\infty \in \mathbb{Z}_{\geq 0}.$$

8.3. MSP correlators. We introduce formal variables q_0, q_∞ and define MSP correlators

$$\langle\!\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle\!\rangle^{\mathrm{MSP},T}_{g,\ell} := \sum_{d_0, 5d_\infty \in \mathbb{Z}_{\geq 0}} \langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle^{\mathrm{MSP},T}_{g,\ell,d_0,d_\infty} q_0^{d_0} q_\infty^{d_\infty}. \tag{46}$$

Then

$$\langle\!\langle \tau_{a_1}\phi_1 \cdots \tau_{a_\ell}\phi_\ell \rangle\!\rangle^{\mathrm{MSP},T}_{g,\ell} \in \left(\mathfrak{t}^{\sum_{j=1}^{\ell}(a_j+b_j-1)+g-1} \mathbb{Q}\left[\!\left[\frac{q_0}{\mathfrak{t}}, \left(\frac{q_\infty}{\mathfrak{t}}\right)^{1/5} \right]\!\right] \right) \cap \mathbb{Q}[\mathfrak{t}]\left[\!\left[q_0, q_\infty^{1/5} \right]\!\right].$$

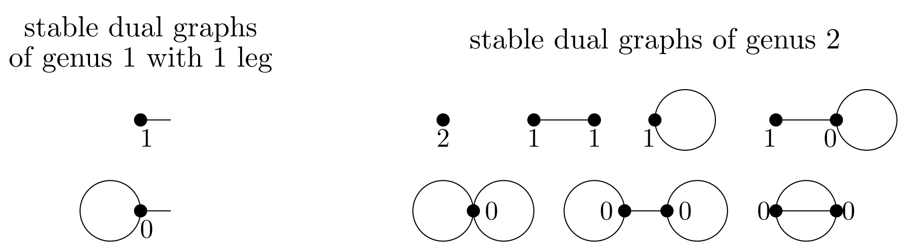
Therefore, the MSP correlator (46) is a homogeneous element in the graded polynomial ring $\mathbb{Q}[\mathfrak{t}, q_0, q_\infty^{1/5}]$ of degree

$$2\Big(\sum_{j=1}^{\ell}(a_j+b_j-1)+g-1\Big),$$

where the grading is given by

$$\deg \mathfrak{t} = \deg q_0 = 2, \quad \deg(q_\infty^{1/5}) = 2/5.$$

To proceed, we recall the notion of stable dual graphs, and stable tripartite dual graphs. Suppose that $2g-2+\ell>0$. Given a genus g, ℓ pointed nodal curve $(C, z_1, \ldots, z_\ell)$, the dual graph of C is a decorated graph Γ where each vertex v corresponds to an irreducible components C_v of C is labelled by the arithmetic genus g_v of C_v, each edge corresponds to a node in C, and each leg corresponds to a marked point. The curve $(C, z_1, \ldots, z_\ell)$ is stable if for each vertex v in its dual graph, $2g_v - 2 + \ell_v > 0$, where ℓ_v is the number of nodes and marked points in C_v, or equivalently, the valency of v. The strata of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,\ell}$ of genus g, ℓ pointed stable curves are in one-to-one correspondence with stable dual graphs of genus g with ℓ legs.



A tripartite stable dual graph is a stable dual graph where each vertex has an additional decoration by an element in $\{0, 1, \infty\}$, so that the set of vertices is a disjoint union $V = V_0 \cup V_1 \cup V_\infty$. If $2g-2+\ell>0$ the set of all tripartite stable graphs of genus g with ℓ legs is a non-empty finite set.

The MSP correlator (46) can be expressed as a finite sum over tripartite stable dual graphs of genus g and with ℓ legs, where contribution from a genus g_v, ℓ_v-valent vertex v in V_0, V_1, V_∞ is a genus g_v, ℓ_v correlator in $\mathrm{GW}(Q)$, GW(point), and $\mathrm{FJRW}(G)$, respectively. A genus g_v, ℓ_v correlator in $\mathrm{GW}(Q)$ (resp. $\mathrm{FJRW}(G)$) can be expressed in terms of $F^Q_{g_v}(q)$ (resp. $F^G_{g_v}(t)$) and its derivatives with respect to $\log(q)$ (resp. t) up to order ℓ_v. The propagators are genus-zero invariants. One may also consider the MSP-$[0,1]$ (resp. MSP-$[1,\infty]$) theory, which is a sub-theory of the MSP theory defined using MSP moduli spaces $\mathcal{W}_{g,\ell,\mathbf{d}=(d_0,0)}$ (resp. $\mathcal{W}_{g,\ell,\mathbf{d}=(0,d_\infty)}$) and insertions from the subspace $H^*(\mathbb{P}^5)$ (resp. $H^*(\mu_5)$) of the MSP state space $\mathcal{H}^{\mathrm{MSP}}$. The correlators in the MSP-$[0,1]$ (resp. MSP-$[1,\infty]$) theory depend only on one-variable q_0 (resp. q_∞), and can be expressed as a sum over bipartite stable dual graphs with $V = V_0 \cup V_1$ (resp. $V = V_1 \cup V_\infty$).

Via direct calculations, one sees that the Yamaguchi-Yau polynomiality conjecture and the BCOV Feynman sum formula pops up without much efforts. To get a real hold of BCOV Feynman sum formula, Chang-Guo-Li-Li introduced the NMSP field theory in [**CGLL**]. In a nutshell, this is via adding N many MSP fields to approximate BCOV Feynman integral. Miraculously, this led to a proof of Yamaguchi-Yau polynomiality conjecture in [**CGL1**], and a proof of BCOV Feynman sum formula (cf. [**CGL2**]).

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