Associative submanifolds and gradient cycles

Simon Donaldson and Christopher Scaduto

Dedicated to Professor S.-T. Yau, for his 70th birthday

Abstract. We discuss a model for associative submanifolds in G_2 manifolds with $K3$ fibrations, in the adiabatic limit. The model involves graphs in a 3-manifold whose edges are locally gradient flow lines. We show that this model produces analogues of known singularity formation phenomena for associative submanifolds. We propose conjectures on the existence of associative and special Lagrangian submanifolds in certain product spaces, corresponding to the vertices of the graphs.

1. Introduction

There is a standard cross product $\mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7$ which is related to the 8-dimensional Cayley algebra in the same way that the familiar cross product on $\mathbf{R}^3$ is related to the quaternion algebra. The Euclidean form on $\mathbf{R}^7$ can be recovered from the cross product by the formula

$$|x|^2 = -\frac{1}{6}\mathrm{Tr}\left(L_x^2\right)$$

where $L_x(y) = x \times y$. One definition of a G_2 structure on a 7-manifold M is a cross product $TM \times TM \to TM$ which is equivalent to the standard model at each point. Applying the formula above in each tangent space, the cross product induces a Riemannian metric on M. The G_2 structure is called torsion-free if the cross product is parallel with respect to the Levi-Civita connection of this Riemannian metric on M. A 3-dimensional submanifold $P \subset M$ is called *associative* if its tangent spaces are closed under the cross-product. Associative submanifolds are interesting from many points of view. They are examples of calibrated submanifolds in the sense of Harvey and Lawson [11] and they are fundamental objects in G_2-geometry. The purpose of this article is to explore a model for associative submanifolds in a class of G_2-structures near to an "adiabatic limit", which was discussed in [5]. (Related ideas were introduced earlier by Gukov, Yau and Zaslow in [10].) In this case the 7-manifold M is equipped with a fibration $\pi : M \to N$ (with

some singular fibres) over a 3-dimensional base and the smooth fibres are diffeomorphic to K3 surfaces. Our model is built on the precise knowledge of complex curves in the fibres coming from the standard theory of K3 surfaces. The basic idea is to describe associative submanifolds via certain graphs in the base N. This idea is in the same vein as other constructions in the literature, in particular of tropical curves in Calabi-Yau manifolds with Strominger-Yau-Zaslow fibrations (see the further discussion in Section 5 below).

We should emphasise at the outset that in this article we only take the first steps towards a more comprehensive theory that one can hope will emerge in the future. In particular we do not prove anything about actual associative submanifolds here. The main purpose of this article is to develop an independent "adiabatic limit theory", involving graphs in 3-manifolds, and to show that it can mimic important known phenomena of singularity formation for associative submanifolds.

Acknowledgements. This work was supported by the Simons Foundation through the Simons Collaboration *Special holonomy in geometry, analysis and physics*. The authors are grateful to Rodrigo Barbosa, Chris Gerig and Andrew Neitzke for helpful discussions.

2. Review of standard theory and Joyce's conjecture

We will now review more systematically some standard material on G_2-geometry.

A G_2 structure on M defines a 3-form ϕ, related to the cross product and metric by

$$\phi(\xi_1, \xi_2, \xi_3) = \langle \xi_1 \times \xi_2, \xi_3 \rangle.$$

We also have the Hodge dual 4-form $*\phi$. If the structure is torsion-free these forms are parallel and hence closed. An alternative formulation of the associative condition for a 3-dimensional submanifold P is that for each point x of P and tangent vector $v \in TM_x$ the contraction $i_v(*\phi)$ restricts to zero on TP_x. This leads to a "Floer type" description of associative submanifolds. Let $P_0 \subset M$ be some compact submanifold and let $\mathcal{P}_0$ be a space of submanifolds close to P_0 in a suitable sense. Then we can define a functional $\mathcal{F}$ on $\mathcal{P}_0$ by

$$\mathcal{F}(P) = \int_W *\phi \tag{1}$$

where W is a 4-chain in a small neighbourhood of P_0 in M with $\partial W = P - P_0$. The facts that $*\phi$ is closed and that we are only working with small deformations of P_0 means that this functional is well-defined, independent of the choice of W. The derivative of $\mathcal{F}$ at P is given by

$$\int_P i_v * \phi,$$

where v is a variation vector field, and we see that the associative submanifolds in $\mathcal{P}_0$ are exactly the critical points of $\mathcal{F}$. Globally, on a whole space of submanifolds $\mathcal{P}$, we do not usually get a well-defined functional but we have a well-defined closed 1-form $d\mathcal{F}$ on $\mathcal{P}$ whose zeros are associative submanifolds, in the familiar way in Floer-type theories. The linearisation of the associative condition is an elliptic differential operator of index zero acting on sections of the normal bundle. (The fact that the index is zero is a consequence of the variational description, which implies that the linearisation is a self-adjoint operator.)

The discussion above applies to any G_2 structure for which the 4-form $*\phi$ is closed. The condition that the 3-form ϕ is closed enters in the calibrated theory: it means that a compact associative submanifold is absolutely volume minimising in its homology class with volume the homological invariant $\langle[\phi],[P]\rangle$. In this case we get at least some partial compactness properties of the set of associative submanifolds in a fixed homology class. A longstanding theme in the literature is the possibility of developing an enumerative theory, "counting" associative submanifolds, or—more ambitiously—defining Floer homology groups. The fundamental difficulty in doing this comes from the possible formation of singularities and corresponding failure of compactness. A detailed understanding of this seems a long way off but there is a standard conjecture in the field, due to Joyce [**13**], which suggests that it may only be necessary to consider three phenomena. (See also the discussion in [**4**].) We consider a generic 1-parameter family $\times_t$, for $t \in [-1,1]$ on a compact 7-manifold M and fix a 3-dimensional homology class in M. We leave imprecise the exact meaning of generic and the exact conditions imposed on the structures. One expects that for all but a discrete set $S \subset [-1,1]$ of parameter values t there is a finite set of associative submanifolds in the given homology class and that as we vary t in $[-1,1] \setminus S$ a signed count of these will be locally constant. The question is, what singularities can develop at the exceptional parameter values $t \in S$? Joyce conjectures that, in generic 1-parameter families, one will only encounter phenomena which we will refer to in this article as:

(1) "multiple covers";
(2) "crossing";
(3) "surgery triples".

In (1) we have in mind a situation where a 1-parameter family of embedded submanifolds $\iota_t : \Pi \to M$ converge as $t \to 0$ to a map $\iota_0 : \Pi \to M$ which is a covering (possibly branched) of its image. Such behaviour has been extensively studied for pseudo-holomorphic curves and there is some work of Doan and Walpuski in the case of associative submanifolds [**4**], but the theory has not yet been developed very far so we will ignore this multiple cover phenomenon in this article.

For (2), observe that for dimensional reasons we expect that generically associative submanifolds do not intersect (or self-intersect). However in a

generic 1-parameter family we can expect to see associative submanifolds P_t, Q_t which intersect at some parameter value $t = 0$. It was predicted by Joyce and confirmed by Nordström (in unpublished work, to appear) that in this situation (if the intersection in the family is transverse) there will be another family $(P\sharp Q)_t$ of associative submanifolds, defined either for $t > 0$ or for $t < 0$ (but not both) described topologically by smoothing the singular union $P_0 \cup Q_0$ into a connected sum. The differential geometric model near the intersection point is given by a "Lawlor neck". Thus in this situation a straightforward count of associative submanifolds in the homology class of $(P\sharp Q)_t$ will change at $t = 0$.

In (3) the differential geometric model is provided by families of special Lagrangian submanifolds in $\mathbf{C}^3$ found by Harvey and Lawson [11]. Take standard complex co-ordinates z_1, z_2, z_3 and for $s \geq 0$ define

$$L_1^s = \{(z_1, z_2, z_3) : \mathrm{Im}(z_1 z_2 z_3) = 0, \mathrm{Re}(z_1 z_2 z_3) \geq 0, |z_1|^2 - |z_2|^2 = s; |z_2|^2 = |z_3|^2\}.$$

When $s = 0$ this is the cone over the standard torus $T^2 \subset S^5$. For $s > 0$ we get a special Lagrangian submanifold in $\mathbf{C}^3$ which, for a standard matching of the structures, is an associative submanifold in $\mathbf{C}^3 \times \mathbf{R}$. Topologically, this submanifold is obtained from the cone by cutting out a neighbourhood of the vertex and gluing $D^2 \times S^1$ to the resulting T^2 boundary. Permuting the co-ordinates gives similar families L_2^s, L_3^s, both equal to the cone when $s = 0$. Topologically, we obtain L_2^s, L_3^s from L_1^s (all for $s > 0$) by performing Dehn surgeries on the circle in L_1^s formed by the core of $D^2 \times S^1$. The manifolds form a "surgery triple", with a cyclic symmetry between them, differing by the way in which $D^2 \times S^1$ is attached to T^2.

Returning to the compact 7-manifold M with a 1-parameter family of structures $\times_t$, Joyce explains in [13] that one could encounter a singular associative P at parameter value $t = 0$ with a singularity modelled on the cone as above. For $t < 0$ there could be an associative submanifold P_1^t, locally modelled on L_1^{-t}, and for $t > 0$ a pair of associative submanifolds P_2^t, P_3^t, locally modelled on L_2^t, L_3^t. Or it could happen that P_2^t, P_3^t exist for $t < 0$ and P_1^t exists for $t > 0$. But either way a straightforward count of associative submanifolds will change across $t = 0$.

3. Kovalev-Lefschetz fibrations

3.1. Topology. We will use the terminology that the *K3 manifold* is the oriented differentiable 4-manifold X underlying any complex K3 surface. We recall that, with its cup product form, $H^2(X; \mathbf{R})$ is isomorphic to $\mathbf{R}^{3,19}$. We write $\Lambda_X \subset \mathbf{R}^{3,19}$ for the integer lattice and $O(\Lambda_X) \subset O(3, 19)$ for its automorphism group. We call a class α in Λ_X with $\alpha^2 = -2$ a -2-*class*. We will frequently use Poincaré duality to identify $H_2(X)$ with $H^2(X)$.

Definition 1. A topological Kovalev-Lefschetz (KL) fibration consists of data $(M, \Phi, N, L, \tilde{L}, \pi)$ where:

- M is a compact 7-manifold, and Φ is a class in $H^3(M, \mathbf{R})$;

- N is a compact oriented 3-manifold, $L \subset N$ is a link (a disjoint union of embedded circles) and $\pi : M \to N$ is a smooth map;
- $\tilde{L} \subset M$ is a submanifold and π restricts to a diffeomorphism from $\tilde{L}$ to L;
- at each point of $M \setminus \tilde{L}$ the derivative of π is surjective;
- around a point of $\tilde{L}$ and the corresponding point of L there are co-ordinates in which π is given by the model $\pi_0 : \mathbf{C}^3 \times \mathbf{R} \to \mathbf{C} \times \mathbf{R}$

$$\pi_0(z_1, z_2, z_3, t) = (z_1^2 + z_2^2 + z_3^2; t) \tag{2}$$

- each fibre of π over points of $N \setminus L$ is diffeomorphic to the K3 manifold X.

It is sometimes convenient to regard N as an orbifold with model at points of L given by the map $(z, t) \mapsto (z^2, t)$ from $\mathbf{C} \times \mathbf{R}$ to $\mathbf{C} \times \mathbf{R}$. (Recall that for such codimension-2 orbifold points the orbifold is also a manifold.)

Given this structure we obtain a flat vector bundle E_0 over $N \setminus L$ with fibre $\mathbf{R}^{3,19}$ and structure group $O(\Lambda_X)$ given by the cohomology along the fibres. The monodromy of this bundle around a small loop about L is of order 2, defined by reflection in a "vanishing cycle"—which is a -2 class in $H^2(X)$. We can extend this flat vector bundle to a flat orbifold vector bundle E over N and we have a corresponding sheaf $\underline{E}$ over N of locally constant sections. The Leray spectral sequence gives an exact sequence

$$0 \to \mathbf{R} = H^3(N; \mathbf{R}) \to H^3(M; \mathbf{R}) \to H^1(N; \underline{E}) \to 0. \tag{3}$$

The class Φ defines a lift of E to an affine orbifold bundle E^+ over N. This is equivalent to saying that we have flat bundle E_0^+ over $N \setminus L$ with structure group A, where A is the affine extension

$$(\mathbf{R}^{3,19}, +) \to A \to O(\Lambda_X),$$

and the monodromy around each component of L maps to a reflection in $O(\Lambda_X)$. To summarise, from any topological KL fibration as above we can obtain data (N, L, E^+). The theme of this article is that we can study such data in 3-dimensions, independent of the 7-dimensional picture which motivated it.

In this article the dual of the sequence (3) will be important. Recall the standard notion in algebraic topology of homology in a local co-efficient system. In our case we have a local co-efficient system E_0 over $N \setminus L$. The group $H_1(N \setminus L, E_0)$ can be defined by 1-cycles as follows. We consider an oriented graph Γ embedded in $N \setminus L$. Each edge γ of the graph is labelled by a constant section α_γ of E_0 over γ and reversing the orientation of γ is equivalent to changing the sign of the label. At a vertex of the graph we have

$$\sum_\gamma \pm \alpha_\gamma = 0 \tag{4}$$

where the sum runs over edges γ incident to the vertex and the sign $\pm$ is determined by ingoing/outgoing orientation. To obtain the homology group

$H_1(N \setminus L, E_0)$ we divide the group of 1-cycles by a subgroup of boundaries, defined in a similar fashion.

To bring in the link L we make one change to the above recipe. We take graphs in N and we allow an edge to terminate on a point of L provided that its label is a multiple of the corresponding vanishing cycle. (To be more precise, the vanishing cycle is defined up to sign in the fibres of E_0 near L and we allow the label to take either sign.) Any intersection point of the graph with the link is assumed to be one of these terminal vertices. In this fashion we obtain a group which we denote by $H_1(N, E)$ which fits into an exact sequence

$$0 \to H_1(N; E) \to H_3(M; \mathbf{R}) \to H_3(N; \mathbf{R}) = \mathbf{R} \to 0 \tag{5}$$

dual to (3).

The class $\Phi \in H^3(M)$ gives a linear map from $H_3(M)$ to $\mathbf{R}$ and hence a linear map $\chi : H_1(N; E) \to \mathbf{R}$. This can be described as follows. Choose any smooth orbifold section u of the affine bundle E^+. That is, in an orbifold chart around a point of L the section is given by a $\mathbf{Z}/2$-equivariant map to $\mathbf{R}^{3,19}$, where $\mathbf{Z}/2$ acts on $\mathbf{R}^{3,19}$ by the local monodromy. Let Γ be a labelled graph as above. For each edge γ of Γ with label α_γ and running from p to q we can define

$$\langle \gamma, u \rangle = \alpha_\gamma (u(p) - u(q)). \tag{6}$$

Then the map $\chi : H_1(N; E) \to \mathbf{R}$ defined by Φ is induced at the chain level by $\Gamma \mapsto \sum_\gamma \langle \gamma, u \rangle$.

3.2. Adiabatic G_2 structures. We recall some material from [5]. One standard model for the cross product on $\mathbf{R}^7$ is obtained by writing $\mathbf{R}^7 = \mathbf{R}^4 \oplus \mathbf{R}^3$ where $\mathbf{R}^4$ is taken with its standard orientation and Euclidean structure and $\mathbf{R}^3$ is identified the 3-dimensional space of self-dual 2-forms on $\mathbf{R}^4$. It is more convenient to work with the 3-form (which determines the cross product). If $\omega_1, \omega_2, \omega_3$ is a standard basis for Λ^+ and y_i are coordinates on $\mathbf{R}^3$, the model 3-form on $\mathbf{R}^7$ is $\phi_0 = -\sum \omega_i dy_i + dy_1 dy_2 dy_3$. (Here we are using a different sign convention from [5], which fits better with the literature.)

We define a *hyperkähler structure* on the K3 manifold X to be a triple of closed 2-forms $\Omega_1, \Omega_2, \Omega_3$ such that

$$\Omega_i \wedge \Omega_j = a_{ij} \mathrm{vol}_X,$$

for some volume form vol_X on X and a constant positive definite matrix (a_{ij}). It is more standard to require that $a_{ij} = \delta_{ij}$ but this can always be achieved by a change of basis and the extra freedom will be convenient. If we have such a structure then we get a torsion-free G_2-structure on $X \times \mathbf{R}^3$ with 3-form

$$-\sum \Omega_i dy_i + dy_1 dy_2 dy_3. \tag{7}$$

The deep fact we need is the *Torelli theorem* for K3 surfaces. In our set-up this can be stated as follows. Suppose that $h_1, h_2, h_3 \in \mathbf{R}^{3,19} = H^2(X;\mathbf{R})$ span a maximal positive subspace H in $\mathbf{R}^{3,19}$. Suppose in addition that there is no class α in Λ_X with $\alpha^2 = -2$ which is orthogonal to H. Then there is a hyperkähler structure (Ω_i) on X with $[\Omega_i] = h_i$ and this is unique up to the action of diffeomorphisms of X which act trivially on $H^2(X)$.

We now return to our topological KL fibration and affine bundle E^+ over the 3-manifold N and let u be a section of E^+. We consider first the situation over a co-ordinate neighbourhood $B \subset N \setminus L$ with co-ordinates y_i. Over B the section u is given by a map $u_B : B \to \mathbf{R}^{3,19}$. We say that u_B is positive if at each point b of B the image of the derivative of u_B is a maximal positive subspace H_b in $\mathbf{R}^{3,19}$. We also assume that there is no -2 class orthogonal to H_b. Then we can apply the Torelli theorem with h_i the image of ∂_{y_i} under the derivative of u_B. So at each point b of B we get a hyperkähler structure (Ω_i) on the fibre $\pi^{-1}(b) \subset M$. A procedure for choosing forms $\tilde{\Omega}_i$ on $\pi^{-1}(B)$ which restrict to Ω_i on the fibres is explained in [5]. We say that u_B is *maximal positive* if it is positive and the image is a "maximal submanifold" of $\mathbf{R}^{3,19}$—i.e. the image satisfies the Euler-Lagrange equation associated to the volume functional, just as for minimal submanifolds in Euclidean spaces. Write the induced volume form on B as $\lambda dy_1 dy_2 dy_3$ and introduce a positive parameter ϵ. Then we call the 3-form on $\pi^{-1}(B) \subset M$

$$\phi_\epsilon = -\epsilon \sum \tilde{\Omega}_i dy_i + \lambda dy_1 dy_2 dy_3 \tag{8}$$

an *adiabatic solution* to the torsion-free G_2-equations. (This is independent of the choice of local co-ordinates.) For a fixed ϵ this is not an exact solution but, roughly speaking, when ϵ is small it can be deformed slightly to an exact solution $\tilde{\phi}_\epsilon$. (For more precise statements we refer to [5].) In the metric induced by ϕ_ϵ the diameter of the fibre is $O(\epsilon^{1/2})$ while the diameter of $\pi^{-1}(B)$ is $O(1)$, so as $\epsilon \to 0$ we are studying a collapsing family of metrics (in the same spirit as the Calabi-Yau metrics discussed by Tosatti in this volume [21]).

The most technical issue here is the behaviour of the section u near a point p of the link L, which we now describe.

- There is an orthogonal decomposition

$$\mathbf{R}^{3,19} = H \oplus \mathbf{R}\delta \oplus K \tag{9}$$

  where H is a maximal positive subspace and δ is the vanishing cycle. We require that $\pm\delta$ are the only -2 classes in $H \oplus \mathbf{R}\delta$.
- There are local co-ordinates (z,t) centred at p (with $z \in \mathbf{C}$ and $t \in \mathbf{R}$) so that L is locally defined by $z = 0$ and the section u is given by a multivalued map $(I(z,t), f(z,t), F(z,t))$ with respect to the decomposition of $\mathbf{R}^{3,19}$ as follows.
  (1) I is the identity under a linear identification $H = \mathbf{C} \oplus \mathbf{R}$;
  (2) F is a smooth single-valued function which vanishes along with its first derivatives at the origin;

(3) f is a 2-valued function (defined up to sign) with

$$f(z,t) = \mathrm{Re}(b(t)z^{3/2}) + \eta(z,t) \tag{10}$$

where b is smooth and nowhere zero and $\eta(z,t) = O(|z|^{5/2})$. More precisely we require that all t derivatives of η are $O(|z|^{5/2})$ and the first derivatives of η in the z factor are $O(|z|^{3/2})$.

In a standard way, we can avoid the use of multivalued functions by passing to an orbifold chart, as explained in [5]. We define a *branched maximal positive section* of E^+ to be a section which satisfies these conditions near L and the conditions described before in local coordinates $B \subset N \setminus L$.

The conclusion is that we have a notion of an *adiabatic G_2-structure* which is a quadruple (N, L, E_+, u) where E_+ is a flat affine orbifold bundle over N with monodromy around L given by reflections and u is a branched maximal positive section.

We should mention that, at the time of writing, there are no real examples of these structures known and it has not been shown that they lead to collapsing families of torsion-free G_2-structures $\tilde{\phi}_\epsilon$ on compact 7-manifolds. But one can hope that this will change.

4. Adiabatic associative submanifolds

Let (Ω_i) be a hyperkähler structure on X. For each nonzero vector $v \in \mathbf{R}^3$ there is a complex structure J_v on X for which $\sum v_i\Omega_i$ is a Kähler form. If $\Sigma \subset X$ is a smooth complex curve with respect to the complex structure J_v then it is easy to check that $\Sigma \times \mathbf{R}v$ is an associative submanifold in the product $X \times \mathbf{R}^3$ with the G_2-structure (7). Let α be a class in the integer lattice $\Lambda_X \subset \mathbf{R}^{3,19}$ with $\alpha^2 \geq -2$ and let $p(\alpha) \in H$ be the orthogonal projection to the subspace H spanned by $[\Omega_i]$ so $p(\alpha) = \sum v_i[\Omega_i]$ for a vector v in $\mathbf{R}^3$. The vector v is not zero: if $\alpha^2 \geq 0$ this follows from the fact that H is a maximal positive subspace and if $\alpha^2 = -2$ it is the condition arising in the Torelli theorem. By construction, α has type $(1,1)$ with respect to the complex structure J_v and it corresponds to a holomorphic line bundle L of positive degree. The Riemann-Roch theorem implies that $\dim H^0(L) \geq \alpha^2/2 + 2 \geq 1$, so α is represented by a J_v-complex curve. In this article we restrict attention to the case when $\alpha^2 = -2$, that is, a -2 class. Then standard theory gives that there is a *unique* J_v complex curve Σ in the class α and this is "generically" a smooth embedded 2-sphere. More precisely, we will say that the pair (α, H) is *irreducible* if we cannot write α as a sum of -2 classes α_i with $p(\alpha_i) = \lambda_i p(\alpha)$ for $\lambda_i > 0$. Then if (α, H) is irreducible the J_v complex representative is a smooth 2-sphere. If (α, H) is not irreducible the representative could have a number of components, as we will discuss further in Section 6.4 below.

Let ϕ_ϵ be the adiabatic 3-form in (8) over $\pi^{-1}(B) \subset M$. To simplify the formulae we work in local co-ordinates centred at a point b in B, chosen so that $a_{ij} = \delta_{ij}$ at b where the integral of the volume form vol_X is 1. So, at

the point b,

$$\int_X \Omega_i \wedge \Omega_j = \delta_{ij}$$

and $\lambda = 1$. Then at this point

$$*\phi_\epsilon = \epsilon \sum \tilde{\Omega}_i dy_j dy_k + \epsilon^2 \mathrm{vol}_X, \tag{11}$$

where the sum runs over cyclic permutations of (123). Thus the family of 4-forms $\epsilon^{-1} * \phi_\epsilon$ has a well-defined limit $\Psi = \sum \Omega_i dy_j dy_k$ as $\epsilon \to 0$. Of course Ψ is defined over all of B, with a more complicated formula in a general co-ordinate system. Recall that the associative condition for a submanifold P in the G_2-structure ϕ_ϵ is that $i_v * \phi_\epsilon$ restricts to 0 on P for all tangent vectors v in the 7-manifold. Thus, while the 4-form Ψ does *not* correspond to a G_2-structure we can consider the same condition. We say that a submanifold P is "Ψ-associative" if $i_v \Psi$ restricts to zero on P for all $v \in TM$.

Let α be a -2 class. In a trivialisation of the bundle E^+ over B the section u is given by a map $u_B : B \to \mathbf{R}^{3,19}$ and we have a function h_α on B defined by

$$h_\alpha(y) = \langle \alpha, u_B(y) \rangle.$$

We also have a Riemannian metric on B induced by the embedding in $\mathbf{R}^{3,19}$. Let $\gamma : (a, b) \to B$ be a *gradient flowline* for the function h_α, *i.e.*

$$\gamma'(s) = (\mathrm{grad} h_\alpha)_{\gamma(s)}\,.$$

Assume that at each point $\gamma(s)$ the pair $(\alpha, H_{\gamma(s)})$ is irreducible, where $H_{\gamma(s)}$ is the positive subspace in $\mathbf{R}^{3,19}$ defined by the image of du_B. We can apply the preceding discussion at each point $\gamma(s)$ and we see that there is a unique embedded 2-sphere $\Sigma_s \subset \pi^{-1}(\gamma(s))$ in the class α which is complex with respect to the complex structure defined by the velocity vector $\gamma'(s)$. Thus we obtain a 3-dimensional submanifold $P_\gamma \subset \pi^{-1}(B)$ which fibres over the flowline γ with fibre over $\gamma(s)$ the 2-sphere Σ_s. Then we have

Proposition 1. *The submanifold P_γ is Ψ-associative.*

To see this we can work in adapted co-ordinates at a point in B as above and assume that the gradient of h_α is a multiple of ∂_{y_1}. From the nature of P_γ the condition that $i_v \Psi$ restricts to 0 is automatic if v is a tangent vector along the fibres or a lift of ∂_{y_1}. Taking lifts of $\partial_{y_2}, \partial_{y_3}$ the Ψ-associative condition is just that Ω_2 and Ω_3 vanish on the fibre of P_γ, which is just the condition that this fibre is a complex curve with respect to the complex structure corresponding to ∂_{y_1}.

5. Calibrated 1-cycles

We will now formulate the main definition of this article. Let (N, L, E^+, u) be an adiabatic G_2 structure. We want to consider two situations: "orbits" and "graphs".

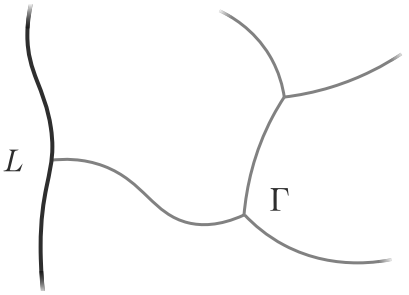

FIGURE 1. Local diagram of gradient cycle Γ which has one univalent vertex terminating at the link L.

Orbits. Let $\Gamma \subset N \setminus L$ be an oriented embedded circle. Suppose that there is a constant -2 section α of the bundle E_0 over Γ, so the pair (Γ, α) is a cycle for the group $H_1(N, E)$ discussed in Section 3.1. Locally, we have a function h_α as considered above. This function is not globally defined but its gradient vector field is defined on a neighbourhood of Γ in $N \setminus L$. We say that (Γ, α) is *a gradient orbit* if it is an integral curve of this vector field (compatible with the orientation of Γ in the obvious way). We also require that for each point q on Γ the pair (α, H_q) is irreducible, where H_q is the positive subspace defined by the derivative of u at q.

Graphs. Let $\hat{\Gamma}$ be an oriented graph with each vertex having valence 1 or 3. Let $\iota : \hat{\Gamma} \to N$ be a continuous embedding which is smooth on each edge and which maps the vertices of valence 1 to L and all other points of $\hat{\Gamma}$ to $N \setminus L$. We call the image Γ of ι an embedding of $\hat{\Gamma}$. (Thus Γ does not depend on the parametrisation ι.) Suppose that for each edge γ of Γ there is a label by a constant -2 section α_γ of E, that the label of the edge containing a vertex of valence 1 is the corresponding vanishing cycle and that the condition (4) holds at each vertex of valence 3. So $(\Gamma, \{\alpha_\gamma\})$ is a cycle for the group $H_1(N, E)$. We say that $(\Gamma, \{\alpha_\gamma\})$ is a *gradient graph* if each edge γ is an integral curve of the gradient vector field defined by α_γ (compatible with the orientations). We also require that the irreducible condition holds at each point of Γ. (At a vertex q of valence 3 this means that (α_i, H_q) is irreducible for each of the -2 classes $\alpha_1, \alpha_2, \alpha_3$ labelling the three edges.) Note that the sign ambiguity in the vanishing cycle near to L is taken care of by the orientation of the graph.

In either case we call Γ (with the appropriate labels) a *gradient cycle*. See Figure 1 for an example.

Discussion.

(1) We could vary the definition of gradient cycles in a number of ways. We could
   (a) allow vertices of higher valence;
   (b) relax the condition that ι is an embedding;

(c) relax the irreducible requirement on (α, H_q);
(d) allow Γ to meet L in interior points of edges.

But the definition we have given simplifies some statements and one expects that for generic sections u the gradient cycles obtained under such weaker hypotheses will satisfy our stronger conditions. The topic of Section 6 below is to study situations where these various conditions fail, in 1-parameter families.

(2) The fundamental idea underlying these definitions is that if the adiabatic G_2-structure (N, L, E^+, u) corresponds to a 1-parameter family $\tilde{\phi}_\epsilon$ of KL fibred torsion-free G_2-structures on a 7-manifold M then for small ϵ a gradient 1-cycle Γ should yield an associative submanifold $\tilde{P}(\Gamma)$ in $(M, \tilde{\phi}_\epsilon)$. In the case of orbits it is relatively straightforward to prove such a result. First, at the topological level, since the irreducible condition holds over Γ we clearly have a compact 3-manifold $P(\Gamma) \subset M$ fibered over Γ with 2-sphere fibres (and in fact $P(\Gamma)$ is diffeomorphic to $S^2 \times S^1$). Then the analysis problem is to show that this can be deformed into an associative submanifold.

In the case of a graph Γ the problem is harder but, as a first step, we will construct a topological model: a submanifold $P(\Gamma) \subset M$. For a vertex of valence 1 on an edge γ which terminates at a point p in L the construction is essentially the well-known "thimble" of a Lefschetz fibration. For the topological discussion in our model around p we can deform γ to be the positive real axis in $\mathbf{C} \subset \mathbf{C} \times \mathbf{R}$. Then the thimble is simply given by $\mathbf{R}^3 \subset \mathbf{C}^3 \subset \mathbf{C}^3 \times \mathbf{R}$ and the fibre of π_0 over a point $\eta > 0$ is the 2-sphere $\{(z_1, z_2, z_3) : z_i \in \mathbf{R}, \sum z_i^2 = \eta\}$.

The case of a vertex of valence 3 is more interesting. We can suppose that all orientations are outgoing so we have three -2 classes $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. This implies that $\alpha_i.\alpha_j = 1$. In the hyperkähler structure on the fibre X these classes are represented by embedded spheres $\Sigma_1, \Sigma_2, \Sigma_3$ which are complex with respect to three complex structures J_1, J_2, J_3. Standard theory of K3 surfaces shows that this configuration of embedded spheres is unique up to diffeomorphisms of X, so for the topological discussion we can take any convenient model. The model we use involves the non-compact manifold X_0 obtained as either a smoothing or resolution of the A_2-singularity which, again by standard theory, can be embedded in the K3 manifold X. We use the "Gibbons-Hawking" description of X_0 (but for our present purposes only at the topological level). For this we take three points A, B, C in $\mathbf{R}^3$ and construct an S^1-bundle $Z \to \mathbf{R}^3 \setminus \{A, B, C\}$ with Chern class 1 on small spheres around the three points. The restriction of Z to such a sphere gives the Hopf fibration of S^3 so we can complete Z to a 4-manifold X_0 which has a circle action with three fixed points

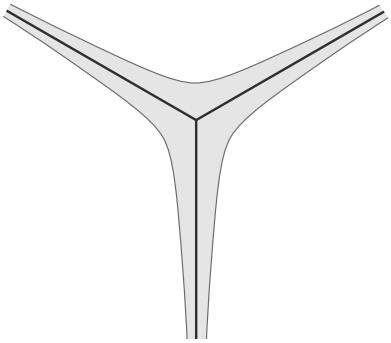

FIGURE 2. Thickening of $Y \subset \mathbf{R}^2$. The rays are spanned by vectors v_1, v_2, v_3 which sum to zero.

and the quotient by the action is a smooth map $\mu : X_0 \to \mathbf{R}^3$. The pre-image by μ of the line segment from A to B is a 2-sphere in X_0 with self-intersection -2 and taking the three sides of the triangle ABC we get a configuration of spheres of the desired kind. We can suppose that A, B and C lie in the standard plane $\mathbf{R}^2 \subset \mathbf{R}^3$. Let $v_1 = A - B$, $v_2 = B - C$, $v_3 = C - A$, so v_i are vectors in $\mathbf{R}^2$ with $\sum v_i = 0$. Let $Y \subset \mathbf{R}^2$ be the union of the three rays $\mathbf{R}^+ v_i$ and let $\Omega \subset \mathbf{R}^2$ be a thickening of Y, as in Figure 2, with three boundary components, asymptotic to the three pairs of rays. Let $F : \Omega \to \mathbf{R}^2$ be a smooth map which takes the three boundary components of Ω to the points A, B, C with the obvious ordering (so, for example, the boundary component asymptotic to v_1 and v_3 is mapped to A). For a suitable large number R the set $\{x \in \Omega : |x| > R\}$ has three connected components, corresponding to the three rays. We choose the map F so that it maps these three components to the three edges of the triangle, again with the obvious ordering. The graph of F is a surface S with boundary in $\mathbf{R}^2 \times \mathbf{R}^2 \subset \mathbf{R}^3 \times \mathbf{R}^2$ and the boundary of S lies in $\{A, B, C\} \times \mathbf{R}^2$. It follows that the preimage

$$P_0 = (\mu \times \mathrm{id})^{-1}(S) \subset X_0 \times \mathbf{R}^2$$

is a 3-dimensional submanifold of $X_0 \times \mathbf{R}^2$. By construction, P_0 has three ends, which are small deformations of $\Sigma_i \times \mathbf{R}^+ v_i$. Finally, taking $\mathbf{R}^2 \subset \mathbf{R}^3$ we can regard P_0 as a submanifold of $X_0 \times \mathbf{R}^3$.

There are many other ways of describing this submanifold $P_0 \subset X_0$. The approach above has the advantage that it makes evident the symmetry between the three 2-spheres. Using this model at each vertex of valence 3 we can construct a 3-dimensional submanifold $P(\Gamma) \subset M$ for any graph Γ.

The problem of constructing a nearby associative submanifold $\tilde{P}(\Gamma)$ motivates the following conjecture.

CONJECTURE 1. *Let $\alpha_1, \alpha_2, \alpha_3$ be -2 classes on the K3 manifold X with $\alpha_1 + \alpha_2 + \alpha_3 = 0$. Let $\mathbf{R}^3 = H \subset H^2(X)$ be a maximal positive subspace corresponding to a hyperkähler structure and v_i be the projection of α_i to H. Assume that the (α_i, H) are irreducible. Then there is an associative submanifold $\Pi \subset X \times \mathbf{R}^3$ with three ends asymptotic to $\Sigma_i \times \mathbf{R}^+ v_i$ where Σ_i is the complex curve representing α_i, for the complex structure defined by v_i, and Π is unique up to the translations of $\mathbf{R}^3$.*

Since the vectors v_i lie in a plane this associative submanifold should in fact be a *special Lagrangian* submanifold in $X \times \mathbf{C}$.

(3) Gradient cycles are the zeros of a closed 1-form on a suitable infinite-dimensional space of cycles. This is simplest to set up in the case of orbits. Let Γ_0 be an embedded circle in $N \setminus L$ with constant section α over Γ_0 and let V_α be the corresponding gradient vector field, defined in a neighbourhood of Γ_0. Let σ be the 2-form on this neighbourhood given by the contraction of the volume form with V_α. Locally we have a function h_α and $\sigma = *dh_\alpha$. Recall that h_α is obtained as the restriction of a linear function on $\mathbf{R}^{3,19}$ to the image of U, which is a maximal submanifold. The maximal condition implies that the restriction of linear functions to the submanifod are harmonic, with respect to the induced metric. So h_α is a harmonic function and σ is a closed 2-form. For any 1-cycle Γ close to Γ_0 choose a 2-chain W of the obvious kind with $\partial W = \Gamma - \Gamma_0$ and define

$$\mathcal{F}(\Gamma) = \int_W \sigma.$$

Then the gradient orbits in this neighbourhood are the critical points of this functional. The derivative is a well-defined 1-form given by

$$\delta\mathcal{F} = \int_\Gamma i_\xi \sigma,$$

where ξ is a variation vector field along Γ.

We can proceed in a similar way for graphs. For each edge γ we consider a variation vector field ξ_γ. These satisfy a matching condition $\xi_{\gamma_i} = \xi_{\gamma_j}$ at a vertex of valence 3 and are tangent to L at vertices of valence 1. We define a 1-form by

$$\delta\mathcal{F} = \sum_\gamma \int_\gamma i_{\xi_\gamma} \sigma_\gamma.$$

where σ_γ is the closed 2-form in a neighbourhood of γ defined as above. The reader can check that this is a closed 1-form on an infinite dimensional space of labelled embedded graphs, with zeros the gradient graphs.

One can go on to develop at least some elements of a "Floer theory", with chain complex generated by gradient graphs and boundary map defined by suitable 2-cycles in $N \times \mathbf{R}$ (which should correspond to Cayley submanifolds in $M \times \mathbf{R}$). In the same vein, one can develop theory for adiabatic co-associative submanifolds in M, but we will not go into these variants further in this article.

There is also an adiabatic analogue of the calibrated property. For a segment $\gamma(s)$ with label α we define the weighted length to be

$$\int |\gamma'(s)||\nabla h_\alpha|_{\gamma(s)} ds.$$

Then the weighted length of any cycle Γ is bounded below by $|\chi[\Gamma]|$ where $\chi : H_1(E) \to \mathbf{R}$ is the map discussed in Section 3.1 above and equality holds if and only if $\pm\Gamma$ is a gradient cycle.

5.1. Related literature. There is a web of connections between the ideas that we discuss in this article and existing literature. To give a proper account of this would go far beyond the space available here and the authors' knowledge, so we will just indicate some of these connections.

- In classical algebraic topology, graphs of gradient lines on a manifold can be used to describe cup products and higher operations in the framework of Morse Theory [3].
- In symplectic topology, similar graphs appear in describing pseudoholomorphic curves [7], [1] in various contexts. These include relations to tropical geometry, Calabi-Yau manifolds with Lagrangian torus fibrations, and the Strominger-Yau-Zaslow approach to mirror symmetry [9].
- Invariants of 3-manifolds, related to Chern-Simons theory, obtained by counting graphs of gradient lines have been studied by Fukaya [6] and Watanabe [22]. This leads to connections with the theory of finite type invariants of 3-manifolds, see e.g. [16].
- In the more specific context of G_2-manifolds fibred over a 3-dimensional base with K3 fibres, the basic idea in this article was discussed in [5] for the particular case of an arc joining two components of the link. Some closely related ideas appear in the earlier paper [18], in the case where the fibre is a hyperkähler ALE 4-manifold. Similar ideas have been considered in the case of Special Lagrangian submanifolds in Calabi-Yau 3-folds fibred over a 2-dimensional base ([15, Section 3], [19, Section 1.4]) and there are connections to the theory of Spectral Networks. A recent article [14] gives some explicit examples of fibrations of noncompact G_2-manifolds, and associative submanifolds.
- In 3-manifold topology, Hutchings defined a "periodic Floer homology" using the integral curves of the gradient vector field of a circle-valued harmonic function, in the case when the vector field

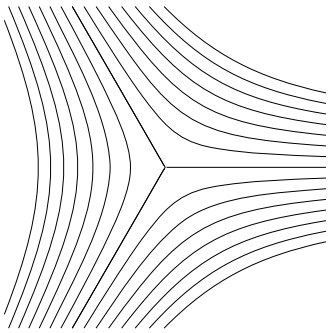

FIGURE 3. Flow lines of the vector field $2\overline{z}^{1/2}$ on $\mathbf{C}$.

has no zeros [12]. This was shown by Lee and Taubes to agree with Seiberg-Witten-Floer homology [17]. In this case the 3-manifold is necessarily a surface bundle over the circle. A general theory in the case of a vector field with zeros does not seem to have been developed yet but it seems likely that the work of Taubes [20] and Gerig [8] on holomorphic curves in near symplectic manifolds would be relevant to that.

6. Theory of gradient cycles

6.1. Terminating manifolds. In this subsection we discuss gradient flow lines corresponding to the vanishing cycle near the link L. We work in the unit ball B^3 in standard local co-ordinates (z,t) as in Section 3.2, so we have maps $f : B^3 \to \mathbf{R}$ and $F : B^3 \to \mathbf{R}^{18}$ such that $f(z,t) = \mathrm{Re}(b(t)z^{3/2}) + \eta(z,t)$, where η, F satisfy the conditions stated in Section 3.2. We have a (non-smooth) Riemannian metric g on B^3 defined by the graph of (f,F) in $\mathbf{R}^{3,19} = \mathbf{C}\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}^{18}$ and a vector field V on B^3 given by the gradient of f in the metric g. Without loss of generality assume that $b(0) = 4/3$ and take the standard branch of $z^{3/2}$ near the positive real axis. Let V_0 be the gradient vector field of $\mathrm{Re}(4/3z^{3/2})$ on $\mathbf{C}\times\mathbf{R}$ with respect to the *Euclidean* metric g_0. Thus $V_0 = (2\overline{z}^{1/2}, 0)$ and there is an integral curve $x_0(s) = (s^2, 0)$ of V_0 through the origin. See Figure 3. Let $s_0 < 1$ be a small number to be fixed later.

PROPOSITION 2. *There is a unique function $h : [0, s_0) \to \mathbf{C}\times\mathbf{R}$ with $|h(s)| \le Cs^3$ and $|h'(s)| \le Cs^2$ such that $x_0 + h$ is an integral curve of V.*

To see this write $V = V_0 + W$ so we want to solve the equation

$$h' = V_0(x_0 + h) - V_0(x_0) + W(x_0 + h). \tag{12}$$

Straightforward calculations show that our hypotheses on η, F imply that

$$|W(z,t)| = O(|z|^{3/2} + |t||z|^{1/2}),$$

with the corresponding estimates for the derivatives. We have an elementary estimate that if $|\zeta| \leq (1/2)|z|$ and if we set

$$2(z+\zeta)^{1/2} - 2z^{1/2} - z^{-1/2}\zeta = q(z,\zeta)$$

then

$$|q(z,\zeta)| \leq C|\zeta|^2|z|^{-3/2}$$

and if $|\zeta_1|, |\zeta_2| \leq (1/2)|z|$ then

$$|q(z,\zeta_1) - q(z,\zeta_2)| \leq C|\zeta_1 - \zeta_2|(|\zeta_1| + |\zeta_2|)|z|^{-3/2}.$$

Thus if we write $h = (h_{\mathbf{C}}, h_{\mathbf{R}})$ for the components in $\mathbf{C} \times \mathbf{R}$ and assume that $|h(s)| \leq s^2/2$ (which is certainly true for small s if h satisfies the conditions of the Proposition), then we have

$$[V_0(x_0 + h) - V_0(x_0)](s) = s^{-1}h_{\mathbf{C}} + Q(h)(s)$$

where

$$|Q(h)(s)| \leq C|h(s)|^2 s^{-3}.$$

And if $h_1(s)$, $h_2(s)$ are two such functions we have

$$|Q(h_1)(s) - Q(h_2)(s)| \leq C|h_1(s) - h_2(s)|(|h_1(s)| + |h_2(s)|)s^{-3}.$$

Our equation (12) becomes

$$Lh = Q(h) + W(x_0 + h) \tag{13}$$

where L is the linear operator $Lh = h' - s^{-1}(h_{\mathbf{C}}, 0)$. We have an inverse operator S to L which acts separately on the $\mathbf{C}$ and $\mathbf{R}$ components:

$$S(\rho_{\mathbf{C}}, \rho_{\mathbf{R}}) = (S_{\mathbf{C}}(\rho_{\mathbf{C}}), S_{\mathbf{R}}(\rho_{\mathbf{R}}))$$

where

$$(S_{\mathbf{C}}\rho_{\mathbf{C}})(\sigma) = \sigma \int_0^\sigma s^{-1}\rho_{\mathbf{C}}(s)ds,$$

and

$$(S_{\mathbf{R}}\rho_{\mathbf{R}})(\sigma) = \int_0^\sigma \rho_{\mathbf{R}}(s)ds.$$

We set $h = S(\rho)$ in (13), so the equation becomes the fixed point equation $\rho = \mathcal{F}(\rho)$ where

$$\mathcal{F}(\rho) = Q(S(\rho)) + W(x_0 + S\rho).$$

For a given s_0 we define a weighted norm on functions on $[0, s_0]$

$$\|\rho\| = \sup s^{-2}|\rho(s)|.$$

Then

$$|S(\rho)(s)| \leq \frac{1}{2}\|\rho\|s^3.$$

Thus if $\|\rho\| \leq 1$ (say) then $h = S(\rho)$ satisfies the condition $|h(s)| \leq s^3/2 \leq s^2/2$ (since $s_0 < 1$). It is now straightforward to find a fixed point using the contracting mapping theorem and we get uniqueness in the standard way. (In fact the proof gives uniqueness of the solution within the larger class of functions h with $|h(s)| \leq s^2/2$.)

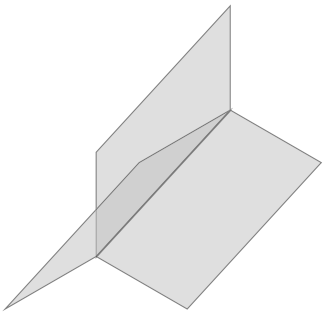

FIGURE 4. Terminal submanifolds meeting at the link.

For small t we have a similar integral curve $(s^2, t) + h_t(s)$ of the vector field V passing through $(0, t)$. For $x \geq 0$ write

$$\Phi(x, t) = (x, t) + h_t(\sqrt{x}).$$

Then Φ is a C^1 map on a neighbourhood of the origin in the manifold with boundary $\mathbf{R}^+ \times \mathbf{R} \subset \mathbf{C} \times \mathbf{R}$ and its image is a C^1-submanifold T_1 with boundary on the axis $z = 0$. Each point of T_1 lies on a gradient line terminating on the axis. There are two other submanifolds T_2, T_3 obtained in the same way starting with the model solutions $\omega s^2, \omega^2 s^2$ where $\omega = e^{2\pi i/3}$. The general picture is much the same as the standard picture for a Morse-Bott function in three dimensions with a 1-dimensional critical submanifold and Hessian of index (1,1) in the normal plane. In that case one would have ascending and descending submanifolds near the critical set. In our case the multivalued nature of the function mans that we cannot consistently differentiate between ascending and descending, so we call these *terminating submanifolds*. See Figure 4.

Using similar techniques one can show that the only flow lines (for the vector field corresponding to the vanishing cycle) which terminate on the link L are those that we have constructed above.

REMARK. Locally on L, there are three terminating submanifolds but it seems possible that these could be permuted as we move around a component of L.

6.2. Expected dimension and transversality. The set of gradient cycles has "expected dimension" 0. This notion can be made precise in various ways. One approach would be to set up a Fredholm theory in suitable infinite dimensional spaces where the expected dimension appears as the index of a linearised operator. The fact that the dimension is 0 is essentially a consequence of the local variational description discussed in Section 5. But since we are studying ordinary differential equations we can also proceed with more elementary methods. For example suppose that we have a gradient cycle formed by a single arc γ running from one component L' of L to another component L''. Fix a disc D transverse to γ at an interior point

q. Near L' the arc γ lies in a 2-dimensional terminating manifold T'. We follow the gradient flow along paths close to γ and extend T' until it intersects D in a 1-dimensional submanifold $S' \subset D$. Similarly we have another submanifold $S'' \subset D$ extending a terminating manifold T'' near L''. By construction S', S'' intersect at q and the intersection points $S' \cap S''$ correspond to gradient cycles near γ. So the meaning of "expected dimension" in this case is that if S', S'' are transverse then their intersection has dimension 0. Of course this is the same as the usual discussion in Morse-Bott theory of gradient flow lines between critical submanifolds. Similarly, for a gradient cycle Γ which is an orbit we take a transverse disc D and represent nearby gradient cycles as the fixed points of a return map $\phi : D \to D$, in a standard way.

Now let Γ be a gradient cycle for a section u_0 which is a graph with all vertices of valence 3. Let V be the set of vertices q_i and E be the set of edges γ_a and let $I \subset E \times V$ be the incidence set. Choose a disc $D_a \subset N$ transverse to the flow line γ_a. Parametrise $\gamma_a(s)$ so that $s = 0$ corresponds to the intersection point with D_a and suppose that the two end points correspond to $s = l_a^+$, $s = -l_a^-$ for $l_a^\pm > 0$. Let

$$\widetilde{D}_a = D_a \times (-\epsilon, \epsilon)^2.$$

For a triple (z, η_1, η_2) in $\widetilde{D}_a$ we have two points in N given by following the flow line through z to parameter values $l_a^+ + \eta_1$ and $-l_a^- + \eta_2$. Let B_i be a small ball around the point in N corresponding to the vertex q_i and for $(a, i) \in I$ let $B_{a,i}$ be a copy of B_i. The construction above gives a map

$$f : \prod_{a \in E} \widetilde{D}_a \to \prod_{(a,i) \in A} B_{a,i}.$$

By hypothesis, for each vertex q_i there are exactly three pairs (a, i) in I say (a_1, i), (a_2, i), (a_3, i). Let Δ_i be the diagonal in $B_{a_1,i} \times B_{a_2,i} \times B_{a_3,i}$, so we have a submanifold

$$\Delta^* = \prod \Delta_i \subset \prod_I B_{a,i}.$$

By construction gradient cycles close to Γ correspond to points in $\prod \tilde{D}_a$ which map by f to Δ^* and we have one such point $\mathbf{O}$ corresponding to Γ. Let n be the number of the vertices in Γ and m the number of edges. Then Δ^* has codimension $6n$ in $\prod_A B_{a,i}$ and $\prod \tilde{D}_\alpha$ has dimension $4m$. But since the graph is trivalent we have $2m = 3n$ so $4m = 6n$ and if f is transverse to Δ^* at $\mathbf{O}$ the set of nearby gradient cycles has dimension 0.

When the transversality condition holds for all gradient cycles of u_0 we say that the set of gradient cycles is "cut out transversally".

We expect that for a single section u_0 the transversality conditions considered above may fail and the set of gradient cycles might have some more complicated structure but, in the usual way in such theories, we can hope to achieve transversality after suitable generic perturbations. There are many kinds of perturbations we could consider; here we will discuss perturbations

of the section u_0, dropping the maximality condition. (Doing this we lose the "Floer-type", variational, description: there are other kinds of perturbations one could use which retain that.) Thus we let $\mathcal{H}$ be a space of small perturbations $u_0 + \sigma$ with σ supported away from the link L. (For most purposes we can work with a suitable large finite dimensional space of perturbations.) Then in the setting above our map f extends to

$$F : \mathcal{H} \times \prod \tilde{D}_a \to \prod_A B_{a,i},$$

where we use the gradient lines defined using $u_0 + \sigma \in \mathcal{H}$.

PROPOSITION 3. *The map F is transverse to Δ^* at $(u_0, \mathbf{O})$.*

To see this we need to compute the derivative of F with respect to u. For this we utilize the "perturbation theorem" for flows (see 32.1 in [2] for example). Let Φ^s be the flow of a smooth vector field X on a manifold, defined near a point x and write $\gamma(s) = \Phi^s(x)$. Consider a vector field Y defined on a neighbourhood of $\gamma(s)$ and let Φ_t^s be the flow of $X + tY$. Then

$$\frac{d}{dt}\Phi_t^s(x)|_{t=0} = \int_0^s (\Phi^s)_*(Y_{\gamma(s-r)})dr. \tag{14}$$

More generally the same formula holds for a flow defined by a 1-parameter family of vector fields X_t with t-derivative Y at $t = 0$.

Let $\gamma(s) = \gamma_a(s)$ be the flow line corresponding to an edge of Γ, as considered above, with $s = 0$ at the intersection point 0 with the transverse disc D_a. Write $l^+ = l_a^+$ so that $q_+ = \gamma(l^+)$ is a vertex of Γ. We consider first perturbations $u_0 + t\sigma$ where σ is supported in a small ball around $\gamma(l^+/2)$ (say). We get perturbed gradient lines $\gamma_t(s)$ passing through the same point 0. For $s \leq 0$ we have $\gamma_t(s) = \gamma(s)$ but for $s > 0$ they are different. Let V be the derivative

$$V = \frac{d}{dt}\gamma_t(l^+).$$

So V is a vector in the tangent space of N at q_+.

LEMMA 1. *We can choose a pair of such perturbations σ_1, σ_2 giving vectors V_1, V_2 in TN_{q_+} such that $V_1, V_2, \gamma'(l^+)$ form a basis of TN_{q_+}.*

Assuming this Lemma we deduce Proposition 3 as follows. Focusing on the vertex q_+ and the edge γ_a, we have a ball B around q_+ and a map

$$F_a^+ : \mathcal{H} \times D_a \times (-\epsilon, \epsilon) \to B,$$

which gives one component of the map F. The derivative of F_a^+ with respect to $\eta \in (-\epsilon, \epsilon)$ is given by $\gamma'(l^+)$ and the derivative with respect to the two variations provided by the Lemma gives V_1, V_2. The derivatives of all other components of the map F with respect to these variations vanish so taking $4m$ similar variations (two for each pair $(a, i) \in A$) we see that the derivative of F is surjective so certainly F is transverse to Δ^*.

We now prove the Lemma. Recall that in a local description, the gradient vector fields we are considering are given by the projection of a fixed vector α in $\mathbf{R}^{3,19}$ to the tangent space of the image of the map u from a neighbourhood B in N to $\mathbf{R}^{3,19}$. We take $u = u_0 + t\sigma$ and choose σ to be a normal variation so that for each $x \in B$ the value $\sigma(x)$ is orthogonal to the image of du in $\mathbf{R}^{3,19}$. One gets the following formula for the t-derivative Y of the gradient vector field X:

$$Y = \mathrm{grad}\langle \sigma, \alpha \rangle - \sigma S X \tag{15}$$

where S is the second fundamental form of the image submanifold. In our application using the formula (14) we only need the value of Y on the curve γ so we choose σ to vanish on γ and this means that the second term in the formula (15) vanishes. The derivative $(\Phi^s)_*$ appearing in the formula (14) preserves the tangent vectors to γ, so for our purposes we only need to consider the component of Y normal to γ. It is straightforward to construct a σ of this kind realising *any* normal vector field and then to use (14) to produce the desired σ_1, σ_2.

The whole discussion above can be adapted without difficulty to general gradient cycles and using standard techniques we obtain:

Corollary 1. *There is a residual set $\mathcal{P}$ of positive sections such that for any $u \in \mathcal{P}$ the set of gradient cycles is a 0-manifold cut out transversally.*

The results of this section can be extended to many situations when the cycles are allowed self-intersections although there are some difficulties in proving a completely general result.

6.3. Crossing. We have now reached the main point of this article, which is to study phenomena for gradient cycles in 1-parameter families analogous to those we outlined in Section 2 for associative submanifolds. In the remainder of Section 6 we will be content to make whatever transversality assumptions are relevant, without analysing the exact meaning of these. In this subsection we study what happens when, in a generic 1-parameter family, two gradient graphs cross in the complement of the link.

Let $\Gamma \subset N$ be a gradient graph for a section u_0 and q a point of Γ which is not a vertex, so q lies in the interior of some gradient flowline γ, an integral curve of a local vector field V. We introduce a notion of "cut" gradient graphs. Let D be a small disc in N, centred at q and transverse to γ. For each z in D there is gradient flowline γ_z through z. The intersection of $\Gamma \setminus \{q\}$ with a small ball centred at q has two components. By a cut gradient graph we mean a small deformation of Γ away from q but near q we allow these two components to deform to possibly different gradient curves $\gamma_z, \gamma_{z'}$. We have a moduli space M of such cut gradient graphs which maps to $D \times D$. Our dimension analysis above shows that M is a 2-manifold and we suppose that the image of M in $D \times D$ intersects the diagonal transversally at (q, q). So we can assume that M is embedded in $D \times D$.

Next suppose that Γ_1, Γ_2 are two such gradient graphs (for the same section u_0) which intersect at a point q (not a vertex of either graph). So q lies on γ_1 and γ_2 which are integral curves of vector fields V_1, V_2 which we assume are linearly independent at q. In our situation we want to assume that the V_i are defined by -2 classes α_i with $\alpha_1.\alpha_2 = 1$ but this condition will not play a fundamental role in the discussion below. We choose transverse discs D_1, D_2 as above and we have moduli spaces M_1, M_2 of cut graphs. Let $I \subset D_1 \times D_2$ be the set corresponding to pairs of intersecting flow lines. It is clear that I is a 3-dimensional submanifold of $D_1 \times D_2$ and the intersection point defines a map f from I to N. In the 8-manifold

$$D_1 \times D_2 \times D_1 \times D_2 = D_1 \times D_1 \times D_2 \times D_2$$

we have two subsets $I \times I$ and $M_1 \times M_2$. Let J be their intersection. We assume that this intersection is transverse, so J is a 2-manifold and $F = (f \times f)$ gives a map $F : J \to N \times N$. By our hypotheses the point (q, q) lies in the image of F: it is the image of the point $Q = (q, q, q, q)$ in J.

We now introduce a 1-parameter family of sections u_t for $|t| < \epsilon$ so Γ_1, Γ_2 deform in families Γ_i^t. We can deform all the constructions above in the family, so for each t we have a 2-manifold J_t and a map $F_t : J_t \to N \times N$. The difference is that for $t \neq 0$ we do not expect that the image of F_t meets the diagonal in $N \times N$. We can choose a family of co-ordinate charts $\psi_t : B^3 \to N$ which linearise the vector fields $V_1^t + V_2^t$. In other words, $V_1^t + V_2^t$ is the image by $d\psi_t$ of a constant vector field n on the unit ball B^3 in $\mathbf{R}^3$. Let $G_t : J_t \to \mathbf{R}^3$ be the composite of $(\psi_t^{-1} \times \psi_t^{-1}) \circ F_t$ with the difference map $(x, y) \mapsto x - y$ from $\mathbf{R}^3 \times \mathbf{R}^3$ to $\mathbf{R}^3$.

Let $\mathcal{J}$ be the 3-manifold formed by the family of 2-manifolds J_t, so we have a map $\pi : \mathcal{J} \to (-\epsilon, \epsilon)$ with fibres the J_t and a map $\mathcal{G} : \mathcal{J} \to \mathbf{R}^3$, equal to G_t on the fibres. By construction we have a point Q in $J_0 \subset \mathcal{J}$ with $\mathcal{G}(Q) = 0$. We make the transversality assumptions:

(1) Q is a regular point of $\mathcal{J}$,
(2) the vector n does not lie in the image of dG_0 at Q.

Let $Z \subset \mathbf{R}^3$ be the ray generated by n. These transversality assumptions imply that the pre-image $\mathcal{G}^{-1}(Z)$ is a 1-manifold with boundary embedded in $\mathcal{J}$ and it intersects the fibres J_t either for small positive t or for small negative t (but not both). Without loss of generality, suppose that the intersection occurs for small positive t. Then, by construction, for such t there is a gradient graph Γ^t of the kind indicated in the right of Figure 5. In other words, in the family a new gradient graph is "born" at $t = 0$.

The above discussion gives an adiabatic analogue of the Joyce-Nordström crossing for associative submanifolds outlined in Section 2. One can see that the topology matches up. Recall that for each gradient graph Γ we have defined, at the topological level, a submanifold $P(\Gamma) \subset M^7$. In the situation above, the fact that $\alpha_1.\alpha_2 = 1$ implies that $P_i = P(\Gamma_i)$ intersect in one point and one can check that the submanifold $P(\Gamma^t)$ corresponding to the graph Γ^t is the connected sum $P_1 \sharp P_2$.

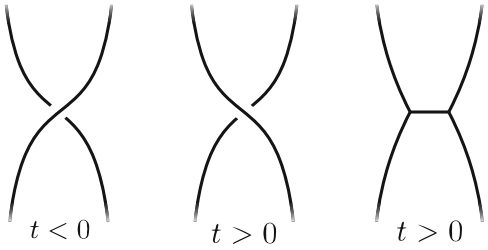

FIGURE 5. An adiabatic analogue of Joyce-Nordström crossing: a new gradient cycle is "born" at $t = 0$.

6.4. Splitting of flowlines and surgery triples. In this subsection we discuss an adiabatic analogue of the "surgery triple" outlined in Section 2. The essential phenomenon arises for general pairs of vector fields on a manifold. Let V_1, V_2 be non-zero vector fields on the open unit ball B^3 and Φ_1^s, Φ_2^s be the local flows they generate. Fix $\delta > 0$ so that Φ_i^s are defined with values in B^3 for $s \leq 2\delta$ on the half-sized ball $\frac{1}{2}B^3$. Define

$$C = \{x \in \frac{1}{2}B^3 : V_1(x) = \lambda V_2(x) \text{ for some } \lambda > 0\},$$

and

$$S = \{(x, x') \in \frac{1}{2}B^3 \times \frac{1}{2}B^3 : \Phi_1^{s_1}(x) = x', \Phi_2^{s_2}(x) = x' \text{ for some } 0 < s_1, s_2 < \delta\}.$$

So we have an inclusion map $F : S \to B^3 \times B^3$ and we can consider C as contained in the diagonal $B^3 \subset B^3 \times B^3$.

The set C corresponds to the intersection of the two sections of the unit sphere bundle defined by $V_i/|V_i|$. We assume that this intersection is transverse, so that C is a 1-dimensional submanifold of B^3.

PROPOSITION 4. *Under this transversality assumption, C lies in the closure $\overline{S}$ of S and near C the closure $\overline{S}$ has the structure of a 2-manifold with boundary C.*

To prove the Proposition, let q be a point of C and choose local coordinates (x_0, x_1, x_2) centred on q so that $V_1 = \frac{\partial}{\partial x_0}$. The statement of the Proposition is unchanged if we multiply V_2 by a positive function so we can suppose that

$$V_2 = \frac{\partial}{\partial x_0} + \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2},$$

where ξ_i are functions of x_0, x_1, x_2 vanishing at the origin. The transversality condition is that the 2×2 matrix $\frac{\partial \xi_i}{\partial x_j}$ $(i, j = 1, 2)$ is invertible. The submanifold C is the common zero set of ξ_1, ξ_2. The local flow Φ_1^s of V_1 is just translation in the x_0 factor and we have

$$\Phi_2^s(0, \mathbf{x}) = (s, \phi_s(\mathbf{x}))$$

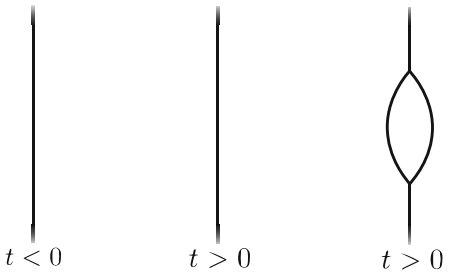

FIGURE 6. An adiabatic analogue of a surgery triple: a new gradient cycle is "born" at $t = 0$.

for a family of diffeomorphisms ϕ_s of neighbourhoods of 0 in $\mathbf{R}^2$. If $a < b$ and $\mathbf{x} \in \mathbf{R}^2$ is a fixed point of the diffeomorphism $\phi_b \circ \phi_a^{-1}$ then the pair $((a, \mathbf{x}), (b, \mathbf{x}))$ lies in S and all points of S arise in this way. The Proposition is now a consequence of the following simple fact. Suppose that ψ_s is a family of diffeomorphisms of a neighbourhood of 0 in $\mathbf{R}^2$ with ψ_0 equal to the identity. Let η be the vector field given by the s derivative of ψ_s at $s = 0$. Suppose that η vanishes at the origin and that the graph of η is transverse to the zero section there. Under these conditions it is standard that for small s there is a unique fixed point of ψ_s close to the origin. Applying this to $\phi_b \phi_a^{-1}$ we get a local parametrisation of S by pairs (a, b) with $a < b$ which extends to the diagonal $a = b$, mapped to C.

Let V be the vector field $V_1 + V_2$. Let γ be the flow line of V through the origin and choose a transversal disc D as before, parametrising nearby flow lines γ_z of V. The quotient by the flow defines a map $p : B^3 \to D$ so we get a map

$$G = (p \times p) \circ F : S \to D \times D.$$

This extends to $\overline{S}$, mapping C to the diagonal in $D \times D$. Let M be a 2-dimensional submanifold of $D \times D$. If $(z, z') \in D \times D$ is an intersection point of M and S we have a configuration of flow lines of the kind seen on the right of Figure 6. This consists of a segment of a flow line γ_z for V ending at a point x, a pair of flow lines of V_1, V_2 from x to another point x' and a segment of the flow line $\gamma_{z'}$ starting from x'.

Suppose now that the origin in B^3 lies in the submanifold C. Let V_1^t, V_2^t be a 1-parameter family of deformations, so for each t we have submanifolds C_t, S_t. Let $\mathcal{C}, \mathcal{S}$ be the corresponding sets in $B^3 \times B^3 \times (-\epsilon, \epsilon)$ so we have a map

$$\mathcal{G} : \overline{\mathcal{S}} \to D \times D \times (-\epsilon, \epsilon),$$

which maps the boundary $\mathcal{C}$ to the diagonal times $(-\epsilon, \epsilon)$. Similarly, a family $M_t \subset D \times D$ defines a 3-dimensional submanifold $\mathcal{M} \subset D \times D \times (-\epsilon, \epsilon)$. By construction the image of $\mathcal{G}$ intersects $\mathcal{M}$ at the point $\mathbf{0} = (0, 0, 0)$. We make the transversality assumption that this intersection is transverse; then the intersection is a 1-dimensional manifold I with boundary $\mathbf{0}$ and the derivative of the projection map $I \to (-\epsilon, \epsilon)$ at $\mathbf{0}$ is non-zero. Depending

on the sign of this derivative there is a configuration of the kind described above for the vector fields V_i^t and submanifolds M_t *either* for small positive t *or* for small negative t, but not both. Without loss of generality we suppose the first case.

In our situation we take B^3 to be a co-ordinate chart centred at a point q in $N \setminus L$ and V to be the gradient vector field defined by a -2 class α. We suppose that q lies on a segment γ of a cycle Γ which satisfies all the conditions to be a gradient cycle except that the irreducibility condition fails at q. We let M be the submanifold parametrising cut gradient cycles, as before. The central assumption is that $\alpha = \alpha_1 + \alpha_2$ for -2 classes α_i giving gradient vector fields V_i with $V_1 = \lambda V_2$ at q. Then we see that as we vary our section u in a 1-parameter family u_t, satisfying the transversality hypotheses, there will be a family of gradient cycles $\tilde{\Gamma}_t$ "born" at $t > 0$. See Figure 6.

We interpret this as the adiabatic analogue of a surgery triple obtained by the three Harvey-Lawson smoothings of a cone singularity. We get evidence for this from the topology of the situation. The condition that $V_1 = \lambda V_2$ at q means that we are in the exceptional case where the complex curve $\Sigma \subset X$ is reducible, a union of two embedded spheres Σ_1, Σ_2 intersecting in a point in X. For a nearby point q' in Γ the corresponding curve $\Sigma' \subset X$ is smooth but "close" to singular, with a small circle $\lambda \subset \Sigma'$ (the vanishing cycle) which shrinks to a point as q' moves towards q. Now consider a small deformation of Γ to Γ_t for $t < 0$. We have a 3-manifold $P_- = P(\Gamma_t)$ in M^7 and we can deform λ slightly to a circle λ_- in P_-. We also have a 3-manifold $\tilde{P}$ defined by $\tilde{\Gamma}_t$ for $t > 0$. We leave the reader to check that $\tilde{P}$ is obtained from P_- by 0-surgery on λ_-. The third 3-manifold in the picture is given by $P_+ = P(\Gamma_t)$ for $t > 0$, which contains a small circle λ_+. In fact P_+ is diffeomorphic to P_- but we can consider a more refined notion. Let $U_\pm$ be small tubular neighbourhoods of $\lambda_\pm$ in $P_\pm$. There is a natural diffeomorphism $F : P_- \setminus U_- \to P_+ \setminus U_+$, well-defined up to a small isotopy. Then, relative to F, the 3-manifolds P_+, P_- are different: i.e. we cannot extend F to a diffeomorphism from P_- to P_+. Relative to F, and with suitable orientations, the manifold P_+ is obtained from P_- by $+1$ surgery on λ_-. This follows from the fact that if we have a standard family of complex structures I_τ on X parametrised by τ in the unit disc in $\mathbf{C}$ such that the -2 class α is represented by a smooth curve for $\tau \neq 0$ and the singular curve $\Sigma_1 \cup \Sigma_2$ for $\tau = 0$ then the monodromy of the family of curves is given by the Dehn twist in the vanishing cycle λ. This Dehn twist is trivial in the mapping class group of S^2 but is non-trivial in the appropriate relative mapping class group.

6.5. Other transitions. In this subsection we outline (without full proofs) two phenomena involving gradient flow lines near the link L. The first is another version of "crossing". Suppose that Γ_1, Γ_2 are two gradient graphs for a section u_0 each of which has a vertex of valence 1 at the same

point p of the link L. As usual we consider a 1-parameter family of sections u_t and suppose that Γ_i deform in families Γ_i^t. So we have vertices $p_1(t)$, $p_2(t)$ say on L which coincide at $t = 0$. We make the transversality assumption that the map $t \mapsto (p_1(t), p_2(t))$ is transverse to the diagonal at $t = 0$. Then one can show that there is another family of gradient graphs $(\Gamma_1 \sharp \Gamma_2)^t$ defined either for small positive t or for small negative t, but not both. The gradient graph is obtained by deforming the union of two flow lines terminating on L into a single flow line. The basic model is given by the gradient curves of the function $\mathrm{Re}(z^{3/2})$ with respect to the Euclidean metric, as in Subsection 6.1. Then, up to parametrisation, there are gradient curves

$$\gamma_\epsilon(s) = (s + i\epsilon)^{2/3},$$

which converge in an obvious sense as $\epsilon \to 0$ to the union of two line segments. This can be seen in any of the three sectors in Figure 3. It is easy to see that $P((\Gamma_1 \sharp \Gamma_2)^t)$ is topologically the connected sum of $P(\Gamma_1)$ and $P(\Gamma_2)$.

For the second phenomenon we consider a gradient cycle $\Gamma_{-\epsilon}$ which contains a segment γ that passes close to a component L_0 of the link L. We have two -2 classes α, δ where α is the label of γ and δ is the vanishing cycle and we suppose that $\alpha.\delta = 1$. The multi-valued setting means that the gradient vector field V associated to α is not well-defined on a neighbourhood of L_0, even up to sign. We can define V on a cut neighbourhood but there will be a jump by the addition of the vector field V_δ associated to δ across the cut. Since V_δ vanishes on L_0 the value of V at points of L_0 is well-defined and is not zero.

The basic phenomenon can be seen in the model case where we work on $\mathbf{C}$ with the gradient vector field V of the multivalued function $\mathrm{Re}(z^{3/2} + iz)$ which we interpret by making a cut along the positive real axis. The jump across the cut is by $V_\delta = \frac{3}{2} z^{1/2}$. There is a family of flow lines γ_t for $t < 0$ given up to parametrisation by $\mathrm{Im}(z^{3/2} + iz) = t$. These do not meet the cut. The function $\mathrm{Im}(z^{3/2} + iz)$ extends continuously across the cut but its derivative does not. If we attempt to extend the definition to $t > 0$ the same equation defines a set which meets the cut at the point $p_t = t$. But p_t also lies on a flow line of V_δ through the origin. Instead of a single flow line we get a configuration of three flow lines meeting at the point p_t. If we change our point of view and define the multivalued function by making a cut along the negative real axis (say) then the flow lines are associated to the three vector fields $V, V_\delta, V + V_\delta$.

In a similar fashion, for a 1-parameter family of sections u_t we may see a gradient cycle $\Gamma_{-\epsilon}$ for $t = -\epsilon$ which deforms in a family Γ_t where Γ_0 meets L_0. This family Γ_t can be extended to $t > 0$ but with a graph of a different topological type, inserting a vertex p_t of valence 3 and a vertex of valence 1 on L_0. The labels of the three edges meeting at p_t, with outgoing orientations, are $-\alpha, \alpha + \delta, -\delta$. The essential point here is that the monodromy of the flat bundle around L_0 takes α to $\alpha + \delta$. For an illustration of the family Γ_t as it changes from $t < 0$ to $t > 0$ see Figure 7.

FIGURE 7. A gradient cycle passes through the link L in a cross-section, where L corresponds to the origin in $\mathbf{C}$.

In this case we do not change the "count" of gradient cycles and the 3-manifolds $P(\Gamma_t)$ are homeomorphic for t positive and negative.

References

[1] M. Abouzaid, *A topological model for the Fukaya category of plumbings*, J. Differential Geometry 87(1), 1–80, 2011. MR 2786590

[2] R. Abraham and J. Robbin, *Transversal mappings and flows*, An appendix by A. Kelley. W. A. Benjamin Inc. New York-Amsterdam, 1967. MR 0240836

[3] R. Cohen and P. Norbury, *Morse field theory*, Asian J. Math. 16(4), 661–711, 2012. MR 3004282

[4] A. Doan and T. Walpuski, *On counting associative submanifolds and Seiberg-Witten monopoles*, Pure Applied Math. Quarterly 15, 1047–1133, 2019. MR 4085667

[5] S. Donaldson, *Adiabatic limits of co-associative Kovalev-Lefschetz fibrations*, *Progr. Math*, vol. 324, pp. 1–29. Birkhäuser/Springer, 2017. MR 3702382

[6] K. Fukaya, *Morse Homotopy and Chern-Simons perturbation theory*, Commun. Math. Phys. 181, 37–90, 1996. MR 1410567

[7] K. Fukaya and Y.-G. Oh, *Zero loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. 1(1), 96–180, 1997. MR 1480992

[8] C. Gerig, *Taming the pseudoholomorphic beasts in* $\mathbf{R} \times S^1 \times S^2$, Geom. Topol. 24(4), 1791–1839, 2020. MR 4173922

[9] M. Gross and B. Siebert, *From real affine geometry to complex geometry*, Annals of Math. 174, 1301–1425, 2011. MR 2846484

[10] S. Gukov, E. Zaslow and S-T. Yau, *Duality and fibrations on G_2-manifolds*, Turkish J. Math. 27, 61–97, 2003. MR 1975332

[11] R. Harvey and H.B. Lawson, *Calibrated geometry*, Acta Math. 148, 47–157, 1982. MR 0666108

[12] M. Hutchings, *An index inequality for embedded pseudoholomorphic curves in symplectizations*, Jour. Eur. Math. Soc. 4, 313–361, 2002. MR 1941088

[13] D. Joyce, *Conjectures on counting associative 3-folds in G_2-manifolds*, Proc. Sympos. Pure Math., vol. 99, pp. 97–160. Amer. Math. Soc., 2018. MR 3838881

[14] S. Karigiannis and J. Lotay, *Bryant-Salamon G_2 manifolds and coassociative fibrations*, J. Geom. Phys. 162, 104–164, 2021. MR 4199397

[15] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, Nuclear Phys. B 477(3), 746–764, 1996. MR 1413475

[16] C. Lescop, *Invariants of links and 3-manifolds from graph configurations*, arXiv:2001.09929.
[17] Y-J. Lee and C.H. Taubes, *Periodic Floer homology and Seiberg-Witten-Floer cohomology*, Jour. Symplectic Geometry 10, 81–164, 2012. MR 2904033
[18] T. Pantev and M. Wijnholt, *Hitchin's equations and M-Theory phenomenology*, J. Geometry and Physics 61, 1223–1247, 2011. MR 2788324
[19] I. Smith, *Quiver algebras as Fukaya categories*, Geom. Topol. 19(5), 2557–2617, 2015. MR 3416110
[20] C.H. Taubes, *A compendium of pseudoholomorphic beasts in* $\mathbf{R} \times S^1 \times S^2$, Geometry and Topology 6, 657–814, 2002. MR 1943381
[21] V. Tosatti, *Collapsing Calabi-Yau manifolds*, arXiv:2003.00673.
[22] T. Watanabe, *Higher order generalization of Fukaya's Morse homotopy invariant of 3-manifolds I. Invariants of homology 3-spheres*, Asian J. Math. 22(1), 111–180, 2018. MR 3805158

Simons Center for Geometry and Physics, Stony Brook and Department of Mathematics, Imperial College, London, UK
Email address: s.donaldson@imperial.ac.uk

Department of Mathematics, University of Miami, Coral Gables, FL, USA
Email address: c.scaduto@math.miami.edu