Collapsing Calabi-Yau manifolds

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Dedicated to Professor Shing-Tung Yau on the occasion of his 70th birthday

ABSTRACT. We survey some recent developments on the problem of understanding degenerations of Calabi-Yau manifolds equipped with their Ricci-flat Kähler metrics, with an emphasis on the case when the metrics are volume collapsing.

1. Introduction

Calabi-Yau manifolds form an important class of compact complex manifolds that enjoys remarkable geometric properties, and have been extensively studied in many fields of mathematics (as well as theoretical physics). The main feature of these manifolds is that they carry Kähler metrics with everywhere vanishing Ricci curvature, thanks to Yau's solution [**112**] of the Calabi Conjecture [**9**]. These Ricci-flat Kähler metrics are not flat (unless we are on a torus or a finite quotient), and do not have an explicit description; rather, they are constructed by solving a fully nonlinear PDE of complex Monge-Ampère type.

In recent years, the study of families of such Ricci-flat Calabi-Yau manifolds has appeared naturally in many contexts, including in the study of moduli spaces and in mirror symmetry, and it is especially interesting to understand the possible ways in which these metrics can degenerate.

In this survey we will give an overview of some aspects of this problem, with a focus on the topics that are closest to the author's interests. Greater emphasis is placed on the case when the Ricci-flat manifolds are volume collapsing, which is by far the hardest case to analyze, hence the title of this survey. We will also describe a number of open problems and conjectures, some well-known and some less so, in the hope to stimulate further progress.

The reader is also encouraged to consult the author's earlier surveys [**100**, **101**], as well as the more recent surveys by Zhang [**116**] and Sun [**95**] on this circle of ideas. The relations with holomorphic dynamics that we will

discuss in section 3.8 is also described in the recent expository papers [**30**] and [**104**].

2. Degenerations of Calabi-Yau manifolds

2.1. Calabi-Yau manifolds. The main object of study in this article are compact Calabi-Yau manifolds. There are several slightly different definitions of Calabi-Yau manifolds in the literature, and we will use the following:

DEFINITION 2.1. A Calabi-Yau manifold is a compact Kähler manifold X whose real first Chern class $c_1(X) \in H^2(X, \mathbb{R})$ vanishes, i.e. $c_1(X) = 0$.

Since $c_1(X) = -c_1(K_X)$, where K_X is the canonical bundle of X, the Calabi-Yau condition is clearly equivalent to K_X being topologically torsion. In fact we have the following result which was established independently in [**3**, Theorem 3], [**38**, Proposition 6.6], [**73**, Theorem 3.13] (see also [**9**, Theorem 2] and [**75**, Theorem 3] for earlier results):

THEOREM 2.2. *Every Calabi-Yau manifold X has K_X holomorphically torsion, i.e. $K_X^{\otimes \ell} \cong \mathcal{O}_X$ for some $\ell \geqslant 1$.*

In particular, some finite étale cover $\tilde{X}$ of X is a Calabi-Yau manifold with (holomorphically) trivial canonical bundle.

2.2. Examples. The classical examples of Calabi-Yau manifolds are the following:

(1) Complex tori, $X = \mathbb{C}^n/\Lambda$ where $\Lambda \cong \mathbb{Z}^{2n}$ is a lattice.
(2) Smooth projective hypersurfaces $X = \{P = 0\} \subset \mathbb{CP}^{n+1}$ of degree $\deg P = n + 2$.
(3) Generalizing the previous example, if Y is a Fano $(n + 1)$-fold, i.e. K_Y^{-1} is ample, and $X \subset Y$ is a smooth hypersurface in the anticanonical linear system $|K_Y^{-1}|$, then X is a Calabi-Yau n-fold.

(4) As another generalization of (3) we have smooth complete intersections in products of projective spaces $X^n = \{P_1 = \cdots = P_m = 0\} \subset \mathbb{CP}^{n_1} \times \cdots \times \mathbb{CP}^{n_k}$ where $n = \sum_{p=1}^{k} n_p - m$ and each P_j is a multi-homogeneous polynomial of multidegree $(d_1^{(j)}, \ldots, d_k^{(j)})$ and we have $\sum_{j=1}^{m} d_p^{(j)} = n_p + 1$ for all $1 \leqslant p \leqslant k$.

(5) A simply connected Calabi-Yau surface (i.e. $\dim_{\mathbb{C}} X = 2$) is called a $K3$ surface. The 2-dimensional examples in (2) and (3) above are projective $K3$ surfaces.

(6) A special class of Calabi-Yau manifolds are hyperkähler (or irreducible holomorphic symplectic) manifolds. These are simply connected compact Kähler manifolds X with $\dim_{\mathbb{C}} X = 2n$, with a holomorphic 2-form Ω which is unique up to scaling and which is symplectic (i.e. it induces an isomorphism $T^{(1,0)}X \cong (T^{(1,0)}X)^*$). These are indeed Calabi-Yau since Ω^n trivializes K_X. Examples of hyperkähler $2n$-manifolds are the Hilbert schemes $X = \mathrm{Hilb}^{[n]}(Y)$ of n points on a $K3$ surface Y, see [1].

It is a classical fact that Calabi-Yau manifolds of complex dimension 1 are elliptic curves (1-dimensional tori). The Kodaira-Enriques classification of complex surfaces implies that every 2-dimensional Calabi-Yau manifold is either a torus, a $K3$ surface, or a finite unramified quotient of these (which are called bielliptic and Enriques surfaces respectively). It is also known that there are only finitely many possible diffeomorphism types (and even complex deformation equivalence classes) for Calabi-Yau manifolds of dimension $\leqslant 2$, but whether this is true also in any given dimension $n \geqslant 3$ is an outstanding well-known open problem:

QUESTION 2.3. *For any given $n \geqslant 3$, is there only a finite number of possible diffeomorphism types of Calabi-Yau n-folds?*

One can also ask the stronger question of finitely many complex deformation equivalence classes, or the weaker questions of finitely many topological types, or even just finitely many values of the topological Euler characteristic $\chi(X)$.

2.3. Ricci-flat Kähler metrics. The fundamental result about Calabi-Yau manifolds, which gives them their name, is Yau's solution [112] of the Calabi Conjecture [9]:

THEOREM 2.4. *Let X^n be a Calabi-Yau manifold and ω a Kähler metric on X. Then there is a unique Kähler metric $\tilde{\omega}$ on X with $[\tilde{\omega}] = [\omega]$ in $H^2(X, \mathbb{R})$ and with vanishing Ricci curvature,*

$$\mathrm{Ric}(\tilde{\omega}) = 0.$$

More precisely, consider the Ricci curvature form $\mathrm{Ric}(\omega)$ of an arbitrary Kähler metric on X. This is a closed real $(1,1)$-form on X, locally given by $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\det(g)$, whose cohomology class in $H^2(X, \mathbb{R})$ equals

$2\pi c_1(X)$, hence it vanishes since X is Calabi-Yau. By the $\partial\overline{\partial}$-Lemma, there is a smooth function F on X, unique up to an additive constant, such that $\mathrm{Ric}(\omega) = \sqrt{-1}\partial\overline{\partial}F$. We fix this additive constant so that

$$\int_X (e^F - 1)\omega^n = 0.$$

By the $\partial\overline{\partial}$-Lemma again the Ricci-flat metric $\tilde{\omega}$ that we seek to construct can be written as $\tilde{\omega} = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ for some $\varphi \in C^\infty(X,\mathbb{R})$ (unique up to an additive constant). It is then elementary to see that $\tilde{\omega}$ is Ricci-flat if and only if it satisfies

$$\tilde{\omega}^n = (\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F\omega^n,$$

which can be thought of as a PDE for the unknown function φ, which of course should satisfy that $\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0$. This is thus a complex Monge-Ampère equation, a fully nonlinear (if $n \geqslant 2$) elliptic second order scalar PDE for φ, which we can write as

$$(2.1) \qquad (\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F\omega^n, \quad \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \sup_X \varphi = 0,$$

by fixing a normalization for φ. Yau's fundamental result [**112**] is then:

THEOREM 2.5. *Let (X^n, ω) be a compact Kähler manifold and $F \in C^\infty(X,\mathbb{R})$ be a given function with $\int_X(e^F - 1)\omega^n = 0$. Then there exists a unique $\varphi \in C^\infty(X,\mathbb{R})$ solving* (2.1).

The Ricci-flat Kähler metrics provided by Theorem 2.4 have restricted holonomy contained in $SU(n)$. If X is a torus, then every Ricci-flat Kähler metric is flat and the holonomy is trivial. If X is simply connected and $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$ then the holonomy is exactly $SU(n)$, while if X is hyperkähler (of dimension $2n$) then the holonomy is exactly $Sp(n)$. The Bogomolov-Beauville-Calabi Decomposition Theorem [**1, 4, 9**] shows that if (X, ω) is a Ricci-flat Calabi-Yau manifold, then some finite cover splits holomorphically and isometrically into the product of factors each of which is either a flat torus, a Calabi-Yau manifold Y with holonomy equal to $SU(\dim Y)$ or a hyperkähler manifold Z with holonomy equal to $Sp(\dim Z/2)$.

2.4. The Kähler cone and its boundary. On a compact Kähler manifold (X, ω) the cohomology class $[\omega]$ lies naturally in the cohomology group $H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$ of real de Rham 2-classes which admit a representative which is a closed real $(1,1)$-form. The vector space $H^{1,1}(X, \mathbb{R})$ has real dimension equal to the Hodge number $h^{1,1}(X)$. The cohomology classes of Kähler metrics form a cone

$$\mathcal{C}_X = \{[\omega] \mid \omega \text{ Kähler metric on } X\} \subset H^{1,1}(X, \mathbb{R}),$$

the *Kähler cone* of X, which is easily seen to be an open and convex cone. Its closure $\overline{\mathcal{C}_X}$ is called the *nef cone*, and is a closed convex and salient cone. It is easily verified that a class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ is nef if and only if for every

$\varepsilon > 0$ it contains a representative α_ε which is a smooth real $(1,1)$-form which satisfies

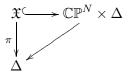$$(2.2) \qquad\qquad \alpha_\varepsilon \geqslant -\varepsilon\omega,$$

on X. The boundary $\partial\mathcal{C}_X$ of the Kähler cone has in general a very complicated geometry, already on $K3$ surfaces, see [**109**] for some explicit examples.

On the other hand, a class $[\alpha]$ which admits a smooth semipositive representative $\alpha \geqslant 0$ (i.e. satisfying (2.2) with $\varepsilon = 0$) is called semipositive. The cone of semipositive classes clearly contains the Kähler cone and is contained in the nef cone, but this last inclusion is in general strict. The first example of a nef $(1,1)$-class with no smooth semipositive representative was constructed by Demailly-Peternell-Schneider [**23**] on a ruled surface, and we will give examples on $K3$ surfaces in section 3.8.

2.5. Degenerations of Ricci-flat Kähler metrics. Thanks to Yau's Theorem 2.4, there is a bijective correspondence between the set of Ricci-flat Kähler metrics on a Calabi-Yau manifold X and the points of its Kähler cone $\mathcal{C}_X$. Because of this result, there are two basic ways in which Ricci-flat Calabi-Yau metrics can degenerate: we can either fix the complex structure and move the Kähler class to the boundary of the Kähler cone, or we can "fix the Kähler class" and degenerate the complex structure in a family. Of course one can also consider more general degenerations in which the complex and Kähler structures vary simultaneously, but we will not consider these here. Instead, let us explain these two basic setups more precisely.

2.6. Setup I: Kähler class degenerations. We assume we have a Calabi-Yau manifold X with a cohomology class $[\alpha] \in \partial\mathcal{C}_X$ and a path $[\alpha_t] \in H^{1,1}(X,\mathbb{R}), 0 \leqslant t \leqslant 1$, with $[\alpha_t] \in \mathcal{C}_X$ for $t > 0$ and $[\alpha_0] = [\alpha]$. Theorem 2.4 gives us a unique Ricci-flat Kähler metric ω_t in the class $[\alpha_t]$ for $t > 0$.

In many cases, we will consider the special path $[\alpha_t] = [\alpha_0] + t[\omega_X]$, where ω_X is a Kähler metric on X.

2.7. Setup II: complex structure degenerations. For simplicity here we restrict to the projective case. The setup is then that we have a family

where $\mathfrak{X}$ is an irreducible $n + 1$-dimensional variety, $\Delta \subset \mathbb{C}$ is the unit disc, π is a surjective holomorphic map with connected fibers which is a proper holomorphic submersion over Δ^* and $X_t = \pi^{-1}(t)$ are Calabi-Yau manifolds for $t \in \Delta^*$. We denote by $\mathfrak{L} \to \mathfrak{X}$ the pullback of $\mathcal{O}_{\mathbb{CP}^N}(1)$ and by $[\alpha_t] = c_1(\mathfrak{L}|_{X_t})$ for $t \in \Delta^*$. This is a Kähler class on X_t so by Theorem 2.4 it contains a unique Ricci-flat Kähler metric ω_t, for $t \in \Delta^*$.

2.8. The main question. The main motivating question is the following:

QUESTION 2.6. *What is the behaviour of the Ricci-flat manifolds* (X, ω_t) *in Setup I and* (X_t, ω_t) *in Setup II when t approaches zero? Do they have a Gromov-Hausdorff limit, or a smooth limit away from a singular set? If so, how large are the singularities limit metric space? Is the limit space homeomorphic to a complex analytic space?*

There is of course a vast literature on the general problem in Riemannian geometry of understanding limits of Einstein manifolds, starting from the general result of Gromov [**41**] which shows that there is some sequence $t_i \to 0$ such that ω_{t_i} converges in the (pointed) Gromov-Hausdorff sense to a metric space, which under certain assumptions can be proven to have additional properties, see e.g. [**12, 13, 14**], but even the uniqueness of the limit space in our setting does not follow from this theory. The philosophy of the work that we will overview is that in the setting of Question 2.6 it is possible to go substantially beyond what the general Riemannian convergence theory gives, using tools from Kähler geometry and the analysis of complex Monge-Ampère equations, and to give definite answers to Question 2.6 in many cases.

3. Degenerating the Kähler class

In this section we will discuss Setup I from §2.6. Recall that we have a Calabi-Yau manifold X^n with a path $[\alpha_t] \in H^{1,1}(X, \mathbb{R}), 0 \leqslant t \leqslant 1$, of classes which are Kähler for $t > 0$ and with $[\alpha_0] = [\alpha] \in \partial \mathcal{C}_X$, and for $t > 0$ we let ω_t be the unique Ricci-flat Kähler metric in the class $[\alpha_t]$.

We can fix a unit-volume Ricci-flat Kähler metric ω_X on X, and then the Ricci-flatness of ω_t is equivalent to

$$(3.1) \qquad \omega_t^n = \left(\int_X \omega_t^n \right) \omega_X^n.$$

3.1. Diameter. The first basic result about the metrics ω_t is the following:

THEOREM 3.1 (T. [**98**], Zhang [**114**]). *There is a constant* $C > 0$ *such that*

$$\mathrm{diam}(X, \omega_t) \leqslant C,$$

for all $0 < t \leqslant 1$.

The idea of the proof of this result is to first use a lemma of Demailly-Peternell-Schneider [**22**] together with (3.1) (and the assumption that $[\omega_t]$ converges to the limiting class $[\alpha]$) to show that for all t there is a unit-radius ω_t-geodesic ball $B_t = B_{\omega_t}(x_t, 1) \subset X$ with ω_t-volume a definite fraction of the total volume, i.e. $\int_{B_t} \omega_t^n \geqslant C^{-1} \int_X \omega_t^n$. On the other hand, an inequality

of Yau [**86**, Theorem I.4.1] (which only uses that $\mathrm{Ric}(\omega_t) \geqslant 0$) gives that for all $0 < R < \mathrm{diam}(X, \omega_t)$ we have

$$\frac{R-1}{4n} \leqslant \frac{\int_{B_{\omega_t}(x_t, 2(R+1))} \omega_t^n}{\int_{B_t} \omega_t^n},$$

and taking $R = \mathrm{diam}(X, \omega_t) - 1$ and using the result above, we obtain Theorem 3.1.

From Theorem 3.1, it follows in particular that any Gromov-Hausdorff limit of some sequence (X, ω_{t_i}) $(t_i \to 0)$ is a compact length metric space.

3.2. Volume. The volume of the Ricci-flat metrics ω_t is readily calculated as (we drop here and in the following the customary factor of $n!$)

$$\mathrm{Vol}(X, \omega_t) = \int_X \omega_t^n = \int_X \alpha_t^n \to \int_X \alpha^n, \quad \text{as } t \to 0,$$

so the behavior of the volume is completely determined by the intersection number $\int_X \alpha^n \geqslant 0$.

If $\int_X \alpha^n > 0$ we will say that the metrics ω_t are non-collapsed, which is justified since by the Bishop-Gromov volume comparison theorem and the diameter bound in Theorem 3.1 we have for every $x \in X$ and $r \leqslant \mathrm{diam}(X, \omega_t)$,

$$\mathrm{Vol}(B_{\omega_t}(x, r), \omega_t) \geqslant C^{-1} r^{2n},$$

for a uniform constant $C > 0$ independent of $t > 0$. We will refer to the non-collapsed setting as Setup I.A.

If on the other hand we have $\int_X \alpha^n = 0$, we will say that the metrics ω_t are volume collapsed. We will call this Setup I.B.

3.3. Setup I.A: Volume non-collapsed. In this section we discuss the volume non-collapsed case of Setup I, which we call Setup I.A: we have a Calabi-Yau manifold X with a cohomology class $[\alpha] \in \partial \mathcal{C}_X$ with $\int_X \alpha^n > 0$ (this is known as "nef and big"), with and a path $[\alpha_t] \in H^{1,1}(X, \mathbb{R}), 0 \leqslant t \leqslant 1$, with $[\alpha_t] \in \mathcal{C}_X$ for $t > 0$ and $[\alpha_0] = [\alpha]$, and we let ω_t be the unique Ricci-flat Kähler metric in the class $[\alpha_t]$ for $t \neq 0$.

EXAMPLE 3.2. The simplest example of Setup I.A is when X is a Kummer $K3$ surface, associated with a 2-torus $T = \mathbb{C}^2/\Lambda$. To construct X we consider the involution $\iota : T \to T$ induced by reflection across the origin in $\mathbb{C}^2$, and take the quotient $Y = T/\iota$. Since ι has 16 fixed points (the 2-torsion points of T), Y is has 16 orbifold points (which are rational double points), and blowing up these 16 points we get $\pi : X \to Y$ where X is our Kummer $K3$ surface. We then take ω_Y any orbifold Kähler metric on Y, and let $[\alpha] = \pi^*[\omega_Y]$. This is a nef and big class on X which is not Kähler since it intersects trivially with the 16 exceptional divisors of π.

Going back to the general Setup I.A, if $V^k \subset X$ is an irreducible k-dimensional closed analytic subvariety, smooth real (k, k)-forms on X can be integrated over V (by restricting to the regular part) to get a finite

number, which vanishes if the form is exact (see e.g. [**40**] for these classical results by Lelong), and in fact the $2k$-dimensional volume of V with respect to ω_t can be calculated as

$$\mathrm{Vol}(V, \omega_t) = \int_V \omega_t^k = \int_V \alpha_t^k \to \int_V \alpha^k, \quad \text{as } t \to 0,$$

so again the behavior of the volume of V is controlled by the intersection number $\int_V \alpha^k \geqslant 0$. We can collect together all the subvarieties whose volume is shrinking to zero and define the *null locus* of $[\alpha]$ as

$$\mathrm{Null}([\alpha]) = \bigcup_{\int_V \alpha^{\dim V} = 0} V,$$

where the (set-theoretic) union is over all positive-dimensional irreducible closed analytic subvarieties $V \subset X$ with vanishing intersection with $[\alpha]$. This object first appeared explicitly in the work of Nakamaye [**80**] in the context of algebraic geometry where X is a projective manifold and $[\alpha] = c_1(L)$ for some holomorphic line bundle $L \to X$.

In the Kummer Example 3.2 we have that $\mathrm{Null}([\alpha])$ equals the union of the 16 exceptional divisors of π.

Despite it being defined as a potentially large union of subvarieties, the null locus is in fact Zariski closed as shown by Collins and the author [**18**]:

THEOREM 3.3. *Let X^n be a compact Kähler manifold and $[\alpha]$ a nef $(1,1)$-class with $\int_X \alpha^n > 0$. Then $\mathrm{Null}([\alpha]) \subset X$ is a proper closed analytic subvariety, which is empty if and only if $[\alpha] \in \mathcal{C}_X$.*

The fact that $\mathrm{Null}([\alpha]) = \emptyset$ characterizes Kähler classes among nef classes is just a restatement of the famous Nakai-Moishezon criterion for Kähler classes by Demailly-Păun [**21**], and the arguments in [**18**] together with [**17**] provide a new proof of it.

Furthermore, inspired by foundational results of Nakamaye [**80**] and Ein-Lazarsfeld-Mustaţă-Nakamaye-Popa [**27**] in algebraic geometry, we showed in [**18**] that the class $[\alpha]$ behaves like a Kähler class on the complement of its null locus, in the following sense:

THEOREM 3.4. *Let (X^n, ω) be a compact Kähler manifold and α a closed real $(1,1)$ form such that $[\alpha]$ is nef with $\int_X \alpha^n > 0$. Then there is a quasi-psh function φ on X, which is smooth on $X \backslash \mathrm{Null}([\alpha])$ and has logarithmic singularities along $\mathrm{Null}([\alpha])$, such that*

$$\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant \varepsilon\omega,$$

in the weak sense of currents on X, for some $\varepsilon > 0$.

Without entering into the technicalities of closed positive currents, all that is important for us here is that outside of $\mathrm{Null}([\alpha])$ we have that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi$ is a smooth Kähler metric with a definite lower bound $\varepsilon\omega > 0$ which does not degenerate near $\mathrm{Null}([\alpha])$, and that the potential φ goes to $-\infty$ uniformly along $\mathrm{Null}([\alpha])$. One can thus think of $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi$ as

a Kähler metric with certain singularities along $\mathrm{Null}([\alpha])$, or in technical terms as a Kähler current in the class $[\alpha]$.

Using Theorems 3.3 and 3.4, the following result was proved in [**98**, **84**, **18**]:

THEOREM 3.5. *Suppose we are in Setup I.A, so X^n is Calabi-Yau and $[\alpha] \in \partial\mathcal{C}_X$ is a nef $(1,1)$-class with $\int_X \alpha^n > 0$. Then there is an incomplete Ricci-flat Kähler metric ω_0 on $X\backslash\mathrm{Null}([\alpha])$ such that $\omega_t \to \omega_0$ in $C^\infty_{\mathrm{loc}}(X\backslash\mathrm{Null}([\alpha]))$. The limit ω_0 is unique, independent of the path $[\alpha_t]$, and (X, ω_t) converges in the Gromov-Hausdorff topology to (Z, d), the metric completion of $(X\backslash\mathrm{Null}([\alpha]), \omega_0)$.*

This theorem applies in particular in the Kummer $K3$ surfaces of Example 3.2, thus recovering earlier results of Kobayashi-Todorov [**59**] and LeBrun-Singer [**67**]. Briefly, the idea is that φ can be used as a barrier function to obtain uniform C^∞ estimates for ω_t on compact subsets away from $\mathrm{Null}([\alpha])$. In the case when $[\alpha] \in H^2(X, \mathbb{Q})$ one can use Nakamaye's Theorem [**80**] in place of Theorems 3.3 and 3.4, which was done by the author in [**98**], and the Gromov-Hausdorff limit in this case was obtained by Rong-Zhang [**84**]. Lastly, Collins and the author [**18**] used Theorems 3.3 and 3.4 to remove the rationality assumption.

Furthermore, if we assume that $[\alpha] \in H^2(X, \mathbb{Q})$, then some multiple of $[\alpha]$ equals $c_1(L)$ for a holomorphic line bundle $L \to X$, and the fact that $c_1(L)$ is nef and big implies that X is projective (since it is Kähler and Moishezon, see e.g. [**21**]) and Kawamata's base-point-free theorem [**58**] implies that L is semiample, so the linear system induced by some high tensor power of L defines a surjective holomorphic birational map $\pi : X \to Y$ with connected fibers onto a normal projective variety Y which is a Calabi-Yau variety with at worst canonical singularities (see e.g. [**98**]). Furthermore, the exceptional locus of π is equal to $\mathrm{Null}([\alpha])$, we have that $L = f^*A$ for some ample line bundle A on Y, and the limiting Ricci-flat metric ω_0 is naturally the pullback of a singular Ricci-flat metric on Y [**28**] in a class proportional to $c_1(A)$. In this case it was proved by Song [**89**] (improving results of Donaldson-Sun [**25**] that do not apply here because of the changing Kähler classes) that the Gromov-Hausdorff limit Z is in fact homeomorphic to Y, and so the non-collapsed Gromov-Hausdorff limit of (X, ω_t) has an algebraic structure. Furthermore, the singular set of Z (in the sense of [**12**]) has real Hausdorff codimension at least 4.

These results give an answer to Question 2.6 in Setup I.A when $[\alpha]$ is rational. We expect that a similar picture should hold even when the class $[\alpha]$ (or any of his multiples $\lambda[\alpha], \lambda > 0$) is not rational:

CONJECTURE 3.6. *Let X^n be a Calabi-Yau manifold and $[\alpha] \in \partial\mathcal{C}_X$ a nef $(1,1)$-class with $\int_X \alpha^n > 0$. Then there is a surjective holomorphic and bimeromorphic map $\pi : X \to Y$ with connected fibers onto a normal compact Kähler analytic space Y such that $[\alpha] = \pi^*[\beta]$ for some Kähler class $[\beta]$ on Y (in the sense of e.g. [**28**]), and $\mathrm{Exc}(\pi) = \mathrm{Null}([\alpha])$.*

In particular, the nef and big class $[\alpha]$ would contain a smooth semipositive representative, which is also an open problem. The following related conjecture is also open:

CONJECTURE 3.7. *Let X^n be a Calabi-Yau manifold and α a closed real $(1,1)$-form such that $[\alpha] \in \partial \mathcal{C}_X$ is nef. Then there is a bounded function $\varphi \in L^\infty(X)$ such that $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$ in the weak sense of currents on X.*

As we just said, if Conjecture 3.6 holds then we can even find such a $\varphi \in C^\infty(X)$ when $\int_X \alpha^n > 0$, so Conjecture 3.7 when $[\alpha]$ is nef and big follows from Conjecture 3.6. Conjecture 3.6 is known when $n = 2$ by work of Filip and the author [31] and when $n = 3$ by Höring [54]. Note also that if $[\alpha] \in H^2(X, \mathbb{Q})$ then every irreducible component of $\mathrm{Null}([\alpha]) = \mathrm{Exc}(\pi)$ is uniruled, as follows for example from [57], and it is natural to conjecture that this property also holds for general nef and big $[\alpha]$.

On the other hand, Conjecture 3.7 when $\int_X \alpha^n = 0$ is not even known for $K3$ surfaces, see also section 3.8.2. Observe that the condition in Conjecture 3.7 is equivalent to the fact that a closed positive current with minimal singularities in the class $[\alpha]$ has bounded potentials, in the language of Demailly [23]. The Calabi-Yau condition is crucial in both conjectures, see e.g. [23] for a counterexample to Conjecture 3.7 where X is a ruled surface. Also, as remarked above, it would follow from Conjecture 3.6 that every nef and big class $[\alpha]$ on a Calabi-Yau manifold admits a smooth semipositive representative, but this is known to be false for nef classes with $\int_X \alpha^n = 0$ as discussed in detail in section 3.8 below.

Going back to the Ricci-flat metrics ω_t, in analogy with Theorem 3.5 and the aforementioned result of [89] we then expect the following:

CONJECTURE 3.8. *In setup I.A, the Gromov-Hausdorff limit Z of the Ricci-flat manifolds (X, ω_t) is homeomorphic to a normal compact Kähler analytic space Y. The singular set of Z (in the sense of [12]) has real Hausdorff codimension at least 4. If Conjecture 3.6 holds, Y should be the same space that appears there.*

Lastly, let us mention that one can ask further questions about the structure of the Gromov-Hausdorff limit space (Z, d) in Theorem 3.5 near its singular points. In particular, it is not known whether the tangent cones at singular points of (Z, d) are unique. This is established in [26] in a related but different situation (with fixed polarization), and in [50] in the case when the contracted space Y has certain types of isolated singularities.

3.4. Setup I.B: volume collapsed. In this section we discuss the volume collapsed case of Setup I, which we call Setup I.B: we have a Calabi-Yau manifold X^n with a cohomology class $[\alpha] \in \partial \mathcal{C}_X$ with $\int_X \alpha^n = 0$, which we approach with the path $[\alpha_t] = [\alpha] + t[\omega_X]$ (where ω_X is a Kähler metric on X), and we let ω_t be the unique Ricci-flat Kähler metric in the class $[\alpha_t]$ for $t > 0$.

The choice of path $[\alpha]+t[\omega_X]$ comes from the work of Gross-Wilson [**47**], and it is also formally analogous to the path traced in cohomology by the Kähler-Ricci flow on compact Kähler manifolds with semiample canonical bundle and intermediate Kodaira dimension [**91**] (in fact, many of the results described below have analogs in the setting of the Kähler-Ricci flow, see [**103**]).

EXAMPLE 3.9. The simplest example of Setup I.B is when X is a $K3$ surface that admits an elliptic fibration $f : X \to \mathbb{CP}^1$. This is a surjective holomorphic map with connected fibers and with all but finitely many fibers equal to elliptic curves. If we take ω_{FS} to be the Fubini-Study metric on $\mathbb{CP}^1$, then $[\alpha] = f^*[\omega_{\mathrm{FS}}]$ is a nef class on X with $\int_X \alpha^2 = 0$. The behavior of the Ricci-flat metrics ω_t on X in this case was first investigated by Gross-Wilson [**47**].

3.5. Fiber spaces. Example 3.9 can be generalized to higher dimension as follows. Let X^n be a Calabi-Yau manifold and suppose that we have $f : X \to Y$ a surjective holomorphic map with connected fibers onto a compact Kähler analytic space Y with $0 < \dim Y = m < \dim X$. The map f will be called a *fiber space*. If $[\omega_Y]$ is a Kähler metric on Y (in the sense of analytic spaces [**28**]) then $[\alpha] = f^*[\omega_Y]$ is a nef class on X with $\int_X \alpha^n = 0$.

Many nef classes $[\alpha]$ arise in this way: for example if X is a $K3$ surface and $0 \neq [\alpha] \in \partial\mathcal{C}_X$ satisfies $\int_X \alpha^2 = 0$ and $[\alpha] \in H^2(X, \mathbb{Q})$ then $[\alpha]$ is the pullback of a Kähler class from the base $\mathbb{CP}^1$ of an elliptic fibration on X, see [**31**]. In higher dimensions, a well-known conjecture in algebraic geometry (see e.g. [**66**]) would imply that if X is any projective Calabi-Yau manifold and $0 \neq [\alpha] \in \partial\mathcal{C}_X$ satisfies $\int_X \alpha^n = 0$ and $[\alpha] \in H^2(X, \mathbb{Q})$ then X admits a fiber space structure $f : X \to Y$ and $[\alpha] = f^*[\omega_Y]$ for some Kähler class $[\omega_Y]$ on Y. On the other hand, the analogous conjecture for $[\alpha] = c_1(D)$ with D a nef $\mathbb{R}$-divisor fails, see Theorem 3.20, so the rationality assumption is crucial here.

It is important to note that fiber spaces in general may have singular fibers. More precisely, if we denote by $S' \subset Y$ the union of the singular locus of Y together with the critical values of f on the regular part of Y, then S' is a closed analytic subvariety of Y, its preimage $S = f^{-1}(S')$ is a closed analytic subvariety of X, and $f : X \backslash S \to Y \backslash S'$ is a proper holomorphic submersion with fibers $X_y = f^{-1}(y)$ Calabi-Yau manifolds of dimension $n - m$. The subvariety S is informally referred to as the union of singular fibers of f.

It is also important to note that the smooth fibers $X_y = f^{-1}(y)$ for $y \in Y \backslash S'$ are all pairwise diffeomorphic (Ehresmann's Lemma) but in general are not biholomorphic. By a result of Fischer-Grauert [**34**], the fibers X_y are all pairwise biholomorphic for all y in some open subset $U \subset Y \backslash S'$ if and only if $f : f^{-1}(U) \to U$ is a holomorphic fiber bundle.

When Y is smooth and f is a submersion everywhere (i.e. with our notation when $S = \emptyset$) then we have the following result proved by Zhang and the author [**106, 108**]:

THEOREM 3.10. *Let X be a compact Calabi-Yau manifold and $f : X \to Y$ a holomorphic submersion with connected fibers onto a lower-dimensional Kähler manifold. Then f is a holomorphic fiber bundle, and both Y and the fiber of f are Calabi-Yau.*

Furthermore, if X is projective, then this fiber bundle is actually the quotient of the trivial bundle by a finite group action. This result shows that to have nontrivial variation of the complex structure of the fibers of f, one must allow singular fibers.

3.6. Setup I.B.1: fiber spaces. The discussion of our main Question 2.6 in the collapsing Setup I.B now splits into two subcases, according to whether the class $[\alpha]$ comes from the base of a fiber space structure on X, or not. We first discuss in detail the case when it does, so we have $f : X \to Y$ a fiber space with $[\alpha] = f^*[\omega_Y]$ for some Kähler class $[\omega_Y]$ on Y, and we will call this Setup I.B.1.

The Ricci-flat Kähler metrics ω_t in the class $[\alpha_t] = f^*[\omega_Y] + t[\omega_X]$ can be written as

$$\omega_t = f^*\omega_Y + t\omega_X + \sqrt{-1}\partial\overline{\partial}\varphi_t, \quad \sup_X \varphi_t = 0,$$

where the normalized Kähler potentials φ_t are uniquely determined and they solve the degenerating family of complex Monge-Ampère equations

$$(3.2) \qquad (f^*\omega_Y + t\omega_X + \sqrt{-1}\partial\overline{\partial}\varphi_t)^n = c_t t^{n-m}\omega_X^n,$$

where c_t is a constant which bounded away from 0 and ∞.

3.6.1. *Behavior away from the singular fibers.* The first general result about (3.2) is a uniform bound for φ_t,

$$\|\varphi_t\|_{L^\infty(X)} \leqslant C,$$

which was proved independently in [**20, 29**], extending classical work of Kołodziej [**63**]. Using this estimate as a starting point, and using also Yau's Schwarz Lemma computations [**113**], the author proved in [**99**] (extending [**90**] which dealt with elliptically fibered $K3$ surfaces):

THEOREM 3.11. *Suppose we are in Setup I.B.1, so X^n is Calabi-Yau, $f : X \to Y$ is a fiber space, $[\alpha] = f^*[\omega_Y]$ as above, $[\alpha_t] = [\alpha] + t[\omega_X]$ and ω_t is the Ricci-flat Kähler metric on X in the class $[\alpha_t]$ for $t > 0$. Then there is a Kähler metric $\omega_0 = \omega_Y + \sqrt{-1}\partial\overline{\partial}\varphi_0$ on $Y \backslash S'$ with*

$$\mathrm{Ric}(\omega_0) = \omega_{\mathrm{WP}} \geqslant 0,$$

such that $\omega_t \to f^* \omega_0$ as $t \to 0$ in the sense that $\varphi_t \to f^* \varphi_0$ in $C^{1,\gamma}_{\text{loc}}(X \backslash S, \omega_X)$ for every $\gamma < 1$. Furthermore, on every compact subset of $X \backslash S$ we have

$$(3.3) \qquad C^{-1}(f^* \omega_Y + t\omega_X) \leqslant \omega_t \leqslant C(f^* \omega_Y + t\omega_X).$$

for a uniform constant $C > 0$.

Thanks to (3.3), the Ricci-flat metrics ω_t shrink the smooth fibers X_y of f to points as $t \to 0$ and collapse to the metric ω_0 on the base in the $C^{1,\gamma}$ topology of Kähler potentials. The Weil-Petersson form ω_{WP} measures the variation of the complex structure of the smooth fibers X_y, and in particular it vanishes whenever f is a holomorphic fiber bundle, and already appears in [**47**].

It is highly desirable to improve the topology of the convergence of ω_t to $f^* \omega_0$ (away from the singular fibers) and the main open question is then the following (cf. [**100, 101**]):

CONJECTURE 3.12. *In Setup I.B.1, we have that $\omega_t \to f^* \omega_0$ in $C^{\infty}_{\text{loc}}(X \backslash S, \omega_X)$.*

In other words, one has to establish a uniform bound for $\|\varphi_t\|_{C^k(K, \omega_X)}$ on every compact subset $K \subset X \backslash S$ for every $k \in \mathbb{N}$. It follows from the work of Fine [**33**] that Conjecture 3.12 holds when $S = \emptyset$, but recall however that thanks to Theorem 3.10 this assumption is very restrictive as it implies that f is a holomorphic fiber bundle. When singular fibers are present, the currently known results are:

THEOREM 3.13. *Conjecture 3.12 holds in any of the following cases:*
- *(Gross-T.-Zhang [**44**], Hein-T. [**52**]) if the smooth fibers X_y are tori (or finite étale quotients of tori). In this case the metrics ω_t have locally uniformly bounded sectional curvature away from S.*
- *(Hein-T. [**53**]) if the smooth fibers X_y are pairwise biholomorphic (but S may be nonempty)*

We will give some ideas of the proof of this theorem in section 3.7 below. On the other hand, without any extra assumptions on the fibers of f, the convergence in Theorem 3.11 was improved by Weinkove, Yang and the author [**105**] to $C^0_{\text{loc}}(X \backslash S, \omega_X)$ convergence of ω_t to $f^* \omega_0$. This was more recently further improved by Hein and the author [**53**]:

THEOREM 3.14. *In Setup I.B.1, we have that $\omega_t \to f^* \omega_0$ in $C^{\gamma}_{\text{loc}}(X \backslash S, \omega_X)$ for every $\gamma < 1$.*

Theorems 3.13 and 3.14 are in fact purely local on the base.

3.6.2. *Adiabatic limits.* The above results show that the Ricci-flat metrics ω_t shrink the smooth fibers and collapse to a metric on the base. The second result that we will discuss, proved by Zhang and the author [**107**], shows that if we "zoom in" to make the fibers have fixed size (and therefore the base of the fibration is getting blown up), then we see a "semi-Ricci-flat" picture. This is known as an "adiabatic limit".

THEOREM 3.15. *In Setup I.B.1, given any $x \in X \backslash S$ fix a neighborhood U of $y = f(x)$ relatively compact in $Y \backslash S'$, Then the pointed spaces $(f^{-1}(U), \frac{\omega_t}{t}, x)$ converge smoothly modulo diffeomorphisms to the cylinder*

$$(\mathbb{C}^m \times X_y, \omega_{\mathbb{C}^m} \oplus \omega_{\mathrm{SRF},y}, p),$$

where $\omega_{\mathbb{C}^m}$ is a Euclidean metric and $\omega_{\mathrm{SRF},y}$ denotes the unique Ricci-flat Kähler metric on X_y cohomologous to $[\omega_X|_{X_y}]$.

We emphasize here that the complex structure on $f^{-1}(U)$ also converges smoothly (modulo diffeomorphisms) to the product complex structure of $\mathbb{C}^m \times X_y$. Furthermore, the diffeomorphisms that appear in this theorem are completely explicit, and are obtained by fixing a smooth trivialization of f (up to shrinking U) and stretching out the base directions while being the identity on the fibers.

As a consequence of Theorem 3.15, if the smooth fibers X_y are not tori or quotients, then the sectional curvatures of ω_t have to blow up (to $+\infty$ and also to $-\infty$) on every fiber X_y, since the rescaled metrics $\frac{\omega_t}{t}$ converge smoothly modulo diffeomorphism to the Ricci-flat (but non-flat) cylinder as above which has both positive and negative sectional curvatures in the fiber directions.

Of course, all of these results were already obtained by Gross-Wilson [**47**] using their gluing construction for elliptically fibered $K3$ surfaces with only I_1 singular fibers. More recently, Li [**68, 69**] proves a similar result also when we zoom in at x on a singular fiber when X is a Calabi-Yau 3-fold, X_y are $K3$ surfaces, and f is a Lefschetz fibration.

3.6.3. *Gromov-Hausdorff limits.* Finally, we discuss the Gromov-Hausdorff limit of (X, ω_t). Recall that thanks to the diameter bound in Theorem 3.1 and to Gromov's precompactness theorem, for every sequence $t_i \to 0$ we can find a subsequence (still denoted by t_i) and a compact metric space (Z, d) such that $(X, \omega_{t_i}) \to (Z, d)$ in the Gromov-Hausdorff topology, as $i \to \infty$.

It is then natural to ask whether (Z, d) depends on the chosen sequence or is in fact unique, and what its relation is to the collapsed limit $(Y \backslash S', \omega_0)$ away from the singular fibers given by Theorem 3.13.

The first general result in this direction follows by combining the work of Gross, Zhang and the author [**44**] and of Weinkove, Yang and the author [**105**]:

THEOREM 3.16. *In Setup I.B.1, suppose that $(X, \omega_{t_i}) \to (Z, d)$ in the Gromov-Hausdorff topology. Then there is a map $F : (Y \backslash S', \omega_0) \to (Z, d)$ which is a homeomorphism with its image $Z_0 \subset Z$, an open dense subset of Z, and F is a local isometry.*

This shows that any collapsed Gromov-Hausdorff limit (Z, d) is in fact a smooth manifold on a dense open set.

The main outstanding question in then the following [**100, 101**]:

CONJECTURE 3.17. *In Setup I.B.1, let (Z, d) be the metric completion of $(Y \backslash S', \omega_0)$, and let $S_Z = Z \backslash (Y \backslash S') \subset Z$. Then*

(a) *(Z, d) is a compact metric space and $(X, \omega_t) \to (Z, d)$ in the Gromov-Hausdorff topology as $t \to 0$*

(b) *S_Z has real Hausdorff codimension at least 2*

(c) *Z is homeomorphic to Y*

The conjecture is known in several cases:

THEOREM 3.18. *Conjecture 3.17 holds in any of the following cases:*

- *(Gross-T.-Zhang [**44, 45**], T.-Weinkove-Yang [**105**]) if X is projective and $\dim Y = 1$*
- *(T.-Zhang [**108**] if X is projective and hyperkähler*
- *(Gross-T.-Zhang [**46**]) if X is projective, Y is smooth and the divisorial components of S' have simple normal crossings*

Furthermore,

- *(Song-Tian-Zhang [**92**]) if X is projective then part (a) holds in general; if furthermore Y is smooth then part (c) holds*

While, as discovered in [**92**], the proof of part (a) does not require a detailed understanding of the blowup behavior of the limiting metric ω_0 near S', so far this seems to be needed to prove part (b), and this is what is achieved in [**45, 108, 46**] under the above assumptions. More specifically, under the assumption that X is projective, Y is smooth and the divisorial components of S' have simple normal crossings, it is proved in [**46**] that near these components ω_0 is quasi-isometric to a *conical metric* (with rational cone angles in $(0, 2\pi]$) up to logarithmically blowing up errors (which are negligible compared to the conical singularities), and this is crucial to prove part (b) of Conjecture 3.17.

It is also interesting to note that in the hyperkähler case, in [**108**] it is proved that the Gromov-Hausdorff limit Z is homeomorphic to the base Y (without assuming that Y is smooth), while a conjecture in algebraic geometry predicts that the base of every nontrivial fiber space $f : X \to Y$ with X^{2n} projective hyperkähler should be isomorphic to $\mathbb{CP}^n$, but this is only known in general when Y is smooth [**55**].

3.7. Smooth collapsing. In this section we give some ideas of the proof of Theorem 3.13 that gives smooth collapsing (away from the singular fibers) in Setup I.B.1 when the fibers are either tori or pairwise biholomorphic.

3.7.1. *Torus fibers.* First, we discuss the easier case of torus fibers [**44, 52**]. In this case, by old results in deformation theory [**111**], the preimage $f^{-1}(U)$ of every sufficiently small ball $U \subset Y \backslash S'$ has universal cover biholomorphic to $U \times \mathbb{C}^{n-m}$, and we denote $p : U \times \mathbb{C}^{n-m} \to f^{-1}(U)$, which satisfies $f \circ p = \mathrm{pr}_U$. In [**44, 48, 52**] one then constructs a holomorphic

period map for the torus fibers (equipped with a Kähler class from the to-
tal space) $Z : U \to \mathfrak{H}_{n-m}$ into the Siegel upper half space of symmetric
$(n-m) \times (n-m)$ complex matrices with positive definite imaginary part.
This is used to define an explicit semi-flat form $\omega_{\mathrm{SF}} \geqslant 0$ on $f^{-1}(U)$ by
defining its pullback via p to be

$$p^* \omega_{\mathrm{SF}} = -\frac{\sqrt{-1}}{4} \partial \overline{\partial} \left(\sum_{i,j=1}^{n-m} (\mathrm{Im} Z(y))_{ij}^{-1} (z_i - \overline{z}_i)(z_j - \overline{z}_j) \right),$$

and checking that it descends to $f^{-1}(U)$, and satisfies the crucial scaling
property that

$$\lambda_t^* p^* \omega_{\mathrm{SF}} = t^{-1} p^* \omega_{\mathrm{SF}}, \quad \lambda_t(y, z) = (y, t^{-\frac{1}{2}} z).$$

The semi-flat form ω_{SF} (in the projective case) was used in the context of
elliptically fibered $K3$ surfaces by Greene-Shapere-Vafa-Yau [**39**], and can
be traced back to Satake's book [**85**, Lemma IV.8.5].

Thanks to (3.3) it follows easily that on $f^{-1}(U)$ we have

$$C^{-1}(f^* \omega_Y + t \omega_{\mathrm{SF}}) \leqslant \omega_t \leqslant C(f^* \omega_Y + t \omega_{\mathrm{SF}}),$$

and using the scaling property of ω_{SF} this implies that

$$C^{-1} p^* (f^* \omega_Y + \omega_{\mathrm{SF}}) \leqslant \lambda_t^* p^* \omega_t \leqslant C p^* (f^* \omega_Y + \omega_{\mathrm{SF}}),$$

holds on $U \times \mathbb{C}^{n-m}$, and so $\lambda_t^* p^* \omega_t$ is uniformly equivalent to the Euclidean
metric on any fixed compact set. Standard higher order estimates (see e.g.
[**53**]) then give locally uniform C^k bounds on $\lambda_t^* p^* \omega_t$, for all k, from which
it is elementary to deduce uniform C^k bounds for ω_t on $f^{-1}(U)$, as well as
a uniform sectional curvature bound, as desired. The appendix of [**19**] also
provides an explicit rate of convergence.

3.7.2. *Isomorphic fibers.* Secondly, we discuss the second case in Theo-
rem 3.13, where the smooth fibers of f are fiberwise biholomorphic, following
[**53**]. By the Fischer-Grauert theorem, f is a holomorphic fiber bundle away
from the singular fibers S, so f can be holomorphically trivialized over any
sufficiently small ball $U \subset Y \backslash S'$, so in this trivialization the fiber space
equals the projection $U \times F \to U$ where $F = X_y$ is a fixed Calabi-Yau
$(n-m)$-fold. We let ω_F be the Calabi-Yau metric on F in the class that cor-
responds to $[\omega_X|_{X_y}]$ under our trivialization, let $\omega_{\mathbb{C}^m}$ be a Euclidean metric
on U, and define a family of shrinking product Kähler metrics $\tilde{\omega}_t = \omega_{\mathbb{C}^m} + t \omega_F$
on $U \times F$.

After a gauge-fixing argument (see [**53**]) we may assume that our Ricci-
flat metrics ω_t can be written as $\omega_t = \tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi_t$, they satisfy the
complex Monge-Ampère equation

$$(3.4) \qquad \omega_t^n = (\tilde{\omega}_t + \sqrt{-1} \partial \overline{\partial} \varphi_t)^n = c_t t^{n-m} \omega_{\mathbb{C}^m}^m \wedge \omega_F^{n-m},$$

and from (3.3)

$$(3.5) \qquad C^{-1}\tilde{\omega}_t \leqslant \omega_t \leqslant C\tilde{\omega}_t,$$

on $U \times F$.

The goal would be to prove that

$$(3.6) \qquad \|\omega_t\|_{C^k(U' \times F, \omega_{\mathbb{C}^m} + \omega_F)} \leqslant C,$$

on a smaller relatively compact subdomain $U' \subset U$ (C here depends on k, on the subdomain U' and on the background data), but in fact we will prove the much stronger estimate

$$(3.7) \qquad \|\omega_t\|_{C^k(U' \times F, \tilde{\omega}_t)} \leqslant C.$$

Note that in the case of torus fibers, the argument above does indeed prove such a stronger estimate (on $U \times \mathbb{C}^{n-m}$) (see also [**56**]).

However, when the fibers are not tori, we lack the "fiberwise scaling trick" above (using λ_t). We also cannot resort to purely local estimates for the Monge-Ampère equation (3.4) since there are simple counterexamples to local higher-order interior estimates for the analogous Monge-Ampère equation on a polydisc in $\mathbb{C}^{n-m} \times \mathbb{C}^m$ satisfying (3.5), see the introduction of [**53**]. These estimates are also false in a global setting where the fibers of f are manifolds with boundary (and Dirichlet conditions are imposed), see again [**53**]. So our argument to prove (3.7) will exploit the fact that the fiber F is compact without boundary.

The uniform estimate (3.5) means that in the collapsed geometry of $\tilde{\omega}_t$ the PDE (3.4) is uniformly elliptic. Nevertheless, (3.7) is not a consequence of any of the existing techniques to obtain higher-order estimates for complex Monge-Ampère equations, since these reference metrics are collapsing in the fiber directions. Instead, our arguments rely on a contradiction and blowup strategy, employing linear and nonlinear Liouville theorems and new sharp Schauder estimates on cylinders. The C^γ bound in Theorem 3.14 is proved in a similar fashion, but with formidable extra difficulties caused by the varying complex structure of the fibers X_y.

Let us discuss the proof of (3.7) for $k = 1$ (the estimate for $k = 0$ is given by (3.5)). We shall identify U with the Euclidean unit ball B_1. Ignoring issues with the boundary of U and with having to replace t by some sequence $t_i \to 0$, we will assume for a contradiction that

$$\mu_t = \sup_{U \times F} |\nabla^{\tilde{\omega}_t} \omega_t|_{\tilde{\omega}_t} \to +\infty,$$

define a stretching map in the base directions $\Psi_t : B_{\mu_t} \times F \to B_1 \times F$ by $\Psi_t(z, y) = (\mu_t^{-1} z, y)$, and let replacing ω_t by $\mu_t^2 \Psi_t^* \omega_t$ (still denoting this by ω_t) and $\tilde{\omega}_t$ by $\mu_t^* \Psi_t^* \tilde{\omega}_t = \omega_{\mathbb{C}^m} + \delta_t^2 \omega_F, \delta_t^2 = t\mu_t^2$ (still denoting this by $\tilde{\omega}_t$), we have the same estimate (3.5) on $B_{\mu_t} \times F$ but now $\sup_{B_{\mu_t} \times F} |\nabla^{\tilde{\omega}_t} \omega_t|_{\tilde{\omega}_t} = 1$, achieved at $(0, y_t)$ say.

Up to passing to subsequences there are 3 cases. In case 1 we have $\delta_t \to \infty$. Here we fix some chart on F containing all the y_t's for t small,

and in this chart times B_{μ_t} we pull back everything via the diffeomorphism $(z, y) \mapsto (z, y_t + \delta_t^{-1} y)$, so that ω_t pulls back to a Ricci-flat metric uniformly equivalent to Euclidean on $B_{\mu_t} \times B_{\delta_t}$, so standard higher order estimates give uniform C^∞ bounds for ω_t, which will subconverge smoothly to a Ricci-flat Kähler metric ω_0 on $\mathbb{C}^n$, uniformly equivalent to the Euclidean metric $\omega_{\mathbb{C}^n}$ with $|\nabla^{\omega_{\mathbb{C}^n}} \omega_0|(0) = 1$. Such a metric cannot exist by the Liouville Theorem for the complex Monge-Ampère equation in $\mathbb{C}^n$ (see [**53**]).

In case 2 (up to scaling) we have $\delta_t \to 1$. Here we argue analogously to case 1, without applying any further diffeomorphism since ω_t is already uniformly equivalent to Euclidean locally, so ω_t will subconverge smoothly to a Ricci-flat Kähler metric on $\mathbb{C}^m \times F$ which is uniformly equivalent (and cohomologous) to $\omega_{\mathbb{C}^m} + \omega_F$ and satisfies $|\nabla^{\omega_{\mathbb{C}^m} + \omega_F} \omega_0|(0, y_0) = 1$. Such a metric cannot exist by the Liouville Theorem for the complex Monge-Ampère equation on the cylinder $\mathbb{C}^m \times F$ of Hein [**49**] (see also [**72**]).

We are left with case 3 where $\delta_t \to 0$, which is where all the work goes. By construction we have

$$\|\omega_t\|_{C^1(B_{\mu_t} \times F, \tilde{\omega}_t)} \leqslant C.$$

We first need to improve this to

$$(3.8) \qquad\qquad \|\omega_t\|_{C^{1,\gamma}(K, \tilde{\omega}_t)} \leqslant C,$$

for any fixed $\gamma < 1$ and compact set $B \subset B_{\mu_t} \times F$, which is done by applying the following new Schauder estimates on an infinite cylinder, proved in [**53**]:

THEOREM 3.19. *Let (F, g_F) be a closed Riemannian manifold and $g_P = g_{\mathbb{R}^d} + g_F$ the product metric on the cylinder $\mathbb{R}^d \times F$. For every $k \geqslant 2, 0 < \gamma < 1$, there is $C > 0$ such that for all $0 < \rho < R$, all basepoints in $\mathbb{R}^d \times F$ and all $f \in C^{k,\gamma}_{\mathrm{loc}}(B_{2R})$ we have*

$$[\nabla^k f]_{C^\gamma(B_\rho)} \leqslant C \left([\nabla^{k-2} \Delta f]_{C^\gamma(B_R)} + (R - \rho)^{-k-\gamma} \|f\|_{L^\infty(B_R)} \right),$$

where here geodesic balls and differential operators are those of g_P.

This estimate is proved using the method of Simon [**88**], by reducing to a linear Liouville Theorem (for harmonic functions) after contradiction and blowup, but unlike in [**88**] (which worked in $\mathbb{R}^d$) here we also encounter 3 cases which blow up to $\mathbb{R}^{d+\dim F}, \mathbb{R}^d \times F$, and $\mathbb{R}^d$ respectively.

To prove (3.8) the idea is then to apply the further stretching $(z, y) \mapsto (\delta_t z, y)$ and scaling the resulting metrics by δ_t^{-2}, so that $\tilde{\omega}_t$ becomes a fixed product metric ω_P and ω_t is uniformly equivalent to it, so by standard higher order estimates it will subconverge smoothly to some Kähler metric on $\mathbb{C}^m \times F$, which can be assumed with loss of generality to be equal to ω_P. In this stretched picture we can then effectively linearize the Monge-Ampère equation at ω_P and apply Theorem 3.19 to obtain the right estimate which after un-stretching and un-scaling gives (3.8).

Now (3.8) implies in particular a $C^{1,\gamma}$ bound with respect to any fixed metric, so Ascoli-Arzelà gives that, up to passing to a sequence, $\omega_t \to \omega_0$

in $C^{1,\gamma}_{\text{loc}}$ where ω_0 is the pullback of a $C^{1,\gamma}$ Kähler metric on $\mathbb{C}^m$, uniformly equivalent to Euclidean.

We then argue that $\sup_{\mathbb{C}^m} |\nabla^{\omega_{\mathbb{C}^m}} \omega_0| = 1$, which however is not a trivial consequence of $|\nabla^{\tilde{\omega}_t} \omega_t|_{\tilde{\omega}_t}(0, y_t) = 1$, since the reference metrics $\tilde{\omega}_t$ are collapsing in the fiber directions. Schematically, in local product coordinates, we have at $(0, y_t)$

$$1 = |\nabla^{\tilde{\omega}_t} \omega_t|_{\tilde{\omega}_t} \sim |\nabla_z g_{zz}| + \delta_t^{-1} |\nabla_z g_{zy}| + \cdots + \delta_t^{-3} |\nabla_y g_{yy}|,$$

where we write $g_{\bullet\bullet}$ for the components of ω_t, and the claim is then that all the terms on the RHS except the first one must go to zero, which then passing to the limit gives us $\sup_{\mathbb{C}^m} |\nabla^{\omega_{\mathbb{C}^m}} \omega_0| = 1$. To see why these go to zero we use the following elementary result (which uses crucially that F is compact without boundary): given a metric vector bundle $E \to F$, and fix a metric on F, then for every $\gamma < 1$ there is $C > 0$ such that for every smooth section σ of E over F we have

$$\|\nabla \sigma\|_{L^\infty(F)} \leqslant C[\nabla \sigma]_{C^\gamma(F)},$$

and we apply it as follows. Fix any $(z, y) \in K \subset B_{\mu_t} \times F$, and let $\sigma = g_{yy}|_{\{z\} \times F}$, a section of $E = (T^*F)^{\otimes 2}$ and apply this estimate to get

$$\sup_{\{z\} \times F} |\nabla_y g_{yy}| \leqslant C[\nabla_y g_{yy}]_{C^\gamma(\{z\} \times F, \omega_F)} \leqslant C\delta_t^{3+\gamma}[\nabla_y g_{yy}]_{C^\gamma(\{z\} \times F, \tilde{\omega}_t)}$$

$$\leqslant C\delta_t^{3+\gamma} \|\omega_t\|_{C^{1,\gamma}(K, \tilde{\omega}_t)} \leqslant C\delta_t^{3+\gamma},$$

using (3.8) and so $\delta_t^{-3} |\nabla_y g_{yy}| \leqslant C\delta_t^\gamma \to 0$ as claimed, and similarly for the other terms.

Lastly, an argument adapted from [**99**] shows that ω_0^m equals a constant times the Euclidean volume form, and so by standard bootstrapping ω_0 is smooth and Ricci-flat. Thus, we again reach a contradiction to the Liouville Theorem for the complex Monge-Ampère equation in $\mathbb{C}^m$ (see [**53**]). This concludes the sketch of proof of (3.7) for $k = 1$.

3.8. Setup I.B.2: no fiber space. In this section we discuss the second subcase of the collapsing Setup I.B, which is the following: we have a Calabi-Yau manifold X^n with a cohomology class $[\alpha] \in \partial \mathcal{C}_X$ with $\int_X \alpha^n = 0$, which we approach with the path $[\alpha_t] = [\alpha] + t[\omega_X]$ (where ω_X is a Kähler metric on X), we let ω_t be the unique Ricci-flat Kähler metric in the class $[\alpha_t]$ for $t > 0$, but now (unlike in Setup I.B.1) we assume that $[\alpha]$ is not the pullback of a Kähler class from the base of a fiber space structure on X. We will call this Setup I.B.2.

The Ricci-flat metrics ω_t are therefore still volume collapsing, but there is no fibration structure so most of the techniques used in Setup I.B.1 are not available anymore.

The simplest examples are "irrational" classes on the boundary of the Kähler cone of a torus $X = \mathbb{C}^n / \Lambda$, which are represented by a constant

Hermitian semipositive $(1,1)$-form β whose kernel foliation has some non-closed leaf (equivalently, $\ker \beta$ viewed as a linear subspace of $\mathbb{C}^n$ is not $\mathbb{Q}$-defined modulo Λ). In this case, the flat metrics ω_t on X simply converge smoothly to β, while the Gromov-Hausdorff limit of (X, ω_t) is a real torus T^k where $2n - k$ is the real dimension of the largest leaf closure.

However, in general, the behavior can be much worse as shown by Filip and the author [**31, 32**]:

THEOREM 3.20. *There are K3 surfaces X with classes $[\alpha] \in \partial \mathcal{C}_X$ with $\int_X \alpha^2 = 0$ that satisfy Setup I.B.2 such that the Ricci-flat metrics ω_t converge weakly as currents to η, the unique closed positive current in $[\alpha]$, but η is not even continuous on the complement of any closed analytic subvariety. In particular, the Ricci-flat metrics ω_t cannot have a uniform $C_{\mathrm{loc}}^\gamma(X \backslash S, \omega_X)$ bound for any $\gamma > 0$ and any closed analytic subvariety $S \subset X$.*

This result in particular implies that such classes $[\alpha]$ do not admit any smooth semipositive representative. There are examples where the K3 surface is projective and the class $[\alpha]$ equals $c_1(D)$ for some nef real $\mathbb{R}$-divisor D.

The negative result in Theorem 3.20 contrasts with the positive results that we discussed in Setup I.B.1, and show that once the fiber space structure is lost, the Ricci-flat metrics can be rather wildly behaved.

3.8.1. *Holomorphic dynamics.* The construction of such X and $[\alpha]$ uses holomorphic dynamics [**10, 77**]: these K3 surfaces support a chaotic automorphism T, in the sense that its topological entropy h is positive, and the class $[\alpha]$ is an eigenclass for T^* acting on $H^{1,1}(X, \mathbb{R})$, with eigenvalue $e^h > 1$

$$T^*[\alpha] = e^h[\alpha].$$

Before we explain the idea of how this is used to prove Theorem 3.20, let us give a few examples of such automorphisms:

EXAMPLE 3.21. Consider the torus $Y = \mathbb{C}^2/\Lambda$ where $\Lambda = (\mathbb{Z} \oplus i\mathbb{Z})^2$ is the "square" lattice, and let $T_Y : Y \to Y$ be the automorphism induced by the linear map $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, which is known as "Arnol'd's cat map". An elementary computation shows that the largest eigenvalue of T_Y^* on $H^{1,1}(Y, \mathbb{R})$ is $e^h = \left(\frac{3+\sqrt{5}}{2}\right)^2$. If we pass to the associated Kummer K3 surface X, then T_Y induces an automorphism T of X with the same value of topological entropy. The eigenclass $[\alpha]$ with eigenvalue e^h (which is unique up to scaling) admits a smooth semipositive representative.

EXAMPLE 3.22. Let X be a K3 surface which is a generic complete intersection of degree $(2, 2, 2)$ in $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$. Then the three projections to $\mathbb{CP}^1 \times \mathbb{CP}^1$ exhibit X as a ramified $2:1$ cover, and X is then equipped with three involutions $\sigma_j, j = 1, 2, 3$, that permute the sheets. The composition $T = \sigma_1 \circ \sigma_2 \circ \sigma_3$ has $e^h = 9 + 4\sqrt{5}$. In this case, the eigenclass $[\alpha]$ can be chosen to be $c_1(D)$ for some nef real $\mathbb{R}$-divisor D. This construction first appeared in [**76**].

EXAMPLE 3.23. By using the Torelli Theorem, McMullen constructed [**77**] examples of non-projective $K3$ surfaces X with an automorphism T with $h > 0$ which admit a Siegel disc, i.e. an open subset $\Delta \subset X$ biholomorphic to the bidisc in $\mathbb{C}^2$ which is preserved by T and where T is holomorphically conjugate to an irrational rotation $(z_1, z_2) \mapsto (az_1, bz_2)$ with $|a| = |b| = 1$ and a, b and ab are not roots of unity.

If now $T : X \to X$ is a $K3$ automorphism with largest eigenvalue of T^* on $H^{1,1}(X, \mathbb{R})$ larger than 1, so equal to $e^h, h > 0$, then Cantat [**10**] shows that there are two nontrivial nef classes $[\eta_\pm] \in \partial \mathcal{C}_X$ with $\int_X \eta_\pm^2 = 0, \int_X \eta_+ \wedge \eta_- = 1$ which satisfy

$$T^*[\eta_\pm] = e^{\pm h}[\eta_\pm],$$

and are not the pullback of a Kähler class from the base of an elliptic fibration, so we are in Setup I.B.2. Note that replacing T with T^{-1} interchanges the roles of $[\eta_+]$ and $[\eta_-]$.

Fix smooth closed real $(1, 1)$-forms $\alpha_\pm$ which represent $[\eta_\pm]$. Then Cantat [**10**] shows that the classes $[\eta_\pm]$ contain unique closed positive currents, which are of the form $\eta_\pm = \alpha_\pm + \sqrt{-1}\partial\overline{\partial}\varphi_\pm \geqslant 0$ for quasi-psh functions $\varphi_\pm$ which belong to $C^\gamma(X)$ for some $\gamma > 0$ by [**24**]. By uniqueness, they satisfy $T^*\eta_\pm = e^{\pm h}\eta_\pm$, where the pullback of currents here is defined by $T^*\eta_\pm = T^*\alpha_\pm + \sqrt{-1}\partial\overline{\partial}(\varphi_\pm \circ T)$.

We now take (as we did above) $[\alpha] = [\eta_+]$, and consider as in Setup I.B.2 the Ricci-flat Kähler metrics ω_t on X with cohomology class $[\alpha] + t[\omega_X], t > 0$. By weak compactness of closed positive $(1, 1)$ currents with bounded cohomology class, from every sequence $t_i \to 0$ one can extract a subsequence (still denoted by t_i) such that ω_{t_i} converge in the weak topology of currents to some closed positive $(1, 1)$ current in the class $[\alpha] = [\eta_+]$. But such a current is unique, as mentioned above, and so we see that the whole family ω_t converges weakly to η_+ as $t \to 0$.

Suppose that both η_+ and η_- were smooth, so they are closed real semi-positive $(1, 1)$-forms. Then their wedge product $\mu = \eta_+ \wedge \eta_-$ would be a T-invariant volume form μ with total mass 1, and Cantat [**10**] shows that it is also T-ergodic. On the other hand, if Ω is a never-vanishing holomorphic 2-form on X, normalized so that $d\text{Vol} = \Omega \wedge \overline{\Omega}$ is a unit-volume volume form, then $d\text{Vol}$ is also easily seen to be T-invariant as well. We can then write $\mu = f d\text{Vol}$ for some smooth nonnegative function f (with integral 1 with respect to $d\text{Vol}$), which is T-invariant and hence constant μ-a.e. by ergodicity of μ. But since $\mu = f d\text{Vol}$, this means that f is constant on the set where it is strictly positive (which is nonempty), and since f is smooth this implies that f is a constant everywhere, and this constant must be 1. We thus conclude that $\mu = d\text{Vol}$. With small changes to the argument, one obtains the same conclusion also by just assuming that $\eta_\pm$ are both continuous off of some closed analytic subvariety of X.

Now, in the Kummer example 3.21 there is no contradiction, since indeed here we have $\mu = \mathrm{dVol}$. However we claim that this is not the case in both examples 3.22 and 3.23, which therefore shows that at least one among η_+ and η_- cannot be continuous on the complement of any subvariety, which would prove Theorem 3.20 up to replacing T by T^{-1}.

For Example 3.23 we argue as follows: it is easy to see in general that for any given Kähler metric the iterates

$$\frac{(T^n)^*\omega}{e^{nh}} \to \eta_+,$$

weakly as $n \to +\infty$ (this is because this holds at the level of cohomology classes, and η_+ is the unique closed positive current it its class). But now on the Siegel disc Δ we have that $T|_\Delta$ is conjugate to a rotation, so on any rotation-invariant compact $K \subset \Delta$ we can choose a T-invariant Riemannian metric g, and so

$$\sup_K \left| \frac{(T^n)^*\omega}{e^{nh}} \right|_g = e^{-nh} \sup_K |\omega|_g \leqslant Ce^{-nh},$$

which converges to 0, thus showing that $\eta_+|_\Delta = 0$ and so $\mu|_\Delta = 0$ which is a contradiction to $\mu = \mathrm{dVol}$.

This argument cannot work for Example 3.22 since these $K3$ surfaces are projective and so T cannot have a Siegel disc, as shown in [**77**]. We use instead the following rigidity result, proved in the projective case by Cantat-Dupont [**11**] and in general by Filip and the author [**32**]:

THEOREM 3.24. *Let $T : X \to X$ be an automorphism of a $K3$ surface with positive topological entropy, and let $\mu = \eta_+ \wedge \eta_-$ and $\mathrm{dVol} = \Omega \wedge \overline{\Omega}$ be the two invariant measures described above. Then the following are equivalent:*

(a) *$\mu \ll \mathrm{dVol}$*
(b) *$\mu = \mathrm{dVol}$*
(c) *X is Kummer and T is induced by an affine linear automorphism of the torus*

To clarify, the measure μ can be defined in general as the wedge product of η_+ and η_- (not just when these are smooth) because these currents have Hölder continuous potentials and their wedge product was defined by Bedford-Taylor [**2**] as a T-invariant Radon measure μ on X.

To conclude our argument, it suffices to note that the $K3$ surfaces in Example 3.22 have Picard number 3 and so cannot be Kummer (which have Picard number at least 16, coming from the exceptional divisors), and so using Theorem 3.24 we obtain a contradiction to $\mu = \mathrm{dVol}$.

3.8.2. *Other boundary classes.* Apart from the negative results that we just discussed, essentially nothing is known about the behavior of the Ricci-flat Kähler metrics in Setup I.B.2 above, where the limiting nef class $[\alpha]$ has vanishing self-intersection and does not come from the base of a fiber space. We will now restrict our attention to the case of $K3$ surfaces. The examples

above coming from dynamics produce certain nef classes $[\alpha]$ as above, which as we said contain only one closed positive $(1,1)$ current which has a Hölder continuous (but in general not smooth) potential.

This uniqueness property is expected to hold for all nef classes $[\alpha]$ with vanishing self-intersection which don't come from the base of an elliptic fibration (or equivalently, none of the rescaled classes $\lambda[\alpha], \lambda \in \mathbb{R}_{>0}$, belongs to $H^2(X, \mathbb{Q})$), see the ongoing work of Sibony-Verbitsky [**87**].

On the other hand, recall that according to Conjecture 3.7 we expect that every such class $[\alpha]$ has a closed positive $(1,1)$ current representing it (which is expected to be unique) with L^∞ potential, i.e. of the form $\alpha + \sqrt{-1}\partial\overline{\partial}\varphi \geqslant 0$ where α is a smooth representative of the class and φ is a bounded quasi-psh function (which can be normalized by $\sup_X \varphi = 0$).

We can also write our Ricci-flat metrics as $\omega_t = \alpha + t\omega_X + \sqrt{-1}\partial\overline{\partial}\varphi_t$, $\sup_X \varphi_t = 0$, for smooth Kähler potentials φ_t, and then if Conjecture 3.7 was settled, then from estimates in pluripotential theory in [**37**] generalizing [**5, 20, 29, 63**] it would follow that $\|\varphi_t\|_{L^\infty(X)} \leqslant C$ as $t \to 0$ (in all dimensions, not just on $K3$ surfaces).

Furthermore, in the case of $K3$ surfaces, we expect the optimal regularity for the limiting potential φ to be C^0 and no better (in general, despite the fact that in the dynamical examples above the potential was Hölder continuous).

Lastly, one would also like to understand the Gromov-Hausdorff limit of (X, ω_t) (in particular, show it exists independent of subsequences). In the examples above from dynamics, this limit is in fact just a point, even though the class $[\alpha]$ is not the trivial class. This is easiest to see in Example 3.22 when we take the sequence of Ricci-flat Kähler metrics $\frac{(T^n)^*\omega_X}{e^{nh}}$ (which for suitably chosen ω_X will move towards $[\alpha] = [\eta_+]$ along the same path as ω_t), since these are obviously isometric to $e^{-nh}\omega_X$ and so shrink to a point.

We then expect the following to hold:

CONJECTURE 3.25. *For every $K3$ surface X and class $[\alpha] \in \partial\mathcal{C}_X$ in Setup I.B.2, the Ricci-flat manifolds (X, ω_t) converge to a point in the Gromov-Hausdorff topology.*

4. Degenerating the complex structure

In this section we will discuss Setup II from §2.7. Recall that we have now a projective family $\pi : \mathfrak{X} \to \Delta$, smooth over Δ^*, with polarization $\mathfrak{L}$ and with Ricci-flat Kähler metrics $\omega_t \in c_1(\mathfrak{L}|_{X_t})$ on $X_t = \pi^{-1}(t)$ for $t \in \Delta^*$. Up to passing to a finite cover, we will assume for simplicity that $K_{X_t} \cong \mathcal{O}_{X_t}$ for all $t \in \Delta^*$.

We can consider the normalization $\nu : \tilde{\mathfrak{X}} \to \mathfrak{X}$ and the composition $\tilde{\pi} = \pi \circ \nu$, and then if $K_{\tilde{\mathfrak{X}}}$ denotes the canonical sheaf then $\tilde{\pi}_* K_{\tilde{\mathfrak{X}}/\Delta}$ is a line bundle on Δ, which therefore admits a trivializing section, which under the isomorphism $H^0(\Delta, \tilde{\pi}_* K_{\tilde{\mathfrak{X}}/\Delta}) \cong H^0(\tilde{\mathfrak{X}}, K_{\tilde{\mathfrak{X}}})$ corresponds to a holomorphic

section $\Omega \in H^0(\tilde{\mathfrak{X}}, K_{\tilde{\mathfrak{X}}})$. Since ν is an isomorphism away from the central fiber, we can restrict Ω to $X_t, t \neq 0$, and obtain a trivialization Ω_t of K_{X_t}.

Then the Ricci-flat Kähler metric ω_t on X_t satisfies

$$(4.1) \qquad \omega_t^n = \left(\frac{\int_{X_t} c_1(\mathfrak{L}|_{X_t})^n}{\int_{X_t} (\sqrt{-1})^{n^2} \Omega_t \wedge \overline{\Omega_t}} \right) (\sqrt{-1})^{n^2} \Omega_t \wedge \overline{\Omega_t}.$$

4.1. Volume. The volume of (X_t, ω_t) equals the intersection number $\int_{X_t} c_1(\mathfrak{L}|_{X_t})^n$ which is a constant independent of t (since π is flat).

4.2. Diameter. Unlike in Setup I, where the diameter of ω_t was always bounded above thanks to Theorem 3.1, the diameter of (X_t, ω_t) in Setup II may go to infinity. In fact, we have the following result, which combines work of Wang [**110**], Takayama [**97**] and the author [**102**] (when the base of the family is a quasi-projective curve, the stated case with base a disc follows by using also [**61**]), see also the exposition in [**117**]:

THEOREM 4.1. *In Setup II, the following are equivalent:*
- $\operatorname{diam}(X_t, \omega_t) \leqslant C$ *as* $t \to 0$
- *There is a new family* $\mathfrak{X}'$*, obtained from* $\mathfrak{X}$ *by a finite base-change and a birational modification along the central fiber, such that* X_0' *is a normal Calabi-Yau variety with at worst canonical singularities*
- $0 \in \Delta$ *is at finite distance from* Δ^* *equipped with the Weil-Petersson pseudometric* $\omega_{\mathrm{WP}} = -\sqrt{-1}\partial\overline{\partial} \log \left(\int_{X_t} (\sqrt{-1})^{n^2} \Omega_t \wedge \overline{\Omega_t} \right)$.

More precisely, the new family $\mathfrak{X}'$ is obtained from $\mathfrak{X}$ by applying semi-stable reduction followed by a relative Minimal Model Program.

Since we prefer to work with Ricci-flat metrics with bounded diameter, so that their Gromov-Hausdorff limits will be compact, so we will consider the rescaled metrics

$$\tilde{\omega}_t = \frac{\omega_t}{\operatorname{diam}(X_t, \omega_t)^2},$$

which have unit diameter. We therefore see that the volume of $(X_t, \tilde{\omega}_t)$ remains bounded away from 0 if and only if one of the equivalent conditions in Theorem 4.1 holds. So, as in Setup I, we have to deal separately with the volume non-collapsed case and with the volume collapsed case.

4.3. Setup II.A: volume non-collapsed. Here we assume that we are in Setup II and that any of the equivalent conditions in Theorem 4.1 holds, thus the Ricci-flat manifolds (X_t, ω_t) have fixed volume and diameter bounded uniformly bounded above (and also from below away from 0 thanks to the Bishop-Gromov volume comparison theorem). Replacing $\mathfrak{X}$ with $\mathfrak{X}'$ (which doesn't change the fibers away from 0), we may assume that X_0 itself is a normal Calabi-Yau variety with at worst canonical singularities.

Thanks to a result of Eyssidieux-Guedj-Zeriahi [**28**], which extends the pioneering work of Yau [**112**], X_0 admits a unique "singular Ricci-flat Kähler metric" ω_0 in $c_1(\mathfrak{L}|_{X_0})$. This can be understood as a weak solution of the

corresponding complex Monge-Ampère equation in the sense of pluripotential theory, but all that is important for us is that on the regular part X_0^{reg}, ω_0 defines an honest Ricci-flat Kähler metric.

Then Rong-Zhang [**84**] prove the following:

THEOREM 4.2. *Suppose in Setup II we assume that X_0 is a normal Calabi-Yau variety with at worst canonical singularities. Then (X_t, ω_t) converge to $(X_0^{\text{reg}}, \omega_0)$ locally smoothly on compact subsets of X_0^{reg}. Furthermore, the metric completion (Z, d) of $(X_0^{\text{reg}}, \omega_0)$ is a compact metric space and $(X_t, \omega_t) \to (Z, d)$ in the Gromov-Hausdorff topology as $t \to 0$.*

To clarify the statement, locally smooth convergence means that there is a smooth fiber-preserving embedding $F : X_0^{\text{reg}} \times \Delta \hookrightarrow \mathfrak{X}$, which is the identity map on $X_0^{\text{reg}} \times \{0\}$, such that $(F|_{X_0^{\text{reg}} \times \{t\}})^* \omega_t \to \omega_0$ in the smooth topology on compact subsets of X_0^{reg}.

The Gromov-Hausdorff limit Z in fact has an algebraic structure, as shown by Donaldson-Sun [**25**] (see also [**115**, Lemma 2.2]):

THEOREM 4.3. *Under the same assumptions as Theorem 4.2, we have that Z is homeomorphic to X_0. The homeomorphism identifies the singular set of Z (in the sense of [**12**]) with $X_0 \backslash X_0^{\text{reg}}$.*

There are of course many examples of families $\mathfrak{X}$ that satisfy our hypothesis on X_0, for example the family of $K3$ surfaces in $\mathbb{CP}^3 \times \Delta$ given (in affine coordinates) by

$$x^4 + x^2 + y^4 + y^2 + z^4 + z^2 = t,$$

whose central fiber X_0 is a nodal $K3$ surface.

These results give an answer to Question 2.6 in Setup II.A.

4.4. Setup II.B: volume collapsed. Here we assume that we are in Setup II and that $\text{diam}(X_t, \omega_t) \to \infty$, i.e. the negation of any of the equivalent conditions in Theorem 4.1. As before, we consider the rescaled Ricci-flat metrics $\tilde{\omega}_t = \text{diam}(X_t, \omega_t)^{-2} \omega_t$, so that the Ricci-flat manifolds $(X_t, \tilde{\omega}_t)$ have unit diameter and their total volume is collapsing to zero.

An algebro-geometric construction due to Kontsevich-Soibelman [**65**], refined in [**79, 81**], associates to our family $\mathfrak{X}$ a connected simplicial complex $Sk(\mathfrak{X})$, the essential skeleton, which is roughly speaking the dual intersection complex of the central fiber X_0' of the family $\mathfrak{X}'$ in Theorem 4.1. The real dimension of the essential skeleton satisfies $0 \leqslant \dim_\mathbb{R} Sk(\mathfrak{X}) \leqslant n$, and we have that $\dim_\mathbb{R} Sk(\mathfrak{X}) = 0$ if and only if X_0' is a normal Calabi-Yau variety with at worst canonical singularities, and that $\dim_\mathbb{R} Sk(\mathfrak{X}) = n$ if and only if $\mathfrak{X}$ is a *maximal degeneration* or *large complex structure limit* in the sense of mirror symmetry, see e.g. [**47, 78**].

The simplest example of a large complex structure limit family is the family of elliptic curves in $\mathbb{CP}^2 \times \Delta$ given (in affine coordinates) by

$$y^2 = x^3 + x^2 + t,$$

whose central fiber X_0 is a nodal elliptic curve. The essential skeleton of this family is S^1. In this case we can see the behavior of $\tilde{\omega}_t$ explicitly: for $t \neq 0$ we can write $X_t = \mathbb{C}/\Lambda_t$ where Λ_t is spanned over $\mathbb{Z}$ by 1 and $\tau(t)$, and an explicit calculation shows that as $t \in \mathbb{R}$ approaches 0 then $\tau(t)$ is approximately equal to $i|\log t|$. This implies that the unit-diameter rescaled flat metrics $\tilde{\omega}_t$ come from the Euclidean metric with a fundamental domain which is approximately a rectangle with sides 1 and $\tau(t)^{-1}$, and so their Gromov-Hausdorff limit is the unit-diameter S^1.

Another standard example of a large complex structure limit family is the one in $\mathbb{CP}^{n+1} \times \Delta$ given (in homogeneous coordinates) by

$$z_0^{n+2} + \cdots + z_{n+1}^{n+2} + \frac{1}{t}z_0 \cdots z_{n+1} = 0,$$

with smooth Calabi-Yau n-folds X_t for $t \neq 0$ and with $X_0 = \{z_0 \cdots z_{n+1} = 0\}$ the union of the $n + 2$ coordinate hyperplanes. Note that the dual intersection complex of X_0 is the boundary of the standard $(n + 1)$-simplex in $\mathbb{R}^{n+2}$, and so homeomorphic to S^n. In fact, this is the essential skeleton of $\mathfrak{X}$, i.e. $Sk(\mathfrak{X}) \cong S^n$.

It is conjectured (see [**60, 62**]) that for large complex structure limits where the Calabi-Yau fibers X_t are simply connected with $H^i(X_t, \mathcal{O}_{X_t}) = 0, 0 < i < n$, the skeleton $Sk(\mathfrak{X})$ should be homeomorphic to S^n, while when X_t is hyperkähler it should be homeomorphic to $\mathbb{CP}^{\frac{n}{2}}$.

4.5. Large complex structure limits. The guiding conjecture about large complex structure limits of Calabi-Yau manifolds is due to Strominger-Yau-Zaslow [**94**] (SYZ):

CONJECTURE 4.4. *Given a large complex structure limit family $\pi : \mathfrak{X} \to \Delta$ of Calabi-Yau n-folds, for $t \neq 0$ sufficiently small the manifold X_t admits a Special Lagrangian T^n-fibration over a half-dimensional base B, with singular fibers lying over a subset $S \subset B$ of codimension* 2.

By taking the fiberwise dual tori one obtains dual T^n-fibrations over $B\backslash S$, which should be compactifiable to the "mirror family" $\check{\mathfrak{X}}$ of $\mathfrak{X}$ (which satisfy certain properties that we will not get into, but see e.g. [**42**] for an introduction). The base B is expected to be homeomorphic to S^n if the Calabi-Yau fibers X_t are simply connected with $H^i(X_t, \mathcal{O}_{X_t}) = 0, 0 < i < n$, and to be homeomorphic to $\mathbb{CP}^{\frac{n}{2}}$ if X_t is hyperkähler (there is also strictly speaking the elementary case when X_t are tori, where B is also T^n).

Conjecture 4.4 remains open, but see [**71**] for a recent result in this direction, and also the Gross-Siebert program [**43**] for an algebro-geometric approach to the construction of $\check{\mathfrak{X}}$.

Around 2000, Kontsevich-Soibelman [**64, 65**], Gross-Wilson [**47**] and Todorov [**74**] proposed that the base B of the conjectural SYZ fibration arises as the collapsed Gromov-Hausdorff limit of Ricci-flat metrics on X_t, exactly in the setup II.B that we have been considering. More precisely, they conjectured:

CONJECTURE 4.5. *Given a large complex structure limit family $\pi : \mathfrak{X} \to \Delta$ of Calabi-Yau n-folds, equip $X_t, t \neq 0$, with the Ricci-flat Kähler metrics $\omega_t \in c_1(\mathfrak{L}|_{X_t})$ and consider the unit-diameter rescaled metrics $\tilde{\omega}_t = \mathrm{diam}(X_t, \omega_t)^{-2}\omega_t$. Then as $t \to 0$, $(X_t, \tilde{\omega}_t) \to (Z, d)$ in the Gromov-Hausdorff topology, where (Z, d) is a compact metric space with an open dense subset Z_0 over which $d|_{Z_0}$ is induced by a Riemannian metric g_0, and the real Hausdorff codimension of $Z \backslash Z_0$ is at least 2. Furthermore, Z_0 carries an integral affine structure and in local affine coordinates $g_0 = \nabla^2 F$ where F is a smooth convex function that satisfies the real Monge-Ampère equation $\det(\nabla^2 F) = c \in \mathbb{R}_{>0}$.*

In other words, (Z_0, g_0) is a Monge-Ampère manifold in the terminology of Cheng-Yau [16]. Of course, Z is expected to be equal to the base B of the conjectural SYZ fibration, and therefore it is also expected to be topologically S^n or $\mathbb{CP}^{\frac{n}{2}}$ (as above), and so it is expected to be homeomorphic to $Sk(\mathfrak{X})$. Furthermore, passing to the dual affine structure, and applying Legendre transform to F, one obtains a dual Monge-Ampère manifold $(Z_0, \check{g}_0)$. From this dual Monge-Ampère manifold it is elementary to construct a family of smooth T^n-fibration over Z_0 with fiber size depending on t, which for t small is expected to compactify across $Z \backslash Z_0$ to the mirror family $\check{X}_t$, see [65].

Going back to Conjecture 4.5, which fits precisely in Question 2.6, it is not hard to see that it holds when X_t are tori (see [82]). For other Calabi-Yau manifolds, it was proved by Gross-Wilson [47] for those large complex structure limits of $K3$ surfaces which arise from hyperkähler rotation from elliptically fibered $K3$ surfaces (as in Setup I.B.1) with only I_1 singular fibers. Later the author with Gross and Zhang [44, 45] gave a new proof of this result, which allows for arbitrary singular fibers (see also [15] for a very recent extension of the Gross-Wilson gluing method), and the author with Zhang [108] proved it for some families of hyperkähler manifolds which arise from hyperkähler rotation from hyperkähler manifolds with a holomorphic Lagrangian fibration; these results were described in the section about Setup I.B.1. In all of these cases, we also proved in [45, 108] that the limiting manifold (Z_0, g_0) is *special Kähler* in the sense of [36, 93]. Using these results, Odaka-Oshima [83] proved Conjecture 4.5 in general for $K3$ surfaces. Lastly, the first step towards a Gross-Wilson type gluing construction for large complex structure limits of Calabi-Yau 3-folds was taken in [70].

In a related direction, Boucksom-Jonsson [7] have investigated the fiber-wise Ricci-flat volume forms ω_t^n on X_t, and they proved that ω_t^n converges in the weak topology to a Lebesgue-type measure μ on the essential skeleton $Sk(\mathfrak{X})$. To make sense of this convergence, they construct a *hybrid space* $\mathfrak{X}^{\mathrm{hyb}}$ over Δ which coincides with $\mathfrak{X}$ over Δ^* but over 0 one has the Berkovich analytification $\mathfrak{X}^{\mathrm{an}}$ associated to the basechange of $\mathfrak{X}$ over Δ^* to the non-Archimedean field $\mathbb{C}((t))$, inside which $Sk(\mathfrak{X})$ naturally sits. On the other hand, Boucksom-Favre-Jonsson [6] have managed to solve the

non-Archimedean analog of the complex Monge-Ampère equation on $Sk(\mathfrak{X})$ with volume form μ (when $\mathfrak{X}$ is defined over an algebraic curve, with [**8**] later removing this assumption), and it is natural to ask about the relation between their solution and Conjecture 4.5, and whether this circle of ideas can be used to attack it.

4.6. Other collapsing limits. Lastly, we consider the case of a family $\mathfrak{X}$ of Calabi-Yau n-folds as in Setup II.B such that $0 < \dim_{\mathbb{R}} Sk(\mathfrak{X}) < n$. In this case the following is expected:

CONJECTURE 4.6. *Given a family $\mathfrak{X} \subset \mathbb{CP}^N \times \Delta$ of Calabi-Yau n-folds with $0 < \dim_{\mathbb{R}} Sk(\mathfrak{X}) < n$, equip $X_t, t \neq 0$, with the Ricci-flat Kähler metrics $\omega_t \in c_1(\mathfrak{L}|_{X_t})$ and consider the unit-diameter rescaled metrics $\tilde{\omega}_t = \mathrm{diam}(X_t, \omega_t)^{-2}\omega_t$. Then as $t \to 0$, $(X_t, \tilde{\omega}_t) \to (Z, d)$ in the Gromov-Hausdorff topology, where (Z, d) is a compact metric space homeomorphic to the essential skeleton $Sk(\mathfrak{X})$.*

The first nontrivial examples are families of $K3$ surfaces with $\dim_{\mathbb{R}} Sk(\mathfrak{X}) = 1$, for example the family $\mathfrak{X}$ in $\mathbb{CP}^3 \times \Delta$

$$(4.2) \qquad\qquad P_1(z)P_2(z) + tP(z) = 0,$$

where P_1, P_2 are generic quadratic polynomials and P is a generic quartic polynomial. The central fiber X_0 has two irreducible components with non-trivial intersection, so its dual intersection complex is homeomorphic to the interval $[0, 1]$, and so is $Sk(\mathfrak{X})$.

The only examples where Conjecture 4.6 is known are the families considered in [**96**], for which the Ricci-flat metrics $\tilde{\omega}_t$ can be obtained via a gluing construction. These include the $K3$ families in (4.2), where collapsing to a 1-dimensional interval is proved. Note that there are earlier gluing constructions of collapsing hyperkähler metrics on $K3$ surfaces in [**51**] which converge to $[0, 1]$ but it is not clear whether some of them come from projective families as in Conjecture 4.6. There are also other examples in [**35**] of hyperkähler metrics on $K3$ surfaces which collapse to $T^3/\mathbb{Z}_2$, and will not arise from Conjecture 4.6. We refer the reader to the recent survey [**95**].

Lastly, we mention one more problem by Kontsevich-Soibelman [**65**] which remains open:

CONJECTURE 4.7. *Suppose that $\mathfrak{X}$ is a Calabi-Yau family as in Setup II, which does not satisfy the equivalent conditions in Theorem 4.1. Then there are constants $A, C > 0$ such that the Ricci-flat Kähler metrics $\omega_t \in c_1(\mathfrak{L}|_{X_t})$ satisfy*

$$(4.3) \qquad\qquad \mathrm{diam}(X_t, \omega_t) \geqslant C^{-1}(-\log|t|)^A,$$

as $t \to 0$.

Of course, by Theorem 4.1 we know that $\mathrm{diam}(X_t, \omega_t) \to +\infty$, so the conjecture asks for an effective lower bound. Note that in [**65**] this is only conjectured for large complex structure limit families, but it seems natural to

extend this as above. Note also that (4.3) holds for the metrics constructed in [**96**] by gluing methods.

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