Unification of the Kähler-Ricci and Anomaly flows

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Dedicated to Professor Shing-Tung Yau on the occasion of his 70th birthday

Abstract. A new formulation of the Anomaly flow in the case of vanishing slope parameter is given, where the dependence on the global section of the canonical bundle appears only in the initial data. This allows a natural unification of the Anomaly flow with the Kähler-Ricci flow.

1. Introduction

The idea of using a geometric flow to implement a cohomological constraint on a metric in the absence of an analogue of the $\partial\bar{\partial}$ lemma was introduced in [**14, 15**]. The specific case of the conformally balanced condition arising from supersymmetric compactifications of the heterotic string was considered there and generalized further in [**18**]. Many other conditions and flows have been introduced since, including dual Anomaly flows [**6**] and flows motivated by Type II A and Type II B string compactifications in [**13, 4**]. Anomaly flows appear to be a flexible and powerful method, as they have led to new proofs of major results in geometry such as Yau's theorem [**23**] on the existence of Kähler Ricci-flat metric and the Fu-Yau solution [**8, 9**] of the Hull-Strominger system [**16, 17, 5**].

A flow is usually given by a vector field on the configuration space and the prescription of an initial data. In the Anomaly flows considered in [**14, 15, 17**], the underlying manifold X is complex and equipped with a non-vanishing top holomorphic form Ω. The form Ω appears explicitly in the vector field on the space of Hermitian metrics defining the flow (see [**15**], eq. (1.9)). This explicit appearance of Ω seems to set Anomaly flows apart from more familiar flows such as the Kähler-Ricci flow, and prevent the direct use of many powerful techniques which had been developed for these flows.

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The main purpose of the present note is to show that, in the simpler case with parameter $\alpha' = 0$, the dependence of the vector field of the Anomaly flow on Ω can be eliminated by a suitable rescaling of the evolving metric:

THEOREM 1. *Let X be a complex manifold of dimension $m \geq 2$ equipped with a nowhere holomorphic $(m,0)$-form Ω. Assume that $t \to \omega(t)$ is a flow of Hermitian metrics satisfying*

$$\partial_t(\|\Omega\|_\omega \omega^{m-1}) = i\partial\bar\partial\omega^{m-2} \tag{1.1}$$

and $d(\|\Omega\|_{\omega(t)}\omega(t)^{m-1}) = 0$ for each t. Set for each t

$$\eta(t) = \|\Omega\|_{\omega(t)}\omega(t). \tag{1.2}$$

Then the Hermitian metrics $\eta(t)$ satisfy the conformally balanced condition $d(\|\Omega\|_\eta^2\eta^{m-1}) = 0$, and they evolve according to

$$\partial_t\eta(t) = -\frac{1}{m-1}(\tilde R_{\bar kj}(\eta) + \frac{1}{2}(T\circ\bar T)_{\bar kj}). \tag{1.3}$$

Here ${R_{\bar qp}}^k{}_j = -\partial_{\bar q}(g^{k\bar m}\partial_p g_{\bar kj}$ is the curvature of the metric $\eta := ig_{\bar kj}dz^j \wedge d\bar z^k$, $\tilde R_{\bar kj} := {R^p}_{p\bar kj} = -g^{p\bar q}g_{\bar km}\partial_{\bar q}(g^{m\bar\ell}\partial_p g_{\bar\ell j})$ is its Chern-Ricci tensor, and $(T\circ\bar T)_{\bar kj} := T_{\bar kpq}\bar T_j{}^{pq}$, where $T = i\partial\eta = \frac{1}{2}T_{\bar kpq}dz^q\wedge dz^p\wedge d\bar z^k$ is its torsion tensor. In our convention, the Ricci tensor $R_{\bar kj}$ is defined by $R_{\bar qp} = {R_{\bar qp}}^k{}_k$.

The form Ω has cancelled out from the vector field $\partial_t\eta$, as desired. From the point of view of $\eta(t)$, the only dependence on Ω of the Anomaly flow resides now in the conformally balanced condition for the initial data $\eta(0)$. Thus the flow defined by the right hand side of (1.3) with an arbitrary initial Hermitian metric can be viewed as a generalization of the Anomaly flow with $\alpha' = 0$ to arbitrary complex manifolds X. When X is compact, it is not difficult to see, as we shall show in detail later, that the flow (1.3) preserves the Kähler property and reduces to the Kähler-Ricci flow if the initial data is Kähler. We can then formulate the following theorem, which is essentially Theorem 1 combined with the uniqueness of solutions of parabolic flows on compact manifolds, and which unifies the Kähler-Ricci flow with the Anomaly flow:

THEOREM 2. *Let X be a compact complex manifold of dimension $m \geq 1$. Consider the flow $t \to \eta(t)$ of Hermitian metrics defined by*

$$\partial_t\eta(t) = -(\tilde R_{\bar kj}(\eta) + \frac{1}{2}(T\circ\bar T)_{\bar kj}), \tag{1.4}$$

with initial data a Hermitian metric $\eta(0)$

(i) *The flow is parabolic, and for any initial data $\eta(0)$, it admits a unique smooth solution in some maximal time interval $[0,T)$ with $T > 0$.*

(ii) *If $\eta(0)$ is Kähler (this includes the general case in dimension $m = 1$), then $\eta(t)$ remains Kähler for all time $t \in [0,T)$, and the flow reduces to the Kähler-Ricci flow,*

$$\partial_t\eta_{\bar kj} = -R_{\bar kj}(\eta). \tag{1.5}$$

(iii) *Assume that $m \geq 2$, and X admits a nowhere vanishing holomorphic $(m,0)$-form Ω. If $\eta(0)$ is conformally balanced in the sense that $d(\|\Omega\|^2_{\eta(0)}\eta^{m-1}(0)) = 0$, then $\eta(t)$ remains conformally balanced for all time $t \in [0,T)$, and the flow reduces to the Anomaly flow (1.1), after rescaling $t \to (m-1)t$.*

Parts (i) and (ii) of Theorem 2 are elementary, and have been noted by Streets and Tian [**20**] who proposed the family of flows of the form

$$\partial_t \eta_{\bar{k}j} = -(\tilde{R}_{\bar{k}j} + Q_{\bar{k}j}(T,\bar{T})) \tag{1.6}$$

as generalizations of the Kähler-Ricci flow to arbitrary complex manifolds, where $Q(T,\bar{T})$ is a $(1,1)$-form which is linear in each factor T and $\bar{T}$. Among these, Ustinovskiy [**22**] has identified the same combination $Q = \frac{1}{2}T \circ \bar{T}$ as in (1.3) as a flow that preserves the Griffiths positivity and the dual Nakano-positivity of the tangent bundle. In the case of the Kähler-Ricci flow, the preservation of the positivity of the bisectional curvature was proven by Bando [**1**] and Mok [**12**] and is a particularly important property of the flow with many applications, see e.g. [**19**]. Ustinovskiy's result [**22**] suggests that some generalizations in (1.6) may be better behaved than others. With Theorem 1, we see that the Anomaly flow with $\alpha' = 0$ also singles out the particular combination $Q = \frac{1}{2}T \circ \bar{T}$.

We note that many other generalizations of the Kähler-Ricci flow to the non-Kähler setting have been proposed in the literature, including in [**10**], [**2**], and [**21**].

2. Proof of Theorem 1

First, we note that the rescaled metric $\eta = \|\Omega\|_\omega \omega$ satisfies

$$\|\Omega\|^{2-m}_\omega = \|\Omega\|^2_\eta, \qquad \|\Omega\|_\omega \omega^{m-1} = \|\Omega\|^2_\eta \eta^{m-1}, \tag{2.1}$$

and hence the Anomaly flow (1.1) can be expressed in terms of η as

$$\partial_t(\|\Omega\|^2_\eta \eta^{m-1}) = i\partial\bar{\partial}\,(\|\Omega\|^2_\eta \eta^{m-2}). \tag{2.2}$$

2.1. Elimination of $\partial_t\|\Omega\|^2_\eta$. Carrying out the differentiation in time gives

$$\|\Omega\|^2_\eta(\partial_t \log \|\Omega\|^2_\eta \eta^{m-1} + (m-1)\partial_t\eta \wedge \eta^{m-2}) = i\partial\bar{\partial}(\|\Omega\|^2_\eta \eta^{m-2}) \tag{2.3}$$

Let Λ be the usual Hodge operator, which is the adjoint of the operator $\phi \to \omega \wedge \phi$. Its precise expression in components and normalization can be found in Appendix C. Since

$$\partial_t \log \|\Omega\|^2_\eta = \partial_t \log \eta^{-m} = -g^{j\bar{k}}\partial_t g_{\bar{k}j} = -(\Lambda\partial_t\eta) \tag{2.4}$$

we obtain

$$\|\Omega\|^2_\eta(-(\Lambda\partial_t\eta)\eta^{m-1} + (m-1)\partial_t\eta \wedge \eta^{m-2}) = i\partial\bar{\partial}(\|\Omega\|^2_\eta \eta^{m-2}). \tag{2.5}$$

We take now the Hodge $\star$ operator of both sides, using the formulas in Appendix B. We find

$$\begin{aligned}\|\Omega\|_\eta^2\Big\{-(m-1)!(\Lambda\partial_t\eta)\eta+(m-1)(m-2)!(-\partial_t\eta+(\Lambda\partial_t\eta)\eta)\Big\}\\ =\star i\partial\bar\partial(\|\Omega\|_\eta^2\eta^{m-2})\end{aligned} \tag{2.6}$$

The term $(\Lambda\partial_t\eta)$ cancels out from the left hand side, and we obtain the following equation

$$(m-1)!\|\Omega\|_\eta^2\partial_t\eta=-\star i\partial\bar\partial(\|\Omega\|_\eta^2\eta^{m-2}). \tag{2.7}$$

2.2. Elimination of $\|\Omega\|_\eta^2$. It is now easy to see that the explicit appearance of the term $\|\Omega\|_\eta^2$ can be eliminated from the flow, using the torsion constraints. Indeed, before taking the Hodge $\star$ operator, the right hand side of the flow can be expressed as

$$\begin{aligned}&i\partial\bar\partial(\|\Omega\|_\eta^2\eta^{m-2})\\ &=i\partial(\|\Omega\|_\eta^2\bar\partial\log\|\Omega\|_\eta^2\eta^{m-2}+\|\Omega\|_\eta^2\bar\partial\eta^{m-2})\\ &=\|\Omega\|_\eta^2\Big\{i\partial\log\|\Omega\|_\eta^2\wedge\bar\partial\log\|\Omega\|_\eta^2\wedge\eta^{m-2}+i\partial\bar\partial\log\|\Omega\|_\eta^2\wedge\eta^{m-2}\\ &\quad-i\bar\partial\log\|\Omega\|_\eta^2\partial\eta^{m-2}+i\partial\log\|\Omega\|_\eta^2\bar\partial\eta^{m-2}+i\partial\bar\partial\eta^{m-2}\Big\}\end{aligned} \tag{2.8}$$

However, from Lemma 4 in [**15**], with $a=2$, we have

$$\tau_\ell=g^{j\bar k}T_{\bar kj\ell}=\partial_\ell\log\|\Omega\|_\eta^2 \tag{2.9}$$

and in general, $Ric(\eta)=\partial\bar\partial\log\|\Omega\|_\eta^2$. Thus the above equation can be rewritten as

$$\begin{aligned}i\partial\bar\partial(\|\Omega\|_\eta^2\eta^{m-2})=\|\Omega\|_\eta^2\Big\{i\tau\wedge\bar\tau\wedge\eta^{m-2}+iRic(\eta)\wedge\eta^{m-2}+i\partial\bar\partial\eta^{m-2}\\ -i\bar\tau\wedge\partial\eta^{m-2}+i\tau\wedge\bar\partial\eta^{m-2}\Big\}.\end{aligned} \tag{2.10}$$

Returning to the Anomaly flow, it reduces now to the following simpler expression

$$\begin{aligned}(m-1)!\partial_t\eta=-\star\Big\{i\tau\wedge\bar\tau\wedge\eta^{m-2}+iRic(\eta)\wedge\eta^{m-2}+i\partial\bar\partial\eta^{m-2}\\ -i\bar\tau\wedge\partial\eta^{m-2}+i\tau\wedge\bar\partial\eta^{m-2}\Big\}.\end{aligned} \tag{2.11}$$

Next, we note that

$$\begin{aligned}\partial\eta^{m-2}&=(m-2)\partial\eta\wedge\eta^{m-3}=-i(m-2)T\wedge\eta^{m-3}\\ \bar\partial\eta^{m-2}&=i(m-2)\bar T\wedge\eta^{m-3}\end{aligned} \tag{2.12}$$

and

$$i\partial\bar\partial\eta^{m-2} = (m-2)i\partial(\bar\partial\eta\wedge\eta^{m-3}) \\ = (m-2)(i\partial\bar\partial\eta\wedge\eta^{m-3} - i(m-3)\bar\partial\eta\wedge\partial\eta\wedge\eta^{m-4}) \\ = (m-2)(i\partial\bar\partial\eta\wedge\eta^{m-3} - i(m-3)\bar T\wedge T\wedge\eta^{m-4}). \tag{2.13}$$

Collecting all the terms, we obtain

$$(m-1)!\partial_t\eta = -\star\Big\{(i\tau\wedge\bar\tau + iRic(\eta))\wedge\eta^{m-2} \\ + (m-2)(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T)\wedge\eta^{m-3} \\ - (m-2)(m-3)i\bar T\wedge T\wedge\eta^{m-4}\Big\} \tag{2.14}$$

2.3. The Hodge $\star$ of the individual terms. Applying the formulas for the Hodge $\star$ operator given in the appendices, we obtain immediately

$$\star\,[(i\tau\wedge\bar\tau + iRic(\eta))\wedge\eta^{m-2}] \\ = (m-2)![-(i\tau\wedge\bar\tau + iRic(\eta)) + (\Lambda(i\tau\wedge\bar\tau + iRic(\eta)))\eta]$$

and

$$\star\,[(m-2)(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T)\wedge\eta^{m-3}] \\ = (m-2)![-\Lambda(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T) \\ + \frac{1}{2}\Lambda^2(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T)\,\eta] \tag{2.15}$$

and

$$\star[(m-2)(m-3)iT\wedge\bar T\wedge\eta^{m-4}] = (m-2)![-\frac{1}{2}\Lambda^2(iT\wedge\bar T) + \frac{1}{6}\Lambda^3(iT\wedge\bar T)\eta] \tag{2.16}$$

The appearance of a common factor $(m-2)!$ in all the terms of the right hand side allows us to cancel this factor, and obtain a generalization of the Anomaly flow including to dimension $m=2$, defined by

$$-(m-1)\partial_t\eta = -(i\tau\wedge\bar\tau + iRic(\eta)) + (\Lambda(i\tau\wedge\bar\tau + iRic(\eta)))\eta \\ - \Lambda(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T) + \frac{1}{2}\Lambda^2(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T)\,\eta \\ - \frac{1}{2}\Lambda^2(iT\wedge\bar T) + \frac{1}{6}\Lambda^3(iT\wedge\bar T)\eta \\ = A + B\eta \tag{2.17}$$

where we have defined the $(1,1)$-form A and the scalar function B by

(2.18)
$$A = -(i\tau\wedge\bar\tau + iRic(\eta)) - \Lambda(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T) - \frac{1}{2}\Lambda^2(iT\wedge\bar T)$$
$$B = \Lambda(i\tau\wedge\bar\tau + iRic(\eta)) + \frac{1}{2}\Lambda^2(i\partial\bar\partial\eta - \bar\tau\wedge T - \tau\wedge\bar T) + \frac{1}{6}\Lambda^3(iT\wedge\bar T).$$

2.4. Evaluation of $i\partial\bar\partial\eta$. We quote from [15], eq. (2.52)

$$(i\partial\bar\partial\eta)_{\bar k j\bar\ell m} = R_{\bar k j\bar\ell m} - R_{\bar k m\bar\ell j} + R_{\bar\ell m\bar k j} - R_{\bar\ell j\bar k m} - g^{s\bar r}T_{\bar r jm}\bar T_{s\bar k\bar\ell} \quad (2.19)$$

It follows that

$$(\Lambda i\partial\bar\partial\eta)_{\bar\ell m} = i^{-1}g^{j\bar k}(i\partial\bar\partial\eta)_{\bar k j\bar\ell m} = i^{-1}(\tilde R_{\bar\ell m} + R_{\bar\ell m} - g^{s\bar r}g^{j\bar k}T_{\bar r jm}\bar T_{s\bar k\bar\ell}) \quad (2.20)$$

or, in terms of forms,

$$\Lambda i\partial\bar\partial\eta = -i\tilde{R}ic(\eta) - iRic(\eta) + i(T\bar T) \quad (2.21)$$

where the $(1,1)$-form $T\bar T)$ is defined by

$$T\bar T = (T\bar T)_{\bar\ell m}dz^m\wedge d\bar z^\ell, \qquad (T\bar T)_{\bar\ell m} = g^{s\bar r}g^{j\bar k}T_{\bar r jm}\bar T_{s\bar k\bar\ell}. \quad (2.22)$$

As a consequence, the Ricci-Chern terms cancel and we obtain

$$iRic(\eta) + \Lambda i\partial\bar\partial\eta = -i\tilde{R}ic(\eta) + i(T\bar T). \quad (2.23)$$

Similarly,

$$\Lambda^2 i\partial\bar\partial\eta = -2R + |T|^2 \quad (2.24)$$

and the scalar curvature cancels between the terms $\Lambda iRic(\eta)$ and $\Lambda^2 i\partial\bar\partial\eta$,

$$\Lambda iRic(\eta) + \frac{1}{2}\Lambda^2 i\partial\bar\partial\eta = \frac{1}{2}|T|^2. \quad (2.25)$$

It is then convenient to isolate torsion and non-torsion terms in the coefficients A and B as follows

$$A = i\tilde{R}ic(\eta) - i(T\bar T) - i\tau\wedge\bar\tau + \Lambda(\tau\wedge\bar T + \bar\tau\wedge T) - \frac{1}{2}\Lambda^2(iT\wedge\bar T) \quad (2.26)$$
$$B = \frac{1}{2}|T|^2 + |\tau|^2 - \frac{1}{2}\Lambda^2(\bar\tau\wedge T + \tau\wedge\bar T) + \frac{1}{6}\Lambda^3(iT\wedge\bar T).$$

2.5. Evaluation of $iT\wedge\bar T$, $\Lambda(iT\wedge\bar T)$, $\Lambda^2(iT\wedge\bar T)$. The components of $iT\wedge\bar T$ can be expressed as, upon antisymmetrization,

$$\begin{aligned}(iT\wedge\bar T)_{\bar k j\bar\beta\alpha\bar\gamma\ell} = -i(&T_{\bar k j\ell}\bar T_{\alpha\bar\beta\bar\gamma} - T_{\bar k\alpha\ell}\bar T_{j\bar\beta\bar\gamma}\\ &- T_{\bar k j\alpha}\bar T_{\ell\bar\beta\bar\gamma} - T_{\bar\beta j\ell}\bar T_{\alpha\bar k\bar\gamma} + T_{\bar\beta\alpha\ell}\bar T_{j\bar k\bar\gamma} + T_{\bar\beta j\alpha}\bar T_{\ell\bar k\bar\gamma}\\ &- T_{\bar\gamma j\ell}\bar T_{\alpha\bar\beta\bar k} + T_{\bar\gamma\alpha\ell}\bar T_{j\bar\beta\bar k} + T_{\bar\gamma j\alpha}\bar T_{\ell\bar\beta\bar k})\end{aligned} \quad (2.27)$$

It follows that

(2.28)
$$\begin{aligned}(\Lambda iT\wedge\bar T)_{\bar k j\bar\beta\alpha} = &-g^{\ell\bar\gamma}(T_{\bar k j\ell}\bar T_{\alpha\bar\beta\bar\gamma} - T_{\bar k\alpha\ell}\bar T_{j\bar\beta\bar\gamma}) + g^{\ell\bar\gamma}(-T_{\bar\beta j\ell}\bar T_{\alpha\bar k\bar\gamma} + T_{\bar\beta\alpha\ell}\bar T_{j\bar k\bar\gamma})\\ &+ g^{\ell\bar\gamma}T_{\bar\gamma j\alpha}\bar T_{\ell\bar\beta\bar k} - T_{\bar k j\alpha}\bar\tau_{\bar\beta} + T_{\bar\beta j\alpha}\bar\tau_{\bar k} - \tau_j\bar T_{\alpha\bar\beta\bar k} + \tau_\alpha\bar T_{j\bar\beta\bar k}\end{aligned}$$

Note that

$$\tau \wedge \bar{T} = \tau_\alpha dz^\alpha \wedge \frac{1}{2} \bar{T}_{j\bar{\beta}\bar{k}} d\bar{z}^k \wedge d\bar{z}^\beta \wedge dz^j = \frac{1}{2^2} (\tau_\alpha \bar{T}_{j\bar{\beta}\bar{k}} - \tau_j \bar{T}_{\alpha\bar{\beta}\bar{k}}) dz^\alpha \wedge d\bar{z}^\beta \wedge dz^j \wedge d\bar{z}^k \tag{2.29}$$

and hence

$$(\tau \wedge \bar{T})_{\bar{k}j\bar{\beta}\alpha} = \tau_\alpha \bar{T}_{j\bar{\beta}\bar{k}} - \tau_j \bar{T}_{\alpha\bar{\beta}\bar{k}}. \tag{2.30}$$

It follows that

$$(\Lambda \tau \wedge \bar{T})_{\bar{\beta}\alpha} = i^{-1} g^{j\bar{k}} (\tau_\alpha \bar{T}_{j\bar{\beta}\bar{k}} - \tau_j \bar{T}_{\alpha\bar{\beta}\bar{k}}) = i\tau_\alpha \bar{\tau}_{\bar{\beta}} + i g^{j\bar{k}} \tau_j \bar{T}_{\alpha\bar{\beta}\bar{k}}, \tag{2.31}$$

and

$$\Lambda^2 \tau \wedge \bar{T} = 2|\tau|^2. \tag{2.32}$$

Returning to the earlier identity, we can now compute $\Lambda^2 iT \wedge \bar{T}$,

$$\begin{aligned} (\Lambda^2 iT \wedge \bar{T})_{\bar{\beta}\alpha} &= i g^{j\bar{k}} \Big\{ g^{\ell\bar{\gamma}} (T_{\bar{k}j\ell} \bar{T}_{\alpha\bar{\beta}\bar{\gamma}} - T_{\bar{k}\alpha\ell} \bar{T}_{j\bar{\beta}\bar{\gamma}}) \\ &\quad + g^{\ell\bar{\gamma}} (-T_{\bar{\beta}j\ell} \bar{T}_{\alpha\bar{k}\bar{\gamma}} + T_{\bar{\beta}\alpha\ell} \bar{T}_{j\bar{k}\bar{\gamma}}) + g^{\ell\bar{\gamma}} T_{\bar{\gamma}j\alpha} \bar{T}_{\ell\bar{\beta}\bar{k}} \Big\} \\ &\quad + (\Lambda(\tau \wedge \bar{T} + \bar{\tau} \wedge T))_{\bar{\beta}\alpha} \\ &= i g^{\ell\bar{\gamma}} (\tau_\ell \bar{T}_{\alpha\bar{\beta}\bar{\gamma}} + \bar{\tau}_{\bar{\gamma}} T_{\bar{\beta}\alpha\ell}) - i(T \circ \bar{T})_{\bar{\beta}\alpha} - 2i(T\bar{T})_{\bar{\beta}\alpha} \\ &\quad + (\Lambda(\tau \wedge \bar{T} + \bar{\tau} \wedge T))_{\bar{\beta}\alpha} \end{aligned} \tag{2.33}$$

where we have defined the $(1,1)$-form $T \circ \bar{T}$ by

$$(T \circ \bar{T})_{\bar{\beta}\alpha} = g^{\ell\bar{\gamma}} g^{j\bar{k}} T_{\bar{\beta}j\ell} \bar{T}_{\alpha\bar{k}\bar{\gamma}}. \tag{2.34}$$

In intrinsic notation, this can be expressed as

$$(\Lambda^2 iT \wedge \bar{T}) = 2\Lambda(\tau \wedge \bar{T} + \bar{\tau} \wedge T) - 2i\tau \wedge \bar{\tau} - iT \circ \bar{T} - 2iT\bar{T}. \tag{2.35}$$

We shall also need

$$\Lambda^3 iT \wedge \bar{T} = 6|\tau|^2 - 3|T|^2. \tag{2.36}$$

2.6. Evaluation of the coefficients A and B. It is now easy to assemble all the terms and arrive at the final formula $B = 0$ and A is given by

$$A = i\tilde{R}ic(\eta) + \frac{i}{2} T \circ \bar{T}. \tag{2.37}$$

We note that a simpler version of some of these identities when $m = 3$ appeared in [6] and was instrumental in the proof of an upper bound for $\|\Omega\|_\omega$. Altogether the evolution equation for η is

$$\partial_t \eta_{\bar{k}j} = -\frac{1}{m-1} \left(\tilde{R}_{\bar{k}j} + \frac{1}{2} (T \circ \bar{T})_{\bar{k}j} \right). \tag{2.38}$$

This is the flow (1.3) stated in Theorem 1. Q.E.D.

3. Proof of Theorem 2

Part (i) follows immediately from the fact that the Chern-Ricci tensor $\tilde{R}_{\bar{k}j}$ can be expressed in local coordinates as

$$\tilde{R}_{\bar{k}j} = -g^{p\bar{q}}\partial_p\partial_{\bar{q}}g_{\bar{k}j} + \cdots \tag{3.1}$$

where $\cdots$ denote terms with fewer derivatives. Part (ii) follows from the fact that, if $\tilde{\eta}(0)$ is Kähler, then the Kähler-Ricci flow $\partial_t\tilde{\eta} = -R_{\bar{k}j}(\tilde{\eta})$ admits a solution $\tilde{\eta}(t)$ which is Kähler for any t in a small time interval near $t = 0$. Since $T(\tilde{\eta}) = 0$ and $\tilde{R}_{\bar{k}j}(\tilde{\eta}) = R_{\bar{k}j}(\eta)$, the same $\tilde{\eta}(t)$ satisfies the flow (1.3) if we take the same initial data $\eta(0) = \tilde{\eta}(0)$. By uniqueness of the solution of the flow (1.3), it follows that $\eta(t) = \tilde{\eta}(t)$ for all time, and $\eta(t)$ is a solution of the Kähler-Ricci flow, as claimed. Part (iii) follows in the same way, using now Theorem 1. Indeed, if $\eta(0)$ is conformally balanced in the sense of Theorem 2, then the corresponding $\omega(0)$ is conformally balanced in the sense of [**15**]. By Theorem 1 of [**17**], the Anomaly flow (1.1) for ω admits a unique smooth solution $\omega(t)$ for some small time interval near 0. By Theorem 1, the corresponding $\eta(t)$ is a smooth, conformally balanced solution to the flow (1.3). By uniqueness, this solution coincides with the solution known to exist by parabolicity. In particular, the conformally balanced condition is preserved for all $\eta(t)$. Q.E.D.

4. Remarks

It may be interesting to find another, more direct, proof of Part (iii) of Theorem 2, namely that the flow (1.3) preserves the conformally balanced condition, instead of appealing to Theorem 1 and the uniqueness of solutions. This does not appear evident, although it can for example be done for Part (ii). By deriving the flow for $|T|^2$ and applying the maximum principle, we can indeed show directly that the Kähler property is preserved. We reproduce the key calculations below, as the flow of the torsion is crucial in non-Kähler geometry, and the resulting formulas may be useful in other contexts. They are also comparatively simpler than the formulas for the flow of the torsion derived in [**15**] under the conformally balanced condition.

4.1. The flow of the torsion. Consider then the flow (1.3), for general metrics η, not necessarily Kähler or conformally balanced. Introduce the notation $\eta = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$, and write the flow (1.3) as

$$\partial_t\eta = -\frac{1}{m-1}i(\tilde{R}ic(\eta) + \frac{1}{2}(T \circ \bar{T})) \tag{4.1}$$

Since $T = i\partial\eta$, this implies immediately

$$\partial_t T = \frac{1}{m-1}(\partial\tilde{R}ic + \frac{1}{2}\partial(T \circ \bar{T})) \tag{4.2}$$

For general Hermitian metrics, we have the following Bianchi identity

$$R_{\bar{\ell}m\bar{k}j} = R_{\bar{\ell}j\bar{k}m} + \nabla_{\bar{\ell}}T_{\bar{k}jm} = R_{\bar{k}j\bar{\ell}m} + \nabla_j\bar{T}_{m\bar{k}\bar{\ell}} + \nabla_{\bar{\ell}}T_{\bar{k}jm} \tag{4.3}$$

and hence

$$\tilde{R}_{\bar{k}j} = R_{\bar{k}j} - \nabla_j \bar{\tau}_{\bar{k}} + \nabla^m T_{\bar{k}jm} \tag{4.4}$$

By Bochner-Kodaira formulas (see Appendix D), we have

$$\partial^\dagger T_{\bar{k}j} = -\nabla^m T_{\bar{k}jm} + \bar{\tau}^m T_{\bar{k}jm} - \frac{1}{2}(T \circ \bar{T})_{\bar{k}j} \tag{4.5}$$

Since the form $Ric(\eta)$ is closed and the form $\nabla_j \bar{\tau}_k$ is ∂-exact, the right hand side of the equation (4.2) can be expressed in terms of $\partial^\dagger T$ and the differentials of $\bar{\tau}^m T_{\bar{k}jm}$ and $(T \circ \bar{T})_{\bar{k}j}$ alone. We find that (4.2) becomes

$$\partial_t T = \frac{1}{m-1}(-\partial\partial^\dagger T + \partial(\bar{\tau} \cdot T)) \tag{4.6}$$

where we have introduced the notation $\bar{\tau} \cdot T$ for the $(1,1)$-form defined by

$$(\bar{\tau} \cdot T)_{\bar{\alpha}\beta} = \bar{\tau}^\gamma T_{\bar{\alpha}\beta\gamma}. \tag{4.7}$$

Since T is ∂-closed, the operator $\partial\partial^\dagger$ on T can be equated with the Hodge Laplacian $\Box = \partial\partial^\dagger + \partial^\dagger\partial$ on $(2,1)$-forms, so we have obtained a parabolic diffusion equation for T.

To show that the Kähler condition is preserved by the flow (1.3), we need the evolution of $|T|^2$. For this, we again use a Bochner-Kodaira formula to convert the Hodge-Laplacian $\Box$ into the Laplacian $\Delta_c = g^{p\bar{q}} \nabla_{\bar{q}} \nabla_p$, modulo lower order terms,

$$\begin{aligned}
(\partial\partial^\dagger T)_{\bar{k}jm} &= \nabla_m(\partial^\dagger T)_{\bar{k}j} - \nabla_j(\partial^\dagger T)_{\bar{k}m} + T^s{}_{mj}(\partial^\dagger T)_{\bar{k}s} \\
&= -\Delta_c T_{\bar{k}jm} + T_{\bar{k}jl}\tilde{R}^l{}_m - T_{\bar{k}ml}\tilde{R}^l{}_j + g^{s\bar{t}}(R_{\bar{t}l\bar{k}m}T^l{}_{sj} - R_{\bar{t}l\bar{k}j}T^l{}_{sm}) \\
&\quad + \partial(\bar{\tau} \cdot T)_{\bar{k}jm} \\
&\quad + \frac{1}{2}\Big(T^l{}_{jm}(T \circ \bar{T})_{\bar{k}l} + g^{s\bar{t}} g^{p\bar{q}}(\bar{T}_{m\bar{t}\bar{q}} \nabla_j T_{\bar{k}sp} - \bar{T}_{j\bar{t}\bar{q}} \nabla_m T_{\bar{k}sp})\Big) \\
&\quad + g^{s\bar{t}} g^{p\bar{q}} T_{\bar{k}sp}(R_{\bar{q}j\bar{t}m} - R_{\bar{q}m\bar{t}j}).
\end{aligned}$$

Using the flow (4.6), we find

$$\begin{aligned}
&((m-1)\partial_t - \Delta_c)|T|^2 \\
&\quad = -|\nabla T|^2 - |\bar{\nabla} T|^2 - \frac{1}{2}|T \circ \bar{T}|^2 + \langle T \circ \bar{T}, T\bar{T} \rangle \\
&\qquad - 2\mathrm{Re}(\bar{T}^{\bar{k}jm}(g^{s\bar{t}} g^{p\bar{q}} \bar{T}_{m\bar{t}\bar{q}} \nabla_j T_{\bar{k}sp} + 2T^s{}_{pj} R^p{}_{s\bar{k}m} + 2T_{\bar{k}sp} R^p{}_j{}^s{}_m)).
\end{aligned} \tag{4.8}$$

In particular,

$$((m-1)\partial_t - \Delta_c)|T|^2 \le C|T|^2(|T|^2 + |Rm|), \tag{4.9}$$

for some constant C. The maximum principle implies that $T \equiv 0$ if initially $T_0 = 0$, i.e., the Kähler condition is preserved.

We observe that it is easy to derive from the flow of T the flows of τ as well as of the primitive component of T. For example, we have

$$(4.10)\qquad (m-1)\partial_t\tau_j = -\Box\tau_j + \nabla_j\left(|\tau|^2 + \frac{1}{2}|T|^2\right) + T^s{}_{pj}R'^{\,p}_{\ s} + g^{p\bar{k}}\bar{\tau}_{\bar{k}}(\nabla_p\tau_j - \nabla_j\tau_p + T^s{}_{pj}\tau_s).$$

In the conformally balanced case, we have $\partial\tau = 0$ and $R'^{p}_{s} = 0$, and this flow reduces to

$$(4.11)\qquad (m-1)\partial_t\tau_j = -\Box\tau_j + \nabla_j\left(|\tau|^2 + \frac{1}{2}|T|^2\right).$$

This results in the following flow for $|\tau|^2$,

$$\begin{aligned}&((m-1)\partial_t - \Delta_c)|\tau|^2\\ &= -|\nabla\tau|^2 - |Ric|^2 + \langle \tilde{R}ic - Ric + \frac{1}{2}T\circ\bar{T}, i\tau\wedge\bar{\tau}\rangle\\ &\quad + \frac{1}{2}\langle\tau,\nabla|T|^2\rangle + \frac{1}{2}\overline{\langle\tau,\nabla|T|^2\rangle} + g^{j\bar{k}}(\bar{\tau}_{\bar{k}}T^s{}_{lj}R^l{}_s + \tau_j\bar{T}^{\bar{t}}{}_{\bar{p}\bar{k}}R_{\bar{t}}{}^{\bar{p}})\end{aligned}$$

The right hand side can only be bounded above by $O(|\tau|)$, which indicates that, unlike the vanishing of the full torsion T, the vanishing of τ is not preserved. Indeed one can verify that the balanced condition $\tau = 0$ is not preserved by running the Anomaly flow on generalized Calabi-Gray manifolds [5] with balanced initial data.

4.2. Shi-type estimates and long-time existence of the flow. Shi-type estimates for the original Anomaly flow were derived in [15]. For the present version (1.3) in terms of the rescaled metric η, they become simpler to establish, and the same arguments as in [15], or the general results in [20], imply the following statement: the flow in η will continue to exist, unless there is a time $T>0$ and a sequence (z_j,t_j) with $t_j\to T$, with

$$(4.12)\qquad |Rm(z_j)|^2_{\eta(t_j)} + |T(z_j)|^2_{\eta(t_j)} + |\nabla T(z_j)|^2_{\eta(t_j)} \to \infty.$$

Now under a conformal change of metrics $\omega_f = e^f\omega$, the torsion and curvature transform as follows

$$(4.13)\qquad \begin{aligned} T^\ell{}_{jk}(\omega_f) &= T^\ell{}_{jk}(\omega) + f_j\delta^\ell{}_k - f_k\delta^\ell{}_j,\\ R_{\bar{k}j\bar{p}q}(\omega_f) &= e^f(R_{\bar{k}j\bar{p}q}(\omega) + f_{\bar{k}j}\omega_{\bar{p}q}).\end{aligned}$$

In the case at hand, $\eta = \|\Omega\|_\omega\omega$, so it is easy to work out the previous conditions, and find that the flow will continue to exist unless there is a time T and a sequence (z_j,t_j) with $t_j\to T$ satisfying

$$(4.14)\qquad \frac{|Rm(z_j)|^2_{\omega(t_j)}}{\|\Omega(z_j)\|^2_{\omega(t_j)}} + \frac{|T(z_j)|^2_{\omega(t_j)}}{\|\Omega(z_j)\|_{\omega(t_j)}} + \frac{|\nabla T(z_j)|^2_{\omega(t_j)}}{\|\Omega(z_j)\|^2_{\omega(t_j)}} \to \infty.$$

This is a more succinct, and perhaps more natural formulation of the criterion for the appearance of singularities found in [15], which involved the four quantities $|Rm(z_j)|_{\omega(t_j)}$, $|T(z_j)|_{\omega(t_j)}$, $|\nabla T(z_j)|_{\omega(t_j)}$, and $\|\Omega(z_j)\|_{\omega(t_j)}$ separately.

4.3. Two questions by Ustinovskiy. In Ustinovskiy's thesis [22], he raised two questions (Question 6.15 and Problem 6.16) about periodic solutions and stationary points of Hermitian curvature flow. As a result of our Theorems 1 and 2, we can answer these questions.

PROPOSITION 1. *All periodic solutions are stationary, which are Ricci-flat Kähler metrics.*

The proof goes as follows: Suppose that we have a periodic solution. It follows from the results of Ustinovskiy and Theorems 1 and 2 that the flow is exactly the Anomaly flow with $\alpha' = 0$ and with conformally balanced initial data. Therefore we have monotonicity formulae as introduced in [6]. More precisely, a direct calculation in [6] gives

$$\partial_t \|\Omega\|_\omega = \frac{1}{2(n-1)}[R - \frac{1}{n-2}|T|^2 - \frac{2(n-3)}{n-2}|\tau|^2] \tag{4.15}$$

This implies for any $\alpha > 2$,

$$\partial_t \int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!} = \int_X \|\Omega\|_\omega^{\alpha-1} \frac{\alpha-2}{2(n-1)}[R - \frac{1}{n-2}|T|^2 - \frac{2(n-3)}{n-2}|\tau|^2]\frac{\omega^n}{n!} \tag{4.16}$$

and hence, in view of the definition of the scalar curvature,

$$R\frac{\omega^n}{n!} = i\partial\bar\partial \log \|\Omega\|_\omega^2 \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{4.17}$$

we find

$$\begin{aligned} &\partial_t \int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!} \\ &\quad = -\frac{(\alpha-1)(\alpha-2)}{2(n-1)} \int_X \|\Omega\|_\omega^{\alpha-1} i\partial \log \|\Omega\|_\omega^2 \wedge \bar\partial \log \|\Omega\|^2 \wedge \frac{\omega^{n-1}}{(n-1)!} \\ &\qquad - \frac{\alpha-2}{2(n-1)(n-2)} \int_X \|\Omega\|_\omega^{\alpha-1}(|T|^2 + 2(n-3)|\tau|^2)\frac{\omega^n}{n!}. \end{aligned} \tag{4.18}$$

Each term on the right hand side is negative. This implies that $\int_X \|\Omega\|_\omega^\alpha \frac{\omega^n}{n!}$ is monotone decreasing. Periodic solutions imply that all the monotone quantities are actually constants in time, which in turn gives us an equation from the monotonicity formula. This equation can only be satisfied by Ricci-flat Kähler metrics, which are the only stationary points of the Anomaly flow. Note that the second part of this proposition has been established in several different ways in the literature, including by an integration by parts, and by many authors including [3], [7], [11], and [17].

Appendix A. Conventions and preliminaries

If η is a Hermitian metric on X, its curvature is defined by $R_{\bar{k}j}{}^{p}{}_{q} = -\partial_{\bar{k}}(g^{p\bar{m}}\partial_j g_{\bar{m}q})$. Its Ricci curvature $R_{\bar{k}j}(\eta)$ is defined by

$$R_{\bar{k}j}(\eta) = -\partial_j\partial_{\bar{k}} \log\, \eta^m = \partial_j\partial_{\bar{k}} \log\, \|\Omega\|^2_\eta \tag{A.1}$$

and the Ricci form $Ric(\eta)$ is defined by

$$Ric(\eta) = R_{\bar{k}j}(\eta)dz^j \wedge d\bar{z}^k = \partial_j\partial_{\bar{k}} \log\, \|\Omega\|^2_\eta dz^j \wedge d\bar{z}^k = \partial\bar{\partial} \log\, \|\Omega\|^2_\eta. \tag{A.2}$$

As in [15], the other notions of Ricci curvature are defined by $\tilde{R}_{\bar{k}j} = R^p{}_{p\bar{k}j}$, $R'_{\bar{k}j} = R_{\bar{k}}{}^p{}_{pj}$, $R''_{\bar{k}j} = R^p{}_{j\bar{k}p}$. Our conventions for the torsion of η are

$$T = i\partial\eta \equiv \frac{1}{2}T_{\bar{k}jm}dz^m \wedge dz^j \wedge d\bar{z}^m \tag{A.3}$$

In particular $T^m{}_{jp} = \Gamma^m_{jp} - \Gamma^m_{pj}$, with $\Gamma^m_{jp} = g^{m\bar{q}}\partial_j g_{\bar{q}p}$. We set

$$\tau_l = g^{j\bar{k}}T_{\bar{k}jl}. \tag{A.4}$$

The norm $\|\Omega\|^2_\omega$ with respect to a given Hermitian metric is defined as usual as

$$i^{m^2}\Omega \wedge \bar{\Omega} = \|\Omega\|^2_\omega \frac{\omega^m}{m!}. \tag{A.5}$$

Appendix B. The Hodge operator Λ

We define the operator Λ^q from (p,p)-forms to $(p-q,p-q)$-forms by

$$(\Lambda^q\Phi)_{\bar{j}_1 k_1 \cdots \bar{j}_{p-q} k_{p-q}} = i^{-q} \prod_{\alpha=p-q+1}^{p} g^{k_\alpha \bar{j}_\alpha} \Phi_{\bar{j}_1 k_1 \cdots \bar{j}_p k_p} \tag{B.1}$$

for

$$\Phi = \frac{1}{(p!)^2}\Phi_{\bar{j}_1 k_1 \cdots \bar{j}_p k_p} dz^{k_p} \wedge dz^{\bar{j}_p} \wedge \cdots dz^{k_1} \wedge d\bar{z}^{j_1} \tag{B.2}$$

Note that Λ maps real forms to real forms, and that $\Lambda^p\Phi = \langle\Phi, \eta^p\rangle = \mathrm{Tr}\,\Phi$. In terms of Λ, the previous torsion component τ can be written as $\tau = i\Lambda T$.

Appendix C. The Hodge $\star$ operator

Let α, Φ, and Ψ be $(1,1)$-forms, $(2,2)$-forms, and $(3,3)$-forms respectively. Then we have the following identities (the detailed derivations can be found in [17])

$$\begin{aligned} \star(\alpha \wedge \eta^{m-2}) &= (m-2)!(-\alpha + (\Lambda\alpha)\eta) \\ \star(\Phi \wedge \eta^{m-3}) &= (m-3)!(-\Lambda\Phi + \frac{1}{2}(\Lambda^2\Phi)\eta) \\ \star(\Psi \wedge \eta^{m-4}) &= (m-4)!(-\frac{1}{2}\Lambda^2\Psi + \frac{1}{6}(\Lambda^3\Psi)\eta) \end{aligned} \tag{C.1}$$

Let τ and T be $(1,0)$-forms and (2,1)-forms respectively. Then we also have

$$\begin{aligned} \star(\tau\wedge\eta^{m-2}) &= -i(m-2)!\tau\wedge\eta \\ \star(T\wedge\eta^{m-3}) &= i(m-3)!(-\Lambda T\wedge\eta+T) \end{aligned} \tag{C.2}$$

Appendix D. The operator $\partial^\dagger$ and Bochner-Kodaira formulas

First we work out the operator $\partial^\dagger$ on various spaces of forms. The basic formula is the following integration by parts formula for general Hermitian metrics

$$\int_X \nabla_j V^j\omega^n = \int_X \tau_j V^j\omega^n \tag{D.1}$$

where V^j is a vector field.

To get e.g. the $\partial^\dagger$ on $(1,0)$-forms, we take $V^j = f\overline{\psi_k}g^{j\bar k}$ where f and ψ_k are respectively an arbitrary scalar function and an arbitrary $(1,0)$-form. Then

$$\int_X \nabla_j(f\overline{\psi_k}g^{j\bar k})\omega^n = \int_X \tau_j f\tau_j\overline{\psi_k}g^{j\bar k}\omega^n \tag{D.2}$$

which can be rewritten as

$$\int_X(\nabla_j f)\overline{\psi_{\bar k}}g^{j\bar k}\omega^n + \int_X f\overline{g^{k\bar j}\nabla_{\bar j}\psi_k}\omega^n = \int_X f\overline{g^{k\bar j}\bar\tau_{\bar j}\psi_k}\omega^n \tag{D.3}$$

This means that

$$\partial^\dagger\psi = -g^{k\bar j}\nabla_{\bar j}\psi_k + g^{k\bar j}\bar\tau_{\bar j}\psi_k \tag{D.4}$$

for any $(1,0)$-form ψ. More generally, we have

Lemma 1. *Suppose α is a (p,q)-form, then*

$$\begin{aligned} (\partial\alpha)_{\bar t_q\ldots\bar t_1 s_{p+1}\ldots s_1} &= \sum_{k=1}^{p+1}(-1)^{k-1}\partial_{s_k}\alpha_{\bar t_q\ldots\bar t_1 s_{p+1}\ldots\widehat{s_k}\ldots s_1} \\ &= \sum_{k=1}^{p+1}(-1)^{k-1}\nabla_{s_k}\alpha_{\bar t_q\ldots\bar t_1 s_{p+1}\ldots\widehat{s_k}\ldots s_1} \\ &\quad + \sum_{l<k}(-1)^k T^s{}_{s_l s_k}\alpha_{\bar t_q\ldots\bar t_1 s_{p+1}\ldots\widehat{s_k}\ldots s_{l+1}s s_{l-1}\ldots s_1}, \end{aligned}$$

and

$$\begin{aligned} (\partial^\dagger\alpha)_{\bar t_q\ldots\bar t_1 s_{p-1}\ldots s_1} &= -g^{s\bar k}\nabla_{\bar k}\alpha_{\bar t_q\ldots\bar t_1 s_{p-1}\ldots s_1 s} + g^{s\bar k}\bar\tau_{\bar k}\alpha_{\bar t_q\ldots\bar t_1 s_{p-1}\ldots s_1 s} \\ &\quad + \frac{1}{2}\sum_{l=1}^{p-1}(-1)^l\bar T_{s_l}{}^{cd}\alpha_{\bar t_q\ldots\bar t_1 cd s_{p-1}\ldots\widehat{s_l}\ldots s_1}. \end{aligned}$$

For example, we have:

• If α is a (1,0)-form, then

$$\begin{aligned}(\partial\alpha)_{pq} &= \nabla_q\alpha_p - \nabla_p\alpha_q + T^s{}_{qp}\alpha_s,\\ \partial^\dagger\alpha &= -g^{p\bar{k}}\nabla_{\bar{k}}\alpha_p + g^{p\bar{k}}\bar{\tau}_{\bar{k}}\alpha_p.\end{aligned}$$

• If β is a (2,0)-form, then

$$(\partial^\dagger\beta)_l = -g^{p\bar{k}}\nabla_{\bar{k}}\beta_{lp} + g^{p\bar{k}}\bar{\tau}_{\bar{k}}\beta_{lp} - \frac{1}{2}\bar{T}_{l\bar{a}\bar{b}}\beta_{cd}g^{c\bar{a}}g^{d\bar{b}}.$$

• If ψ is a $(2,1)$-form, then

$$(\partial^\dagger\psi)_{\bar{\alpha}\beta} = -g^{\gamma\bar{j}}\nabla_{\bar{j}}\psi_{\bar{\alpha}\beta\gamma} + g^{\gamma\bar{j}}\bar{\tau}_{\bar{j}}\psi_{\bar{\alpha}\beta\gamma} - \frac{1}{2}\bar{T}_{\beta\bar{j}\bar{m}}\psi_{\bar{\alpha}\gamma\delta}g^{\gamma\bar{j}}g^{\delta\bar{m}} \tag{D.5}$$

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