The space of cycles, a Weyl law for minimal hypersurfaces and Morse index estimates

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Abstract. In this note, prepared for the occasion of the Journal of Differential Geometry (JDG) 50th birthday Conference, we will discuss a Weyl law conjectured by Gromov and proved by the authors with Liokumovich in [13], and work of the authors ([19], [20]) on the characterization of the Morse index of minimal hypersurfaces produced by min-max methods.

The last section is an update on dramatic developments obtained since the time of the conference. This includes the proof of Yau's Conjecture (about the existence of infinitely many minimal surfaces) for generic metrics, by establishing density of minimal hypersurfaces, obtained by the authors with Irie [11], the proof of equidistribution of minimal hypersurfaces for generic metrics by the authors with Song [21], the proof of the authors' Multiplicity One Conjecture and Morse Index Conjecture in dimension three by Chodosh and Mantoulidis ([3]), and the full resolution of Yau's Conjecture by Song [26].

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1. Introduction

In 1911, Hermann Weyl [27] proved a beautiful formula that determines the asymptotic behavior of the eigenvalues of the Laplacian purely in terms

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of the volume of the manifold. One formulation in the closed case is:

$$\lim_{p\to\infty} \lambda_p(M) p^{-\frac{2}{n+1}} = c(n) \mathrm{vol}\,(M)^{-\frac{2}{n+1}},$$

where (M^{n+1}, g) is a closed $(n+1)$-dimensional Riemannian manifold and

$$\{\lambda_1(M) < \lambda_2(M) \le \cdots \le \lambda_p(M) \le \cdots\}$$

is the spectrum of the Laplace-Beltrami operator Δ_g.

The story goes that Lorentz proposed this problem during some distinguished lectures he gave at Göttingen. Hilbert was in the audience and predicted he would not see a solution in his lifetime. Weyl, who was also in the audience, was a graduate student at the time and came up with a solution just one year later. To be precise, Weyl proved the formula for the eigenvalues of the Laplacian of a bounded domain in Euclidean space subject to the Dirichlet boundary condition.

Remarkably there is a Weyl law [**13**] in which the eigenvalues of the Laplacian are replaced by the areas of minimal hypersurfaces that are constructed by minimax methods. In order to explain the analogy, recall that λ_p can be given a min-max characterization:

$$\lambda_p(M) = \inf_{\dim Q = p+1} \sup_{f\in Q, f\neq 0} E(f),$$

where $Q \subset W^{1,2}(M)$ and $E(f) = \frac{\int_M |\nabla f|^2 dM}{\int_M f^2 dM}$ is the Rayleigh functional. Note that $E(c\cdot f) = E(c)$ for any constant $c \in \mathbb{R}\setminus\{0\}$ and so the Rayleigh functional descends to the projectivization of $W^{1,2}(M)$:

$$E : \mathbb{P}W^{1,2}(M) \to \mathbb{R}.$$

A $p+1$-plane in $W^{1,2}(M)$ becomes a p-projective space in $\mathbb{P}W^{1,2}(M)$ and one should think of $\mathbb{P}W^{1,2}(M)$ as an $\mathbb{RP}^\infty$. Hence we have the analogous characterization:

$$\lambda_p(M) = \inf_{\mathbb{RP}^p \subset \mathbb{P}W^{1,2}(M)} \sup_{[f]\in\mathbb{RP}^p} E(f).$$

The area functional and the Rayleigh functional seem dramatically different but it follows from the work of Almgren [**1**] that the space of unoriented closed hypersurfaces $\mathcal{Z}_n(M, \mathbb{Z}_2)$ in M can also be thought of as an $\mathbb{RP}^\infty$ (as we shall see later). The fact that $\mathcal{Z}_n(M, \mathbb{Z}_2)$ and $\mathbb{P}W^{1,2}(M)$ share some basic topological principles helps explaining why there is a Weyl law for both functionals.

In the late 1980s, Gromov [**7**] wrote a paper in which he first mentions the analogies explained above and explores applications of the classical Borsuk-Ulam Theorem. This theorem states that for any continuous map $f : S^k \to \mathbb{R}^k$, there is always a point $x \in S^k$ such that $f(x) = f(-x)$. Here is one such application. Take a bounded domain $\Omega \subset \mathbb{R}^{n+1}$, an integer $k \in \mathbb{N}$, a disjoint collection $\Omega_1, \ldots, \Omega_k$ of subdomains of Ω and a vector subspace $E \subset C^\infty(\Omega)$

of $\dim E = k + 1$. Then for every $u \in S^k_E$, where $S^k_E \subset E$ is the unit sphere with respect to some norm, consider

$$F(u) = (\mathrm{vol}\,\{u < 0\} \cap \Omega_1, \cdots, \mathrm{vol}\,\{u < 0\} \cap \Omega_k) \in \mathbb{R}^k.$$

The Borsuk-Ulam Theorem implies the existence of a function $u_0 \in E$ such $F(u_0) = F(-u_0)$, which means that the zero set $Z(u_0) = \{x \in \Omega : u(x) = 0\}$ bisects each Ω_i into two regions of equal volume.

From this we can derive an estimate for the area of $Z(u_0)$. Choose a cube $C \subset \Omega$ and let $l = \lfloor k^{\frac{1}{n+1}} \rfloor$. Denote by a the length of the sides of C. Divide C into l^{n+1} subcubes of size a/l. Since $k \geq l^{n+1}$, there exists $u_0 \in E$ such that the zero set $Z(u_0) = \{x \in \Omega : u(x) = 0\}$ bisects each subcube C_i into two regions of equal volume. The relative isoperimetric inequality then implies area $Z(u_0) \cap C_i \geq d(n)a^n l^{-n}$ for every i. Hence

$$\mathrm{area}\, Z(u_0) \geq d(n)a^n l \geq C(n,a)k^{\frac{1}{n+1}}.$$

This is another instance of some kind of similarity with the eigenvalue problem.

In Section 2, we will explain this $\mathbb{RP}^\infty$ structure and discuss a Weyl law conjectured by Gromov (and proved in [**13**]) in which the Rayleigh energy is replaced by the area functional. In Section 3, we will discuss Yau's conjecture on the existence of infinitely many minimal hypersurfaces. In Section 4, we will describe recent developments in the field obtained since the time of the 50th birthday JDG Conference. This includes the proof of density of minimal hypersurfaces for generic metrics obtained by the authors with Irie [**11**], proof of equidistribution of minimal hypersurfaces by the authors with Song [**21**], proof of the authors' Multiplicity One Conjecture and Morse Index Conjecture in dimension three by Chodosh and Mantoulidis ([**3**]), and the full resolution of Yau's Conjecture by Song [**26**].

2. The space of cycles and a Weyl law for the volume spectrum

We denote by $\mathcal{Z}_n(M, \mathbb{Z}_2)$ the space of n-dimensional modulo two flat chains with no boundary endowed with the flat topology. This space can have more than one connected component when $H_n(M, \mathbb{Z}_2) \neq 0$, but the components are homeomorphic to each other and each is weakly homotopically equivalent to $\mathbb{RP}^\infty$. Note that if $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ is in the connected component of the zero cycle then $T = \partial U$ for some $(n+1)$-chain U.

Almgren [**1**] proved there is a canonical isomorphism between the homotopy group $\pi_l(\mathcal{Z}_k(M, \mathbb{Z}_2))$ and $H_{k+l}(M, \mathbb{Z}_2)$ (he did it for integer coefficients but the same proof applies to coefficients in $\mathbb{Z}_2$). In the codimension one case this implies

$$\pi_1(\mathcal{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2,$$

and

$$\pi_k(\mathcal{Z}_n(M, \mathbb{Z}_2)) = 0$$

for all $k \geq 2$. This is precisely the list of homotopy groups of $\mathbb{RP}^\infty$.

There is another way of interpreting this $\mathbb{RP}^\infty$ structure (see Section 5 of [20] for more details). Consider the boundary map

$$\partial : I_{n+1}(M^{n+1}, \mathbb{Z}_2) \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2),$$

where $I_{n+1}(M^{n+1}, \mathbb{Z}_2)$ is the space of $(n+1)$-dimensional flat chains modulo two. If $T = \partial U$, then also $T = \partial(M - U)$. The chains U and $M - U$ are, by the Constancy Theorem of Geometric Measure Theory ([25]), the only two chains with boundary equal to T and so the boundary map is a two cover. Furthermore, the space of top-dimensional chains $I_{n+1}(M^{n+1}, \mathbb{Z}_2)$ is contractible, like the infinite dimensional sphere S^∞. To see this, choose a Morse function $f : M \to [0,1]$ and define the deformation $H_t(U) = U \cap f^{-1}([0,t])$. The involution

$$\alpha : I_{n+1}(M^{n+1}, \mathbb{Z}_2) \to I_{n+1}(M^{n+1}, \mathbb{Z}_2)$$

defined by $\alpha(U) = M - U$ plays the role of the antipodal map and the boundary map is a two-cover just like the standard projection $\mathbb{S}^\infty \to \mathbb{RP}^\infty$. Choosing coefficients in $\mathbb{Z}_2$ is crucial here. If we take integer coefficients, we will be talking about integral currents but it turns out that the corresponding space of cycles $\mathcal{Z}_n(M^{n+1}, \mathbb{Z})$ is weakly homotopic to the circle S^1 (in the case M is orientable) which is a bit disappointing.

The $\mathbb{RP}^\infty$ structure suggests there are nontrivial multiparameter families of hypersurfaces. In fact, for each $k \in \mathbb{N}$ we can define explicitly a homotopically non-trivial map $\Lambda : \mathbb{RP}^k \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ in the following way. Choose an arbitrary Morse function $f : M \to \mathbb{R}$ and define

$$\Lambda([a_0 : a_1 : \cdots : a_k]) = \partial\{x \in M : a_0 + a_1 f(x) + \cdots + a_k f(x)^k < 0\}.$$

Hence there are nontrivial k-parameter sweepouts for every $k \in \mathbb{N}$.

We would like to replace $\mathbb{P}W^{1,2}(M)$ by the space of cycles $\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ and the Rayleigh functional $E(f)$ of a function by the n-area or the mass (in the language of geometric measure theory) $M(T)$ of a cycle T. In the case of the eigenvalues of the Laplacian there is an underlying Hilbert space structure that allows us to consider the min-max characterization over vector subspaces. We lose this linear structure when we go to the setting of the space of cycles so we have to find a nonlinear generalization. We do that by looking at the cohomology.

The cohomology ring $H^*(\mathbb{RP}^\infty, \mathbb{Z}_2)$ is the polynomial ring $\mathbb{Z}_2[\overline{\lambda}]$ where $\overline{\lambda}$ is the generator of $H^1(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2$. In particular, $H^k(\mathbb{RP}^\infty, \mathbb{Z}_2) = \{0, \overline{\lambda}^k\}$ where $\overline{\lambda}^k = \overline{\lambda} \cup \cdots \cup \overline{\lambda}$ is the cup product power. We use the same notation $\overline{\lambda}$ to denote the generator of $H^1(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$.

We make the following definitions.

Definition. A family of cycles $\mathcal{S} \subset \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is called a *k-sweepout* if it detects the cohomology class $\overline{\lambda}^k$ in the sense that

$$\overline{\lambda}^k_{|\mathcal{S}} \neq 0 \in H^k(\mathcal{S}, \mathbb{Z}_2).$$

Definition. The *k-width* of M is the number

$$\omega_k(M) = \inf_{\{\mathcal{S}\,:\,\overline{\lambda}^k_{|\mathcal{S}} \neq 0\}} \sup_{\Sigma \in \mathcal{S}} M(\Sigma).$$

Definition. The *volume spectrum* of M is the sequence of numbers:

$$\{\omega_1(M) \leq \omega_2(M) \leq \cdots \leq \omega_k(M) \leq \cdots\}.$$

Similarly as before, given any k-sweepout $\mathcal{S}$ and any family of k disjoint open sets in M there is a cycle $\Sigma \in \mathcal{S}$ that bisects each open set into two pieces of equal volume. This gives a lower bound for ω_k but in fact one also has an upper bound:

Theorem 2.1 (Gromov [**8**], Guth [**10**]). *There exist constants $c_1, c_2 > 0$ depending on M such that*

$$c_1 k^{\frac{1}{n+1}} \leq \omega_k(M) \leq c_2 k^{\frac{1}{n+1}}.$$

for every $k \in \mathbb{N}$.

The upper bound can be proven by the bend-and-cancel technique ([**10**]) and is precisely the rate obtained from families of zero sets of polynomials of degree less than or equal to d on the unit sphere.

In [**8**], Gromov made the following conjecture:

Conjecture (Gromov, 2003). *$\{\omega_k(M)\}$ obeys a Weyl law.*

In [**13**], with Liokumovich, we proved this conjecture:

Theorem 2.2. *There exists a dimensional constant $a(n) > 0$ such that*

$$\lim_{k\to\infty} \omega_k(M) k^{-\frac{1}{n+1}} = a(n)\mathrm{vol}(M)^{\frac{n}{n+1}}.$$

The constant $a(n)$ can be estimated but it is not known explicitly, even for $n = 1$. This is in stark contrast with standard Weyl Law where the fact that the universal constant is known is used in the proof. It is also worthwhile to point out that the proof of the Weyl Law for closed manifolds, due to Minakshisundaram and Pleijel, only appeared more than 30 years after Weyl's proof (which applies only to regions of space) and uses techniques for which no analogue in the space of mod 2 cycles has yet been found.

The definition of the volume spectrum also makes sense for manifolds with boundary, by considering multiparameter sweepouts of mod 2 relative (to the boundary) cycles.

The proof of our Weyl law is based on a result inspired by Lusternik-Schnirelmann theory that we proved in [**13**]:

Theorem 2.3. *Let $\{\Omega_1, \ldots, \Omega_p\}$ be a disjoint collection of subregions of Ω. Then*

$$\omega_k(\Omega) \geq \sum_{i=1}^{p} \omega_{k_i}(\Omega_i)$$

as long as $\sum_{i=1}^{p} k_i \leq k$.

The proof uses the cup product structure and goes as follows. Let $\mathcal{S} \subset \mathcal{Z}_n(\Omega, \partial\Omega, \mathbb{Z}_2)$ be a k-sweepout of mod 2 relative cycles of Ω. This means $\overline{\lambda}^k_{|\mathcal{S}} \neq 0$.

Define $\mathcal{S}_i = \{\Sigma \in \mathcal{S} : M(\Sigma \cap \Omega_i) < \omega_{k_i}(\Omega_i)\}$. The map $T \mapsto T \cap \Omega_i$ from relative cycles of Ω to relative cycles of Ω_i preserves the fundamental cohomology class. Hence $\overline{\lambda}^{k_i}_{|\mathcal{S}_i} = 0$. A basic property of the cup product implies $\overline{\lambda}^{k_1+\cdots+k_p}_{|\mathcal{S}_1\cup\cdots\cup\mathcal{S}_p} = 0$. Because $\sum_{i=1}^p k_i \leq k$, we get $\overline{\lambda}^k_{|\mathcal{S}_1\cup\cdots\cup\mathcal{S}_p} = 0$. This implies there must exist $\Sigma \in \mathcal{S} \setminus (\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_p)$. Since

$$M(\Sigma) \geq M(\Sigma \cap \Omega_1) + \cdots + M(\Sigma \cap \Omega_p) \geq \sum_{i=1}^p \omega_{k_i}(\Omega_i),$$

we are done.

3. Existence theory of minimal hypersurfaces

Our motivation to study the volume spectrum comes from the following conjecture in minimal surface theory:

Conjecture (Yau, 1982 [**30**]). *Every compact three-dimensional Riemannian manifold should contain infinitely many smooth, closed, immersed, minimal surfaces.*

Until recently, the best result on this conjecture was the existence of one minimal surface:

Theorem 3.1 (Almgren [**2**], Pitts [**23**], Schoen-Simon [**24**]). *Every compact Riemannian manifold M^{n+1} contains at least one closed minimal hypersurface Σ that is smooth embedded outside a set of Hausdorff dimension less than or equal to $n-7$. In particular, Σ is smooth if $(n+1) \leq 7$.*

Although Yau's conjecture was stated for dimension three only, we expect the existence of infinitely many minimal hypersurfaces with possible singular sets of codimension 7 to be true in any dimension.

Almgren ([**2**], 1965) devised a general min-max theory that succeeded in proving the existence of stationary integral varifolds of any dimension, and Pitts ([**23**], 1981) proved smoothness of the varifold in the codimension one case for $(n+1) \leq 6$. Regularity for higher dimensions was proven by Schoen and Simon ([**24**], 1981).

A few years ago, we proved ([**17**]):

Theorem 3.2.
(a) *For any compact Riemannian manifold M^{n+1}, there exist at least $(n+1)$ closed minimal hypersurfaces.*
(b) *If M has positive Ricci curvature, then there exists infinitely many closed minimal hypersurfaces.*

In order to prove this, we used mod 2 coefficients and applied Almgren-Pitts min-max theory to k-sweepouts. The theory gives (together with the index estimates of [**19**], see Theorem 3.3 below) that, for every $k \in \mathbb{N}$, there exists a disjoint collection of closed minimal hypersurfaces $\{\Sigma_1^{(k)}, \dots, \Sigma_{q_k}^{(k)}\}$ and positive integers $\{m_1^{(k)}, \dots, m_{q_k}^{(k)}\}$ such that

$$\omega_k(M) = m_1^{(k)} \cdot \text{area}(\Sigma_1^{(k)}) + \dots + m_{q_k}^{(k)} \cdot \text{area}(\Sigma_{q_k}^{(k)}).$$

The possibility of integer multiplicities is one of the basic difficulties of the theory. The simple knowledge that ω_k grows to infinity as $k \to \infty$ does not suffice to prove Yau's conjecture because one could be producing the same minimal hypersurface with higher and higher multiplicity.

If the Ricci curvature is positive, Frankel's theorem ([**4**]) implies the min-max minimal hypersurface is connected. In this case counting arguments can be applied, which exploit the sublinear growth of $\{\omega_k\}$, together with ideas from Lusternik-Schnirelmann theory to prove part (b) of last theorem.

Although Almgren-Pitts theory was successful in producing an existence result, it did not provide any information on the Morse index of the min-max minimal hypersurface. Heuristically, as in finite-dimensional Morse theory, one should expect that generically the Morse index should be equal to the dimension of the cohomology class detected by the sweepouts.

We were able to prove in [**19**] an upper bound for the Morse index:

THEOREM 3.3. *If* $\Sigma = m_1^{(k)} \cdot \Sigma_1^{(k)} + \dots + m_{q_k}^{(k)} \cdot \Sigma_{q_k}^{(k)}$ *is the minimal hypersurface produced by min-max over k-sweepouts, then*

$$\text{index}(\Sigma_1^{(k)}) + \dots + \text{index}(\Sigma_{q_k}^{(k)}) \leq k.$$

For the multiplicities, we conjecture:

MULTIPLICITY ONE CONJECTURE. *For a generic (bumpy) metric, every component of a min-max minimal hypersurface is two-sided and has multiplicity one.*

The Multiplicity One Conjecture has appeared in slightly different formulations in some of our previous papers (for instance, [**19**]). The above strong version was proposed in the talk on the occasion of the JDG 50th birthday conference in April 2017.

A metric is *bumpy* if no closed minimal hypersurface admits nontrivial Jacobi fields, i.e. if every closed minimal hypersurface is a nondegenerate critical point of the area functional. White ([**28**], [**29**]) showed that being bumpy is a generic property in the Baire sense.

We proved the Multiplicity One Conjecture for two-sided components in the one-parameter case and we were also able to rule out one-sided components with multiplicity in some settings ([**19**], [**12**], see also [**16**], [**31**], [**32**]).

Under the multiplicity one assumption, we gave in [**20**] a complete characterization of the Morse index:

THEOREM 3.4. *For a generic metric, if* $m_1^{(k)} = \dots = m_{q_k}^{(k)} = 1$ *then*

$$\text{index}(\Sigma_1^{(k)}) + \dots + \text{index}(\Sigma_{q_k}^{(k)}) \geq k.$$

This Theorem implies together with the Multiplicity One Conjecture the following Morse-theoretic description of the space of minimal hypersurfaces:

Morse Index Conjecture. *For a generic metric on M, there should be a sequence $\{\Sigma_k\}$ of minimal hypersurfaces such that*

(a) $\operatorname{index}(\Sigma_k) = k$,

(b) $\lim_{k\to\infty} \operatorname{area}(\Sigma_k) k^{-\frac{1}{n+1}} = a(n)\operatorname{vol}(M)^{\frac{n}{n+1}}$.

4. Recent developments

Since the time of the 50th birthday JDG Conference, there have been dramatic developments in the field. First we were able to settle Yau's conjecture for generic metrics with Irie [11] by proving that a much stronger property holds true:

Theorem 4.1. *Suppose $3 \le (n+1) \le 7$. For a generic metric on M^{n+1}, the union of all closed, smooth, embedded, minimal hypersurfaces of M is dense in M.*

This is an application of our Weyl law. Given a Riemannian metric on M and an open set $U \subset M$ such that no minimal hypersurface intersects U, the basic idea consists in perturbing the metric inside U so that the total volume goes up and using the Weyl law to conclude some k-width also goes up. From that we infer that there must be a new minimal hypersurface for the perturbed metric that intersects U.

Then, with Song [21], we were able to make the argument more quantitative and proved an equidistribution property:

Theorem 4.2. *Suppose $3 \le (n+1) \le 7$. For a generic metric on M^{n+1}, there is a sequence $\{\Sigma_i\}$ of closed, connected, smooth, embedded, minimal hypersurfaces of M such that*

$$\lim_{p\to\infty} \frac{\sum_{i=1}^p \int_{\Sigma_i} f\, d\Sigma_i}{\sum_{i=1}^p \operatorname{area}(\Sigma_i)} = \frac{\int_M f\, dM}{\operatorname{vol}(M)}$$

for any continuous function $f : M \to \mathbb{R}$.

Our Multiplicity One Conjecture (for the Allen-Cahn variant of min-max theory) was proven in remarkable work by Chodosh and Mantoulidis [3] in dimension three:

Theorem 4.3. *For any compact (M^3, g), any Allen-Cahn min-max minimal hypersurface that occurs with multiplicity or is one-sided has a positive Jacobi field (on the two-sided double cover in the second case).*

Their work implies the Morse Index Conjecture in dimension three with (b) replaced by

(b1) $C^{-1}k^{\frac{1}{n+1}} \le \operatorname{area}(\Sigma_k) \le Ck^{\frac{1}{n+1}}$ for some $C > 0$.

The Allen-Cahn variant can be seen as an ε-regularization of the Almgren-Pitts min-max theory and was first studied by Guaraco [9] to give an alternative proof of the existence of one closed minimal hypersurface. Multiparameter sweepouts in the Allen-Cahn setting were introduced by Gaspar and Guaraco [5]. In [6], Gaspar and Guaraco proved a Weyl law for the Allen-Cahn volume spectrum and were able to reproduce the proofs of density and equidistribution of minimal hypersurfaces for generic metrics. Combined with Chodosh-Mantoulidis [3], this establishes item (b) of the Morse Index Conjecture in dimension three perhaps with a different dimensional constant. This carries out completely in dimension three (for the Allen-Cahn variant) a program (see [20]) proposed by the authors in a series of papers ([15], [18], [19], [22]).

Finally, Yau's conjecture was proven in its general formulation in outstanding new work of Song [26]:

THEOREM 4.4. *Suppose* $3 \leq (n+1) \leq 7$. *For any Riemannian metric on a compact manifold* M^{n+1}, *there exist infinitely many closed, smooth, embedded, minimal hypersurfaces.*

Song's proof builds on the methods initially developed by the authors in [17]. Song considers the situation when there are stable minimal hypersurfaces, which can be assumed to exist by Theorem 3.2, and succeeds in proving the existence of infinitely many minimal hypersurfaces confined (or trapped) inside a region with stable boundary. He does so by introducing the volume spectrum of a noncompact (and non-smooth) cylindrical extension, which grows linearly in contrast with the sublinear growth of the volume spectrum of a compact manifold. In order to deal with the noncompact setting, Song takes an exhaustion and uses the theory of Li and Zhou [14] for relative cycles in manifolds with boundary.

It is an interesting question to determine whether density (see Theorem 4.1) and equidistribution (see Theorem 4.2) hold for all Riemannian metrics.

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