Period integrals and tautological systems

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Abstract. Tautological systems are Picard-Fuchs type systems arising from varieties with large symmetry. In this survey, we discuss recent progress on the study of tautological systems. This includes tautological systems for vector bundles, a new construction of Jacobian rings for homogenous vector bundles, and relations between period integrals and zeta functions.

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1. Introduction

Period integrals connect Hodge theory, number theory, mirror symmetry, and many other important areas of math. The study of periods has a long history dating back to Euler, Legendre, Gauss, Abel, Jacobi and Picard in the form of special functions, which are periods of curves. Period integrals first appeared in the form of hypergeometric functions. The name *hypergeometric functions* appeared to have been introduced by John Wallis in his book *Arithmetica Infinitorum* (1655). Euler introduced the theory of elliptic integrals. Gauss studied their differential equations, now known as Euler-Gauss hypergeometric equations. Legendre made the first connection to geometry of hypergeometric equations through the theory of elliptic integrals. He showed that periods of the Legendre family of elliptic curves were in fact solutions to a special hypergeometric equation. Euler obtained the power series solutions to the Euler-Gauss equations and also wrote them in terms of contour integrals. He introduced the important idea that instead of considering one elliptic integral at a time, one should look at a family of such integrals, and consider the corresponding differential equations. This appeared to be the first example of variation of Hodge structures and Picard-Fuchs equations.

The theory of deformations of higher dimensional complex varieties was pioneered by Kodaira and Spencer [**22, 21, 23**]. The modern study of variation of Hodge structures started from the work of Griffiths [**12**], and continued by Deligne [**3, 4, 5**], Schmid [**33**], and others. For example, periods of hypersurfaces in $\mathbb{P}^n$ were studied in [**12**] from the point of view of variation of Hodge structures. Thanks to the relations between pole order filtrations and Hodge filtrations, there is a so-called reduction of pole method for computing Picard-Fuchs systems for hypersurfaces in $\mathbb{P}^n$.

Two natural generalizations of $\mathbb{P}^n$ are toric varieties and homogenous Fano varieties. For toric varieties, Gel'fand, Kapranov and Zelevinski [**8**] found a multivariable PDE systems, now known as GKZ hypergeometric system, that annihilates periods of toric hypersurfaces. More recently, Lian, Song, and Yau [**28, 29**] established a holonomic differential system they called a *tautological system* for varieties with large symmetry, giving a vast generalization of GKZ systems. The machinery of $\mathcal{D}$-module is introduced in tautological systems and opens new ways to study hypersurfaces in homogenous Fano varieties. For instance, in [**2, 17**], Huang, Lian, Yau, together with their collaborators, obtained general solution rank formulas, which recovered the formula for GKZ system, and also proved the completeness of tautological systems for homogenous Fano varieties. More interestingly, new candidates for large complex structure limits appeared naturally as the projected Richardson varieties. For GKZ systems, there are also new results inspired by this approach. For example, in [**18**], the solutions of GKZ systems are realized as chain integrals or semiperiods.

There are already many interesting phenomena in toric hypersurfaces, especially Calabi-Yau hypersurfaces, in which GKZ system plays an important role. For instance, the famous mirror symmetry between genus zero Gromov-Witten invariants and periods of mirror toric Fano varieties in the sense of Batyrev [**1**], was proved independently by Givental [**9, 10**] and Lian-Liu-Yau [**25, 26, 27**]. On the B-side of mirror symmetry, the solutions to GKZ systems are explicitly given by a cohomology-valued function introduced by Hosono-Lian-Yau [**13**] which they called *the B-series*. Aside from giving a close formula for power series of periods of toric Calabi-Yau hypersurfaces, this function has many interesting number theoretic properties – integrality, divisibility, modularity, etc. Similar problems for periods of Calabi-Yau hypersurfaces in homogenous Fano varieties are still open and in need of further explorations.

The theory of tautological systems behaves very nicely for homogenous varieties. For instance, the solutions can all be realized as period integrals along homology classes of the complement of the hypersurfaces. On the other hand the solutions to GKZ systems correspond to period integrals, or Euler integrals on complement inside the affine tori. In some sense, the solutions forget the compactifications of algebraic tori. A conjectural formula, proposed by Hosono-Lian-Yau [**13**] – the hyperplane conjecture – provides an algorithm to exclude those extra solutions and characterizes periods from the B-series. In the case of projective spaces, the conjecture has recently been proved by Lian-Zhu [**30**]. The proof uses both completeness of the tautological system for $\mathbb{P}^n$ as a homogenous variety, and the B-series formula as the solutions to the GKZ system for $\mathbb{P}^n$ as a toric variety. The hyperplane conjecture in general case remains open.

Here is an outline of this paper. In section 2, we discuss the construction of tautological systems [**29**] and Riemann-Hilbert type results [**17**] to homogenous vector bundles in [**16**]. In section 3, we study the Jacobian ring description of variation of Hodge structures arising from the family appeared in section 2. Section 4 is devoted to some arithmetic properties of period integrals of hypersurfaces in both toric and flag varieties. In particular, we proved a conjecture made by Vlasenko [**35**] on an algorithm to compute the unit root part of the F-crystal associated to toric hypersurfaces.

1.1. Notations. We first fix the following notations in the following discussions.

(1) Let G be a complex Lie group and $\mathfrak{g} = Lie(G)$
(2) Let X^n be a smooth projective variety together with action of G.
(3) Let E^r be a G-equivariant vector bundle on X with rank r.
(4) Assume $V^\vee = H^0(X, E)$ has basis $a_1, \cdots, a_N$ and dual basis $a_i^\vee$.
(5) Let $f \in V^\vee$ be an section and Y_f the zero locus of f. We further assume Y_f is smooth with codimension r.

2. Tautological systems for homogenous vector bundles

Definition 2.1 (Period integrals). *We first start with the definition of period integrals. Consider a family $\pi\colon \mathcal{Y} \to B$ of d-dimensional compact Kähler manifolds. We fix a choice of holomorphic d-form $\omega \in H^0(B, \pi_* K_{\mathcal{Y}/B})$. In many cases, we have a canonical choice of ω from residue formula. For any parallel cycle σ, the period integral is defined to be $\int_\sigma \omega$, which is a holomorphic function defined on B.*

The key tool to study period integral is the linear differential operators annihilating the period integral, called Picard-Fuchs operators. Now we construct the holomorphic d-form ω and Picard-Fuchs operators when $\mathcal{Y} \to B$ is the zero loci of vector bundle sections. For simplicity, we only discuss the Calabi-Yau family.

Let X^n be a smooth n-dimensional Fano variety and E be a vector bundle of rank r over X. Denote the dual space of global sections by $V = H^0(X, E)^\vee$. Assume that any generic section $s \in V^\vee$ defines a nonsingular subvariety $Y_s = \{s = 0\}$ in X with codimension r, which gives a family of compact Kähler manifolds $\pi\colon \mathcal{Y} \to B = V^\vee - D$ on the complement of the discriminant locus D. If we further assume $\det E \cong K_X^{-1}$, the adjunction formula implies that

$$K_{Y_s} \cong K_X \otimes \det E|_{Y_s} \cong \mathcal{O}_{Y_s}. \tag{2.1}$$

A section s of $K_X \otimes \det E \cong \mathcal{O}_X$ gives a family of holomorphic top forms ω_s on Y_s corresponding to constant section 1 of $\mathcal{O}_{Y_s}$, also called the residue of s.

If E splits as a direct sum of line bundles, the residue map is defined on line bundles and generalized to E by induction. In the nonsplitting case, we can apply the residue formula in the splitting case locally and glue it together to get a global residue formula. It is not clear how to derive the global information from these local formulas and the Dwork-Griffiths reduction of pole algorithm turns out to be complicated for multivariable families. We follow the idea in [**28, 29**] by introducing the global residue formula on principal bundle.

2.1. Calabi-Yau bundles and adjunction formulas. We first explain the idea in the classical case when X is $\mathbb{P}^n$. We view $P = \mathbb{C}^{n+1} \setminus \{0\}$ as $\mathbb{C}^*$ principal bundle. There is an n-form on $\Omega = \sum_i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_n$ on $\mathbb{C}^{n+1} \setminus \{0\}$. Any section f of $\mathcal{O}(n+1)$ is a degree $(n+1)$-polynomial on $\mathbb{C}^{n+1} \setminus \{0\}$. Then $\frac{\Omega}{f}$ vanishes along the vertical direction and invariant under $\mathbb{C}^*$ action. So it can be descended to X with simple pole along Y_f and $\omega = \operatorname{Res} \frac{\Omega}{f}$. In this example, we need the construction of holomorphic n-form Ω on principal bundle, and the transformation factor cancels the one from K_X. The notion of Calabi-Yau bundles is introduced in [**29**] is used to construct such an Ω and induces an adjunction formula on principal bundles. The canonical sections of holomorphic top forms used in period integral are given by this construction. First we recall the definition of Calabi-Yau bundles in [**29**]

Definition 2.2 (Calabi-Yau bundle). *Denote H and G to be complex Lie groups. Let $p\colon P \to X$ be a principal H-bundle with G-equivariant action. A Calabi-Yau bundle structure on (X, H) says that the canonical bundle of X is the associated line bundle with character $\chi\colon H \to \mathbb{C}^*$. The following short exact sequence*

$$0 \to \operatorname{Ker} p_* \to TP \to p^*TX \to 0 \tag{2.2}$$

induces an isomorphism

$$K_P \cong p^*K_X \otimes \det(P \times_{ad(H)} \mathfrak{h}^\vee). \tag{2.3}$$

Fixing an isomorphism $P \times_H \mathbb{C}_\chi \cong K_X$, the isomorphism (2.3) implies that K_P is a trivial bundle on P and has a section ν which is the tensor product of nonzero elements in $\mathbb{C}_\chi$ and $\det \mathfrak{h}^\vee$. This holomorphic top form satisfies that

$$h^*(\nu) = \chi(h)\chi_{\mathfrak{h}}^{-1}(h)\nu, \tag{2.4}$$

where $\chi_{\mathfrak{h}}$ is the character of H on $\det \mathfrak{h}$ by adjoint action. The tuple (P, H, ν, χ) satisfying (2.4) is called a Calabi-Yau bundle.

Conversely, any section ν satisfying (2.4) determines an isomorphism $P \times_H \mathbb{C}_\chi \cong K_X$. Since the only line bundle automorphism of $K_X \to X$ fixing X is rescaling when X is compact, such ν is determined up to rescaling (Theorem 3.12 in [**29**]). So the equivariant action of G on $P \to X$ changes ν according to a character β^{-1} of G. We say the Calabi-Yau bundle is (G, β^{-1})-equivariant.

When E is the line bundle K_X^{-1}, the following is the residue formula for Calabi-Yau bundles:

Theorem 2.3 ([**29**], Theorem 4.1). *If (P, H, ν, χ) is a Calabi-Yau bundle over a Fano manifold X, the middle dimensional variation of Hodge structure $R^d\pi_*(\underline{\mathbb{C}})$ associated with the family $\pi\colon \mathcal{Y} \to B$ of Calabi-Yau hypersurfaces has a canonical section of the form*

$$\omega = Res \frac{\iota_{\xi_1} \cdots \iota_{\xi_m} \nu}{f}. \tag{2.5}$$

Here $\xi_1, \cdots, \xi_m$ are independent vector fields generating the distribution of H-action on P, and $f\colon B \times P \to \mathbb{C}$ is the function representing the universal section of $P \times_H \mathbb{C}_{\chi^{-1}} \cong K_X^{-1}$.

If E is a direct sum of line bundles associated to characters of H, the residue formula is similar to (2.5) by induction. For general vector bundle, we need Cayley's trick to study the residue forms.

2.2. Cayley's trick. Let $\mathbb{P} = P(E^\vee)$ be the projectivation of $E^\vee$ and $\mathcal{O}(1)$ be the hyperplane section bundle on $\mathbb{P}$. The projection map is denoted by $\pi\colon \mathbb{P} \to X$. From now on, we assume E is ample and by definition, is equivalent to $\mathcal{O}(1)$ being ample. We collect the propositions relating the geometry of X and $\mathbb{P}$ in the following. Proposition 2.4 and 2.6 are from [**34**],[**24**] and [**31**].

Proposition 2.4.
(1) *There is a natural isomorphism $H^0(X, E) \cong H^0(\mathbb{P}, \mathcal{O}(1))$. The corresponding section in $H^0(\mathbb{P}, \mathcal{O}(1))$ is also denoted by f.*
(2) *Let $\tilde{Y}$ be the zero locus of f in $\mathbb{P}$. Then $\tilde{Y}$ is smooth if and only if Y is smooth with codimension r or empty.*
(3) *There is an natural isomorphism $K_{\mathbb{P}} \cong \pi^*(K_X \otimes \det E) \otimes \mathcal{O}(-r)$*

From now on, we assume Y is smooth with codimesion $r \geq 2$.

Definition 2.5. *The variable cohomology $H^{n-r}_{var}(Y)$ is defined to be cokernel of $H^{n-r}(X) \to H^{n-r}(Y)$.*

Proposition 2.6. *There is an isomorphism*

$$H^{n+r-1}(\mathbb{P} - \tilde{Y}) \cong H^{n+r-2}_{var}(\tilde{Y})(-1) \cong H^{n-r}_{var}(Y)(-r) \tag{2.6}$$

We now consider the Calabi-Yau case, equivalently $\det E \cong K_X^{-1}$, for simplicity. Then we have the vanishings in the Hodge filtration $F^{n+r-1-k} = 0$ for $k < r-1$ and isomorphisms

$$\begin{aligned} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}) \to H^0(\mathbb{P}, K_{\mathbb{P}} \otimes \mathcal{O}(r)) \to F^n H^{n+r-1}(\mathbb{P} - \tilde{Y}_f) \\ \to F^{n-r} H^{n-r}(Y_f). \end{aligned} \tag{2.7}$$

Consider the principle bundle adjunction formula for base space $\mathbb{P}$. Let (P, H, ν, χ) is a Calabi-Yau bundle over $\mathbb{P}$. The image of 1 in $H^0(\mathbb{P}, K_{\mathbb{P}} \otimes \mathcal{O}(r))$ has the form $\frac{\Omega}{f^r}$ on principle bundle P. This corresponds to the residue form ω via the above isomorphisms.

2.3. Tautological systems. Straightforward calculations from the residue formula $\frac{\Omega}{f^r}$ gives the following theorem.

THEOREM 2.7. *Let $I(\mathbb{P}(E^\vee), V))$ be the ideal of the image of the map $\mathbb{P} \to \mathbb{P}(V)$. Denote $Z_x = \sum Z(x)_{ij} a_i \partial_j$ for $x \in \mathfrak{g} \mapsto Z(x)_{ij} a_i a_j^\vee \in End(V)$. Then period integral I_γ satisfies the following system of differential equations:*

(1) *Geometric constraints* $Q(\partial_a) I_\gamma = 0 \qquad (Q \in I(\mathbb{P}(E^\vee), V))$
(2) *Symmetry operators* $(Z_x + \beta(x)) I_\gamma = 0 \qquad (x \in \mathfrak{g})$
(3) *Euler operator* $(\sum_i a_i \frac{\partial}{\partial a_i} + r) I_\gamma = 0$

We call the differential system in Theorem 2.7 tautological system for (X, E, H, G). It's the same as the cyclic D-module $\tau(G, \mathbb{P}(E^\vee), \mathcal{O}(-1), \hat{\beta})$ defined in [28] [29] by

$$\tau = D_{V^\vee} / D_{V^\vee}(J(\mathbb{P}(E^\vee)) + Z(x) + \hat{\beta}(x), x \in \hat{\mathfrak{g}}). \tag{2.8}$$

Here $J(\mathbb{P}(E^\vee)) = \{Q(\partial_a) | Q \in I(\mathbb{P}(E^\vee))\}$, $\hat{G} = G \times \mathbb{C}^*$ with Lie algebra $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e$ and $\hat{\beta} = (\beta, r)$.

We can apply the holonomicity criterion for tautological system in [28],[29].

THEOREM 2.8. *If the induced action of G on $\mathbb{P}(E^\vee)$ has finite orbits, the corresponding tautological system τ is regular holonomic.*

EXAMPLE 2.9 (Complete intersections). *When $E = \oplus_1^r L_i$ is a direct sum of very ample line bundles, the above system recovers the tautological system for complete intersections in [29]. This case is equivalent to say that the structure group of E is reduced to the complex torus $(\mathbb{C}^*)^r$. So we have symmetry group $(\mathbb{C}^*)^r$ acting on the fibers of E. This gives the usual Euler operators in [29]. Let $\hat{X}_i$ be the cone of X inside $V_i = H^0(X, L_i)^\vee$ under the linear system of L_i. The cone of $\mathbb{P}(E^\vee)$ inside $V = \oplus_{i=1}^r V_i$ is fibered product $\hat{X}_i$ over X. So the geometric constrains are the same as [29]. Assume X is a G-variety consisting of finite G-orbits and L_i are G-homogenous bundles. Then $\mathbb{P}(E^\vee)$ admits an action of $\tilde{G} = G \times (\mathbb{C}^*)^{r-1}$ with finite orbits. This is the same holonomicity criterion as [29] for complete intersections.*

Example 2.10 (Homogeneous varieties). *Let G be a semisimple complex Lie group and $X = G/P$ is a generalized flag variety quotient by a parabolic subgroup P. This forms a principal P-bundle over X. We assume E to be a homogenous vector bundle from a representation of P and the action of G on $\mathbb{P}(E^\vee)$ is transitive. Then the projectivation of $\mathbb{P}(E^\vee)$ is also a generalized flag variety for a parabolic subgroup $P' \subset P$. Hence the G-action on $\mathbb{P}(E^\vee)$ is transitive. If $\mathcal{O}(1)$ is very ample on $\mathbb{P}(E^\vee)$, the defining ideal of $\mathbb{P}(E^\vee)$ in $\mathbb{P}(V)$ is given by the Kostant-Lichtenstein quadratic relations. Furthermore, any character of G is trivial, hence β is zero. So the differential system is regular holonomic and explicitly given in this case.*

2.4. Solution rank. Now we discuss the solution rank for the system. There are two versions of solution rank formula for hypersurfaces. One is in terms of Lie algebra homology, see [**2**]. One is in terms of perverse sheaves on X, see [**17**]. Here we have similar description for zero loci of vector bundle sections.

2.4.1. *Lie algebra homology description.* We fix some notations. Let $R = \mathbb{C}[V]/I(\mathbb{P}(E^\vee))$ be the coordinate ring of $\mathbb{P}$. Let $Z\colon \hat{\mathfrak{g}} \to \mathrm{End}(V)$ be the extended representation by e acting as identity. We extend the character $\beta\colon \hat{\mathfrak{g}} \to \mathbb{C}$ by assigning $\beta(e) = r$.

Definition 2.11. *We define $\mathcal{D}_{V^\vee}$-module structure on $R[a]e^f \cong R[a_1, \cdots a_N]$ as follows. The functions a_i acts as left multiplication on $R[a_1, \cdots a_N]$. The action of ∂_{a_i} on $R[a_1, \cdots a_N]$ is $\partial_{a_i} + a_i^\vee$.*

Then we have the following $\mathcal{D}_{V^\vee}$-module isomorphism.

Theorem 2.12. *There is a canonical isomorphism of $\mathcal{D}_{V^\vee}$-module*

$$\tau \cong R[a]e^f / Z^\vee(\hat{\mathfrak{g}})R[a]e^f \tag{2.9}$$

This leads to the Lie algebra homology description of (classical) solution sheaf

Corollary 2.13. *If the action of G on $\mathbb{P}(E^\vee)$ has finitely many orbits, then the stalk of the solution sheaf at $b \in V^\vee$ is*

$$sol(\tau) \cong Hom_{\mathcal{D}}(Re^{f(b)}/Z^\vee(\hat{\mathfrak{g}})Re^{f(b)}, \mathcal{O}_b) \cong H_0(\hat{\mathfrak{g}}, Re^{f(b)}) \tag{2.10}$$

2.4.2. *Perverse sheaves description.* We follow the notations in [**17**].

(1) Let $\mathbb{L}^\vee$ be the total space of $\mathcal{O}(1)$ and $\mathring{\mathbb{L}}^\vee$ the complement of the zero section.
(2) Let $ev\colon V^\vee \times \mathbb{P} \to \mathbb{L}^\vee, \quad (a, x) \mapsto a(x)$ be the evaluation map.
(3) Assume $\mathbb{L}^\perp = \ker(ev)$ and $U = V^\vee \times \mathbb{P} - \mathbb{L}^\perp$. Let $\pi\colon U \to V^\vee$. Notice that U is the complement of the zero locus of the universal section.
(4) Let $\mathcal{D}_{\mathbb{P},\beta} = (\mathcal{D}_{\mathbb{P}} \otimes k_\beta) \otimes_{U\mathfrak{g}} k$, where k_β is the 1-dimensional $\mathfrak{g}$-module with character β and k is the trivial $\mathfrak{g}$-character.
(5) Let $\mathcal{N} = \mathcal{O}_{V^\vee} \boxtimes \mathcal{D}_{\mathbb{P},\beta}$.

We have the following theorem. See Theorem 2.1 in [**17**].

Theorem 2.14. *There is a canonical isomorphism*

$$\tau \cong H^0\pi_+^\vee(\mathcal{N}|_U) \tag{2.11}$$

A direct corollary is the following

Corollary 2.15. *If* $\beta(\mathfrak{g}) = 0$*, there is a canonical surjective map*

$$\tau \to H^0\pi_+^\vee\mathcal{O}_U. \tag{2.12}$$

In terms of period integral, we have an injective map

$$H_{n+r-1}(X - Y_b) \to Hom(\tau, \mathcal{O}_{V^\vee,b}) \tag{2.13}$$

given by

$$\gamma \mapsto \int_\gamma \frac{\Omega}{f^r} \tag{2.14}$$

We have similar solution rank formula. We assume G-action on $\mathbb{P}$ has finitely many orbits. Let $\mathcal{F} = Sol(\mathcal{D}_{\mathbb{P},\beta}) = RHom_{\mathcal{D}^{an}}(\mathcal{D}^{an}_{\mathbb{P},\beta}, \mathcal{O}^{an}_{\mathbb{P}})$ be a perverse sheaf on $\mathbb{P}$.

Corollary 2.16. *Let* $b \in V^\vee$*. Then the solution rank of* τ *at* b *is* $\dim H^0_c(U_b, \mathcal{F}|_{U_b})$*.*

Now we apply the solution rank formulas to different cases.

2.4.3. *Irreducible homogeneous vector bundles.* In the following, we assume X is homogeneous G-variety and the lifted G-action on $\mathbb{P}$ is also transitive. In other words, we have $X = G/P$ and $\mathbb{P} = G/P'$ with $P/P' \cong \mathbb{P}^{r-1}$. Then we have the following corollary

Corollary 2.17. *If* $\beta(\mathfrak{g}) = 0$*, then the solution rank of* τ *at point* $b \in V^\vee$ *is given by* $\dim H^{n-r}_{var}(Y_b)$*.*

Example 2.18. *Let* $X = G(k,l)$ *be Grassmannian and* F *be the tautological bundle of rank* k*. Then* $E \cong F^\vee \otimes \mathcal{O}(\frac{l-1}{k})$ *is an ample vector bundle with* $\det E \cong K_X^{-1}$*. The corresponding* $\mathbb{P}$ *is homogenous under the action of* $SL(l+1)$

2.4.4. *Complete intersections.*

Corollary 2.19. *If* E *is direct sum of line bundles* L_i *on partial flag variety* $X = G/P$*. Consider the group action* $G \times \mathbb{G}_m^{r-1}$*, then the solution rank is*

$$\dim H_{n+r-1}(\mathbb{P} - \tilde{Y}_b - (\cup_i D_i))$$

where D_i *are coordinate hypersurfaces in* $\mathbb{P}$

This is not satisfying because the final cohomology is not directly related to X. Let $Y_1, \cdots, Y_r$ be the zero locus of the L_i component of section s_b. From the geometric realization of some solutions as period integral as rational forms along the cycles in the complement of $Y_1 \cup \cdots \cup Y_r$, we have the following conjecture:

Conjecture 2.20. *There is a natural isomorphism of solution sheave as period integrals*

$$Hom_{\mathcal{D}_{V^\vee}}(\tau_V, \mathcal{O})|_b \cong H_n(X - (Y_1 \cup \cdots \cup Y_r)) \tag{2.15}$$

3. Jacobian rings for homogenous vector bundles

In this section, we examine an explicit description of Jacobian rings for homogenous vector bundles.

3.1. Line bundles. In this subsection, we consider the case that E is an ample line bundle L on partial flag variety $X = G/P$. The Hodge structure of Y_f is determined by ambient space X by Lefschetz hyperplane theorem except the middle dimension. Let $U_f = X - Y_f$ and $H^{n-1}_{van}(Y)$ is the kernel of Gysin morphism $H^{n-1}(Y) \to H^{n+1}(X)$. Hodge structures of Y and U are related by the Gysin sequence

$$0 \to H^n_{prim}(X) \to H^n(U) \to H^{n-1}_{var}(Y)(-1) \to 0 \tag{3.1}$$

Here $H^n_{prim}(X)$ is the corker of the Gysin morphism $H^{n-2}(Y) \to H^n(X)$.

Definition 3.1. *Let R be the graded ring $R = \oplus_{k\geq 0} H^0(X, L^k)$. The generalized Jacobian ideal J is the graded ideal generated by $f, L_Z f$ for $Z \in \mathfrak{g}$. Here the Lie derivative $L_Z f$ is from the natural $\mathfrak{g}$-action on $H^0(X, L)$. Then $M = \oplus_{k\geq 0} H^0(X, K_X \otimes L^{k+1})$ is a graded R-module.*

This definition gives Green's Jacobian ring [11] under suitable vanishing conditions, see Proposition 3.7. The vanishing conditions we will consider is as follows.

$$H^p(X, \Omega^q_X \otimes L^l) = 0 \text{ with } p > 0, q \geq 0, \ l \geq 1 \tag{3.2}$$

$$H^1(X, (G \times_{adP} \mathfrak{p}) \otimes L^k \otimes K_X) = 0 \text{ for } k \geq 1 \tag{3.3}$$

Theorem 3.2. *Let $0 \leq k \leq n-1$ and assume L satisfies conditions (3.3) and (3.2) for (p, q, l) in the following range $\{1 \leq p \leq k, q = n-p, l = k-p+1\} \cup \{1 \leq p \leq k-1, q = n-p-1, l = k-p\} \cup \{1 \leq p \leq k-1, q = n-p, l = k-p\}$, then the map*

$$(M/JM)^k \to F^{n-k} H^n(U)/F^{n-k+1} H^n(U)$$

is an isomorphism. It is compatible with multiplication map $H^0(X, L) \times (M/JM)^k \to (M/JM)^{k+1}$ and the Higgs field from Gauss-Manin connection

The proof follows the argument in Theorem 6.5 [36]. The spaces and maps involved in this description are in terms of representations of

Remark 3.3. *If L and K_X are multiples of the same ample line bundle L', then R and M can be embedded in the coordinate ring $R' = \oplus H^k(X, (L')^k)$. We can define the Jacobian ideal J' to be the ideal generated by $f, L_Z f$ in R'. Then the degree-k summand $(M/JM)^k$ is the corresponding* *summand in R'/J. When $X = \mathbb{P}^n$, we can take L' to be $\mathcal{O}(1)$ bundle. Let $L = \mathcal{O}(d)$. The vanishing conditions are satisfied except $kd = n$ for condition 3.3. When $kd \neq n$, this is the same Jacobian ring description for Hodge structures of hypersurfaces in projective spaces. More specifically, the Jacobian ideal defined here are generated by $x_j \frac{\partial f}{\partial x_i}$ if we view f as polynomial of homogenous coordinates $[x_0, \cdots, x_n]$. The usual Jacobian ideal are generated by $\frac{\partial f}{\partial x_i}$. When $kd \neq n$, the corresponding degree part in the usual Jacobian ring are quotients of elements in the form $\sum_i g_i \frac{\partial f}{\partial x_i}$ with g_i homogenous with degree greater than 0, which are the same degree part of ideal generated by $x_j \frac{\partial f}{\partial x_i}$.*

Remark 3.4. *In order to get similar description as $\mathbb{P}^n$ and coordinate ring $\mathbb{C}[a_1, \cdots a_{n+1}]$, we consider M to be the total space of H-principal bundle over X with G-equivariant action. The character $\chi\colon H \to \mathbb{C}^*$ is associated with the line bundle L. In many cases, the total space M is embedded in affine space $\bar{M}$ as Zariski open set and global sections of structure sheaf of M is extended to $\bar{M}$. Assume that the G-action also extends to $\bar{M}$. For example, when $X = G(a, b)$ is the Grassmannian, we can take M to be the Stiefel manifold and $\bar{M}$ the affine space $\mathbb{A}^{ab}$. The coordinate ring R is identified with $\mathbb{C}[\bar{M}]^{H,\chi}$, which is the functions that are equivariant under the H action by characters $m\chi$. The basis can be given by standard monomials. Then the sections f and $L_Z f$ are identified with elements in $\mathbb{C}[\bar{M}]$ similar as the $\mathbb{P}^n$ case.*

Remark 3.5. *For hypersurfaces in irreducible Hermitian symmetric spaces, the Jacobian ring defined here is already given by Saito in [**32**]. See Lemma 4.1.12 [**32**].*

Remark 3.6. *The Kodaira-Spencer map $H^0(X, L) \to H^1(Y, T_Y)$ has kernel equal to J if $H^1(X, T_X \otimes L^{-1}) = 0$, and is surjective if $H^2(X, T_X \otimes L^{-1}) = 0$. In this case, the multiplication map $H^0(X, L)/J \times (M/JM)^k \to (M/JM)^{k+1}$ gives the Higgs field in universal deformation family of Y. For example this holds for Grassmannians $G(a, b)$ with $b \geq 5$ with any ample line bundle L.*

Now we discuss the relation to Green's Jacobian ring [**11**]. See Saito's identification of two definitions for Hermitian symmetric spaces, Lemma 3.2.3 [**32**]. First we recall Green's definition of Jacobian ring. Let Σ_L be the first prolongation bundle. It is the bundle of first order differential operators on L. The differentiation of f gives a section $df \in H^0(X, \Sigma_L^* \otimes L)$. This induces a map $H^0(X, L^k \otimes K_X \otimes \Sigma_L) \to H^0(X, L^{k+1} \otimes K_X)$. The Jacobian ring R_k is the cokernel of this map.

Proposition 3.7. *If L is ample and satisfies condition 3.3, then $(M/JM)^k \cong R_k$.*

3.2. Vector bundles. When E is vector bundle, we apply Cayley's trick to reduce the calculation to the hypersurface case. The corresponding vanishing conditions are

$$
\begin{aligned}
& H^p(X, \Omega_X^{q-a} \otimes \wedge^a E \otimes S^{l-a} E) = 0 \\
& \qquad \text{for } p > 0, q \geq 0 \text{ and } 0 \leq a \leq l-1 \\
\text{or } & H^p(X, \Omega_X^{q-a} \otimes \wedge^{a+1} E \otimes S^{l-a-1} E) = 0 \\
& \qquad \text{for } p > 0, q \geq 0 \text{ and } 0 \leq a \leq l-1
\end{aligned} \tag{3.4}
$$

and

$$H^1(X, (G \times_{adP} \mathfrak{p}) \otimes S^k(E) \otimes \det E \otimes K_X) = 0 \text{ for } k \geq r. \tag{3.5}$$

DEFINITION 3.8. *Let $S^k(E)$ be the symmetric product of E. Then the coordinate ring of $\mathbb{P}$ is $R = \oplus_{k\geq 0} H^0(X, S^k(E))$ graded by k. Let $M = \oplus_{k\geq r-1} H^0(X, S^{k+1-r}(E) \otimes \det E \otimes K_X)$ be a graded R-module with gradings k. The Jacobian ideal J is the ideal in R generated by f and $L_Z f, Z \in \mathfrak{g}$. Denote $N_k = H^0(X, E^\vee \otimes S^{k+1-r}(E) \otimes \det E \otimes K_X)$. There is a map from N_k to M_k defined by pairing the $E^\vee$ component with f.*

THEOREM 3.9. *If E satisfies the vanishing conditions (3.5), (3.4) for p, q, l in the range $\{1 \leq p \leq k+r-1, q = n+r-1-p, l = k+r-p\} \cup \{1 \leq p \leq k+r-2, q = n+r-p-2, l = k+r-1-p\} \cup \{1 \leq p \leq k+r-2, q = n+r-1-p, l = k+r-1-p\}$ and (3.4) with $p = 1, q-a = n, a = r-2, l-a = k-r+2$, then we have the following description of the Hodge structure of Y*

$$H_{var}^{n-r-k,k}(Y) \cong M^{k+r-1}/N^{k+r-1} f + J M^{k+r-2} \tag{3.6}$$

REMARK 3.10. *When X is Grassmannian and E splits as direct sum of line bundles, similar characterization of Hodge groups are carried out independently by Enrico Fatighenti and Giovanni Mongardi in [7]. They also obtained the Hodge numbers appearing in these examples via this description.*

For the methods checking these vanishing conditions, see [**15**].

4. Hasse-Witt invariants and unit roots

The relations between Picard-Fuchs systems and zeta functions were pioneered in Dwork's study of zeta functions for hypersurfaces. In particular, for the Dwork/Fermat family

$$X_0^{n+1} + \cdots X_n^{n+1} - (n+1)tX_0 \cdots X_n = 0, \tag{4.1}$$

The fundamental period is

$$
\begin{aligned}
I_\gamma = F(\lambda) &= {}_nF_{n-1}\left(\begin{matrix} \frac{1}{n+1}, \frac{2}{n+1}, \cdots, \frac{n}{n+1} \\ 1, 1, \cdots, 1 \end{matrix} ; \lambda \right) \\
&= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{n+1}\right)_r \left(\frac{2}{n+1}\right)_r \cdots \left(\frac{n}{n+1}\right)_r}{(r!)^n} \lambda^r.
\end{aligned} \tag{4.2}
$$

The Hasse-Witt matrix H-W$_p$ for the family mod prime p given by the truncation of $F(\lambda)$.

$$\text{(4.3)} \qquad \text{H-W}_p(\lambda) = {}^{(p-1)}F(\lambda) = \sum_{r=0}^{p-1} \frac{\left(\frac{1}{n+1}\right)_r \left(\frac{2}{n+1}\right)_r \cdots \left(\frac{n}{n+1}\right)_r}{(r!)^n} \lambda^r.$$

According to a theorem of Igusa-Manin-Katz, the Hasse-Witt matrices are solutions to Picard-Fuchs equations mod p. In this example, the mod p solutions in some sense"approximate" the solution in characteristic zero. Furthermore, when the Hasse-Witt matrix is invertible, there is exactly one p-adic unit root in the interesting factor of zeta-function of the variety over finite field. This factor is also determined by the fundamental period [**6, 37**].

We generalize this example to multivariable toric hypersurfaces and also hypersurfaces in G/P. The main results are that for Calabi-Yau hypersurfaces in toric and flag varieties, the fundamental period determines the Hasse-Witt matrices and unit roots of the zeta functions for the family mod p.

More specifically, we proved the following conjecture by Vlasenko [**35**]. Let k be a perfect field of characteristic p. Let $W(k)$ be the ring of Witt vectors of k. Denote $\sigma\colon W \to W$ be the absolute Frobenius automorphism of W. For any W-scheme Z, let $Z_0 = Z \otimes_W k$ be the reduction mod p. Let $S = \mathrm{Spec}(R)$ be an affine W-scheme. Let $R_\infty = \varprojlim R/p^sR$ and $S_\infty = Spf(R_\infty)$. We fix a Frobenius lifting on R and also denote it by σ, which is a ring endomorphism $\sigma\colon R \to R$ such that $\sigma(a) = a^p \mod pR$. Let X be a smooth complete toric variety defined by a fan. The 1-dimensional primitive vectors $v_1, \cdots v_N$ correspond to toric divisors D_i. Assume $L = \mathcal{O}(\sum k_i D_i)$ with $k_i \geq 1$. Let $\Delta = \{v \in \mathbb{R}^n | \langle v, v_i\rangle \geq -k_i\}$ and $\mathring{\Delta}$ the interior of Δ. Then $H^0(X, L)$ has a basis corresponding to $u_I \in \Delta \cap \mathbb{Z}^n$ and $H^0(X, L \otimes K_X)$ has basis $e_i^\vee$ identified with $u_i \in \mathring{\Delta} \cap \mathbb{Z}^n$. Let $f = \sum a_I t^{u_I}, a_I \in R$ be a Laurent series representing a section of $H^0(X, L)$. Let $(\alpha_s)_{i,j}$ be a matrix with ij-th entry equal to the coefficient of $t^{p^s u_j - u_i}$ in $(f(t))^{p^s-1}$. The endmorphism σ is also extended entry-wisely to matrices. It is proved in [**35**] that α_s satisfies the following congruence relations

Theorem 4.1 (Theorem 1 in [**35**]). (1) *For* $s \geq 1$,

$$\alpha_s \equiv \alpha_1 \cdot \sigma(\alpha_1) \cdots \sigma^{s-1}(\alpha_1) \mod p.$$

(2) *Assume* α_1 *is invertible in* R_∞. *Then*

$$\alpha_{s+1} \cdot \sigma(\alpha_s)^{-1} \equiv \alpha_s \cdot \sigma(\alpha_{s-1})^{-1} \mod p^s.$$

(3) *Under the condition of (2), for any derivation* $D\colon R \to R$, *we have*

$$D(\sigma^m(\alpha_{s+1})) \cdot \sigma^m(\alpha_{s+1})^{-1} \equiv D(\sigma^m(\alpha_s)) \cdot \sigma^m(\alpha_s)^{-1} \mod p^{s+m}.$$

Suppose that f defines a smooth hypersurface $\pi\colon Y \to S$. Under suitable conditions guaranteeing the existence of unit root part U_0 of the F-crystal

$H^{n-1}_{cris}(Y_0/S_\infty$, the Frobenius matrix and connection matrix of U_0 are conjectured to be the limits of matrices in Theorem 4.1.

CONJECTURE 4.2 ([35]). *The Frobenius matrix is the p-adic limit*

$$F = \lim_{s\to\infty} \alpha_{s+1}\sigma(\alpha_s)^{-1}. \tag{4.4}$$

The connection matrix is given by

$$\nabla_D = \lim_{s\to\infty} D(\alpha_s)(\alpha_s)^{-1}. \tag{4.5}$$

We prove the above conjecture under mild assumptions on toric varieties in [14]. Especially, the assumptions holds for hypersurfaces of $\mathbb{P}^n$ in general positions. The main idea of the proof uses the local expansion method by Katz [19, 20] adapted to log geometry setting. At the same time, the relations to period integrals are carried out by explicit calculations of fundamental periods via local expansions. Similar results holds for partial flag varieties. The key ingredient is torus charts on G/P via Bott-Samelson desingularization of Schubert varieties.

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