Speculations on homological mirror symmetry for hypersurfaces in $(\mathbb{C}^*)^n$

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Abstract. Given an algebraic hypersurface $H = f^{-1}(0)$ in $(\mathbb{C}^*)^n$, homological mirror symmetry relates the wrapped Fukaya category of H to the derived category of singularities of the mirror Landau-Ginzburg model. We propose an enriched version of this picture which also features the wrapped Fukaya category of the complement $(\mathbb{C}^*)^n \setminus H$ and the Fukaya-Seidel category of the Landau-Ginzburg model $((\mathbb{C}^*)^n, f)$. We illustrate our speculations on simple examples, and sketch a proof of homological mirror symmetry for higher-dimensional pairs of pants.

Contents

1. Introduction

Let $H = f^{-1}(0) \subset (\mathbb{C}^*)^n$ be a smooth algebraic hypersurface (close to a maximal degeneration limit), whose defining equation is a Laurent polyno-

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mial of the form

$$f(x_1,\dots,x_n)=\sum_{\alpha\in A} c_\alpha \tau^{\rho(\alpha)} x_1^{\alpha_1}\dots x_n^{\alpha_n}, \tag{1.1}$$

where A is a finite subset of $\mathbb{Z}^n$, $c_\alpha\in\mathbb{C}^*$, $\tau\in\mathbb{R}_+$ is assumed to be sufficiently small, and $\rho: A\to\mathbb{R}$ is a convex function.

More precisely, we require that ρ is the restriction to A of a convex piecewise linear function $\hat\rho$ defined on the convex hull $\mathrm{Conv}(A)\subset\mathbb{R}^n$. The maximal domains of linearity of $\hat\rho$ define a polyhedral decomposition $\mathcal{P}$ of $\mathrm{Conv}(A)$, whose set of vertices is required to be exactly A. We further assume that all the cells of $\mathcal{P}$ are congruent under the action of $GL(n,\mathbb{Z})$ to standard simplices; this ensures that the limit $\tau\to 0$ corresponds to a maximal degeneration, and that the mirror is smooth.

It was first proposed by Hori and Vafa [15] that H should arise as a mirror to a toric Calabi-Yau manifold Y, or more precisely, a toric Landau-Ginzburg model (Y,W). A careful construction of the mirror following the philosophy of the Strominger-Yau-Zaslow conjecture is given in [6]. The outcome can be described as follows.

Consider the piecewise linear function $\varphi:\mathbb{R}^n\to\mathbb{R}$ obtained by "tropicalizing" f,

$$\varphi(\xi)=\max\{\langle\alpha,\xi\rangle-\rho(\alpha)\,|\,\alpha\in A\}, \tag{1.2}$$

and the (noncompact) polytope $\Delta_Y\subseteq\mathbb{R}^{n+1}$ defined by

$$\Delta_Y=\{(\xi,\eta)\in\mathbb{R}^n\oplus\mathbb{R}\,|\,\eta\geq\varphi(\xi)\}. \tag{1.3}$$

Let Y be the (noncompact) $(n+1)$-dimensional toric variety defined by the moment polytope Δ_Y. Equivalently, Y is described by the fan $\Sigma_Y\subseteq\mathbb{R}^n\oplus\mathbb{R}$ whose rays are generated by the vectors $(-\alpha,1)$, $\alpha\in A$, and in which the vectors $(-\alpha_1,1),\dots,(-\alpha_k,1)$ span a cone if and only if $\alpha_1,\dots,\alpha_k$ span a cell of $\mathcal{P}$. Finally, we define

$$W=-z^{(0,0,\dots,0,1)}\in\mathcal{O}(Y). \tag{1.4}$$

The irreducible toric divisors of Y are indexed by the elements of A; denote by Z_α the divisor which corresponds to the ray $(-\alpha,1)$ of Σ_Y, i.e. to the facet of Δ_Y given by the graph of φ over the region where the maximum in (1.2) is achieved by α. The superpotential W is then (up to sign) the toric monomial which vanishes to order 1 on each toric divisor Z_α. Hence $W^{-1}(0)=\bigcup_{\alpha\in A} Z_\alpha$ (the union of all toric strata).

The direction of homological mirror symmetry that we shall concern ourselves with predicts an equivalence between the (derived) wrapped Fukaya category of H [8, 3] and the derived category of singularities of the Landau-Ginzburg model (Y,W) [19], i.e. the quotient $D^b_{sg}(Z):=D^b\mathrm{Coh}(Z)/\mathrm{Perf}(Z)$ of the derived category of coherent sheaves on the singular fiber $Z:=W^{-1}(0)=\bigcup_{\alpha\in A} Z_\alpha$ by the full triangulated subcategory of perfect complexes:

Conjecture 1.1 (Homological mirror symmetry).

$$\mathcal{W}(H) \simeq D^b_{sg}(Z). \tag{1.5}$$

(The other direction of homological mirror symmetry, relating coherent sheaves on H to the Fukaya category of the Landau-Ginzburg model (Y, W), is established in work in progress of the author with Mohammed Abouzaid [**4**]; the methods used to approach the two directions are completely unrelated.)

It is possible, and even likely, that Conjecture 1.1 should in fact be stated at the level of the idempotent completions of the derived categories on each side of (1.5); for simplicity we ignore this issue here.

The wrapped Fukaya category $\mathcal{W}(H)$ depends only on the set $A \subset \mathbb{Z}^n$, not on the coefficients c_α or the function ρ in (1.1), since the hypersurfaces corresponding to different choices are deformation equivalent Liouville (or Stein) submanifolds of $(\mathbb{C}^*)^n$. Meanwhile, Y depends on the polyhedral decomposition $\mathcal{P}$ of $\mathrm{Conv}(A)$, so different choices of ρ can yield different mirrors; however these mirrors are birational to each other (related by flops), and so the resulting derived categories of singularities are expected to be equivalent.

So far, homological mirror symmetry as stated in Conjecture 1.1 has only been established in the 1-dimensional case, i.e. for $H \subset (\mathbb{C}^*)^2$: the case of the pair of pants (and other punctured spheres) is established in [**5**], and higher genus Riemann surfaces are treated in Heather Lee's thesis [**16**]. In higher dimensions, the first step is to consider (generalized) pairs of pants. With the current technology, the computation of the wrapped Fukaya category requires quite a bit of work; we sketch a possible approach in § 9. (Contrast with Sheridan's computations for compact exact Lagrangians [**24**].) We also note Nadler's recent introduction of "wrapped microlocal sheaves", ultimately expected to be equivalent to the wrapped Fukaya category; the analogue of Conjecture 1.1 for wrapped microlocal sheaves has already been verified for higher-dimensional pairs of pants [**18**].

Since $D^b_{sg}(Z)$ is by definition a quotient of $D^b\mathrm{Coh}(Z)$, it is natural to ask for an interpretation of the latter category on the symplectic side. We propose:

Conjecture 1.2. *$Z = \bigcup_\alpha Z_\alpha \subset Y$ is mirror to the complement $(\mathbb{C}^*)^n \setminus H$, and there is a commutative diagram*

$$\begin{array}{ccc} \mathcal{W}((\mathbb{C}^*)^n \setminus H) & \xrightarrow{\simeq} & D^b\mathrm{Coh}(Z) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{W}(H) & \xrightarrow{\simeq} & D^b_{sg}(Z) \end{array} \tag{1.6}$$

where ρ is a restriction functor (see § 4), q is the projection to the quotient, and the horizontal equivalences are predicted by homological mirror symmetry.

We note that the categories in the top row are $\mathbb{Z}$-graded, whereas those in the bottom row are only $\mathbb{Z}/2$-graded unless some additional data is chosen.

Roughly speaking, the restriction functor ρ singles out the ends of a Lagrangian submanifold of $(\mathbb{C}^*)^n \setminus H$ which lie on the missing divisor H. More precisely, ρ is the composition of restriction to a neighborhood of H isomorphic to the product of H with a punctured disc $\mathbb{D}^*$, and "projection" from $H \times \mathbb{D}^*$ to H; see § 4.

Two comments are in order. First, the top row of (1.6) fits into the general philosophy that removing a divisor from a symplectic manifold (here $(\mathbb{C}^*)^n$) should correspond to a degeneration of its mirror (in our case $(\mathbb{C}^*)^n$) to a singular space (namely Z); the level sets of W provide exactly such a degeneration. Seidel and Sheridan's formalism of relative Fukaya categories [25] exhibits $\mathcal{W}((\mathbb{C}^*)^n)$ as a deformation of a full subcategory of $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ (consisting of Lagrangians with no ends on H, i.e. annihilated by ρ), just as the derived categories of the regular fibers of W arise as deformations of a full subcategory of $D^b\mathrm{Coh}(Z)$ (in fact, $\mathrm{Perf}(Z)$).

Second, $(\mathbb{C}^*)^n \setminus H$ can itself be viewed as a hypersurface in $(\mathbb{C}^*)^{n+1}$, defined by

$$\hat{f}(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_n) + x_{n+1} = 0.$$

(This is in fact one way to define the Liouville structure on the complement of H). The tropicalization of $\hat{f}$ is $\hat{\varphi}(\xi_1, \dots, \xi_{n+1}) = \max(\varphi(\xi_1, \dots, \xi_n), \xi_{n+1})$, and thus the construction in [6] predicts that the mirror to this hypersurface is the toric Landau-Ginzburg model $(\hat{Y}, \hat{W}) = (\mathbb{C} \times Y, yW)$ (where y is the coordinate on the $\mathbb{C}$ factor). On the other hand, Orlov's "Knörrer periodicity" result [20] implies that the derived category of singularities of the Landau-Ginzburg model $(\hat{Y}, \hat{W})$ is equivalent to the derived category of coherent sheaves of $W^{-1}(0) = Z \subset Y$. Thus, the two predictions for the mirrors of $(\mathbb{C}^*)^n \setminus H$, namely the Landau-Ginzburg model $(\hat{Y}, \hat{W})$ and the singular variety Z, are consistent with each other.

We can further enrich the picture by considering the Fukaya-Seidel category of the Landau-Ginzburg model $((\mathbb{C}^*)^n, f)$, using f to view $(\mathbb{C}^*)^n \setminus H$ as the total space of a fibration over $\mathbb{C}^*$. Specifically, assume that $0 \in A$, so that f has a non-trivial constant term. The version of the Fukaya-Seidel category that we consider is essentially that introduced by Abouzaid in [1, 2], at least when 0 is in the interior of $\mathrm{Conv}(A)$; specifically, the objects are admissible Lagrangian submanifolds of $(\mathbb{C}^*)^n$ with boundary on a fiber of f, e.g. $f^{-1}(0) = H$, which are moreover required to lie in the subset of $(\mathbb{C}^*)^n$ where the constant term dominates all the other monomials in f. Due to this latter restriction, our category is often smaller than that defined by Seidel; to avoid confusion, we denote the restricted version by $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$. One notable difference from Abouzaid's setup is that when 0 is not in the interior of $\mathrm{Conv}(A)$ the region where the constant term dominates is non-compact and the category we consider involves some wrapping. See § 5.

There are "acceleration" functors α_0 and α_∞ from $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ to $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$. The functor α_0 takes admissible Lagrangian submanifolds in $((\mathbb{C}^*)^n, f)$ with boundary in $f^{-1}(0) = H$ and views them as Lagrangian submanifolds of $(\mathbb{C}^*)^n \setminus H$. Acceleration then "turns on" wrapping around the central fiber $H = f^{-1}(0)$. By construction, $\rho \circ \alpha_0 : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}(H)$ is expected to coincide with the "restriction to the fiber" functor. Meanwhile, α_∞ takes admissible Lagrangians with boundary in some other fiber $f^{-1}(c_0)$, and extends them by parallel transport along a path from c_0 to infinity in order to obtain properly embedded Lagrangian submanifolds of $(\mathbb{C}^*)^n$ which avoid H altogether. The construction of α_∞ is not canonical, however if the following assumption holds:

(1.7)
$0 \in A$ is a vertex of every maximal cell of the polyhedral decomposition $\mathcal{P}$,

then there is a distinguished choice; see § 6.1. The two types of acceleration functors are manifestly different, as $\rho \circ \alpha_\infty$ is identically zero.

The interpretation of the acceleration functors α_0 and α_∞ under mirror symmetry is as follows. The element $0 \in A$ corresponds to a distinguished irreducible toric divisor Z_0 of Y. When Z_0 is compact (which corresponds to 0 being an interior point of $\mathrm{Conv}(A)$), it follows from Abouzaid's thesis [**2**] that the Fukaya-Seidel category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ considered here is derived equivalent to $D^b\mathrm{Coh}(Z_0)$. In fact Abouzaid's argument can be adapted to show that the equivalence still holds in the non-compact case. There is a natural functor $i_* : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(Z)$ induced by the inclusion $i : Z_0 \hookrightarrow Z$. On the other hand, there is sometimes a preferred projection $\pi : Z \to Z_0$; this is e.g. the case when (1.7) holds, which causes Y to be isomorphic to the total space of a line bundle over Z_0. We then have a pullback functor $\pi^* : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(Z)$, whose image is contained in $\mathrm{Perf}(Z)$ since the maximal degeneration assumption implies that Z_0 is smooth, i.e. $\mathrm{Perf}(Z_0) = D^b\mathrm{Coh}(Z_0)$.

Conjecture 1.3. *Under homological mirror symmetry,*

1. *the acceleration functor* $\alpha_0 : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H)$ *corresponds to the inclusion pushforward* $i_* : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(Z)$*;*
2. *if* (1.7) *holds, then the functor* $\alpha_\infty : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H)$ *corresponds to the pullback* $\pi^* : D^b\mathrm{Coh}(Z_0) \to \mathrm{Perf}(Z) \subset D^b\mathrm{Coh}(Z)$*.*

The functors α_0 and α_∞ have a host of further properties, which can ultimately be interpreted in terms of push-pull adjunctions for $i : Z_0 \to Z$ and $\pi : Z \to Z_0$ on the mirror side. For example, there is a distinguished natural transformation between the functors α_∞ and α_0, whose mapping cone involves a "lifting" functor

$$j : \mathcal{W}(H) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H).$$

The functor j is induced by the parallel transport of Lagrangian submanifolds of H over an arc connecting 0 to infinity in $\mathbb{C}^*$ (avoiding the critical values of f). By construction, j is a right (quasi)inverse to the restriction functor ρ, i.e. $\rho \circ j \simeq \mathrm{id}$; assuming (1.7), the functor j should correspond under mirror symmetry to an explicit splitting of the quotient $q : D^b\mathrm{Coh}(Z) \to D^b_{sg}(Z)$. See § 6.2.

REMARK 1.4. While the defining equation of the hypersurface H can be rescaled by any Laurent monomial, the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ depends very much on the choice of normalization, and so do the functors $\alpha_0, \alpha_\infty, j$ discussed above. Given $\alpha \in A$, considering $x^{-\alpha} f$ instead of f causes the distinguished component of Z to become Z_α instead of Z_0. We then get one instance of Conjecture 1.3 for each component of Z.

REMARK 1.5. As pointed out by Zack Sylvan, there is another way to shed light on the relationship between $\mathcal{F}((\mathbb{C}^*)^n, f)$ and $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$, by viewing $(\mathbb{C}^*)^n \setminus H$ as the outcome of gluing together the Landau-Ginzburg models $((\mathbb{C}^*)^n, f)$ and $(\mathbb{C}^* \times H, z)$ (where z is the coordinate on the first factor) along their common fiber H. This is an instance of gluing together Liouville domains with *stops* and their partially wrapped Fukaya categories [**26**], and one expects a pushout diagram [**27, 14**]

$$\begin{array}{ccc} \mathcal{W}(H) & \longrightarrow & \mathcal{F}((\mathbb{C}^*)^n, f) \\ \big\downarrow & & \big\downarrow{\scriptstyle i_2} \\ \mathcal{F}(\mathbb{C}^* \times H, z) & \xrightarrow{\ i_1\ } & \mathcal{W}((\mathbb{C}^*)^n \setminus H). \end{array}$$

Because the Fukaya category of $(\mathbb{C}^*, z)$ is generated by one object with endomorphism algebra $\mathbb{C}[t]$ (it is mirror to the affine line), the category $\mathcal{F}(\mathbb{C}^* \times H, z)$ is related to $\mathcal{W}(H)$ by "extension of scalars" from $\mathbb{C}$ to $\mathbb{C}[t]$. (This is *not* the left edge of the pushout diagram, which amounts to tensoring with the torsion module $\mathbb{C}[t]/t$ rather than $\mathbb{C}[t]$.) Up to this, i_1 is essentially the functor j discussed above. Meanwhile, when (1.7) holds, there is no difference between $\mathcal{F}((\mathbb{C}^*)^n, f)$ and $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$, and i_2 coincides with α_∞. (Otherwise $\mathcal{F}^\circ$ is strictly smaller). The pushout diagram then implies that $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ is generated by the images of the functors j and α_∞.

To illustrate the various constructions and conjectures, we will primarily consider two families of examples:

EXAMPLE 1.6 (Pairs of pants).

$$f(x_1, \dots, x_n) = x_1 + \dots + x_n + 1,$$

$H =: \Pi_{n-1}$ is the $(n-1)$-dimensional pair of pants, and its complement is isomorphic to the n-dimensional pair of pants Π_n. The mirror is $(Y, W) \simeq (\mathbb{C}^{n+1}, -z_1 \dots z_{n+1})$, and $Z = \{z_1 \dots z_{n+1} = 0\}$ is the union of the $n+1$ coordinate hyperplanes.

EXAMPLE 1.7 (Local $\mathbb{P}^n$).

$$f(x_1,\dots,x_n) = x_1 + \dots + x_n + \frac{\tau}{x_1\dots x_n} + 1,$$

Y is isomorphic to the total space of the anticanonical bundle $\mathcal{O}(-(n+1)) \to \mathbb{P}^n$, and $Z \subset Y$ is the union of the zero section $Z_0 \simeq \mathbb{P}^n$ and the total spaces of $\mathcal{O}(-(n+1))$ over the $n+1$ coordinate hyperplanes of $\mathbb{P}^n$.

We will in particular see in §§ 8–9 that Conjectures 1.2 and 1.3 provide a blueprint for understanding the wrapped Fukaya categories of pairs of pants by induction on dimension, using the fact that $\Pi_n \simeq (\mathbb{C}^*)^n \setminus \Pi_{n-1}$.

The rest of this paper is organized as follows. The first two sections are expository: in Sect. 2 we briefly review the definition of the wrapped Fukaya category, and Sect. 3 illustrates the definition by considering the case of the (1-dimensional) pair of pants treated in [**5**], with an eye towards Conjecture 1.2. The next three sections describe the various categories and functors that appear in Conjectures 1.2 and 1.3: in Sect. 4 we introduce the wrapped category $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ and the restriction functor ρ; in Sect. 5 we discuss the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ and its main properties; and Sect. 6 is devoted to the functors α_0, α_∞ and j. Sections 7 and 8 illustrate these constructions for local $\mathbb{P}^n$ and for higher-dimensional pants; finally, Sect. 9 sketches an approach to the computation of $\mathcal{W}(\Pi_n)$.

2. Background: the wrapped Fukaya category

Let $(X, \omega = d\lambda)$ be a Liouville manifold, i.e. an exact symplectic manifold such that the flow of the Liouville vector field Z defined by $\iota_Z\omega = \lambda$ is complete and outward pointing at infinity. In other terms, X is the completion of a compact domain X^{in} with contact boundary $(\partial X^{in}, \alpha = \lambda_{|\partial X^{in}})$, and the Liouville flow identifies $X \setminus X^{in}$ with the positive symplectization $(1, +\infty) \times \partial X^{in}$ endowed with the exact symplectic form $\omega = d(r\alpha)$ (where r is the coordinate on $(1, +\infty)$). In this model, $Z = r\partial_r$.

The objects of the wrapped Fukaya category $\mathcal{W}(X)$ are properly embedded exact Lagrangian submanifolds which are conical at infinity, i.e. any non-compact ends are modelled on the product of $(1, +\infty)$ with some Legendrian submanifold of $(\partial X^{in}, \alpha)$.

The main feature of the wrapped Fukaya category is that Floer theory is modified by suitable Hamiltonian perturbations so as to include not only Lagrangian intersections inside X^{in}, but also Reeb chords between the Legendrians in ∂X^{in}. There are two main ways to carry out the construction, which we briefly review. (We will mostly use the first one.) We assume general familiarity with Lagrangian Floer homology and ordinary Fukaya categories; see [**11, 22**].

2.1. Construction via quadratic Hamiltonian perturbations.

This setup for wrapped Fukaya categories is described in detail in [**3**]. In this version, the wrapped Floer complex of a pair of objects L_0, L_1 is defined using a specific class of Hamiltonians which grow quadratically at infinity, say $H = \frac{1}{2}r^2$ outside of a compact set. Given two objects L_0, L_1, the generating set $\mathcal{X}(L_0, L_1)$ of $CW(L_0, L_1) = CW(L_0, L_1; H)$ consists of time 1 trajectories of the Hamiltonian vector field X_H which start on L_0 and end on L_1, i.e. points of $\phi_H^1(L_0) \cap L_1$. Since at infinity X_H is r times the Reeb vector field of $(\partial X^{in}, \alpha)$, the generators in the cylindrical end can also be thought of as Reeb chords (of arbitrary positive length) from L_0 to L_1 at the contact boundary. (In practice one may need to perturb H slightly in order to achieve transversality.)

The differential $\partial = \mu^1$ on $CW(L_0, L_1)$ counts solutions to Floer's equation

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0, \tag{2.1}$$

where $u : \mathbb{R} \times [0,1] \to X$ is subject to the boundary conditions $u(s,0) \in L_0$ and $u(s,1) \in L_1$ and a finite energy condition. Given two generators $x_-, x_+ \in \mathcal{X}(L_0, L_1)$, the coefficient of x_- in ∂x_+ is a (signed) count of index 1 solutions of (2.1) (up to reparametrization by translation) which converge to $x_\pm$ as $s \to \pm\infty$.

Floer's equation can be recast as a plain Cauchy-Riemann equation by the following trick: consider $\tilde{u}(s,t) = \phi_H^{1-t}(u(s,t))$, where ϕ_H^{1-t} is the flow of X_H over the interval $[t, 1]$. Then (2.1) becomes

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}(t,\tilde{u})\,\frac{\partial \tilde{u}}{\partial t} = 0,$$

where $\tilde{J}(t) = (\phi_H^{1-t})_*(J(t))$. Hence solutions to Floer's equation correspond to honest $\tilde{J}$-holomorphic strips with boundaries on $\phi_H^1(L_0)$ and L_1.

The Floer product μ^2 and higher compositions

$$\mu^k : CW(L_{k-1}, L_k; H) \otimes \cdots \otimes CW(L_0, L_1; H) \to CW(L_0, L_k; H)[2-k]$$

are constructed similarly, with an important subtlety. The k-fold product μ^k counts rigid solutions to a perturbed Cauchy-Riemann equation of the form

$$\left(du - X_H \otimes \beta\right)_J^{0,1} = 0, \tag{2.2}$$

where u is a map from a domain D biholomorphic to a disc with $k+1$ boundary punctures (viewed as strip-like ends) to X and β is a closed 1-form on D such that $\beta_{|\partial D} = 0$ and β is standard in each strip-like end.

(Rather than the usual punctured discs, a convenient model for the domain D which makes the strip-like ends readily apparent is to take D to be a strip $\mathbb{R} \times [0,k]$ with $k-1$ slits $(s_j, +\infty) \times \{t_j\}$ removed. Away from the boundary of the moduli space one can moreover take $t_j = j$. The conformal parameters are then simply $s_1, \dots, s_{k-1}$ up to simultaneous translation, and we can take $\beta = dt$.)

The issue is that counting solutions of (2.2) with boundary on $L_0, \dots, L_k$ naturally yields a map with values in $CW(L_0, L_k; kH)$, whose generators are time k (rather than time 1) trajectories of X_H from L_0 to L_k. While the usual construction of a continuation map from $CW(L_0, L_k; kH)$ to $CW(L_0, L_k; H)$ fails due to lack of energy estimates, a map can nonetheless be constructed via a rescaling trick [**3**]. Namely, the time $\log k$ flow of the Liouville vector field Z, which is conformally symplectic and rescales the r coordinate by a factor of k, conjugates time k and time 1 trajectories of X_H. Denoting this flow by ψ^k, we have a natural isomorphism

$$CW(L_0, L_k; H, J) \cong CW(\psi^k(L_0), \psi^k(L_k); k^{-1}(\psi^k)^* H, \psi^k_* J), \tag{2.3}$$

and since $k^{-1}(\psi^k)^* H = kH$ at infinity, there is a well-defined continuation map from $CW(L_0, L_k; kH, J)$ to the latter complex. (This is easiest when the Lagrangians under consideration are globally invariant under the Liouville flow, as will be the case for our main examples; in general $\psi^k(L_i)$ differs from L_i by a compactly supported Hamiltonian isotopy, which is annoying but does not pose any technical difficulties.) However, to ensure that the A_∞-relations hold, the continuation homotopy should be incorporated directly into (2.2), making the Hamiltonians, almost-complex structures, and boundary conditions depend on s so that the solutions converge at $s \to -\infty$ to generators of the right-hand side of (2.3) rather than $CW(L_0, L_k; kH, J)$; see [**3**].

Assuming the Lagrangians under consideration are invariant under the Liouville flow, we can use the same trick as above to recast (2.2) as an unperturbed Cauchy-Riemann equation with respect to a different almost-complex structure. For instance, given generators x_1 of $CW(L_0, L_1)$, x_2 of $CW(L_1, L_2)$, and y of $CW(L_0, L_2)$ (viewed as points of $\phi^1_H(L_i) \cap L_j$), the coefficient of y in $\mu^2(x_2, x_1)$ can be viewed as a count of pseudo-holomorphic discs with boundary on $\phi^2_H(L_0)$, $\phi^1_H(L_1)$, and L_2, whose strip-like ends converge to $\phi^1_H(x_1) \in \phi^2_H(L_0) \cap \phi^1_H(L_1)$, $x_2 \in \phi^1_H(L_1) \cap L_2$, and the inverse image $\tilde{y} \in \phi^2_H(L_0) \cap L_2$ of y under the rescaling map ψ^2. See § 2.3 for an example.

2.2. Construction by localization. A different setup for wrapped Floer theory, which is especially useful for comparisons with Fukaya-Seidel categories and for the construction of restriction functors, uses finite wrapping and localization with respect to certain continuation morphisms. The construction we sketch here lies somewhere in between the original one due to Abouzaid-Seidel [**8**] and the works in progress by Abouzaid-Seidel and Abouzaid-Ganatra [**9, 7**].

We now consider a Hamiltonian h which has linear growth. For instance, one could require that $h = r$ at infinity. However, when the contact boundary (or part thereof) comes equipped with an open book structure or more generally a compatible S^1-valued projection (such as that induced by a Lefschetz fibration on its vertical boundary), it is often advantageous to tweak the setup in order to arrange for the time t flow generated by h to wrap "by t turns". In any case, the time t flow ϕ_h^t preserves the class of Lagrangians which are conical at infinity.

For every pair of objects L_0, L_1 under consideration, we assume that the set of times t for which $L_0^t = \phi_h^t(L_0)$ and L_1 fail to intersect transversely is discrete. The Floer complex $CF(L_0^t, L_1)$ can be thought of as a truncation of the previously considered wrapped Floer complex, where the generators in the cylindrical end correspond only to Reeb chords of length at most t from L_0 to L_1 at the contact boundary.

For $\tau > 0$ sufficiently small, $HF(L_0^{t+\tau}, L_0^t)$ contains a distinguished element, called "quasi-unit", generally defined via Floer continuation (in the simplest cases it is the sum of the generators of the Floer complex which correspond to the minima of h on L_0). The wrapped Fukaya category is then defined by localization with respect to the class of quasi-units [**9, 7**]; at the level of cohomology, this means that

$$HW(L_0, L_1) := \varinjlim_{t\to\infty} HF(L_0^t, L_1), \tag{2.4}$$

where the Floer cohomology groups $HF(L_0^t, L_1)$ form a direct system in which the connecting maps are given by multiplication with quasi-units. The chain-level construction of the quotient A_∞-category is rather cumbersome in general, and tends to be explicitly computable only in situations where the continuation maps end up being chain-level isomorphisms or inclusions of complexes for sufficiently large t. (See [**8**] for a more geometric approach to the construction of the direct limit at chain level via continuation maps.)

The comparison between the two versions of wrapped Floer theory is well beyond the scope of this survey. We simply note that, since H grows faster than h at infinity, there are well-defined continuation maps from the Floer complexes with linear Hamiltonians to those with quadratic Hamiltonians; these chain maps are compatible with the quasi-units, and induce maps from the direct limit (2.4) to the wrapped Floer cohomology defined in the previous section. The reverse direction can be constructed by filtering the wrapped Floer complex by action (i.e., length of Reeb chords) and approximating H over arbitrarily large portions of the cylindrical ends with linear-growth Hamiltonians.

2.3. First examples: $\mathbb{C}^*$ and $(\mathbb{C}^*)^n$. As a warm-up, we consider $X = \mathbb{C}^*$, identified with $\mathbb{R} \times S^1$ via $z = \exp(r + i\theta)$, with the symplectic form $\omega = dr \wedge d\theta = d\lambda$ where $\lambda = r\,d\theta$ is the standard Liouville form of the cotangent bundle T^*S^1; the Liouville vector field is $Z = r\partial_r$. We view X as the completion of $X^{in} = [-1, 1] \times S^1$, whose boundary $\partial X^{in} = \{\pm 1\} \times S^1$ carries the contact form $\alpha = \lambda_{|\partial X^{in}} = \pm d\theta$, and identify $X \setminus X^{in}$ with the positive symplectization $(1, +\infty) \times \partial X^{in}$. (We abusively denote by r both the real coordinate on the whole space X and the real positive coordinate on the symplectization of ∂X^{in}, which is in fact $|r|$).

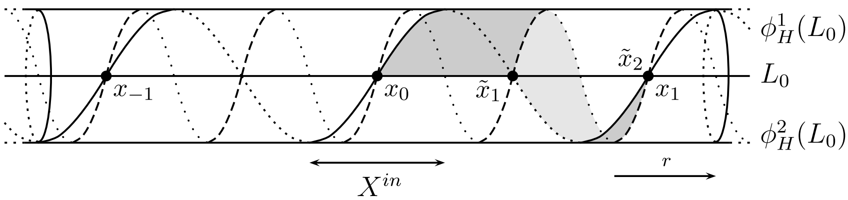

FIGURE 1. The wrapped Floer cohomology of $L_0 = \mathbb{R} \times \{1\}$ in $X = \mathbb{R} \times S^1$

We calculate the wrapped Floer cohomology of $L_0 = \mathbb{R} \times \{1\} \subset \mathbb{R} \times S^1$ (i.e., the real positive axis of $\mathbb{C}^*$). The time 1 flow of the quadratic Hamiltonian $H = \frac{1}{2}r^2$ is given by $\phi_H^1(r, \theta) = (r, \theta + r)$, and the generators of the wrapped Floer complex $CW(L_0, L_0)$, i.e. the points of $\phi_H^1(L_0) \cap L_0$, are evenly spaced along the real axis (at integer values of r); we accordingly label them by integers: $\mathcal{X}(L_0, L_0) = \{x_i, i \in \mathbb{Z}\}$. The generator x_0 which lies at $r = 0$ (the minimum of H) is an interior intersection point, whereas the other generators x_i, $i \neq 0$ correspond to Reeb chords (in one cylindrical end or the other depending on the sign of i).

There is a natural grading on $CW^*(L_0, L_0)$ (using the "obvious" trivialization of TX), for which the generators x_i all have degree zero. This implies immediately that the Floer differential μ^1 and the higher products μ^k for $k \geq 3$ vanish identically. The vanishing of the differential can also be checked on Fig. 1: it is readily apparent that L_0 and $\phi_H^1(L_0)$ do not bound any non-trivial pseudo-holomorphic strips. (Recall that, in complex dimension 1, regardless of the almost-complex structure, rigid pseudo-holomorphic curves correspond to immersed polygons with locally convex boundary.)

Since L_0 is invariant under the Liouville flow (which rescales the r coordinate), we can use the trick described at the end of §2.1 and view the product μ^2 on the wrapped Floer complex as a count of pseudo-holomorphic discs with boundary on $\phi_H^2(L_0)$, $\phi_H^1(L_0)$, and L_0. It is easy to check that, for any $i, j \in \mathbb{Z}$, $\phi_H^1(x_i) \in \phi_H^2(L_0) \cap \phi_H^1(L_0)$ and $x_j \in \phi_H^1(L_0) \cap L_0$ are the vertices of a unique immersed triangle, whose third vertex $\tilde{x}_{i+j} \in \phi_H^2(L_0) \cap L_0$ is mapped to $x_{i+j} \in \phi_H^1(L_0) \cap L_0$ under the Liouville rescaling $r \mapsto 2r$. Hence,

$$\mu^2(x_j, x_i) = x_{i+j}. \tag{2.5}$$

For instance, the triangle shown on Fig. 1 contributes to $\mu^2(x_0, x_1) = x_1$.

Renaming the generator x_j to x^j, we conclude that

$$HW^*(L_0, L_0) \simeq \mathbb{C}[x, x^{-1}] \tag{2.6}$$

as algebras (or in fact as A_∞-algebras with $\mu^k = 0$ for $k \neq 2$).

Instead of the quadratic Hamiltonian H, one could instead use the approach of § 2.2 with linear Hamiltonians. The resulting picture looks like a truncation of Fig. 1: the angular coordinate θ increases from $-t$ at one end of $L_0^t = \phi_h^t(L_0)$ to $+t$ at the other end, so $CF(L_0^t, L_0)$ only accounts for the generators x_i with $|i| < t$. Taking the limit as $t \to \infty$, one recovers (2.6).

Either way, we find that $HW^*(L_0, L_0)$ is isomorphic to $\mathrm{Ext}^*(\mathcal{O}, \mathcal{O}) \simeq \mathbb{C}[x, x^{-1}]$ for the structure sheaf on the mirror $X^\vee = \mathbb{C}^* = \mathrm{Spec}\,\mathbb{C}[x, x^{-1}]$.

By a result of Abouzaid, L_0 generates the wrapped Fukaya category $\mathcal{W}(X)$: this means that every object is quasi-isomorphic to an iterated mapping cone built from (finitely many) copies of L_0. (A weaker notion, that of "split-generation", adds formal direct summands in such iterated mapping cone; it is not needed here). Similarly, the structure sheaf $\mathcal{O}$ generates $\mathrm{Coh}(X^\vee)$. By standard homological algebra, this implies that there is a derived equivalence between the wrapped Fukaya category of $X = \mathbb{R} \times S^1$ and the category of coherent sheaves on $X^\vee = \mathbb{C}^*$.

In fact, $\mathcal{W}(X)$ and $D^b\mathrm{Coh}(X^\vee)$ are both equivalent to the category of perfect complexes of modules over the algebra $\mathcal{A} = \mathbb{C}[x, x^{-1}]$, via Yoneda embedding: on the symplectic side, the A_∞-module associated to an object $\Theta \in \mathcal{W}(X)$ is the wrapped Floer complex $CW^*(L_0, \Theta)$ viewed as an A_∞-module over $CW^*(L_0, L_0)$ (where the structure maps of the module are induced by those of the wrapped Fukaya category), while on the mirror side, the module structure just comes from multiplication by regular functions.

The argument extends in a straightforward manner to the case of $X = (\mathbb{C}^*)^n \simeq T^*T^n$, whose wrapped Fukaya category is generated by $L_0 = (\mathbb{R}_+)^n$. Indeed, the standard quadratic Hamiltonian ($H = \frac{1}{2}\sum r_i^2$, where $r_i = \log|z_i|$) preserves the product structure, and holomorphic triangles with boundary on $\phi_H^2(L_0)$, $\phi_H^1(L_0)$ and L_0 can be studied by projecting to each coordinate. One finds that

$$HW^*(L_0, L_0) = CW^*(L_0, L_0) \simeq \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \tag{2.7}$$

which agrees with the ring of functions of the mirror $X^\vee = (\mathbb{C}^*)^n$. Viewing $\mathcal{W}((\mathbb{C}^*)^n)$ and $D^b\mathrm{Coh}((\mathbb{C}^*)^n)$ in terms of perfect complexes of modules over $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, homological mirror symmetry follows.

3. Example: the pair of pants

In this section, we consider the pair of pants $X = \mathbb{P}^1 \setminus \{0, -1, \infty\} = \mathbb{C}^* \setminus \{-1\}$. Homological mirror symmetry for this example has been studied in [5]; we review the results of that paper from a slightly different perspective.

The details of the Liouville structure on X are not particularly important, except as a warmup for the general setup considered in the following sections. Viewing X as the hypersurface in $(\mathbb{C}^*)^2$ defined by the equation $x_1 + x_2 + 1 = 0$, we use the Liouville structure induced by that of $(\mathbb{C}^*)^2$. Namely, writing $x_j = \exp(r_j + i\theta_j)$, we set $\omega = \frac{1}{2}dd^c(r_1^2 + r_2^2)$, and the Liouville form is $\lambda = \frac{1}{2}d^c(r_1^2 + r_2^2) = r_1\,d\theta_1 + r_2\,d\theta_2$. (In terms of the coordinate z on $\mathbb{C}^* \setminus \{-1\}$, $r_1 = \log|z|$ and $r_2 = \log|z+1|$.)

One could instead view X as the complement of the hypersurface defined by $f(z) = z + 1 = 0$ in $\mathbb{C}^*$. A natural choice of Kähler potential is then $\frac{1}{2}(\log|z|)^2 + \frac{1}{2}(\log|f|)^2$, which gives exactly the same formula. However,

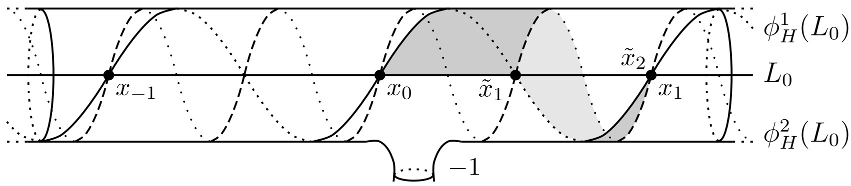

FIGURE 2. The wrapped Floer cohomology of $L_0 = \mathbb{R}_+$ in $X = \mathbb{C}^* \setminus \{-1\}$

we will often prefer to modify this prescription, using cut-off functions so that the Kähler potential is in fact equal to $\frac{1}{2}(\log|f|)^2$ near -1 and to $\frac{1}{2}(\log|z|)^2$ outside of a neighborhood of -1. This offers the advantage that the Liouville structure is the same as that of $\mathbb{C}^*$ outside of a neighborhood of the deleted hypersurface, and "standard" near the hypersurface. Likewise, the Hamiltonian used to define the wrapped Fukaya category of X can be chosen to coincide with that used for $\mathbb{C}^*$ away from a neighborhood of -1.

In the same vein, when viewing X as the complement of a hypersurface in $\mathbb{C}^*$ the most natural choice of gradings in Floer theory uses the trivialization of the tangent bundle induced by that of $\mathbb{C}^*$. This means that the puncture at -1 is graded differently from those at 0 and ∞, across which the trivialization does not extend.

With this understood, we consider again $L_0 = \mathbb{R}_+ \subset X$. Since L_0 (and its image under the flow generated by H) stay away from the puncture at $z = -1$, the calculation of the wrapped Floer complex closely parallels the case of the cylinder (cf. § 2.3). Namely, the generators of $CW^*(L_0, L_0)$ are still evenly spaced along the real positive axis, $\mathcal{X}(L_0, L_0) = \{x_i,\ i \in \mathbb{Z}\}$, with $\deg(x_i) = 0$, the operations μ^k ($k \neq 2$) vanish for degree reasons, and the calculation of the product structure μ^2 proceeds as before; see Fig. 2. The only difference with the case of the cylinder is that only immersed triangles that do not pass through the puncture at -1 contribute to μ^2. Observing that the triangles of Fig. 1 that pass through the central region containing -1 are exactly those whose inputs lie in opposite ends of the cylinder, we find that

$$\mu^2(x_j, x_i) = \begin{cases} x_{i+j} & \text{if } ij \geq 0, \\ 0 & \text{if } ij < 0. \end{cases} \tag{3.1}$$

Thus, renaming x_{-j} to z_1^j and x_j to z_2^j for $j > 0$, we have:

$$HW^*(L_0, L_0) \simeq \mathbb{C}[z_1, z_2]/(z_1 z_2 = 0). \tag{3.2}$$

Denoting this algebra by $\mathcal{A}$, this suggests that a mirror to X might be

$$X^\vee = \operatorname{Spec} \mathcal{A} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}. \tag{3.3}$$

This is indeed the case, but not for any obvious reason as far as we know. Indeed, L_0 does not split-generate the wrapped Fukaya category, and it is not readily evident that the Yoneda functor from $\mathcal{W}(X)$ into the derived category of A_∞-modules over $\mathcal{A}$, given by $\Theta \mapsto CW^*(L_0, \Theta)$ on objects, is fully faithful and its image (which does not solely consist of perfect complexes) agrees with the derived category of coherent sheaves on $X^\vee$.

Observe by the way that, if we view L_0 as an object of the *relative* wrapped Fukaya category of $(\mathbb{C}^*, \{-1\})$ in the sense of Seidel, i.e. if we count holomorphic curves in $\mathbb{C}^*$ which pass k times through -1 with a coefficient of t^k, then the wrapped Floer cohomology of L_0 becomes $HW^*(L_0, L_0) \simeq \mathbb{C}[z_1, z_2, t]/(z_1 z_2 = t)$. This exhibits the mirror to the complement $X = \mathbb{C}^* \setminus \{-1\}$, given by (3.3), as the central fiber of a degeneration of the mirror to $\mathbb{C}^*$. This is a general feature, as noted in the discussion after Conjecture 1.2.

Returning to our study of the wrapped Fukaya category, one can show that $\mathcal{W}(X)$ is (split) generated by the three components of the real locus of X, namely $L_0 = \mathbb{R}_+$, $L_1 = (-\infty, -1)$, and $L_2 = (-1, 0)$. (In fact, any two of these suffice.) Calculating their wrapped Floer complexes and the product structures is a simple exercise similar to the above case of L_0; the outcome is as follows (see [5] for details). First, we have:

$$HW^*(L_0, L_0) \simeq \mathbb{C}[z_1, z_2]/(z_1 z_2 = 0),$$
$$HW^*(L_1, L_1) \simeq \mathbb{C}[z_2, z_0]/(z_2 z_0 = 0),$$
$$HW^*(L_2, L_2) \simeq \mathbb{C}[z_1, z_0]/(z_1 z_0 = 0),$$

as (formal) graded (A_∞-)algebras. Here we denote by z_1 (resp. z_2, z_0) the generators corresponding to Reeb chords that wrap once around 0 (resp. $\infty, -1$). With our choice of trivialization of TX, $\deg z_1 = \deg z_2 = 0$, whereas $\deg z_0 = 2$. For $i \neq j$, $HW^*(L_i, L_j)$ is an (A_∞) bimodule over $HW^*(L_i, L_i)$ and $HW^*(L_j, L_j)$. As such it is generated by a single generator u_{ij} corresponding to a Reeb chord that wraps halfway around the common cylindrical end, and we have

$$HW^*(L_0, L_1) \simeq \mathbb{C}[z_2]\, u_{01}, \qquad HW^*(L_1, L_0) \simeq \mathbb{C}[z_2]\, u_{10},$$
$$HW^*(L_1, L_2) \simeq \mathbb{C}[z_0]\, u_{12}, \qquad HW^*(L_2, L_1) \simeq \mathbb{C}[z_0]\, u_{21},$$
$$HW^*(L_2, L_0) \simeq \mathbb{C}[z_1]\, u_{20}, \qquad HW^*(L_0, L_2) \simeq \mathbb{C}[z_1]\, u_{02},$$

with the bimodule structure implied by the notations (any variable not present in the notation acts by zero), and vanishing higher module maps. (Here u_{12} and u_{21} have degree 1 and the other generators have degree 0.)

Moreover, for $\{i, j, k\} = \{0, 1, 2\}$ we have $\mu^2(u_{ji}, u_{ij}) = z_k$ and $\mu^2(u_{jk}, u_{ij}) = 0$, whereas $\mu^3(u_{ki}, u_{jk}, u_{ij}) = -\mathrm{id}_{L_i}$. In particular there are two exact triangles

$$L_2 \xrightarrow{u_{20}} L_0 \xrightarrow{u_{01}} L_1 \xrightarrow{u_{12}} L_2[1] \quad \text{and} \quad L_1 \xrightarrow{u_{10}} L_0 \xrightarrow{u_{02}} L_2 \xrightarrow{u_{21}} L_1[1]. \tag{3.4}$$

It is shown in [5] that this completely determines the A_∞-structure up to homotopy.

The corresponding calculation on $X^\vee$ is as follows: we consider the structure sheaf $\mathcal{O}$, and the structure sheaves $\mathcal{O}_A$ and $\mathcal{O}_B$ of the two irreducible components of $X^\vee$, $A : \{z_1 = 0\}$ and $B : \{z_2 = 0\}$. In the language of modules over $\mathcal{A} = \mathbb{C}[z_1, z_2]/(z_1z_2)$, these correspond to $\mathcal{A}$, $\mathcal{A}/(z_1)$, and $\mathcal{A}/(z_2)$. To calculate Ext groups between these objects, we use the (infinite, 2-periodic) projective resolution

$$\cdots \xrightarrow{z_1} \mathcal{O} \xrightarrow{z_2} \mathcal{O} \xrightarrow{z_1} \mathcal{O} \longrightarrow \mathcal{O}_A \to 0$$

and similarly for $\mathcal{O}_B$ (exchanging z_1 and z_2). For example, applying $\mathrm{Hom}(-, \mathcal{O}_A)$ to this resolution we find that $\mathrm{Ext}^*(\mathcal{O}_A, \mathcal{O}_A)$ is given by the cohomology of

$$0 \to \mathcal{A}/(z_1) \xrightarrow{0} \mathcal{A}/(z_1) \xrightarrow{z_2} \mathcal{A}/(z_1) \xrightarrow{0} \dots$$

This gives $\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathbb{C}[z_2]$, and $\mathrm{Ext}^{2k}(\mathcal{O}_A, \mathcal{O}_A) = \mathbb{C}$ for all $k \geq 1$. Denoting by z_0 the generator of $\mathrm{Ext}^2(\mathcal{O}_A, \mathcal{O}_A)$, further calculations show that, as an algebra, $\mathrm{Ext}^*(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathbb{C}[z_2, z_0]/(z_2z_0 = 0)$. Similarly for the other Ext groups and module structures; the outcomes match exactly the calculations on the symplectic side. Moreover, there are two short exact sequences

$$0 \to \mathcal{O}_A \to \mathcal{O} \to \mathcal{O}_B \to 0$$

(where the first map is the homomorphism from $\mathcal{A}/(z_1)$ to $\mathcal{A}$ given by multiplication by z_2, and the second map is the projection from $\mathcal{A}$ to $\mathcal{A}/(z_2)$) and

$$0 \to \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_A \to 0,$$

which give rise to two exact triangles in the derived category. This in turn suffices to conclude that $\mathcal{W}(X) \simeq D^b\mathrm{Coh}(X^\vee)$.

Given that the pair of pants has a 3-fold symmetry that is not apparent on the mirror $X^\vee$, it is natural to ask for a more symmetric version of mirror symmetry. The answer comes in the form of the Landau-Ginzburg model $(Y = \mathbb{C}^3, W = -z_0z_1z_2)$: namely, $\mathcal{W}(X)$ is also equivalent to the triangulated category of singularities of (Y, W) [**19**], i.e. the quotient of the derived category of coherent sheaves of the zero fiber $Z = W^{-1}(0) = \{z_0z_1z_2 = 0\}$ by the subcategory of perfect complexes, $D^b_{sg}(Z) = D^b\mathrm{Coh}(Z)/\mathrm{Perf}(Z)$. (Since Z is affine, $\mathrm{Perf}(Z)$ is generated by $\mathcal{O}_Z$.)

The category $D^b_{sg}(Z)$ is generated by $\mathcal{O}_{Z_0}, \mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}$, where Z_i are the irreducible components of Z, i.e. the hyperplanes $\{z_i = 0\}$. The short exact sequence of sheaves

$$0 \to \mathcal{O}_{Z_0} \to \mathcal{O}_Z \to \mathcal{O}_{Z_1 \cup Z_2} \to 0$$

induces an isomorphism $\mathcal{O}_{Z_1 \cup Z_2} \simeq \mathcal{O}_{Z_0}[1]$ in $D^b_{sg}(Z)$, and hence we have two exact triangles in $D^b_{sg}(Z)$,

(3.5)

$$\mathcal{O}_{Z_1} \to \mathcal{O}_{Z_0}[1] \to \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1}[1] \quad \text{and} \quad \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_0}[1] \to \mathcal{O}_{Z_1} \to \mathcal{O}_{Z_2}[1].$$

Note that the natural grading on $D^b_{sg}(Z)$ is only by $\mathbb{Z}/2$. ($\mathbb{Z}$-gradings manifestly exist in this instance, but break the symmetry between the coordinates.) The most efficient method of computation of morphisms in $D^b_{sg}(Z)$

is via 2-periodic resolutions, as

$$\mathrm{Hom}^i_{D^b_{sg}(Z)}(\mathcal{E}_1,\mathcal{E}_2) \cong \mathrm{Ext}^{2k+i}_{D^b\mathrm{Coh}(Z)}(\mathcal{E}_1,\mathcal{E}_2) \qquad \text{for } k \gg 0. \tag{3.6}$$

(see Proposition 1.21 of [19]).

The calculations for the generators $\mathcal{O}_{Z_i}$ are carried out in [5], and yield the same answers as the corresponding calculations in $D^b\mathrm{Coh}(X^\vee)$ and $\mathcal{W}(X)$. Hence, we have equivalences $\mathcal{W}(X) \simeq D^b\mathrm{Coh}(X^\vee) \simeq D^b_{sg}(Z)$, under which the generators L_0, L_1, L_2 of $\mathcal{W}(X)$ correspond to $\mathcal{O}, \mathcal{O}_A, \mathcal{O}_B$ in $D^b\mathrm{Coh}(X^\vee)$, and to $\mathcal{O}_{Z_0}[1], \mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}$ in $D^b_{sg}(Z)$.

The equivalence between $D^b\mathrm{Coh}(X^\vee)$ and $D^b_{sg}(Z)$ is a special case of Orlov's Knörrer periodicity result [19, 20].

From our perspective, the Landau-Ginzburg model $(\mathbb{C}^3, -z_0z_1z_2)$ is the mirror to the pair of pants viewed as a hypersurface $x_1 + x_2 + 1 = 0$ in $(\mathbb{C}^*)^2$, and the equivalence $\mathcal{W}(X) \simeq D^b_{sg}(Z)$ is an instance of Conjecture 1.1. Meanwhile, $X^\vee = \{z_1z_2 = 0\} \subset \mathbb{C}^2$ is the zero fiber of the Landau-Ginzburg model $(\mathbb{C}^2, -z_1z_2)$, which our construction associates to $\{-1\}$ viewed as a hypersurface in $\mathbb{C}^*$. Thus, viewing the pair of pants as the complement $\mathbb{C}^* \setminus \{-1\}$, we are now in the setting of Conjecture 1.2, and the equivalence $\mathcal{W}(X) \simeq D^b\mathrm{Coh}(X^\vee)$ is the top row of the diagram (1.6). The bottom row is the equivalence $\mathcal{W}(\{-1\}) \simeq D^b_{sg}(X^\vee)$, which maps the generator (the Lagrangian consisting of the point itself) to the generator of $D^b_{sg}(X^\vee)$ ($\simeq D^b\mathrm{Vect}$). It is then easy to check that the diagram (1.6) commutes. Indeed, the restriction functor ρ maps L_0 (which avoids the puncture at -1) to the zero object, while L_1 and L_2 (which each have one end at -1) map to the generator of $\mathcal{W}(\{-1\})$ and its shift by one. This is in agreement with the images of $\mathcal{O}$, $\mathcal{O}_A$ and $\mathcal{O}_B$ under the quotient functor $D^b\mathrm{Coh}(X^\vee) \to D^b_{sg}(X^\vee)$.

4. The complement of H and the restriction functor ρ

Let $H = f^{-1}(0) \subset (\mathbb{C}^*)^n$ be a smooth algebraic hypersurface as in the introduction. The standard Liouville structure on H is induced by that of $(\mathbb{C}^*)^n$ as follows. Expressing the coordinates on $(\mathbb{C}^*)^n$ in the form $x_j = \exp(r_j + i\theta_j)$, the standard Kähler form of $(\mathbb{C}^*)^n$ is given by the Kähler potential $\Phi = \frac{1}{2}\sum r_j^2$, i.e. $\omega = dd^c\Phi$, and the standard Liouville form is $\lambda = d^c\Phi = \sum r_j\, d\theta_j$. The Kähler potential, symplectic form and Liouville form of H are then simply the restrictions of Φ, ω and λ to H, given by the same formulas in coordinates. (However, one may also choose a different Liouville structure in the same deformation class as convenient for calculations.)

The natural choice of Liouville structure on the complement $(\mathbb{C}^*)^n \setminus H$ is given by the Kähler potential $\hat{\Phi} = \Phi + \frac{1}{2}(\log|f|)^2$, i.e. the Liouville form is $\hat{\lambda} = d^c\hat{\Phi} = \sum r_j\, d\theta_j + \log|f|\, d\arg(f)$. Observing that $(\mathbb{C}^*)^n \setminus H$ is isomorphic to the hypersurface in $(\mathbb{C}^*)^{n+1}$ defined by $f(x_1,\dots,x_n) + x_{n+1} = 0$, the

Kähler potential $\hat{\Phi}$ and Liouville form $\hat{\lambda}$ are exactly those induced by the standard choices on $(\mathbb{C}^*)^{n+1}$.

In order to construct the restriction functor $\rho : \mathcal{W}((\mathbb{C}^*)^n \setminus H) \to \mathcal{W}(H)$, it is advantageous to deform the Liouville structure (which does not modify the wrapped Fukaya category) in order to make it apparent that $(\mathbb{C}^*)^n \setminus H$ contains a Liouville subdomain equivalent to the product of H with a punctured disc $\mathbb{D}^*$. Namely, for $K \gg 0$ sufficiently large, the potential $\hat{\Phi}_K = \Phi + \frac{1}{2}(\log|f| + K)^2$ defines the same Kähler form on $(\mathbb{C}^*)^n \setminus H$ (since $dd^c \log|f| = 0$), but the corresponding Liouville form is $\hat{\lambda}_K = \hat{\lambda} + K\, d\arg(f)$ and the Liouville vector fields differ by $K\,\nabla \log|f|$. Thus, for K sufficiently large, the Liouville vector field of $\hat{\lambda}_K$ is transverse and outward pointing along arbitrarily large compact subsets of the hypersurface $|f| = \epsilon$ (for fixed ϵ with $e^{-K} \ll \epsilon \ll 1$). We note that this modification amounts to rescaling f to $e^K f$.

With this understood, intersecting the subset of $(\mathbb{C}^*)^n \setminus H$ where $|f| < \epsilon$ with a large compact subset of $(\mathbb{C}^*)^n$ defines a Liouville subdomain which is a topologically trivial fibration over the punctured disc, and Liouville deformation equivalent to the product of $H^{in} \subset H$ with a punctured disc. The completion of this subdomain is (up to Liouville deformation equivalence) $H \times \mathbb{C}^*$. In this setting, the work of Abouzaid and Seidel [8] yields a restriction functor

$$r : \mathcal{W}((\mathbb{C}^*)^n \setminus H) \to \mathcal{W}(H \times \mathbb{C}^*). \tag{4.1}$$

The diagram of Conjecture 1.2 relies on the use of a particular $\mathbb{Z}$-grading on $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$, defined by a choice of trivialization of the determinant line bundle of the tangent bundle of $(\mathbb{C}^*)^n \setminus H$. We use the trivialization obtained by restricting to the complement of H the standard trivialization for $(\mathbb{C}^*)^n$. Hence, the $\mathbb{Z}$-grading that we consider on $\mathcal{W}(H \times \mathbb{C}^*)$ is not the "usual" one, but rather comes from a trivialization that extends to $H \times \mathbb{C}$, which shifts by $2k$ the degree of Reeb chords that wrap k times around the origin in $\mathbb{C}^*$ (whereas in the H factor we have the trivialization induced by that of $(\mathbb{C}^*)^n$ via interior product with df).

The second step in the construction of ρ is to define a $\mathbb{Z}/2$-graded "projection" functor $p : \mathcal{W}(H \times \mathbb{C}^*) \to \mathcal{W}(H)$ as an adjoint to the inclusion $i : \mathcal{W}(H) \to \mathcal{W}(H \times \mathbb{C}^*)$ which maps ℓ to $i(\ell) = \ell \times \mathbb{R}_+$. Given any two objects $\ell_1, \ell_2 \in \mathcal{W}(H)$, we have

$$CW^*(\ell_1 \times \mathbb{R}_+, \ell_2 \times \mathbb{R}_+) \cong CW^*(\ell_1, \ell_2) \otimes_{\mathbb{C}} \mathbb{C}[z^{\pm 1}], \tag{4.2}$$

where $\deg(z) = 2$, with all A_∞-operations extended linearly. This allows us to define i on morphisms by $i(x) = x \otimes 1$; the higher terms vanish. We first define a version of p which takes values in a module category,

$$\hat{p} : \mathcal{W}(H \times \mathbb{C}^*) \to \text{mod-}\mathcal{W}(H).$$

Given an object L of $\mathcal{W}(H \times \mathbb{C}^*)$, the $\mathcal{W}(H)$-module $\hat{p}(L)$ associates to $\ell \in \mathcal{W}(H)$ the chain complex $\hat{p}(L)(\ell) := CW^*(\ell \times \mathbb{R}_+, L)$. The structure maps of the A_∞-module $\hat{p}(L)$ come from the A_∞-operations in $\mathcal{W}(H\times\mathbb{C}^*)$ (via the inclusion i). The definition of $\hat{p}$ on morphisms is tautological and parallels the construction of the Yoneda embedding.

Given $\ell \in \mathcal{W}(H)$, the cohomological unit $e_\ell \in CW^0(\ell,\ell)$ gives rise to a degree 2 automorphism $e_\ell \otimes z \in CW^2(\ell\times\mathbb{R}_+,\ell\times\mathbb{R}_+)$. In particular, multiplication by $e_\ell\otimes z$ induces quasi-isomorphisms

$$CW^*(\ell\times\mathbb{R}_+,L) \xrightarrow{\simeq} CW^{*+2}(\ell\times\mathbb{R}_+,L) \tag{4.3}$$

for all ℓ and L, so that the modules $\hat{p}(L)$ are 2-periodic.

Identifying the graded pieces of $\hat{p}(L)$ of given parity via the quasi-isomorphisms (4.3), we arrive at a $\mathbb{Z}/2$-graded module that we denote by $\bar{p}(L)$. In fact, $\hat{p}$ induces a functor $\bar{p}$ from $\mathcal{W}(H\times\mathbb{C}^*)$ to a category of $\mathbb{Z}/2$-graded modules over $\mathcal{W}(H)$. Constructing $\bar{p}$ carefully involves a significant amount of work; conceptually, the key point is that the isomorphisms $e_\ell\otimes z$ are part of a natural transformation from the identity functor of $\mathcal{W}(H\times\mathbb{C}^*)$ to the shift functor [2] induced by rotation of the $\mathbb{C}^*$ factor.

Next, we observe that the wrapped Fukaya category of $H\times\mathbb{C}^*$ is split-generated by products $\ell\times\mathbb{R}_+$. Moreover, the isomorphism (4.2) implies that, as a $\mathbb{Z}/2$-graded module, $\bar{p}(\ell\times\mathbb{R}_+)$ is isomorphic to the Yoneda module of ℓ. It follows that $\bar{p}$ is representable, i.e. there is a functor

$$p : \mathcal{W}(H\times\mathbb{C}^*) \to \mathcal{W}(H) \tag{4.4}$$

into the ($\mathbb{Z}/2$-graded, split-closed derived) wrapped Fukaya category of H such that $\bar{p}$ is the composition of p with Yoneda embedding. Finally, we set

$$\rho = p\circ r : \mathcal{W}((\mathbb{C}^*)^n\setminus H) \to \mathcal{W}(H). \tag{4.5}$$

Remark 4.1. The family of closed orbits of the Reeb vector field which wrap once around H in unit time determines a class $\theta \in SH^2((\mathbb{C}^*)^n\setminus H)$ which, via the closed-open map, induces a natural transformation $\Theta : \mathrm{id}\to [2]$ acting on $\mathcal{W}((\mathbb{C}^*)^n\setminus H)$. The restriction of Θ to the subdomain $H\times\mathbb{C}^*$ is exactly the degree 2 natural transformation used to construct $\bar{p}$ from $\hat{p}$. With this understood, ρ can be characterized in terms of localization along the natural transformation Θ: for $L_1, L_2 \in \mathcal{W}((\mathbb{C}^*)^n\setminus H)$,

$$HW^i(\rho(L_1),\rho(L_2)) \cong \varinjlim_{k\to\infty} HW^{2k+i}(L_1,L_2), \tag{4.6}$$

where the direct limit on the right-hand side is with respect to multiplication by $[\Theta_{L_1}] \in HW^2(L_1,L_1)$ (or equivalently, $[\Theta_{L_2}]\in HW^2(L_2,L_2)$). This can be viewed both as a "global" version of (4.2) and as a mirror counterpart to (3.6).

Remark 4.2. Many of the Lagrangian submanifolds of $(\mathbb{C}^*)^n\setminus H$ that we will consider below are "framed", i.e. have the property that $\arg(f)$ is equal to zero (or some other fixed constant value) near H. Near H such a Lagrangian is obtained by parallel transport of a Lagrangian submanifold of H in the fibers of f over a radial arc. Thus, the restriction to the subdomain $H\times\mathbb{C}^*$ is a product $\ell\times\mathbb{R}_+$, and the image under ρ is simply ℓ.

5. The Fukaya category of the Landau-Ginzburg model $((\mathbb{C}^*)^n, f)$

From now on, we assume that the Laurent polynomial f has a non-trivial constant term; rescaling f is necessary we will assume that the constant term is equal to 1. We briefly review Abouzaid's version of the Fukaya-Seidel category of $((\mathbb{C}^*)^n, f)$ [**1, 2**], modified to suit our purposes. (Various other constructions are also worth mentioning: see [**23, 26, 8**]. For our purposes each of these brings with it some desirable features and some unwanted complications.)

Fix a regular value c_0 of f, and a simply connected domain $\Omega \subset \mathbb{C}$ such that c_0 lies on the boundary of Ω. (Typically we require Ω to contain all the critical values of f.) The objects of $\mathcal{F}((\mathbb{C}^*)^n, f)$ are properly embedded *admissible* exact Lagrangian submanifolds of $(\mathbb{C}^*)^n$ with boundary in the fiber $f^{-1}(c_0)$. A Lagrangian submanifold L is said to be admissible if $f(L) \subset \Omega$ and, in a neighborhood of ∂L, $f_{|L}$ takes values in a smoothly embedded arc γ (e.g. a straight half-line) whose tangent vector at c_0 points into the interior of Ω. Note that, near its boundary, an admissible Lagrangian L is obtained by parallel transport of $\partial L \subset f^{-1}(c_0)$ in the fibers of f over the arc γ.

If the objects of interest include Lagrangian submanifolds which are non-compact in the fiber direction, we further assume that $f^{-1}(c_0)$ is preserved by the Liouville flow, and that the wrapping Hamiltonian H is well-behaved on admissible Lagrangians. (If necessary this can be ensured by a modification of the Liouville structure to a local product model near $f^{-1}(c_0)$.)

We say that a pair of admissible Lagrangians (L_0, L_1) projecting to arcs γ_0, γ_1 near their boundary is in positive position, and write $L_0 < L_1$, if the tangent vector to γ_1 at c_0 (pointing into the interior of Ω) points "to the left" (counterclockwise) from that of γ_0. It is always possible to perturb L_0 or L_1 by a Hamiltonian isotopy supported near $f^{-1}(c_0)$ in order to ensure that the pair lies in positive position.

If (L_0, L_1) are in positive position, then the morphism space $\hom(L_0, L_1)$ in the Fukaya-Seidel category $\mathcal{F}((\mathbb{C}^*)^n, f)$ is the portion of the (wrapped) Floer complex spanned by generators that lie away from the boundary fiber $f^{-1}(c_0)$. Similarly, given a collection of admissible Lagrangians $L_0, \dots, L_k$ such that $L_0 < L_1 < \dots < L_k$, the A_∞-products are defined as usual by counts of (perturbed) holomorphic discs (only involving generators outside of $f^{-1}(c_0)$).

Starting with this partial definition for objects in positive position, there are two ways to define morphism spaces and A_∞-operations for arbitrary objects. One option is to define $\mathcal{F}((\mathbb{C}^*)^n, f)$ by localization with respect to a suitably defined class of morphisms from any admissible Lagrangian to its admissible pushoffs in the positive direction. The other option is to strengthen the admissibility condition to fix the tangent direction to the arc γ at c_0 (so in fact all objects are required to approach $f^{-1}(c_0)$ from the same direction), and perturb Floer's equation by an auxiliary Hamiltonian

h that rotates a neighborhood of $f^{-1}(c_0)$ in the negative direction so that $\phi_h^1(L_0) < L_1$ for every pair of objects.

The Fukaya category $\mathcal{F}((\mathbb{C}^*)^n, f)$ is invariant under deformations of the domain Ω and does not depend on the choice of the reference point $c_0 \in \partial\Omega$, as long as no critical value or other "special fiber" of f crosses into Ω or out of it during the deformation. This justifies omitting these choices from the notation. (However, we note that the equivalence induced by an isotopic deformation of (Ω, c_0) to some other choice (Ω', c_0') does depend on the choice of isotopy.)

Next, recall that in our case f is a Laurent polynomial of the form (1.1), near the tropical limit. The image of H (or more generally $f^{-1}(c_0)$ for fixed $c_0 \neq 1$ independent of τ) under the logarithm map

$$\mathrm{Log} : (x_1, \dots, x_n) \mapsto \frac{1}{|\log \tau|}(\log|x_1|, \dots, \log|x_n|)$$

converges as $\tau \to 0$ to the tropical hypersurface $\Gamma \subset \mathbb{R}^n$ defined by the tropicalization φ, i.e. the set of points where the maximum in (1.2) is not unique. The components of $\mathbb{R}^n \setminus \Gamma$ correspond to the regions where the different terms in (1.2) achieve the maximum. For $\alpha \in A$, we denote by Δ_α the component of $\mathbb{R}^n \setminus \Gamma$ on which α achieves the maximum in (1.2), and focus our attention on the component Δ_0 corresponding to the constant term. Note that $\mathcal{U}_0 = \mathrm{Log}^{-1}(\Delta_0) \subset (\mathbb{C}^*)^n$ is the set of points where the constant term dominates all the other monomials that appear in f. Enlarging Δ_0 slightly, let $\Delta_0^+ \subset \mathbb{R}^n$ be the δ-neighborhood of Δ_0 for fixed $\delta \ll 1$. For τ small enough, the portion of the amoeba $\mathrm{Log}(H)$ (resp. $\mathrm{Log}(f^{-1}(c_0))$) that converges to $\partial\Delta_0 \subset \Gamma$ is contained inside Δ_0^+, and it makes sense to consider admissible Lagrangians which are entirely contained inside $\mathcal{U}_0^+ = \mathrm{Log}^{-1}(\Delta_0^+) \subset (\mathbb{C}^*)^n$.

Definition 5.1. *We denote by $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ the full subcategory of $\mathcal{F}((\mathbb{C}^*)^n, f)$ whose objects are admissible Lagrangians supported inside $\mathcal{U}_0^+$.*

In fact, Abouzaid only considers Lagrangians which are sections of the logarithm map over the appropriate component of $\mathbb{R}^n \setminus \mathrm{Log}(f^{-1}(c_0))$; these are expected to split-generate $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$.

Recalling that Lefschetz thimbles of critical points of f are an important source of objects of the Fukaya-Seidel category, restricting to $\mathcal{F}^\circ \subset \mathcal{F}((\mathbb{C}^*)^n, f)$ basically amounts to discarding all the critical points of f where the constant terms is not one of the dominant monomials in (1.1). In most cases these correspond to the critical values which tend to infinity as $\tau \to 0$, so it is quite often the case that $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ can be constructed directly by choosing Ω to be a suitable bounded domain – for example, the unit disc centered at 1 in the complex plane, or a slight enlargement thereof, is a natural choice.

The introduction of the restricted Fukaya-Seidel category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ is motivated by homological mirror symmetry. Returning to the setup in the introduction, since the components Δ_α of $\mathbb{R}^n \setminus \Gamma$ arise as facets of the moment polytope Δ_Y defined by (1.3), they can be identified with the moment polytopes for the irreducible toric divisors Z_α of the toric variety Y. In particular, Δ_0 is the moment polytope for the distinguished divisor Z_0 considered in the introduction.

Assume that Z_0 is compact, i.e. the component Δ_0 of $\mathbb{R}^n \setminus \Gamma$ is bounded, which happens precisely when 0 is an interior point of $\mathrm{Conv}(A)$. In this case, ignoring slight differences in setup, Abouzaid's thesis [**2**] essentially shows that

$$\mathcal{F}^\circ((\mathbb{C}^*)^n, f) \simeq D^b\mathrm{Coh}(Z_0). \tag{5.1}$$

In fact, Abouzaid's strategy of proof can be extended to the case where Δ_0 is unbounded, and one expects that (5.1) continues to hold in full generality. (This is by no means non-trivial, but it should follow in a fairly straightforward way from the construction of the wrapped category by localization.)

In general the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ is a strict subcategory of $\mathcal{F}((\mathbb{C}^*)^n, f)$. For example, let

$$f(x_1, x_2) = 1 + x_1^{-1} + x_2^{-1} + \tau x_2 + \tau^{k+1} x_1 x_2^k \quad \text{for } k \geq 3,$$

in which case Z_0 is the non-Fano Hirzebruch surface $\mathbb{F}_k = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-k))$. By [**12**, §5], in this case f has $k-2$ critical points outside of $\mathcal{U}_0^+$, and $\mathcal{F}((\mathbb{C}^*)^n, f)$ is strictly larger than $D^b\mathrm{Coh}(\mathbb{F}_k)$, whereas Abouzaid's result holds for $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$.

When assumption (1.7) holds, we expect the two categories to coincide:

Lemma 5.2. *Assume that $0 \in A$ is a vertex of every maximal cell of the polyhedral decomposition $\mathcal{P}$. Then, for τ sufficiently small, all the critical points of f lie in $\mathcal{U}_0$, and the critical values of f converge to* 1.

Proof. Let x be a critical point of f. Denote by $B \subseteq A$ the set of leading order terms of f near x, i.e. those α which come close to achieving the maximum in (1.2) at $\xi = \mathrm{Log}(x)$. The elements of B are the vertices of some cell of the polyhedral decomposition $\mathcal{P}$, and if τ is sufficiently small the other terms in f are much smaller than those indexed by $\alpha \in B$. Since the cells of $\mathcal{P}$ are simplices (by the maximal degeneration assumption), the assumption that every maximal cell contains 0 as a vertex implies that the non-zero elements of B are linearly independent. Using logarithmic derivatives, the critical points are the solutions of

$$\sum_{\alpha \in A} \left(c_\alpha \tau^{\rho(\alpha)} x^\alpha \right) \alpha = 0.$$

Assume that $B \neq \{0\}$. Then the leading order terms in this equation at x correspond to $\alpha \in B \setminus \{0\}$. However these terms are linearly independent in $\mathbb{R}^n$ and hence cannot cancel out. This leads to a contradiction if τ is sufficiently small. Thus $B = \{0\}$, i.e. x lies in the region where the constant term dominates. □

An alternative argument based on tropical geometry is that, when (1.7) holds, the tropicalization of $f-c$ is "tropically smooth" and combinatorially similar to that of f whenever c is sufficiently different from 1; this implies that for small enough τ the fibration f is locally trivial outside of a small disc centered at 1.

We finish this discussion by recalling two important functors relating $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ to other Fukaya categories. The first one is *restriction to the reference fiber*,

$$\cap : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}(f^{-1}(c_0)).$$

Given an admissible Lagrangian L in $(\mathbb{C}^*)^n$, we define $\cap(L) = \partial L \subset f^{-1}(c_0)$. On morphisms, $\cap$ is defined by counting (perturbed) holomorphic discs with boundary on given admissible Lagrangians (isotoped to lie in positive position), with inputs mapped to given generators in the interior (away from $f^{-1}(c_0)$) and output a generator which lies on the boundary (in $f^{-1}(c_0)$).

A folklore statement (which has so far only been verified in specific cases but should in this setting be well within reach) is as follows (see also [**10**, §5] for related considerations).

Conjecture 5.3. *Let D_0 be the union of all the irreducible toric divisors of Z_0, and denote by $i_{D_0} : D_0 \hookrightarrow Z_0$ the inclusion. Assume that* (1.7) *holds. Then $f^{-1}(c_0)$ is mirror to D_0, and the functor $\cap : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}(f^{-1}(c_0))$ corresponds under mirror symmetry to the restriction functor $i^*_{D_0} : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(D_0)$.*

Remark 5.4. When (1.7) holds, Y is isomorphic to the total space of the canonical bundle of Z_0, and Orlov's results [**20, 21**] give an equivalence $D^b_{sg}(Z) \simeq D^b\mathrm{Coh}(D_0)$. Thus, the statement that $f^{-1}(c_0)$ is mirror to D_0 is consistent with Conjecture 1.1.

The *acceleration* functor $\alpha : \mathcal{F}((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n)$, meanwhile, amounts to completing admissible Lagrangians to properly embedded Lagrangians in $(\mathbb{C}^*)^n$ via parallel transport over an arc η that connects c_0 to infinity in the complement of Ω.

Constructing α in our setting is less straightforward than in some other approaches to the Fukaya-Seidel category [**23, 9, 26**]. One option is to set up the completion of admissible Lagrangians in such a way that the generators which lie inside $f^{-1}(\Omega)$ form a subcomplex of the wrapped Floer complex. Namely, choosing the arc η suitably and/or modifying the Liouville structure, we can assume that $f^{-1}(\eta)$ is preserved by the Liouville flow. Given an admissible Lagrangian $L \subset f^{-1}(\Omega)$ with boundary in $f^{-1}(c_0)$, we get a properly embedded Lagrangian $\hat{L}$ by attaching to L the cylinder obtained by parallel transport of $\partial L \subset f^{-1}(c_0)$ over the arc η (and rounding the corners at c_0 if the projections do not match smoothly). Furthermore, we perturb the wrapping Hamiltonian by a term that pushes $f^{-1}(c_0)$ slightly in the positive direction along the boundary of $f^{-1}(\Omega)$. This has the effect of getting rid of the intersections in $f^{-1}(c_0)$, whose existence would

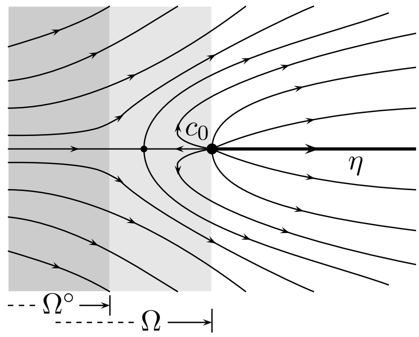

FIGURE 3. Acceleration as restriction to a Liouville subdomain

prevent the interior intersections from forming a subcomplex. With this understood, given admissible Lagrangians $L_0 < \cdots < L_k$ and their completions $\hat{L}_0, \ldots, \hat{L}_k$, the wrapped Floer complexes $CW^*(\hat{L}_i, \hat{L}_j)$ contain two types of generators: those which lie in $f^{-1}(\Omega)$, and those which lie over the arc η. Setting up the Liouville structure carefully over $f^{-1}(\eta)$, one can ensure (using e.g. a maximum principle and the local structure near Reeb chords) that the output of a perturbed J-holomorphic disc with inputs in $f^{-1}(\Omega)$ also lies in $f^{-1}(\Omega)$. Thus, $CW^*(\hat{L}_i, \hat{L}_j)$ contains a subcomplex quasi-isomorphic to $\hom_{\mathcal{F}((\mathbb{C}^*)^n, f)}(L_i, L_j)$, and the inclusion of these subcomplexes is part of an A_∞-functor.

An alternative and perhaps more elegant construction of the acceleration functor is to set things up so that $f^{-1}(\Omega)$ contains a Liouville subdomain whose completion is Liouville deformation equivalent to the total space (here, $(\mathbb{C}^*)^n$). For example, one can arrange for the Liouville structure in a neighborhood of $f^{-1}(\eta)$ to be a product one, where in the base of the fibration f the Liouville flow is as depicted in Fig. 3. Then $f^{-1}(\Omega^\circ)$ is a Liouville subdomain (since the Liouville flow is everywhere transverse to its boundary); the total space of the fibration contains additional cancelling pairs of handles which are not present in the completion of $f^{-1}(\Omega^\circ)$, but the two are nonetheless deformation equivalent as Liouville manifolds. Requiring admissible Lagrangians to approach $f^{-1}(c_0)$ along the horizontal axis of Fig. 3, their restrictions to $f^{-1}(\Omega^\circ)$ are properly embedded and define objects of the wrapped Fukaya category. In this context, α is simply Abouzaid and Seidel's restriction functor [8] to $\mathcal{W}(f^{-1}(\Omega^\circ)) \simeq \mathcal{W}((\mathbb{C}^*)^n)$.

6. Acceleration, restriction, and lifting

6.1. The acceleration functors α_0 and α_∞. In this section we define two acceleration functors α_0 and α_∞ from $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ to $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$.

First we observe that the critical values of f relevant to the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ converge to 1 as $\tau \to 0$, by the proof of Lemma 5.2. (Indeed, going over the proof, since $\mathcal{F}^\circ$ only considers the region where the constant term of f is among the largest monomials, it is a given that $0 \in B$ and the assumption of the lemma is not necessary in order to conclude that $B = \{0\}$.) It follows that the domain Ω to which admissible Lagrangians are required to project can be chosen to be a neighborhood of 1; our preferred choice is the unit disc centered at 1, with $c_0 = 0 \in \partial\Omega$, or a slightly smaller disc, with c_0 on the positive real axis near the origin.

The simplest way to construct the acceleration functor α_0 is to view admissible Lagrangian submanifolds of $(\mathbb{C}^*)^n$ with boundary in $f^{-1}(0) = H$ as properly embedded Lagrangian submanifolds of $(\mathbb{C}^*)^n \setminus H$. This determines α_0 on objects. On morphisms, α_0 is defined by Floer-theoretic continuation maps from the Hamiltonian perturbation used to construct $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ to the quadratic Hamiltonian for $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$; these are well-defined because, even after correcting the former to account for the change in symplectic structure, the latter Hamiltonian has a faster growth rate near H.

More precisely, say that we construct $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ by modifying the Liouville structure of $(\mathbb{C}^*)^n$ to a product one near the hypersurface H, considering only admissible Lagrangians whose projection under f approaches the origin from a fixed direction (e.g. the real positive axis), and using an auxiliary Hamiltonian perturbation that rotates a neighborhood of the origin in the clockwise direction in order to ensure positive position. As seen in § 4, removing H from $(\mathbb{C}^*)^n$ entails a change in the Liouville structure. With respect to the new symplectic structure, the "positive position" perturbation is achieved by a Hamiltonian which grows linearly, with a small positive slope, with respect to the radial coordinate of the cylindrical end near H. The Hamiltonians used to define morphisms and compositions in $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ have a faster growth rate along H, and hence there are well-defined continuation maps.

Another way to construct α_0, which fits into the general framework of acceleration functors discussed at the end of the previous section, is to set up $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ using a domain Ω which stays away from the origin, say a disc of radius $1-\epsilon$ centered at 1, and a reference point located near (but not at) the origin, say $c_0 = \epsilon \in \partial\Omega$. Since the origin lies outside of Ω, we can just as well remove the fiber over zero and work in $(\mathbb{C}^*)^n \setminus H$ with the Liouville structure constructed in § 4 (suitably modified near $f^{-1}(\epsilon)$ for the needs of the construction of $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$). Viewing the restriction of f to $(\mathbb{C}^*)^n \setminus H$ as a fibration over $\mathbb{C}^*$ (instead of $\mathbb{C}$), we choose an arc η_0 connecting $c_0 = \epsilon$ to the origin (instead of infinity); the canonical choice is the interval $(0, \epsilon]$ in the real axis. We then construct α_0 as in § 5, either by extending admissible Lagrangians with boundary in $f^{-1}(\epsilon)$ by parallel transport in the fibers of f over the interval $(0, \epsilon]$, or by restriction to a Liouville subdomain (disjoint from $f^{-1}((0, \epsilon]))$ whose completion is deformation equivalent to $(\mathbb{C}^*)^n \setminus H$.

To define the other acceleration functor α_∞, we construct $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ by setting Ω to be the disc of radius $1-\epsilon$ centered at 1, and observe again that, since the origin lies outside of Ω, we can just as well work with the restriction $f : (\mathbb{C}^*)^n \setminus H \to \mathbb{C}^*$. Choose an arc η_∞ that connects $c_0 \in \partial\Omega$ to

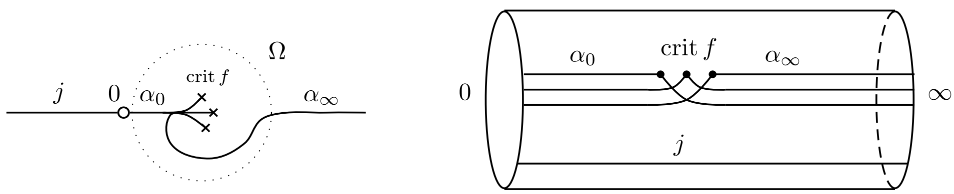

FIGURE 4. The fibration $f : (\mathbb{C}^*)^n \setminus H \to \mathbb{C}^*$ and the functors $\alpha_0, \alpha_\infty, j$

infinity in the complement of Ω (avoiding the origin and any critical values of f that may lie outside of Ω). When there are no critical values outside of Ω (e.g. when (1.7) holds), the most natural choice is to take $c_0 = 2 - \epsilon$ and η_∞ the interval $[2 - \epsilon, \infty)$ in the real axis. In the general case, when f has additional critical values near infinity, the functor α_∞ genuinely depends on the choice of the arc η_∞, as we shall see on an explicit example in § 7.

In any case, the construction described at the end of § 5 then provides an acceleration functor $\alpha_\infty : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H)$. By construction, $\rho \circ \alpha_\infty = 0$, since the objects in the image of α_∞ remain away from a neighborhood of H.

For the purpose of comparing α_0 and α_∞ as we will do in § 6.2 below, it is useful to have both functors defined on the same model of the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$, i.e. choose $c_0 = \epsilon$ rather than $c_0 = 2 - \epsilon$. Whenever necessary, we identify the categories corresponding to the choices $c_0 = \epsilon$ and $c_0 = 2 - \epsilon$ via the isotopy that moves the reference point along the lower half of the boundary of the disc Ω.

The functors α_0 and α_∞ are very similar to each other at first glance, as should be apparent by viewing $(\mathbb{C}^*)^n \setminus H$ as the total space of a fibration over $\mathbb{C}^*$ (the restriction of f). One can then consider admissible Lagrangians whose projections under f go towards either end of the cylinder, and the corresponding acceleration functors; see Fig. 4. Thus, the constructions of α_0 and α_∞ extend in a straightforward manner to more general symplectic fibrations over the cylinder. However, in our case an important feature that breaks the symmetry between the two ends of Fig. 4 is that the monodromy around 0 is trivial, which is a crucial feature needed to define the restriction functor ρ, whereas the monodromy around ∞ is not.

As mentioned in the introduction, it should follow from the construction of α_0 that its composition with the restriction functor $\rho : \mathcal{W}((\mathbb{C}^*)^n \setminus H) \to \mathcal{W}(H)$ introduced in § 4 coincides with the "restriction to the fiber" functor described in § 5:

$$\rho \circ \alpha_0 = \cap : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}(H). \tag{6.1}$$

While the statement is clear at the level of objects, the proof requires some work due to the differences between the two constructions. A possible approach is to consider Floer theory for admissible Lagrangians perturbed by suitably chosen Hamiltonians on $(\mathbb{C}^*)^n \setminus H$ with linear growth near H, whose flow extends over H and wraps around it by a finite number of turns t. In this setting, one can count rigid solutions to Floer's equation with inputs away from H and outputs in H as in the definition of $\cap$. For $0 < t < 1$ this gives $\cap$, while every time t passes through an integer there is a bifurcation and the map changes by composition with a degree 2 element of the (truncated) wrapped Floer cohomology; observing that this element is yet another instance of the natural transformation Θ discussed in Remark 4.1, the statement should then follow by taking the limit as $t \to \infty$.

The interpretation of (6.1) under homological mirror symmetry is as follows. By Conjectures 1.2 and 1.3, we expect a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}^\circ((\mathbb{C}^*)^n, f) & \xrightarrow{\alpha_0} & \mathcal{W}((\mathbb{C}^*)^n \setminus H) & \xrightarrow{\rho} & \mathcal{W}(H) \\ \simeq\downarrow & & \simeq\downarrow & & \downarrow\simeq \\ D^b\mathrm{Coh}(Z_0) & \xrightarrow{i_*} & D^b\mathrm{Coh}(Z) & \xrightarrow{q} & D^b_{sg}(Z) \end{array} \tag{6.2}$$

where the vertical equivalences are instances of homological mirror symmetry, and in the bottom row i_* is the pushforward by the inclusion map $i : Z_0 \hookrightarrow Z$ and q is the quotient by $\mathrm{Perf}(Z)$. When (1.7) holds, Y is the total space of the canonical bundle of Z_0, and Orlov's work [**20, 21**] gives an equivalence $\varepsilon : D^b_{sg}(Z) \xrightarrow{\simeq} D^b\mathrm{Coh}(D_0)$ where D_0 is the union of the irreducible toric divisors of Z_0. It is not hard to check that the composition $\varepsilon \circ q \circ i_* : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(D_0)$ coincides with pullback by the inclusion $i_{D_0} : D_0 \hookrightarrow Z_0$. Using (6.1), the diagram (6.2) then reduces to

$$\begin{array}{ccc} \mathcal{F}^\circ((\mathbb{C}^*)^n, f) & \xrightarrow{\rho\alpha_0=\cap} & \mathcal{W}(H) \\ \simeq\downarrow & & \downarrow\simeq \\ D^b\mathrm{Coh}(Z_0) & \xrightarrow{\varepsilon q i_* = i^*_{D_0}} & D^b\mathrm{Coh}(D_0) \end{array}$$

which is precisely the content of Conjecture 5.3.

On the other hand, the identity $\rho \circ \alpha_\infty = 0$ expresses the property that the counterpart of $\alpha_\infty : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H)$ under mirror symmetry is a functor from $D^b\mathrm{Coh}(Z_0)$ to $D^b\mathrm{Coh}(Z)$ whose image is annihilated by the quotient $q : D^b\mathrm{Coh}(Z) \to D^b_{sg}(Z)$, i.e. it is contained in the subcategory $\mathrm{Perf}(Z)$.

The simplest instance of such a functor from $D^b\mathrm{Coh}(Z_0)$ to $\mathrm{Perf}(Z)$ is the pullback by some projection map $\pi : Z \to Z_0$, when one exists. For example, when (1.7) holds, Y is the total space of the canonical bundle over Z_0 and we can take π to be the restriction to Z of the projection $Y \to Z_0$. Conjecture 1.3 postulates that in this case α_∞ as constructed above does indeed correspond to the pullback π^*.

6.2. The lifting functor j. We now construct a functor $j : \mathcal{W}(H) \to \mathcal{W}((\mathbb{C}^*)^n \setminus H)$ as follows. Choose a properly embedded arc γ in $\mathbb{C}^*$ that connects 0 to infinity and avoids the critical values of f. Deforming the Liouville structure if necessary, we arrange for the Liouville flow to be tangent to $f^{-1}(\gamma)$ and pointing away from some interior fiber $f^{-1}(c_0)$, $c_0 \in \gamma$, which is also preserved by the wrapping Hamiltonian.

With this understood, given an object L of $\mathcal{W}(H)$, we use parallel transport in the fibers of f over the arc γ to obtain a properly embedded Lagrangian submanifold $j(L)$ in $(\mathbb{C}^*)^n \setminus H$. Moreover, we can ensure that, for any pair of objects $L_1, L_2 \in \mathcal{W}(H)$, the generators of the wrapped Floer complex $CW^*(j(L_1), j(L_2))$ which lie in the fiber $f^{-1}(c_0)$ form a subcomplex isomorphic to $CW^*(L_1, L_2)$. We define j on morphisms via these inclusions of wrapped Floer complexes.

The functor j is not canonical, as it depends on the choice of the arc γ; the set of choices is essentially the same as for α_∞. When (1.7) holds there is a preferred choice, namely we can take γ to be the negative real axis $(-\infty, 0)$; see Fig. 4. In the general case, we choose the arc γ to be homotopic to the concatenation of the arcs η_0 and η_∞ used to define α_0 and α_∞ via a homotopy that does not cross any critical value of f.

By construction (and for a suitable choice of grading conventions), $\rho j \simeq \mathrm{id}$, i.e. j is a right (quasi)inverse to the restriction functor. Moreover, we expect to have an exact triangle

$$j\rho\alpha_0[-1] \to \alpha_\infty \to \alpha_0 \to j\rho\alpha_0. \tag{6.3}$$

Indeed, given any admissible Lagrangian $L \in \mathcal{F}^\circ((\mathbb{C}^*)^n, f)$, with boundary $\partial L = \rho\alpha_0(L) \in \mathcal{W}(H)$, near 0 (resp. ∞) the ends of $\alpha_0(L)$ (resp. $\alpha_\infty(L)$) and $j(\partial L)$ are modelled on the products of ∂L with the positive and negative real axes. With our grading conventions, the family of Reeb chords that wrap halfway around the cylindrical end gives rise to an element $\nu_L^0 \in CW^0(\alpha_0(L), j(\partial L))$, resp. $\nu_L^\infty \in CW^1(j(\partial L), \alpha_\infty(L))$. Meanwhile, the continuation map for the isotopy ψ of $f^{-1}(\Omega)$ induced by moving the reference fiber for $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ from ϵ to $2-\epsilon$ counterclockwise along the lower half of $\partial\Omega$ determines an element of $CF^0(\psi(L), L)$, which upon acceleration yields an element $\mu_L \in CW^0(\alpha_\infty(L), \alpha_0(L))$. It is then not hard to check that

$$j(\partial L)[-1] \xrightarrow{\nu_L^\infty} \alpha_\infty(L) \xrightarrow{\mu_L} \alpha_0(L) \xrightarrow{\nu_L^0} j(\partial L) \tag{6.4}$$

is an exact triangle in $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$. This can be viewed as an instance of the surgery exact triangle, observing that $\alpha_\infty(L)$ is Hamiltonian isotopic to the nontrivial component of the Lagrangian obtained by wrapping $\alpha_0(L)$ halfway around the puncture at zero so that it intersects $j(\partial L)$ cleanly along a copy of ∂L, and performing Lagrangian surgery along these intersections (see Fig. 4).

We expect that the morphisms ν_L^∞, μ_L and ν_L^0 are part of natural transformations between the functors $j\rho\alpha_0$, α_∞ and α_0, and the exact triangles (6.4) assemble into the exact triangle (6.3).

On the mirror side, let $Z_{\neq 0} = \bigcup_{\alpha\neq 0} Z_\alpha$, and observe that we have a short exact sequence of sheaves

$$0 \to \mathcal{O}_{Z_{\neq 0}}(-Z_0) \to \mathcal{O}_Z \to \mathcal{O}_{Z_0} \to 0, \tag{6.5}$$

coming from the decomposition $Z = Z_{\neq 0} \cup Z_0$. We note that $Z_{\neq 0} \cap Z_0 = D_0$ is the union of the irreducible toric divisors of Z_0.

Assume that (1.7) holds. Then, recalling that Y is the total space of the canonical line bundle over Z_0 and observing that $Z_{\neq 0}$ is the restriction of this line bundle to $D_0 \subset Z_0$, (6.5) gives rise to an exact triangle of functors

$$\kappa \to \pi^* \to i_* \to \kappa[1]$$

where $\kappa : D^b\mathrm{Coh}(Z_0) \to D^b\mathrm{Coh}(Z)$ is the composition of restriction to the anticanonical divisor $D_0 = Z_{\neq 0} \cap Z_0$, pullback under the projection $\pi_{|Z_{\neq 0}} : Z_{\neq 0} \to D_0$, twisting by $\mathcal{O}(-Z_0)$, and pushforward by the inclusion of $Z_{\neq 0}$ into Z. In this setting, $D^b_{sg}(Z) \simeq D^b\mathrm{Coh}(D_0)$, and restriction from Z_0 to D_0 corresponds to $\rho\alpha_0$. The functor from $D^b\mathrm{Coh}(D_0)$ to $D^b\mathrm{Coh}(Z)$ consisting of the remaining steps (pullback to $Z_{\neq 0}$, twisting by $\mathcal{O}(-Z_0)$, and pushforward to Z) is thus our conjectural counterpart to j under mirror symmetry, up to a grading shift. (This construction is very closely related to Orlov's proof of the equivalence between $D^b\mathrm{Coh}(D_0)$ and $D^b_{sg}(Z)$ [**20, 21**].)

6.3. Framings and gradings. As mentioned in Remark 1.4, the construction of the functors $\alpha_0, \alpha_\infty, j$ can be carried out using a different "framing" of H, i.e. considering the defining equation $x^{-\alpha}f$ instead of f, for any $\alpha \in A$. On the mirror side this amounts to considering the component Z_α of Z instead of Z_0; this suggests that, taken together, the Fukaya categories $\mathcal{F}^\circ((\mathbb{C}^*)^n, x^{-\alpha}f)$ for varying choices of framings give a substantial amount of insight into $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$.

Changing the framing of H does not affect the $\mathbb{Z}$-grading on $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$, since the chosen trivialization of the tangent bundle by restriction from $(\mathbb{C}^*)^n$ did not involve the Laurent polynomial f. On the other hand, it does modify the preferred choice of grading on $\mathcal{W}(H)$, as the preferred trivialization of $\det(TH)$ is induced from that of $\det(T(\mathbb{C}^*)^n)$ by interior product with df.

The analogue of this under mirror symmetry is the observation that, even though $D^b\mathrm{Coh}(Z)$ has a canonical $\mathbb{Z}$-grading, its quotient $D^b_{sg}(Z)$ does not. However, the choice of a $\mathbb{C}^*$-action on Y (for which the superpotential W has weight 2) determines a $\mathbb{Z}$-grading on $D^b_{sg}(Z)$ [**21**]. For each choice of $\alpha \in A$, the divisor Z_α (or equivalently the corresponding ray in the fan of Y) determines a $\mathbb{C}^*$-action, and hence a $\mathbb{Z}$-grading on $D^b_{sg}(Z)$. It is part of our general conjectural setup that these gradings on $D^b_{sg}(Z)$ match up with those on $\mathcal{W}(H)$.

Even though the restriction functor ρ is only $\mathbb{Z}/2$-graded, it admits a $\mathbb{Z}$-graded enhancement if one only considers framed Lagrangians in the sense of Remark 4.2. For instance, the composition $\rho\alpha_0 : \mathcal{F}^\circ((\mathbb{C}^*)^n, f) \to \mathcal{W}(H)$,

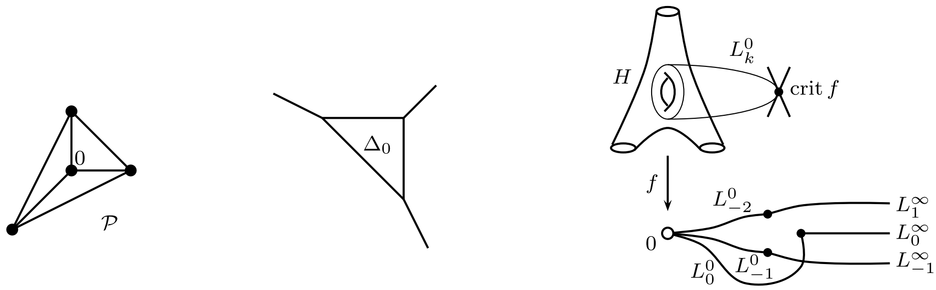

Figure 5. Example: $f(x_1, x_2) = x_1 + x_2 + \dfrac{\tau}{x_1 x_2} + 1$ and $Y = \mathcal{O}(-3) \to \mathbb{P}^2$

which only involves framed Lagrangians, is compatible with the $\mathbb{Z}$-gradings. Similarly, the quotient functor $q : D^b\mathrm{Coh}(Z) \to D^b_{sg}(Z)$ is only $\mathbb{Z}/2$-graded, but its composition with the inclusion pushforward, $qi_* : D^b\mathrm{Coh}(Z_0) \to D^b_{sg}(Z)$, is compatible with the $\mathbb{Z}$-grading on $D^b_{sg}(Z)$ determined by the divisor Z_0.

7. Example: local $\mathbb{P}^n$

In this section, we consider the example where $H \subset (\mathbb{C}^*)^n$ is the hypersurface defined by the Laurent polynomial

$$f(x_1, \dots, x_n) = x_1 + \cdots + x_n + \frac{\tau}{x_1 \dots x_n} + 1.$$

The polyhedral decomposition $\mathcal{P}$ (which gives the fan for Y), the tropicalization of f (which gives the moment polytope for Y), and the Lefschetz fibration $f : (\mathbb{C}^*)^n \to \mathbb{C}$ are depicted (for $n = 2$) on Fig. 5.

One easily checks that Y is the total space of the canonical bundle $\mathcal{O}_{\mathbb{P}^n}(-(n+1))$ over $\mathbb{P}^n$. The facet Δ_0 is a standard simplex, and $Z_0 \simeq \mathbb{P}^n$ is the zero section, while the other components of Z correspond to the total space of $\mathcal{O}(-(n+1))$ over the various coordinate hyperplanes of $\mathbb{P}^n$, whose union forms the anticanonical divisor $D_0 = \{z_0 \dots z_n = 0\} \subset \mathbb{P}^n$.

In [1], Abouzaid constructs admissible Lagrangian submanifolds L_k of $(\mathbb{C}^*)^n$ which are sections of the logarithm map over $\Delta_0 \subset \mathbb{R}^n$, with boundary in

$$f^{-1}(2) = \{x_1 + \cdots + x_n + \tau x_1^{-1} \dots x_n^{-1} - 1 = 0\}.$$

In the tropical limit $\tau \to 0$, L_k is defined by $\arg(x_j) = -2\pi k \log(|x_j|)$ for $j = 1, \dots, n$ (where logarithms are taken in base τ^{-1}). Abouzaid shows the existence of an equivalence $\mathcal{F}^\circ((\mathbb{C}^*)^n, f) \simeq D^b\mathrm{Coh}(\mathbb{P}^n)$ under which L_k corresponds to $\mathcal{O}(k)$.

An alternative description in terms of Lefschetz thimbles is as follows. The Laurent polynomial f has $n+1$ critical points, located at $x_1 = \cdots = x_n = \tau^{1/(n+1)} e^{2\pi i k/(n+1)}$, and the corresponding critical values are $c_k = 1 + (n+1)\tau^{1/(n+1)} e^{2\pi i k/(n+1)}$. The Lagrangian L_k is then Hamiltonian isotopic to the Lefschetz thimble associated to the arc γ_k which runs from the critical value c_k to the reference point 2 by first moving radially away from 1, then clockwise by an angle of $2\pi k/(n+1)$ to reach the real positive axis, then radially outwards again. (The category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ is generated by the exceptional collection $L_0, \dots, L_n$, but one can just as well consider L_k for all $k \in \mathbb{Z}$.)

The properly embedded Lagrangian submanifolds $L_k^\infty = \alpha_\infty(L_k)$ can then be described as Lefschetz thimbles for arcs γ_k^∞ isotopic to the union of γ_k with the interval $[2, +\infty)$ in the real axis; see Fig. 5 right. In particular, L_0^∞ is simply the real positive locus, $L_0^\infty = (\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n \setminus H$, which is consistent with our expectation that it corresponds under mirror symmetry to the pullback of $\mathcal{O}_{\mathbb{P}^n}$ under the projection $\pi : Z \to Z_0 = \mathbb{P}^n$, i.e. $\mathcal{O}_Z$. More generally, for $|k| < (n+1)/2$ the arc γ_k^∞ is isotopic to a radial straight line from c_k to infinity, so we can take $L_k^\infty = (e^{2\pi i k/(n+1)}\,\mathbb{R}_+)^n$.

Meanwhile, the Lagrangian submanifolds $L_k^0 = \alpha_0(L_k)$ are Lefschetz thimbles for arcs γ_k^0 isotopic to the union of γ_k with the lower half of the unit circle centered at 1, i.e., running from the critical value c_k to the origin by moving radially away from 1 then clockwise by an angle of $\pi + 2\pi k/(n+1)$, as shown in Fig. 5. In fact, L_k^0 can be described directly as a section of the logarithm map over Δ_0, with boundary in $f^{-1}(0) = H$, by modifying Abouzaid's construction to account for the sign change. Namely, in the tropical limit, L_k^0 is the Lagrangian section over Δ_0 defined by

$$\arg(x_j) = -(2k+n+1)\pi \log(|x_j|) + \pi \text{ for } j = 1, \dots, n.$$

Conjecture 1.3 predicts that L_k^∞ corresponds under mirror symmetry to the pullback $\pi^*\mathcal{O}_{\mathbb{P}^n}(k)$, which is a line bundle over Z that we denote by $\mathcal{O}_Z(k)$, while L_k^0 corresponds to $i_*\mathcal{O}_{\mathbb{P}^n}(k) = \mathcal{O}_{Z_0}(k)$. Calculations of the wrapped Floer cohomology groups of these Lagrangians inside $(\mathbb{C}^*)^n \setminus H$ (which are fairly straightforward using knowledge of homological mirror symmetry for $\mathbb{P}^n$ and the Lefschetz thimble descriptions) confirm these predictions. For instance, we have ring isomorphisms

$$\begin{aligned} HW^*(L_0^\infty, L_0^\infty) &\simeq \bigoplus_{d\geq 0} \mathbb{C}[z_0, \dots, z_n]_{(n+1)d}/(z_0 \dots z_n) \\ &\simeq \bigoplus_{d\geq 0} H^0(D_0, \mathcal{O}(d(n+1))) \simeq H^0(Z, \mathcal{O}_Z). \end{aligned}$$

Namely, the homogeneous polynomials of degree $(n+1)d$ correspond to the Reeb chords from L_0^∞ to itself that wrap d times around infinity under projection by f. Indeed, the Reeb chords are the same as in $(\mathbb{C}^*)^n$, where the image of L_0 under wrapping d times around infinity is isotopic to $L_{(n+1)d}$. By Abouzaid [1] the boundary intersections between these two admissible Lagrangians correspond to monomials of degree $(n+1)d$ which are not divisible by $z_0 \dots z_n$. However, when computing the product structure in $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ one should discard all holomorphic discs whose projection under f passes through the origin; these correspond exactly to all product operations in $\mathcal{W}((\mathbb{C}^*)^n)$ where the projection under f of the output Reeb chord wraps around infinity fewer times than the sum of the inputs, i.e. all those cases where the product of two monomials is divisible by $z_0 \dots z_n$. A similar argument shows that

$$HW^*(L_j^\infty, L_k^\infty) \simeq H^*(\mathbb{P}^n, \mathcal{O}(k-j)) \oplus \bigoplus_{d>0} H^*(D_0, \mathcal{O}(d(n+1)+k-j)) \simeq H^*(Z, \mathcal{O}_Z(k-j)).$$

Meanwhile, a similar calculation around the puncture at the origin (recalling that the monodromy around zero is trivial up to a grading shift by 2) shows that

$$HW^*(L_j^0, L_k^0) \simeq H^*(\mathbb{P}^n, \mathcal{O}(k-j)) \oplus \bigoplus_{d>0} H^{*-2d}(D_0, \mathcal{O}(k-j)) \simeq \mathrm{Ext}^*(\mathcal{O}_{Z_0}(j), \mathcal{O}_{Z_0}(k)),$$

while

$$HW^*(L_j^\infty, L_k^0) \simeq H^*(\mathbb{P}^n, \mathcal{O}(k-j)) \simeq \mathrm{Ext}^*(\mathcal{O}_Z(j), \mathcal{O}_{Z_0}(k)).$$

Let us now modify the framing and treat H as the zero set of

$$f' = \tau^{-1} x_1 \dots x_n f = \tau^{-1} x_1 \dots x_n (x_1 + \dots + x_n + 1) + 1.$$

The only critical point of f' is at $x_1 = \dots = x_n = -\frac{1}{n+1}$, which lies outside of the region where the constant term dominates. However, the fibers of f', which are degree $n+1$ affine hypersurfaces (e.g. three-punctured elliptic curves for $n=2$), degenerate at the special value 1, and $f'^{-1}(1) = \{x_1 + \dots + x_n + 1 = 0\}$ is an $(n-1)$-dimensional pair of pants.

Choosing $f'^{-1}(2)$ as reference fiber for $\mathcal{F}^\circ((\mathbb{C}^*)^n, f')$, Abouzaid's construction provides admissible Lagrangian submanifolds L'_k which are sections of the logarithm map over the (unbounded) region $\Delta'_0 \subset \mathbb{R}^n$ where the constant term of f' dominates (on Fig. 5, this is the lower-left component). In the tropical limit, L'_k corresponds again to $\arg(x_j) = -2\pi k \log(|x_j|)$. Writing the standard symplectic form of $(\mathbb{C}^*)^n$ as $\omega_0 = \sum dr_j \wedge d\theta_j$, where $x_j = \exp(r_j + i\theta_j)$, L'_k is the image of $L'_0 \subset (\mathbb{R}_+)^n$ by the time 1 flow of the Hamiltonian $\varphi = -\pi k \sum r_j^2$. This Hamiltonian has quadratic growth, and hence for $k \neq 0$ L'_k is not conical at infinity (as $x_j \to 0$).

This can be fixed by the judicious use of cut-off functions to achieve linear growth. Namely, splitting φ into the sum of $\varphi_1 = -\pi k \sum (r_j + \frac{1}{n+1})^2$ and $\varphi_2 = \frac{2\pi k}{n+1} \sum r_j + \frac{n\pi k}{n+1}$, we replace φ_1 by a function $\tilde{\varphi}_1 = \tilde{\varphi}_1(|\vec{r} - \vec{r}_0|)$ of the Euclidean distance between $\vec{r} = (r_1, \dots, r_n)$ and $\vec{r}_0 = (-\frac{1}{n+1}, \dots, -\frac{1}{n+1})$, which grows quadratically up to a certain point and linearly at infinity. Because $\tilde{\varphi}_1$ only depends on $|\vec{r} - \vec{r}_0|$, it is still the case that along the unbounded facet of Δ'_0 where $2r_1 + r_2 + \dots + r_n = -1$, we have the equality $\arg(x_1^2 x_2 \dots x_n) = \partial_{(2,1,\dots,1)}(\tilde{\varphi}_1 + \varphi_2) = 2\pi k$, which is the key property needed to ensure that $\partial L'_k \subset f'^{-1}(2)$. Similarly for the other facets of Δ'_0. The linear growth of $\tilde{\varphi}_1$ along radial straight lines from $\vec{r}_0$ (and the overall linearity of φ_2, whose effect is simply to rotate θ_j by $\frac{2\pi k}{n+1}$) implies that, after this modification,

L'_k is conical at infinity with respect to the Liouville structure $\lambda = \sum(r_j + \frac{1}{n+1})\,d\theta_j$.

A Floer homology calculation (wrapping by a Hamiltonian that grows quadratically with $|\vec{r} - \vec{r}_0|$, and otherwise imitating Abouzaid's arguments [**1, 2**]) shows that the subcategory of $\mathcal{F}^\circ((\mathbb{C}^*)^n, f')$ generated by the admissible Lagrangians L'_k is equivalent to $D^b\mathrm{Coh}(Z'_0)$, where Z'_0 is the distinguished component of Z, i.e. the total space of the line bundle $\mathcal{O}(-(n+1)) \to \mathbb{P}^{n-1}$. The Lagrangian L'_k corresponds under this equivalence to the line bundle $\mathcal{O}_{Z'_0}(k)$.

We can extend L'_k to a properly embedded Lagrangian $L'^{\infty}_k \subset (\mathbb{C}^*)^n \setminus H$ which is a section of the logarithm map over all of $\mathbb{R}^n$ (rather than just Δ'_0), by setting $\arg(x_j) = -2k\pi \log(|x_j|)$ over a large bounded subset of $\mathbb{R}^n$, and arranging for L'^{∞}_k to be conical at infinity by the same cut-off trick as for L'_k. By construction, each monomial of f' is real positive along the tropical hypersurface, and hence the leading order terms of f' cannot cancel out; this in turn implies that L'^{∞}_k is disjoint from H.

It is not hard to check that L'^{∞}_k is Hamiltonian isotopic to L^{∞}_k, and hence isomorphic to it as an object of $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$. Thus, L'^{∞}_k corresponds to the unique way (for $n \geq 2$) to extend $\mathcal{O}_{Z'_0}(k)$ to a line bundle over all of Z, $\mathcal{O}_Z(k)$. However, the images of L'^{∞}_k under f' go to infinity along paths that lie in different homotopy classes relative to the critical value of f'; it might be the case that no choice of arc η_∞ in the construction of the acceleration functor α_∞ gives $\alpha_\infty(L'_k) \simeq L'^{\infty}_k$ for all k.

The reason for this is particularly apparent in the case $n = 1$. where the admissible Lagrangians L'_k are arcs connecting the origin to the point of $f'^{-1}(2)$ which lies near $x = \tau$, inside the region of $\mathbb{C}^*$ where $|x| \leq \tau$. These arcs are all isotopic and represent isomorphic objects of $\mathcal{F}^\circ(\mathbb{C}^*, f') \simeq D^b\mathrm{Coh}(\mathbb{C})$, as expected given that $Z'_0 \simeq \mathbb{C}$ and hence $\mathcal{O}_{Z'_0}(k) \simeq \mathcal{O}_{Z'_0}$. On the other hand, the arcs L'^{∞}_k that connect the origin to infinity in $\mathbb{C}^* \setminus H$ are definitely not isotopic to each other: for example, the intersection number of L'^{∞}_k with the portion of the negative real axis that lies between the two points of H (located near -1 and $-\tau$) is equal to k. This is consistent with mirror symmetry, since the line bundles $\mathcal{O}_Z(k)$ are pairwise non-isomorphic. And, of course, no functor from $\mathcal{F}^\circ(\mathbb{C}^*, f')$ to $\mathcal{W}((\mathbb{C}^*) \setminus H)$ (resp. $D^b\mathrm{Coh}(Z'_0)$ to $D^b\mathrm{Coh}(Z)$) can map the isomorphic objects L'_k (resp. $\mathcal{O}_{Z'_0}(k)$) to the non-isomorphic L'^{∞}_k (resp. $\mathcal{O}_Z(k)$).

On the other hand, this issue does not arise for the other acceleration functor α_0. Namely, the Lagrangians $L'^0_k = \alpha_0(L'_k)$ can be constructed by modifying the sign conventions in the definition of L'_k in order to obtain admissible Lagrangians with boundary in $f'^{-1}(0) = H$. Specifically, L'^0_k is again a section of the logarithm map over $\Delta'_0 \subset \mathbb{R}^n$; in the tropical limit, we take L'^0_k to be defined by

$$\arg(x_j) = -(2k-1)\pi \log(|x_j|) \quad \text{for } j = 1, \dots, n$$

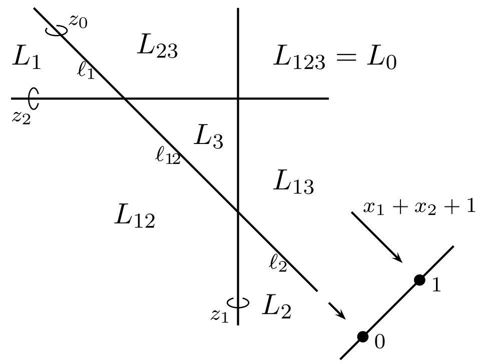

FIGURE 6. $\Pi_2 \simeq (\mathbb{C}^*)^2 \setminus \Pi_1$ and the components of its real locus

over a large bounded subset of Δ'_0, and use cut-off functions as in the construction of L'_k in order to make L'^0_k conical at infinity.

For $n = 1$ the arcs L'^0_k connecting the origin to the point of H which lies near $x = -\tau$ are all isotopic to each other in $\mathbb{C}^* \setminus H$, as expected, whereas for $n \geq 2$ these are genuinely different objects of $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$. For $n = 2$ one can check explicitly that L'^k_0 corresponds under mirror symmetry to $i_*\mathcal{O}_{Z'_0}(k)$; presumably this remains true for all n.

8. Higher dimensional pairs of pants

8.1. Setup and notations. We consider the pair of pants $H = \Pi_{n-1} \subset (\mathbb{C}^*)^n$, defined by the equation

$$f(x_1, \dots, x_n) = x_1 + \dots + x_n + 1 = 0.$$

One easily checks that our construction gives $(Y, W) = (\mathbb{C}^{n+1}, -z_1 \dots z_{n+1})$. Thus Z is the union of the coordinate hyperplanes $Z_i : \{z_i = 0\}$, where the distinguished component (previously called Z_0) that corresponds to the constant term in f is Z_{n+1}, while the other components $Z_1, \dots, Z_n$ correspond to the monomials $x_1, \dots, x_n$. For $I \subseteq \{1, \dots, n+1\}$, we set $Z_I = \bigcup_{i \in I} Z_i$.

The complement $(\mathbb{C}^*)^n \setminus \Pi_{n-1}$ is isomorphic to the higher dimensional pants Π_n, whose wrapped Fukaya category we aim to study using the ideas introduced in Sects. 4–6. The key Lagrangian submanifolds of interest to us will be the $2^{n+1} - 1$ connected components L_I of the real locus of $\Pi_n = (\mathbb{C}^*)^n \setminus \Pi_{n-1}$, which we label by proper non-empty subsets $I \subset \{0, \dots, n+1\}$, up to the equivalence relation that identifies each subset with the complementary subset of $\{0, \dots, n+1\}$. We usually choose the representative which does not contain the element 0 as the "canonical" label, except in the case of $\{1, \dots, n+1\} \sim \{0\}$. See Fig. 6 for the case $n = 2$.

The labelling is as follows: (1) the positive orthant $(\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n \setminus \Pi_{n-1}$ is labelled $L_{\{1,\dots,n+1\}} = L_{\{0\}}$; (2) whenever two components of the real locus are adjacent to each other across the hyperplane $x_i = 0$ $(1 \leq i \leq n)$, their labelling sets differ exactly by adding or removing the element i; (3)

whenever two components are adjacent to each other across Π_{n-1}, their labels differ by adding or removing the element 0 (or equivalently, they are labelled by complementary subsets of $\{1,\dots,n+1\}$).

A more symmetric viewpoint embeds the picture into $\mathbb{P}^{n+1}$ with homogeneous coordinates $(x_1:\dots:x_{n+1}:x_0)$ as follows. We can realize Π_n as the intersection of the hyperplane $\{x_1+\dots+x_{n+1}+x_0=0\}\subset\mathbb{P}^{n+1}$ with the complement of the $n+2$ coordinate hyperplanes $x_i=0$; the identification with $(\mathbb{C}^*)^n\setminus H$ is by mapping $(x_1:\dots:x_n:1:x_0)$ to $(x_1,\dots,x_n)$. For $I\subset\{0,\dots,n+1\}$, L_I is the set of real points that admit homogeneous coordinates $(x_1:\dots:x_{n+1}:x_0)$ (with their sum equal to zero) satisfying $x_i>0$ for $i\in I$ and $x_i<0$ for $i\notin I$.

For $I\subseteq\{1,\dots,n+1\}$, the image of L_I under the logarithm map covers exactly those components of $\mathbb{R}^n\setminus\mathrm{Log}(\Pi_{n-1})$ that correspond to the elements of I (recalling that $n+1$ corresponds to the region where the constant term of f dominates). Thus, as a general principle we expect that under mirror symmetry L_I corresponds to an object of $D^b\mathrm{Coh}(Z)$ which is supported on $Z_I=\bigcup_{i\in I}Z_i$ and whose restriction to each Z_i, $i\in I$ is a (trivial) line bundle:

Conjecture 8.1. *There is an equivalence $\mathcal{W}(\Pi_n)\simeq D^b\mathrm{Coh}(Z)$ under which the objects L_I map to $\mathcal{O}_{Z_I}$ for all non-empty $I\subseteq\{1,\dots,n+1\}$.*

We now explore how this prediction fits with Conjectures 1.2 and 1.3; a proof of Conjecture 8.1 is sketched in Sect. 9 below.

8.2. Restriction and lifting. For non-empty $I\subseteq\{1,\dots,n\}$, denote by $\ell_I\subset\Pi_{n-1}$ the components of the real locus of the $(n-1)$-dimensional pair of pants, labelled as above. More precisely, embedding Π_{n-1} into $\mathbb{P}^n$ with homogeneous coordinates $(y_1:\dots:y_n:y_0)$ as the intersection of the hypersurface $\{y_1+\dots+y_n+y_0=0\}$ with the complement of the $n+1$ coordinate hyperplanes, we define ℓ_I to be the set of real points where $y_i>0$ iff $i\in I$. Meanwhile, embedding Π_n into $\mathbb{P}^{n+1}$ as above, f can be expressed in homogeneous coordinates as $(x_1:\dots:x_{n+1}:x_0)\mapsto -x_0/x_{n+1}$. The zero set $\Pi_{n-1}=f^{-1}(0)$ then corresponds to setting $x_0=0$ (while still requiring the other coordinates to be non-zero), i.e. in projective coordinates we use the embedding

$$(y_1:\dots:y_n:y_0)\mapsto(x_1:\dots:x_{n+1}:x_0)=(y_1:\dots:y_n:y_0:0). \tag{8.1}$$

By construction, for $I\subseteq\{1,\dots,n\}$, the portion of the boundary of L_I that lies on the hyperplane $x_0=0$ is exactly ℓ_I, and similarly for $L_{I\cup\{0\}}$; see Fig. 6. Using Remark 4.2, the images of the objects L_I under the ($\mathbb{Z}/2$-graded) restriction functor $\rho:\mathcal{W}(\Pi_n)\to\mathcal{W}(\Pi_{n-1})$ defined in § 4 are therefore as follows:

Lemma 8.2. *For all non-empty $I\subseteq\{1,\dots,n\}$, $\rho(L_I)\cong\ell_I$, while $\rho(L_{I\cup\{0\}})\cong\ell_I[1]$, and $\rho(L_{\{0\}})=0$.*

This is consistent with Conjecture 1.2, given our expectation that, for $I\subseteq\{1,\dots,n\}$, L_I corresponds to $\mathcal{O}_{Z_I}\in D^b\mathrm{Coh}(Z)$. Indeed, recall that $D^b_{sg}(Z)\simeq$

$D^b\mathrm{Coh}(D)$, where $D = Z_{\{1,\dots,n\}} \cap Z_{n+1} = \{(z_1,\dots,z_n,0) \in \mathbb{C}^n \times 0 \mid z_1 \dots z_n = 0\}$. For $I \subseteq \{1,\dots,n\}$, we set $D_I = Z_I \cap Z_{n+1} \subseteq D$. Orlov's construction [**20, 21**] identifies $[\mathcal{O}_{Z_I}] \in D^b_{sg}(Z)$ with $\mathcal{O}_{D_I} \in D^b\mathrm{Coh}(D)$. Thus, denoting by $q : D^b\mathrm{Coh}(Z) \to D^b_{sg}(Z)$ the quotient functor, and by $\epsilon : D^b_{sg}(Z) \to D^b\mathrm{Coh}(D)$ Orlov's equivalence, we find that $\varepsilon \circ q(\mathcal{O}_{Z_I}) \cong \mathcal{O}_{D_I}$. Given that our mirror symmetry ansatz for pairs of pants matches $\mathcal{O}_{D_I}$ with ℓ_I and $\mathcal{O}_{Z_I}$ with L_I, this is consistent with $\rho(L_I) \cong \ell_I$.

Moreover, let $I' = \{1,\dots,n+1\} \setminus I$, so that $I \cup \{0\} \sim I'$. We expect that $L_{I\cup\{0\}} = L_{I'}$ corresponds to $\mathcal{O}_{Z_{I'}} \in D^b\mathrm{Coh}(Z)$. Observing that $Z = Z_I \cup Z_{I'}$, we have a short exact sequence $0 \to \mathcal{O}_{Z_I} \to \mathcal{O} \to \mathcal{O}_{Z_{I'}} \to 0$, which implies that $\mathcal{O}_{Z_{I'}}$ is isomorphic to $\mathcal{O}_{Z_I}[1]$ in the quotient category $D^b_{sg}(Z)$, hence $\varepsilon \circ q(\mathcal{O}_{Z_{I'}}) \cong \mathcal{O}_{D_I}[1]$. This is again consistent with $\rho(L_{I'}) \cong \ell_I[1]$. Finally, $L_{\{0\}} = L_{\{1,\dots,n+1\}}$ corresponds to $\mathcal{O}_Z \in D^b\mathrm{Coh}(Z)$, which is annihilated by q, in agreement with $\rho(L_{\{0\}}) = 0$.

Next we consider the lifting functor $j : \mathcal{W}(\Pi_{n-1}) \to \mathcal{W}(\Pi_n)$ of Sect. 6.2. Since x_0 and x_{n+1} have the same sign on L_I for $I \subseteq \{1,\dots,n\}$, under f the Lagrangian L_I projects to the real negative axis, and coincides with the parallel transport of $\ell_I \subset \Pi_{n-1} = f^{-1}(0)$ in the fibers of f over $\gamma = (-\infty, 0)$. Thus:

$$j(\ell_I) \cong L_I \qquad \text{for } I \subseteq \{1,\dots,n\}. \tag{8.2}$$

In particular, $\rho \circ j \cong \mathrm{id}$ as expected.

8.3. The Fukaya-Seidel category. We now consider the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$. While $f = x_1 + \dots + x_n + 1$ does not have any critical points, the special fiber $f^{-1}(1)$ is the complement of $n+1$ hyperplanes through the origin in $\mathbb{C}^{n-1}$, which is diffeomorphic to $\mathbb{C}^* \times \Pi_{n-2}$, whereas the regular fibers are complements of $n+1$ affine hyperplanes in generic position in $\mathbb{C}^{n-1}$, i.e. isomorphic to Π_{n-1}. This degeneration gives rise to a (non-compact) "vanishing cycle", and the Lagrangian obtained by parallel transport of this vanishing cycle over the interval $(1, +\infty)$ is simply the real positive locus $L_{\{0\}} = (\mathbb{R}_+)^n \subset (\mathbb{C}^*)^n$.

Taking the reference fiber to be $f^{-1}(2)$, the category $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ is generated by the admissible Lagrangian L_{adm} obtained by truncating $L_{\{0\}}$ (namely, the portion of $\mathbb{R}^n_+$ where $x_1 + \dots + x_n \le 1$), which is a section of the logarithm map over the appropriate region in $\mathbb{R}^n$ (essentially the negative orthant). A calculation then shows that the endomorphisms of L_{adm} form a polynomial algebra $\mathbb{C}[z_1,\dots,z_n]$, where the generator z_i corresponds to a Reeb chord that wraps once around the hyperplane $x_i = 0$. This computation is consistent with mirror symmetry, as it matches $\mathrm{End}(\mathcal{O})$ in the derived category of $Z_{n+1} = \mathbb{C}^n$.

By construction, $\alpha_\infty(L_{adm}) = L_{\{0\}}$, in agreement with the behavior of pullback on the mirror. Namely, the pullback of $\mathcal{O}_{Z_{n+1}}$ under the map $\pi : Z \to Z_{n+1}$ defined by $\pi(z_1,\dots,z_{n+1}) = (z_1,\dots,z_n,0)$ is indeed $\mathcal{O}_Z$.

Meanwhile, $\alpha_0(L_{adm})$ is (up to isotopy) the Lagrangian obtained by parallel transport of the "vanishing cycle" of f over the interval $(0,1)$, i.e., the portion of $(\mathbb{R}_-)^n$ where $x_1+\cdots+x_n+1>0$. Therefore, $\alpha_0(L_{adm}) = L_{\{n+1\}}$. This is in agreement with Conjecture 1.3, as the image of the structure sheaf under pushforward by the inclusion $i : Z_{n+1} \hookrightarrow Z$ is indeed $\mathcal{O}_{Z_{n+1}}$.

Finally, since ρ maps $L_{\{n+1\}} = L_{\{0,1,\dots,n\}}$ to $\ell_{\{1,\dots,n\}}[1]$ (cf. Lemma 8.2), we have $j\rho\alpha_0(L_{adm}) \simeq L_{\{1,\dots,n\}}[1]$, and the exact triangle (6.3) takes the form

$$L_{\{1,\dots,n\}} \to L_{\{0\}} \to L_{\{n+1\}} \to L_{\{1,\dots,n\}}[1].$$

Under mirror symmetry, this corresponds to the exact triangle in $D^b\mathrm{Coh}(Z)$ induced by the short exact sequence of sheaves

$$0 \to \mathcal{O}_{Z_1\cup\dots\cup Z_n} \to \mathcal{O}_Z \to \mathcal{O}_{Z_{n+1}} \to 0.$$

9. Computing $\mathcal{W}(\Pi_n)$

We now sketch an approach to the calculation of $\mathcal{W}(\Pi_n)$, relying on a mix of explicit computations and the structural considerations introduced in the previous section. Conjecture 8.1 should then follow as a corollary.

9.1. Liouville structure and wrapping Hamiltonian. Embedding Π_n into $\mathbb{P}^{n+1}$ as the hyperplane $\Sigma = \{x_1+\cdots+x_{n+1}+x_0=0\}$ minus the $n+2$ coordinate hyperplanes, it is clear that the labels $0,\dots,n+1$ should play symmetric roles, with the important exception of gradings. (Since our preferred trivialization of the tangent bundle is inherited from $(\mathbb{C}^*)^n$ and extends across the hyperplane $x_0=0$ but not across the others, Reeb orbits that wrap around the hyperplane $x_0=0$ are graded differently from those that wrap around the other coordinate hyperplanes.) Apart from this, one would like all calculations to be invariant under the action of $\mathfrak{S}_{n+2}$ by permutation of the coordinates. Thus, it is desirable to choose the Liouville structure on Π_n and the wrapping Hamiltonian to be $\mathfrak{S}_{n+2}$-invariant. (We note that the Lagrangians L_I are conical at infinity, and in fact invariant under the Liouville flow, for *any* choice of complex conjugation anti-invariant Liouville structure on Π_n.)

With this in mind, we stratify the hyperplane Σ depending on which coordinates vanish, namely for I a proper subset of $\{0,\dots,n+1\}$ we set Σ_I to be the codimension $|I|$ subset of Σ where $x_i=0$ exactly for $i\in I$; the pair of pants Π_n is then the open stratum $\Sigma_\emptyset$. We claim that the Liouville structure on Π_n can be chosen in such a way that, in a neighborhood of Σ_I, there are $|I|$ commuting S^1-actions, generated by Hamiltonians $h_{I,i}$ $(i\in I)$, each of which essentially acts by rotating one of the coordinates x_i, $i\in I$ around the origin. Moreover, observing that for $I\subset J$ we have $\Sigma_J\subset\overline{\Sigma}_I$, we require that $h_{I,i}$ and $h_{J,i}$ agree near Σ_J for all $i\in I\subset J$. We then take our (quadratic) wrapping Hamiltonian to be $H_I=\frac{1}{2}\sum_{i\in I} h_{I,i}^2$ near Σ_I.

One possible approach to the construction is as follows. Near Σ_I, we define local affine coordinates $x_{I,i}$ by

$$x_{I,i} = \Big(\pm \prod_{j\notin I} x_j\Big)^{-1/(n+2-|I|)} x_i, \tag{9.1}$$

for some local choice of the $(n+2-|I|)$-th root. (Since we will only use $|x_{I,i}|$ and $d\log x_{I,i}$, the choice of root is not important.) We note that Σ_I still corresponds to the locus where $x_{I,i} = 0$ for all $i \in I$, and (for suitable choices of signs and roots) these coordinates are still real-valued on the components of the real locus of Π_n. These coordinates patch as follows: if $I \subset J$, then $\Sigma_J \subset \overline{\Sigma}_I$, and (up to a root of unity)

$$x_{I,i} = \Big(\prod_{j\in J\setminus I} x_{J,j}\Big)^{-1/(n+2-|I|)} x_{J,i}$$

and conversely

$$x_{J,i} = \Big(\prod_{j\in J\setminus I} x_{I,j}\Big)^{1/(n+2-|J|)} x_{I,i}. \tag{9.2}$$

Writing $x_{I,i} = \exp(r_{I,i} + i\theta_{I,i})$, we set up the Liouville structure so that, near Σ_I and away from all the lower-dimensional strata, the dominant term in the Kähler potential and in the Hamiltonian is

$$H_I = \Phi_I = \frac{1}{2}\sum_{i\in I}(r_{I,i} + K_I)^2 \tag{9.3}$$

where $K_I = K_{|I|} > 0$ is some fixed constant depending only on $|I|$. As one approaches a lower-dimensional stratum Σ_J $(I \subset J)$, this is patched together with the expression H_J by making K_I a function of the variables $r_{I,j}$ (or equivalently $r_{J,j}$) for $j \in J \setminus I$, and also by introducing quadratic terms in the variables $r_{J,j}$.

In light of (9.2), near $\Sigma_J \subset \overline{\Sigma}_I$ (and away from other smaller strata) we want to set $K_I = \frac{1}{(n+2-|J|)}\sum_{j\in J\setminus I} r_{I,j} + K_{|J|}$ when the quantities $r_{I,j}$, $j \in J \setminus I$ are sufficiently negative. In other terms, over a neighborhood of Σ_I our Hamiltonian H is expressed in terms of the $x_{I,i}$ as some smooth approximation of

(9.4)
$$H = \frac{1}{2}\sum_{i=0}^{n+1}\Big[\min\Big(r_{I,i} + \min_{I\subseteq J,\ |J|\le n}\Big\{\frac{1}{(n+2-|J|)}\sum_{j\in J\setminus I} r_{I,j} + K_{|J|}\Big\}, 0\Big)\Big]^2.$$

where the smoothing of the minimum still only depends on the values of $r_{I,j}$ for $j \notin I$. Choosing the positive constants $K_{|I|}$ sufficiently large, the term involving $r_{I,i}$ is supported in a small neighborhood of the hyperplane $x_i = 0$. Near the zero-dimensional strata the same formula can be used for the Kähler potential, but along higher dimensional strata we need to add a term involving only the coordinates $x_{I,j}$ for $j \notin I$ (essentially, a Kähler form on the $(n-|I|)$-dimensional pair of pants Σ_I).

For $|I| = n$, i.e. near a zero-dimensional stratum, the Kähler potential is given by (9.3), and the Hamiltonian $h_{I,i} = r_{I,i} + K_I$ generates the vector field $\partial/\partial\theta_{I,i}$ which rotates $x_{I,i}$ while leaving all other $x_{I,j}$, $j \in I$ unchanged. (Meanwhile, the two remaining homogeneous coordinates, which are the largest, vary slightly as needed to preserve the condition $\sum x_j = 0$.) Even when the "constants" K_I in (9.3) are allowed to vary and depend on $\{r_{I,j},\ j \in J \setminus I\}$ for some $J \supset I$ with $|J| = n$, to arrive at an expression of the form (9.4), and the Kähler potential includes an extra term depending only on $x_{I,j}$ for $j \in J \setminus I$, it remains true that for $i \in I$ the Hamiltonian $h_{I,i} = r_{I,i} + K_I$ generates the vector field $\partial/\partial\theta_{I,i}$ which rotates $x_{I,i}$ while leaving $x_{I,j}$ unchanged for $j \in J \setminus \{i\}$.

Further away from all the zero-dimensional strata there is no longer a preferred n-element subset of the $r_{I,j}$'s on which we can assume the Kähler potential (or even the term K_I) solely depends. Nonetheless, we can arrange for that, near Σ_I, for $i \in I$ the Hamiltonian $h_{I,i} = r_{I,i} + K_I$ still generates an S^1-action which rotates $x_{I,i}$ while preserving the other coordinates $x_{I,j}$, $j \in I \setminus \{i\}$; a priori none of the coordinates $x_{I,j}$, $j \notin I$ are preserved, though we can arrange for them to vary only by small amounts. For simplicity we still denote these vector fields by $\partial/\partial\theta_{I,i}$.

Putting everything together, we find that near Σ_I and away from lower-dimensional strata the vector field generated by the quadratic Hamiltonian H takes the form $\sum_{i\in I} h_{I,i}\, \partial/\partial\theta_{I,i} = \sum_{i\in I}(r_{I,i} + K_I)\, \partial/\partial\theta_{I,i}$. (Note that $h_{I,i}$ tends to $-\infty$ as x_i approaches zero, i.e. we wrap clockwise around the coordinate hyperplanes).

9.2. Wrapped Floer cohomology. Given any subset $I \subset \{0, \dots, n+1\}$, denote by $\overline{I} = \{0, \dots, n+1\} \setminus I$ the complementary subset. We will consider various quotients of the polynomial ring $\mathbb{C}[z_0, \dots, z_{n+1}]$, graded with $\deg(z_0) = 2$ and $\deg(z_i) = 0$ for $i \geq 1$. For convenience, we define $z_I = \prod_{i\in I} z_i$. (By convention, $z_\emptyset = 1$.)

Proposition 9.1. *Given a non-empty proper subset $I \subset \{0, \dots, n+1\}$, as a graded ring we have*

$$HW^*(L_I, L_I) \simeq \mathbb{C}[z_0, \dots, z_{n+1}]/(z_I, z_{\overline{I}}). \tag{9.5}$$

Proof. Recall that L_I is the component of the real locus of Π_n where the coordinates x_i are positive for $i \in I$ and negative for $i \in \overline{I}$. The closure of L_I in Σ intersects the stratum Σ_J if and only if neither I nor $\overline{I}$ is a subset of J.

For such J, near $\overline{L}_I \cap \Sigma_J$ the local coordinates $x_{J,j}$, $j \in J$ define a local projection to $\mathbb{C}^{|J|}$ under which L_I maps to an orthant in the real locus, whereas the wrapping Hamiltonian flow rotates each $x_{J,j}$ clockwise by increasing amounts as $|x_{J,j}| \to 0$. Thus, for each tuple of positive integers

$(k_j)_{j\in J}$, along $\overline{L}_I \cap \Sigma_J$ (and away from lower-dimensional strata) we have a family of time 1 trajectories of X_H from L_I to itself that wraps k_j times around the hyperplane $x_j = 0$.

To make things non-degenerate, we pick a "convex" bounded Morse function on $\overline{L}_I$ which reaches its maximum at the corners and whose restriction to each stratum $\overline{L}_I \cap \Sigma_J$ has a single critical point which is a minimum. (Such a function is easy to construct using the contractibility of $\overline{L}_I$ and all of its strata.) After perturbing the Hamiltonian by a small positive multiple of this function, there is a single non-degenerate time 1 chord of X_H from L_I to itself which wraps k_j times around each hyperplane $x_j = 0$ near $\overline{L}_I \cap \Sigma_J$. We label the corresponding generator of $CW^*(L_I, L_I)$ by the monomial $\prod_{j\in J} z_j^{k_j}$, and note that its degree (using our chosen trivialization of the tangent bundle) is equal to $2k_0$ if $0 \in J$, and zero otherwise.

Letting J vary over all subsets of $\{0, \ldots, n+1\}$ which contain neither I nor $\overline{I}$ (including the empty subset, which gives rise to a single generator $z_\emptyset = 1$ at the minimum of the chosen Morse function), the generators of $CW^*(L_I, L_I)$ are labelled by all the monomials in $\mathbb{C}[z_0, \ldots, z_{n+1}]$ which are divisible neither by z_I nor by $z_{\overline{I}}$. Moreover, since all the generators have even degree, the Floer differential necessarily vanishes; thus (9.5) holds as an isomorphism of graded vector spaces.

Next we observe that $H_1(\Pi_n, L_I; \mathbb{Z}) \simeq H_1(\Pi_n, \mathbb{Z}) \simeq \mathbb{Z}^{n+2}/(1, \ldots, 1)$, where the generators of $\mathbb{Z}^{n+2}$ correspond to meridian loops around the coordinate hyperplanes. Under this isomorphism, the generator $\prod z_j^{k_j}$ of $CW^*(L_I, L_I)$ represents the homology class $(k_0, \ldots, k_{n+1}) \bmod (1, \ldots, 1)$; these homology classes are all distinct. Whenever there is a perturbed holomorphic curve contributing to the Floer product, the relative homology class of the output chord must be equal to the sum of those of the input chords. Moreover, the degree of the output generator must be the sum of those of the input generators. Since $\deg(z_0) = 2$, we can use the grading to lift the translation ambiguity: there is an isomorphism $(2\mathbb{Z}) \times H_1(\Pi_n, \mathbb{Z}) \simeq \mathbb{Z}^{n+2}$ under which the degree and homology class of the generator $\prod z_j^{k_j}$ map to the tuple $(k_0, \ldots, k_{n+1}) \in \mathbb{Z}^{n+2}$.

Any generator which appears in the expression of the product of the generators $\prod z_j^{k_j}$ and $\prod z_j^{\ell_j}$ of $HW^*(L_I, L_I)$ must have degree $2(k_0 + \ell_0)$ and represent the homology class $(k_0 + \ell_0, \ldots, k_{n+1} + \ell_{n+1}) \bmod (1, \ldots, 1)$. This implies that the product must be zero if $k_j + \ell_j > 0$ for all $j \in I$ or for all $j \in \overline{I}$ (for lack of a suitable generator of $HW^*(L_I, L_I)$), and otherwise it must be a multiple of the generator $\prod z_j^{k_j+\ell_j}$.

In the latter case, to determine the number of solutions to the perturbed holomorphic curve equation, we observe that since the set of j such that $k_j + \ell_j > 0$ contains neither all of I nor all of $\overline{I}$, it must have at most n elements, and there exists J with $|J| = n$, containing neither I nor $\overline{I}$, such that $k_j = \ell_j = 0$ whenever $j \notin J$. We claim that we can determine the product $\left(\prod z_j^{k_j}\right)\cdot\left(\prod z_j^{\ell_j}\right)$ by working in a local model near the 0-dimensional stratum Σ_J, using the affine coordinates $x_{J,j}$, $j \in J$.

In these coordinates, the wrapping Hamiltonian near Σ_J is modelled on a standard product Hamiltonian on a neighborhood of the origin in $(\mathbb{C}^*)^n$, given by $H = \frac{1}{2}\sum h_{J,j}^2$, where $h_{J,j} = \min(r_{J,j} + K, 0)$ up to some smoothing near $r_{J,j} + K = 0$ (recall that $r_{J,j} = \log|x_{J,j}|$), and the Lagrangian L_I is one of the orthants in the real locus. We perturb the degenerate minimum of $h_{J,j}^2$ to achieve non-degeneracy, in a manner such that the minimum of the perturbed Hamiltonian lies within the local coordinate chart. Concretely, in the local chart we can take the perturbed Hamiltonian to be

$$H_\epsilon = \frac{1}{2}\sum_{j\in J}\Big(\min(r_{J,j} + K, 0)^2 + \epsilon r_{J,j}\Big)$$

where $\epsilon > 0$ is small. (This choice of perturbation clearly depends on the choice of J, and is different from the perturbation used above to compute the overall chain complex; this is not an issue, since we are only interested in a cohomology level computation of the product structure, and the isomorphism induced by continuation between the two choices of perturbations is the obvious one.)

With this understood, L_I and its image under the flow generated by H_ϵ are product Lagrangians in the local coordinates $(x_{J,j})_{j\in J}$, and in each factor the picture looks exactly like the left half of Fig. 1 (up to just past the midpoint). The chord from L_I to itself which is labelled by the monomial $\prod z_j^{k_j}$ wraps k_j times around each coordinate hyperplane, and corresponds to the intersection point labelled x_{-k_j} in the left half of Fig. 1 for each of the n coordinate factors. Similarly for $\prod z_j^{\ell_j}$.

The maximum principle implies that any perturbed holomorphic disc which contributes to the product of these two generators must remain entirely within the local chart. Moreover, the projection to each coordinate factor $x_{J,j}$ is a perturbed holomorphic disc in (a neighborhood of the origin in) $\mathbb{C}^*$ with boundary on the appropriate arcs. Conversely, every tuple of index 0 perturbed holomorphic discs in the coordinate factors lifts to an index 0 perturbed holomorphic disc in the total space of the local chart. Recall from § 2.3 that, on the cylinder, the generators x_{-k_j} and $x_{-\ell_j}$ are the inputs of a unique triangle contributing to the Floer product, whose output is $x_{-k_j-\ell_j}$. Thus, we conclude that

$$\left(\prod z_j^{k_j}\right)\cdot\left(\prod z_j^{\ell_j}\right) = \prod z_j^{k_j+\ell_j}.$$

Hence the ring structure on $HW^*(L_I, L_I)$ is as expected. □

This calculation of $HW^*(L_I, L_I)$ agrees with the expectation from mirror symmetry. Indeed, switching I and $\overline{I}$ if needed, we can assume that $0 \notin I$, and L_I is expected to correspond to $\mathcal{O}_{Z_I} \in D^b\mathrm{Coh}(Z)$, or equivalently, since $Z = \operatorname{Spec} R$ for $R = \mathbb{C}[z_1, \dots, z_{n+1}]/(z_1 \dots z_{n+1})$, the R-module $R/(z_I)$. Set

$I' = \{1, \dots, n+1\} \setminus I$. Using the resolution

$$\dots \longrightarrow R \xrightarrow{z_I} R \xrightarrow{z_{I'}} R \xrightarrow{z_I} R \longrightarrow R/(z_I) \to 0, \tag{9.6}$$

we find that

$$\mathrm{Ext}^{2k}(R/(z_I), R/(z_I)) \simeq \begin{cases} R/(z_I) & \text{for } k = 0, \\ (R/(z_I, z_{I'}))z_0^k & \text{for } k > 0 \end{cases}$$

where z_0 is a generator of $\mathrm{Ext}^2(R/(z_I), R/(z_I))$ as a module over $\mathrm{End}(R/(z_I))$.

Next, we consider pairs of objects, and show:

PROPOSITION 9.2. *Given non-empty proper subsets $I, J \subset \{0, \dots, n+1\}$, we have*

$$\begin{aligned} HW^*(L_I, L_J) \simeq \ & \mathbb{C}[z_0, \dots, z_{n+1}]/(z_{I \cap J}, z_{\overline{I} \cap \overline{J}}) \cdot u_{\overline{Q}} \\ & \oplus \mathbb{C}[z_0, \dots, z_{n+1}]/(z_{I \cap \overline{J}}, z_{\overline{I} \cap J}) \cdot u_Q \end{aligned} \tag{9.7}$$

as a graded $(HW^*(L_I, L_I), HW^*(L_J, L_J))$-bimodule, where we set $Q = (I \cap J) \cup (\overline{I} \cap \overline{J})$ and $\overline{Q} = (I \cap \overline{J}) \cup (\overline{I} \cap J)$, and the generator u_Q (resp. $u_{\overline{Q}}$) has degree 1 if $0 \in Q$ (resp. $0 \in \overline{Q}$), and 0 otherwise.

As an additional piece of notation, we formally set

$$u_Q = z_Q^{1/2} = \prod_{j \in Q} z_j^{1/2}$$

and similarly for $u_{\overline{Q}}$. This allows us to view generators of $HW^*(L_I, L_J)$ as monomials in $z_0, \dots, z_{n+1}$ with half-integer exponents (and is consistent with gradings).

PROOF. The argument is similar to the case of Proposition 9.1. First, we find a criterion for the closures of L_I and L_J in Σ to intersect along the stratum Σ_K for some $K \subset \{0, \dots, n+1\}$. In terms of the homogeneous coordinates $(x_1 : \ldots : x_{n+1} : x_0)$, the points of $\overline{L}_I \cap \Sigma_K$ are those where x_i is positive for $i \in I \cap \overline{K}$, zero for $i \in K$, and negative for $i \in \overline{I} \cap \overline{K}$, or vice-versa exchanging I and $\overline{I}$. Moreover, since the sum of the coordinates is zero, there must be at least one positive and one negative coordinate. Thus, $\overline{L}_I \cap \overline{L}_J \cap \Sigma_K$ is non-empty in precisely two cases:

(1) $I \cap \overline{K} = J \cap \overline{K} \neq \emptyset$ and $\overline{I} \cap \overline{K} = \overline{J} \cap \overline{K} \neq \emptyset$, or
(2) $I \cap \overline{K} = \overline{J} \cap \overline{K} \neq \emptyset$ and $\overline{I} \cap \overline{K} = J \cap \overline{K} \neq \emptyset$.

In case (1), K must contain the symmetric difference of I and J, i.e. $\overline{Q} \subseteq K$; but none of $I, \overline{I}, J, \overline{J}$ can be a subset of K. Similarly for case (2), K must contain the symmetric difference of I and $\overline{J}$, i.e. Q. Thus, we can reformulate our criterion as:

(1) $\overline{Q} \subseteq K$, but K contains neither $I \cap J$ nor $\overline{I} \cap \overline{J}$, or
(2) $Q \subseteq K$, but K contains neither $I \cap \overline{J}$ nor $\overline{I} \cap J$.

With this understood, in case (1), near Σ_K the coordinates $x_{K,j}$, $j \in K$ define a local projection to $\mathbb{C}^{|K|}$ in which L_I and L_J map to orthants in the real locus; these orthants correspond to real points whose coordinates have the same signs for $j \in Q \cap K$ and different signs for $j \in \overline{Q}$. Thus, given any tuple $(k_j)_{j\in K}$ with $k_j \in \mathbb{Z}_{>0}$ for $j \in Q \cap K$ and $k_j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ for $j \in \overline{Q}$, near Σ_K there is a family of time 1 trajectories of X_H from L_I to L_J that wrap k_j times around the hyperplane $x_j = 0$ for each $j \in K$. After perturbing the Hamiltonian slightly as in the proof of Proposition 9.1, there is a single non-degenerate such trajectory, and we label the corresponding generator of $CW^*(L_I, L_J)$ by the monomial $\prod_{j\in K} z_j^{k_j} = \prod_{j\in K} z_j^{\lfloor k_j \rfloor} u_{\overline{Q}}$.

Similarly in case (2), near Σ_K the Lagrangians L_I and L_J map to orthants where the coordinates have the same signs for $j \in \overline{Q} \cap K$ and different signs for $j \in Q$, and there are time 1 trajectories of X_H from L_I to L_J that wrap k_j times around the hyperplane $x_j = 0$, with $k_j \in \mathbb{Z}_{>0}$ for $j \in \overline{Q} \cap K$ and $k_j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ for $j \in Q$. The corresponding generator of $CW^*(L_I, L_J)$ is denoted by $\prod_{j\in K} z_j^{k_j} = \prod_{j\in K} z_j^{\lfloor k_j \rfloor} u_Q$.

In all cases, with our choice of trivialization of the tangent bundle the degree of these generators is $2k_0$ if $0 \in K$, and zero otherwise. Letting K vary over all subsets which satisfy (1) or (2), we obtain that $CW^*(L_I, L_J)$ is isomorphic as a graded vector space to the right-hand side of (9.7).

Next, we observe that, since L_I and L_J are contractible, by choosing base points $*_I \in L_I$, $*_J \in L_J$, and $* \in \Pi_n$, and reference paths from $*$ to $*_I$ and from $*$ to $*_J$, we can use the reference paths to complete any arc connecting L_I to L_J into a closed loop in Π_n, uniquely up to homotopy. In other terms, the space of homotopy classes of paths from L_I to L_J is a torsor over the fundamental group $\pi_1(\Pi_n, *)$, and can be identified (non-canonically) with it. Passing to homology, we can use this to assign elements of $H_1(\Pi_n, \mathbb{Z}) \simeq \mathbb{Z}^{n+2}/(1, \ldots, 1)$ to the generators of $CW^*(L_I, L_J)$. A more canonical choice in our case shifts by $\frac{1}{2}$ the entries corresponding to elements of Q or $\overline{Q}$, and takes values in the subset Γ_Q of $(\frac{1}{2}\mathbb{Z})^{n+2}/(\frac{1}{2}, \ldots, \frac{1}{2})$ consisting of those tuples whose non-integer entries correspond exactly to the elements of either Q or $\overline{Q}$. (Note that Γ_Q is an $H_1(\Pi_n, \mathbb{Z})$-torsor, and additively, $\Gamma_I + \Gamma_Q = \Gamma_J$.)

With this understood, the class associated to the generator of $CW^*(L_I, L_J)$ that lies near Σ_K and wraps $k_j > 0$ times around the hyperplane $x_j = 0$ for each $j \in K$ (and setting $k_j = 0$ for $j \notin K$) is $(k_0, \ldots, k_{n+1}) \in \Gamma_Q \subset (\frac{1}{2}\mathbb{Z})^{n+2}/(\frac{1}{2}, \ldots, \frac{1}{2})$.

As before, the grading on the Floer complex can be used to avoid quotienting by the diagonal subgroup. Namely, since the above-mentioned generator has degree $2k_0$, its degree and homology class can be encoded simultaneously by the tuple of half-integers $(k_0, \ldots, k_{n+1}) \in (\frac{1}{2}\mathbb{Z})^{n+2}$ (where the non-integer entries correspond exactly to either Q or $\overline{Q}$; we denote the $\mathbb{Z} \times H_1(\Pi_n, \mathbb{Z})$-torsor of such elements by $\hat{\Gamma}_Q$).

With this understood, any two generators of $CW^*(L_I, L_J)$ related by the Floer differential must represent the same homology class, while their degrees differ by 1, hence the corresponding tuples must differ by $(\frac{1}{2}, \dots, \frac{1}{2})$. However, since the subsets Q satisfying conditions (1) or (2) above have at most n elements, all the generators of $CW^*(L_I, L_J)$ correspond to tuples in $\hat{\Gamma}_Q \subset (\frac{1}{2}\mathbb{Z})^{n+2}$ in which all entries are non-negative and at least two are zero. Two such tuples cannot differ by $(\frac{1}{2}, \dots, \frac{1}{2})$. Thus, the Floer differential must vanish identically, and (9.7) holds as an isomorphism of graded vector spaces.

The statement about module structures is a special case of Proposition 9.3, which we state and prove below. □

This calculation again agrees with the mirror symmetry prediction. Without loss of generality we can assume that $0 \notin I$ and $0 \notin J$. $\mathrm{Ext}^*(R/(z_I), R/(z_J))$ can then be computed using the resolution (9.6); the outcome of the calculation matches the right-hand side of (9.7).

Proposition 9.3. *Indexing generators by monomials in $z_0, \dots, z_{n+1}$ with half-integer exponents as in Proposition 9.2, the Floer product $HW^*(L_J, L_K) \otimes HW^*(L_I, L_J) \to HW^*(L_I, L_K)$ is simply given by multiplication of monomials (and quotienting by the appropriate ideals).*

Proof. Recall from the proof of Proposition 9.2 that the degree and homology class of each generator of $HW^*(L_I, L_J)$ can be encoded by an element of $\hat{\Gamma}_{Q=Q(I,J)} \subset (\frac{1}{2}\mathbb{Z})^{n+1}$, which we call its *class*, and that this corresponds to the labelling by monomials. Namely, the class of the generator of $HW^*(L_I, L_J)$ denoted by $\prod z_j^{k_j}$ is $(k_0, \dots, k_{n+1})$. We also recall that the entries of this tuple are non-negative, and that the set K of its non-zero entries must contain either $Q(I, J)$ or its complement (one of which corresponds to the half-integer entries) but cannot entirely contain any of $I, \overline{I}, J, \overline{J}$.

Whenever there is a perturbed holomorphic curve contributing to the Floer product, the relative homology class of the output chord must equal the sum of those of the inputs, and its degree must also be the sum of those of the input. It follows that the class of the output must be the sum of those of the inputs. (Here we recall that, under addition of tuples, $\hat{\Gamma}_{Q(I,J)} + \hat{\Gamma}_{Q(J,K)} = \hat{\Gamma}_{Q(I,K)} \subset (\frac{1}{2}\mathbb{Z})^{n+2}$.)

Thus, given generators $\prod z_j^{k_j} \in HW^*(L_I, L_J)$ and $\prod z_j^{\ell_j} \in HW^*(L_J, L_K)$, their product must be

(a) a multiple of $\prod z_j^{k_j+\ell_j} \in HW^*(L_I, L_K)$ if there is a generator of $HW^*(L_I, L_K)$ representing the class $(k_0 + \ell_0, \dots, k_{n+1} + \ell_{n+1})$,

(b) zero otherwise.

Case (b) obviously agrees with our expected product formula; so it is enough to consider case (a). Let $S = \{j \in \{0, \dots, n+1\} \mid k_j + \ell_j > 0\} = \{j \mid k_j > 0\} \cup \{j \mid \ell_j > 0\}$. By assumption, S contains either $Q(I, J)$ or $\overline{Q}(I, J)$, and it contains either $Q(J,K)$ or $\overline{Q}(J,K)$; but since we are in case (a), it does not contain any of $I,\overline{I},K,\overline{K}$. Since $Q(I,J)$ (resp. $\overline{Q}(I,J)$) is the symmetric difference of I and $\overline{J}$ (resp. I and J), this implies that S does not contain J or $\overline{J}$ either. Moreover, by the same argument, for arbitrary elements $i_1 \in I\cap\overline{S}$ and $i_2 \in \overline{I}\cap\overline{S}$, the n-element subset $T = \overline{\{i_1,i_2\}} \supseteq S$, which does not contain I or $\overline{I}$ by construction, also fails to contain any of $J,\overline{J},K,\overline{K}$.

The holomorphic curves contributing to the product can then be determined by working in a local model near the 0-dimensional stratum Σ_T, using the affine coordinates $x_{T,j}$, $j\in T$, and reducing to a product situation, exactly as in the proof of Proposition 9.1. The only difference is that L_I, L_J, L_K now correspond to different orthants, hence inside each $\mathbb{C}^*$ factor they project to arcs that may be either $\mathbb{R}_+$ or $\mathbb{R}_-$. In $\mathbb{C}^*$, the generators of $CW^*(\mathbb{R}_+,\mathbb{R}_-)$ or $(\mathbb{R}_-,\mathbb{R}_+)$ are naturally labelled by half-integers rather than integers. Nonetheless, for each pair of inputs x_{-k_j} and $x_{-\ell_j}$ there is a unique triangle contributing to the Floer product, whose output is $x_{-k_j-\ell_j}$. Thus, in the product $(\mathbb{C}^*)^n$ there is a unique contribution to the Floer product, and we find that $\left(\prod z_j^{k_j}\right)\cdot\left(\prod z_j^{\ell_j}\right) = \prod z_j^{k_j+\ell_j}$ as expected. □

9.3. Exact triangles and generators. We expect that the A_∞-category $\mathcal{W}(\Pi_n)$ is entirely determined by the cohomology-level computations in Propositions 9.1–9.3 and the existence of certain exact triangles that follow from the general framework introduced in the previous sections.

PROPOSITION 9.4. *Given any partition $\{0,\dots,n+1\} = I\sqcup J\sqcup K$ into three non-empty disjoint subsets, with $0\in K$, there is an exact triangle*

$$L_I \xrightarrow{u_J} L_K \xrightarrow{u_I} L_J \xrightarrow{u_K} L_I[1]. \tag{9.8}$$

Note that $L_K = L_{I\sqcup J}$. Thus, under mirror symmetry the exact triangle (9.8) corresponds to the triangle in $D^b\mathrm{Coh}(Z)$ induced by the short exact sequence

$$0\to \mathcal{O}_{Z_I}\to\mathcal{O}_{Z_{I\sqcup J}}\to\mathcal{O}_{Z_J}\to 0.$$

SKETCH OF PROOF. The easiest way to establish the existence of an exact triangle relating L_I, L_K, L_J in $\mathcal{W}(\Pi_n)$ is by induction on dimension, using symmetry and the lifting functor j. The case $n=1$ holds by [**5**], as reviewed in Sect. 3 (see (3.4)).

Assume first that $\{0,n+1\}\subset K$, so that I and J are subsets of $\{1,\dots,n\}$. Then, as noted in § 8, the lifting functor $\rho:\mathcal{W}(\Pi_{n-1})\to\mathcal{W}(\Pi_n)$ coming from the identification $\Pi_n\simeq(\mathbb{C}^*)^n\setminus\Pi_{n-1}$ maps the objects ℓ_I, ℓ_J and $\ell_{I\sqcup J}$ of $\mathcal{W}(\Pi_{n-1})$ to L_I, L_J, and $L_{I\sqcup J}$. Assuming the conjecture holds for Π_{n-1}, in $\mathcal{W}(\Pi_{n-1})$ we have an exact triangle

$$\ell_I \xrightarrow{v_J} \ell_{I\sqcup J} \xrightarrow{v_I} \ell_J \xrightarrow{v_{K'}} \ell_I[1],$$

where $K' = \{0,\dots,n\}\setminus(I\cup J)$, and we change the notation for the generators of the Floer complexes in $\mathcal{W}(\Pi_{n-1})$ to $v_I, v_J, v_{K'}$ to avoid confusion. Since

A_∞-functors are automatically exact, the image by j of this exact triangle in $\mathcal{W}(\Pi_{n-1})$ is an exact triangle relating L_I, $L_{I\sqcup J}$ and L_J in $\mathcal{W}(\Pi_n)$.

Moreover, recall that the action of j on morphisms comes from the inclusion of some fiber of $f = -x_0/x_{n+1}$ along which the wrapping Hamiltonian reaches its minimum; for example, we can take the fiber above -1. Also recall that we use the embedding (8.1) to match the pictures for Π_{n-1} and Π_n; in this setting, the embedding into the fiber $f^{-1}(-1)$ that gives rise to the functor j is $(y_1:\ldots:y_n:y_0) \mapsto (y_1:\ldots:y_n:y_0:y_0)$.

With this understood, it is not hard to check that $j(v_I) = u_I$ and $j(v_J) = u_J$. Meanwhile, because $v_{K'}$ wraps halfway around the hyperplane $y_0 = 0$, which maps to the base locus of f, its image under the embedding is a trajectory that wraps halfway around *both* of the hyperplanes $x_0 = 0$ and $x_{n+1} = 0$. Hence $j(v_{K'}) = u_{K'\cup\{n+1\}} = u_K$. This completes the proof in the case where $\{0, n+1\} \subset K$.

The remaining cases follow by symmetry under the action of $\mathfrak{S}_{n+2}$. Namely, for $n \geq 2$ at least one of the subsets I, J, K must have cardinality greater than one. Observing that a cyclic permutation of (I, J, K) amounts simply to a rotation of the exact triangle (9.8), and relaxing the setup to allow 0 to be in any of I, J, K, we can assume without loss of generality that $|K| \geq 2$. We can then use $\mathfrak{S}_{n+2}$-symmetry to relabel the elements of $\{0, \ldots, n+1\}$ (with a grading change as needed if the permutation does not fix 0) in order to reduce to the case where $\{0, n+1\} \subset K$. ☐

The two remaining ingredients in the proof of homological mirror symmetry for the pair of pants Π_n are:

Conjecture 9.5. *$\mathcal{W}(\Pi_n)$ is split-generated by the objects $L_{\{i\}}$, $i = 0, 1, \ldots, n+1$.*

This statement follows from Zack Sylvan's work in progress [**27**], as discussed in Remark 1.5. Namely, $\mathcal{W}((\mathbb{C}^*)^n \setminus H)$ is generated by the images of the functors i_1 and i_2 in the pushout diagram; translating to our setup, the image of i_1 coincides with that of the lifting functor j, while $i_2 = \alpha_\infty$. By induction on dimension, assuming that $\mathcal{W}(\Pi_{n-1})$ is split-generated by $\ell_{\{0\}} = \ell_{\{1,\ldots,n\}}, \ell_{\{1\}}, \ldots, \ell_{\{n\}}$, the image of j is split-generated by $j(\ell_{\{1,\ldots,n\}}) = L_{\{1,\ldots,n\}} = L_{\{0,n+1\}}$ and $j(\ell_{\{i\}}) = L_{\{i\}}$, $i = 1, \ldots, n$. Meanwhile, as seen in §8, α_∞ maps the generator of $\mathcal{F}^\circ((\mathbb{C}^*)^n, f)$ to $L_{\{0\}}$. Thus, $\mathcal{W}(\Pi_n)$ is split-generated by $L_{\{0,n+1\}}$, $L_{\{1\}}, \ldots, L_{\{n\}}$, and $L_{\{0\}}$. Finally, by Proposition 9.4 the objects $L_{\{0\}}$, $L_{\{0,n+1\}}$ and $L_{\{n+1\}}$ are related by an exact triangle, so any two of them generate the third one.

Conjecture 9.6. *Up to homotopy, there is a unique A_∞-structure on the algebra $\bigoplus_{I,J} HW^*(L_I, L_J)$ which is compatible with the grading and satisfies the two conditions:*

(1) *any generator appearing in the output of a higher product represents a relative homology class in $H_1(\Pi_n, \mathbb{Z})$ equal to the sum of those of the inputs, and*

(2) $\mu^3(u_I, u_J, u_K) = \pm\mathrm{id}$ *for all* $I \sqcup J \sqcup K = \{0, \dots, n+1\}$ (*as implied by the exact triangles* (9.8)).

Representing generators by monomials in $z_0, \dots, z_{n+1}$ with half-integer exponents as in Proposition 9.3, condition (1) and compatibility with the grading can be restated as: the class of the output of a Floer product μ^k differs from the sum of those of its inputs by $(2-k)(\frac{1}{2}, \dots, \frac{1}{2})$. In other terms, given generators $\gamma_i = \prod z_j^{\ell_{i,j}} \in HW^*(L_{I_{i-1}}, L_{I_i})$ for $i = 1, \dots, k$, the product $\mu^k(\gamma_k, \dots, \gamma_1)$ must be a multiple of $\gamma_{out} = \prod z_j^{\ell_{out,j}}$, where $\ell_{out,j} = \sum_{i=1}^k \ell_{i,j} + \frac{2-k}{2}$ for $j = 0, \dots, n+1$, if $HW^*(L_{I_0}, L_{I_k})$ contains such a generator, and zero otherwise. The conjecture states that, given these constraints, the A_∞-structure is entirely determined by the μ^3 in condition (2). The case $n = 1$ is established in [5] by an explicit Hochschild cohomology calculation.

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