Constant mean curvature surfaces

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Abstract. In this article we survey recent developments in the theory of constant mean curvature surfaces in homogeneous 3-manifolds, as well as some related aspects on existence and descriptive results for H-laminations and CMC foliations of Riemannian n-manifolds.

Contents

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1. Introduction

We present here a survey of some recent developments in the theory of constant mean curvature surfaces in homogeneous 3-manifolds and some related topics on the geometry and existence of H-laminations and CMC foliations of Riemannian n-manifolds. For the most part, the results presented in this manuscript are related to work of the authors. However, in Section 5, we include a brief discussion of some outstanding results described below:

(1) The solution of the Lawson Conjecture (the Clifford Torus is the unique embedded minimal torus up to congruencies in the 3-sphere $\mathbb{S}^3$) by Brendle [**17, 18**]. More generally, Brendle [**16**] proved that Alexandrov embedded constant mean curvature tori in $\mathbb{S}^3$ are rotational (also see Andrews and Li [**7**]).

(2) The result of Marques and Neves [**96, 97**] that a closed embedded minimal surface in $\mathbb{S}^3$ of positive genus has area at least $2\pi^2$, which is a key tool in their proof of the Willmore Conjecture (the Clifford Torus is the unique minimizer of the Willmore energy among tori in $\mathbb{S}^3$).

(3) The classification of properly embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ by Hauswirth, Kilian and Schmidt [**66**], from which it follows that such annuli intersect each level set sphere $\mathbb{S}^2 \times \{t\}$ in a circle.

We begin by pointing out two theorems in the classical setting of $\mathbb{R}^3$. The first theorem concerns the classification of properly embedded minimal planar domains in 3-dimensional Euclidean space $\mathbb{R}^3$:

Theorem 1.1. *The plane, the helicoid, the catenoid and the one-parameter family $\{\mathcal{R}_t\}_{t>0}$ of Riemann minimal examples are the only complete, properly embedded, minimal planar domains in $\mathbb{R}^3$.*

The proof of Theorem 1.1 depends primarily on work of Colding and Minicozzi [**37, 38**], Collin [**40**], López and Ros [**95**], Meeks, Pérez and Ros [**132**] and Meeks and Rosenberg [**138**]. The second theorem concerns the classification of complete, simply connected surfaces embedded in $\mathbb{R}^3$ with non-zero constant mean curvature:

Theorem 1.2. *Complete, simply connected surfaces embedded in $\mathbb{R}^3$ with non-zero constant mean curvature are compact, and thus, by the classical results of Hopf* [**72**] *or Alexandrov* [**3**]*, are round spheres.*

Theorem 1.2 was proven by Meeks and Tinaglia [**143**] and depends on obtaining curvature and radius estimates for embedded disks of non-zero constant mean curvature. We will cover in some detail the proof of this result and will explain how they lead to a deeper understanding of the geometry of complete constant mean curvature surfaces embedded in Riemannian 3-manifolds.

In the setting of homogeneous 3-manifolds X, we will cover results on the uniqueness of constant mean curvature spheres, as described in the next problem.

Hopf Uniqueness Problem: *If S_1, S_2 are immersed spheres in X with the same constant mean curvature, does there exist an isometry I of X with $I(S_1) = S_2$?*

This uniqueness question gets its name from Hopf [**72**], who proved that an immersed sphere in $\mathbb{R}^3$ of constant mean curvature H is a round sphere of radius $1/|H|$. This problem is further motivated by the result of Abresch and Rosenberg [**1, 2**] that constant mean curvature spheres in homogeneous 3-manifolds X with a four-dimensional isometry group are spheres of revolution, from which it can be shown that a positive answer to the Hopf Uniqueness Problem holds in this special setting. More recently, the combined results of Daniel and Mira [**48**] and of Meeks [**107**] gave a positive solution to the Hopf Uniqueness Problem in the case that X is isometric to the solvable Lie group Sol_3 equipped with one of its most symmetric left invariant metrics. In Section 13, we will cover in some detail the approach of Meeks, Mira, Pérez and Ros in [**108, 109, 110**] to solving the Hopf Uniqueness Problem in the remainder of the possible homogeneous geometries for X. Their approach includes classification theorems for the moduli space $\mathcal{M}_X$ of immersed constant mean curvature spheres in X in terms of the Cheeger constant of the universal cover of X.

Another fundamental problem that we will cover is the *Calabi-Yau problem* for complete, constant mean curvature surfaces in locally homogeneous 3-manifolds X, especially in the classical case $X = \mathbb{R}^3$. This problem in the case that the surface is *embedded* asks the following question.

Embedded Calabi-Yau Problem: *Does there exist a complete, non-compact surface of fixed constant mean curvature that is embedded in a given compact subdomain Ω of X?*

Some versions of the Embedded Calabi-Yau Problem also restrict the topology of the surface and/or assert that such a surface can be chosen to be proper in the interior of Ω and/or weaken the condition that the surface be contained in a compact domain to that of being non-proper in X.

We will also discuss the theory of constant mean curvature H-laminations and CMC foliations of Riemannian n-manifolds. By *CMC foliation*, we mean a transversely oriented, codimension-one foliation $\mathcal{F}$ of a Riemannian n-manifold X (not necessarily orientable), such that all of the leaves of $\mathcal{F}$ are two-sided hypersurfaces of constant mean curvature, and where the value of the constant mean curvature can vary from leaf to leaf. Notable results on this subject include the following ones by Meeks, Pérez and Ros: the Stable Limit Leaf Theorem [**131**], the Local Removable Singularity for H-laminations [**121, 133**], the Dynamics Theorem for properly embedded minimal surfaces [**122**], curvature estimates for CMC foliations of Riemannian 3-manifolds [**121**] and the application of these results to classify the CMC foliations of $\mathbb{R}^3$ and $\mathbb{S}^3$ with a closed countable set of singularities [**121**]. In this final section we will give an outline of the proof by Meeks and Pérez [**112**] that a smooth closed n-manifold X admits a smooth CMC foliation for some Riemannian metric if and only if its Euler characteristic vanishes, a result that was proved previously when X is orientable by Oshikiri [**164**].

Henceforth for clarity of exposition, we will call an oriented surface M immersed in a Riemannian 3-manifold X an *H-surface* if it is *connected*, *embedded* and it has *non-negative constant mean curvature H*; our convention of mean curvature gives that a sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ of radius 1 has $H = 1$ when oriented by the inward pointing unit normal to the ball that it bounds. If we say that M is an *immersed* H-surface in X, then that indicates that the surface might *not* be embedded. We will call an H-surface an *H-disk* if the surface is homeomorphic to a closed unit disk in the Euclidean plane.

We now elaborate further on the results mentioned so far and on the organization of the paper. The theory of H-surfaces in $\mathbb{R}^3$ has its roots in the calculus of variations developed by Euler and Lagrange in the 18-th century and in later investigations by, among others, Delaunay, Enneper, Scherk, Schwarz, Riemann and Weierstrass in the 19-th century. During the years, many great mathematicians have contributed to this theory: besides the above mentioned names that belong to the 19-th century, we find fundamental contributions by Bernstein, Courant, Douglas, Hopf, Morrey, Morse, Radó and Shiffman in the first half of the last century. Several global questions and conjectures that arose in this classical subject have only recently been addressed.

The next two classification results give solutions to long standing conjectures. Concerning the first one, several mathematicians pointed out to us that Osserman was the first to ask the question about whether the plane and the helicoid were the only simply connected, complete 0-surfaces; Osserman described this question as potentially the most beautiful extension and explanation of Bernstein's Theorem. For a complete outline of the proof of the second result below, including Riemann's original proof of the classification of minimal surfaces foliated by circles and lines in parallel planes, see the historical account by the first two authors presented in [**119**].

THEOREM 1.3. *A complete, simply connected H-surface in $\mathbb{R}^3$ is a plane, a sphere or a helicoid.*

THEOREM 1.4 (Meeks, Pérez and Ros [**132**]). *Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in $\mathbb{R}^3$ is a plane, a helicoid, a catenoid or one of the Riemann minimal examples. In particular, for every such surface there exists a foliation of $\mathbb{R}^3$ by parallel planes, each of which intersects the surface transversely in a connected curve which is a circle or a line.*

To understand the context and implications of the next theorem, first note that every simply connected homogenous 3-manifold X that is not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, where κ is the non-zero Gaussian curvature of $\mathbb{S}^2$, is isometric to a *metric Lie group*, i.e., a Lie group equipped with a left invariant metric; see [**117**] for a proof of this fact. In particular, if X is compact, simply connected and homogeneous, then it is isometric to the Lie group

$$\mathrm{SU}(2) = \{A \in \mathcal{M}_2(\mathbb{C}) \mid A^t\overline{A} = I_2,\ \det(A) = 1\} \tag{1}$$

with a left invariant metric. When X is homogenous and diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ or more generally when X has a four-dimensional isometry group, Abresch and Rosenberg [**1, 2**] proved that for every $H \geq 0$, there exists a unique immersed H-sphere in X and this sphere is embedded when X is diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$; they obtained these results by first proving that every such sphere is a surface of revolution and then, using this symmetry property, they classified the examples. In the classical setting of $X = \mathbb{R}^3$, Hopf [**72**] earlier proved that an immersed H-sphere is a round sphere of radius $1/H$. Motivated by these results, the uniqueness up to ambient isometry question for immersed H-spheres in X became known as the previously mentioned Hopf Uniqueness Problem in homogeneous 3-manifolds; since spheres are simply connected and lift to the universal cover of X, we henceforth will only consider this uniqueness problem with the additional condition that the homogeneous 3-manifold X be simply connected.

The next theorem by Meeks, Mira, Pérez and Ros [**109**] gives a complete solution to the Hopf Uniqueness Problem and to the classification of immersed H-spheres when the homogeneous manifold X is diffeomorphic to $\mathbb{S}^3$. These authors are confident that they also have a proof of the classification for the moduli space of immersed H-spheres in a general simply connected, homogeneous 3-manifold X, and this is work in progress in [**108**]. Their proposed classification result depends on their characterization in [**110**] of the Cheeger constant of X as being twice the value of the infimum of the mean curvatures of immersed closed H-surfaces in the space. See Theorem 13.3 in Section 13 for a more complete version of the next theorem and for further explanations.

Theorem 1.5 (Compact case of the Hopf Uniqueness Problem). *Let X be* SU(2) *equipped with a left invariant metric and let $\mathcal{M}_X$ be the moduli space of immersed constant mean curvature spheres in X identified up to left translations. Then for every $H \in [0,\infty)$ there exists an oriented immersed H-sphere S_H in X and S_H is the unique immersed H-sphere in X up to left translations. Hence, $\mathcal{M}_X$ is naturally parameterized by the interval $[0,\infty)$ of all possible mean curvature values.*

The proofs of Theorems 1.3, 1.4 and 1.5 depend on a series of new results and theory that have been developed over the past decade. The purpose of this article is two-fold. The first goal is to explain these results and the history behind them in a manner accessible to a graduate student interested in Differential Geometry or Geometric Analysis, and the second goal is to explain how these results and theory transcend their application to the proofs of Theorems 1.3, 1.4 and 1.5 and enhance the understanding of the theory, giving rise to new theorems and conjectures. Since much of this material for minimal surfaces is well-documented in the survey [**116**] and book [**118**] by the first two authors, we will focus somewhat more of our attention here on the case when $H > 0$ and we refer the interested reader to [**116, 118, 119**] for further background on the minimal surface results that we mention here.

Before proceeding, we make a few general comments on the proof of Theorem 1.3 that we feel can suggest to the reader a visual idea of what is going on. The most natural motivation for understanding this theorem, Theorem 1.4 and other results presented in this survey is to try to answer the following heuristic question:

What are the possible shapes of surfaces which satisfy a variational principle and have a given topology?

For instance, if the variational equation expresses the critical points of the area functional with respect to compactly supported volume preserving variations, and the requested topology is the simplest one of a disk, then Theorem 1.3 says that the possible shapes for complete non-compact examples are the trivial one given by a plane and (after a rotation) an infinite double spiral staircase, which is a visual description of a vertical helicoid; in particular there are no non-compact examples which are not minimal.

A more precise description of the double spiral staircase nature of a vertical helicoid is that this surface is the union of two infinite-sheeted multi-valued graphs, which are glued along a vertical axis. Crucial in the proof of Theorem 1.3 are local and global results of Colding and Minicozzi on 0-disks [**34, 35, 37, 38**], global results of Meeks and Rosenberg for complete 0-disks [**138**], and generalizations of them by Meeks and Tinaglia to the $(H = 1)$-setting [**142, 143, 146, 148**]. The local results of Colding and Minicozzi describe the structure of compact embedded minimal disks (with boundary) as essentially being modeled by the plane or the helicoid, i.e., either they are graphs or pairs of finitely sheeted multi-valued graphs glued along an "axis". In the case of 1-disks, the recent work of Meeks and Tinaglia demonstrates that 1-disks are modeled only by graphs away from their boundary curves, in other words, there exist curvature estimates for 1-disks at points at any fixed positive intrinsic distance from their boundary curves.

THEOREM 1.6 (Curvature Estimates, Meeks, Tinaglia [**143**]). *Given δ, $\mathcal{H} > 0$, there exists a $K(\delta, \mathcal{H}) \geq \sqrt{2}\mathcal{H}$ such that any H-disk M in $\mathbb{R}^3$ with $H \geq \mathcal{H}$ satisfies*

$$\sup_{\{p \in M \mid d_M(p, \partial M) \geq \delta\}} |A_M|(p) \leq K(\delta, \mathcal{H}),$$

where $|A_M|$ is the norm of the second fundamental form and d_M is the intrinsic distance function of M.

We wish to emphasize that the curvature estimates for H-disks given in Theorem 1.6 depend only on the fixed *lower* positive bound $\mathcal{H}$ for their mean curvature, and we next explain a simple but important consequence of this observation. Recall that the *radius* of a compact Riemannian surface with boundary is the maximum intrinsic distance of points in the surface to its boundary; we claim that the radius of a 1-disk must be less than $K(1,1)$, where $K(1,1)$ is the constant given in the above theorem with $\delta = 1, \mathcal{H} = 1$. To see this, let Σ be a 1-disk and let $\widehat{\Sigma} = \frac{1}{K(1,1)} \cdot \Sigma$ be the homothetic scaling of Σ by the factor $\frac{1}{K(1,1)}$. Note that the mean curvature of $\widehat{\Sigma}$ is $K(1,1) \geq \sqrt{2} > 1$ and thus, the classical inequality $\text{Trace}(A) \leq \sqrt{2}|A|$ valid for every symmetric 2×2 real matrix A implies that

$$\inf_{p \in \widehat{\Sigma}} |A_{\widehat{\Sigma}}|(p) \geq \sqrt{2} \cdot K(1,1) > K(1,1). \tag{2}$$

Therefore, the radius of $\widehat{\Sigma}$ must be less than 1, otherwise $\{p \in \widehat{\Sigma} \mid d_{\widehat{\Sigma}}(p, \partial\widehat{\Sigma}) \geq 1\} \neq \emptyset$, and then Theorem 1.6 with $\delta = 1, \mathcal{H} = 1$ would give

$$\sup_{\{p \in \widehat{\Sigma} \,|\, d_{\widehat{\Sigma}}(p, \partial\widehat{\Sigma}) \geq 1\}} |A_{\widehat{\Sigma}}|(p) \leq K(1,1),$$

contradicting (2). This contradiction implies that the radius of $\Sigma = K(1,1) \cdot \widehat{\Sigma}$ is less than $K(1,1)$, which proves our claim. With these considerations in mind, it is perhaps not too surprising that the proof of the curvature estimates in Theorem 1.6 is intertwined with the proof of the following result on the existence of radius estimates for $(H > 0)$-disks.

Notice that the next theorem implies that there do not exist complete $(H > 0)$-planes in $\mathbb{R}^3$, since such planes contain topological disks of arbitrarily large radius, which resolves the $H > 0$ case in the proof of Theorem 1.3.

THEOREM 1.7 (Radius Estimates, Meeks, Tinaglia [**143**]). *There exists an $\mathcal{R} \geq \pi$ such that any H-disk in $\mathbb{R}^3$ with $H > 0$ has radius less than $\mathcal{R}/H$.*

Another important result in the proof of Theorem 1.3, as well as in the proofs of Theorems 1.4, 1.6 and 1.7, concerns global aspects of limits of H-disks and genus-zero H-surfaces, which were first described by Colding and Minicozzi in their Lamination Theorem for 0-Disks and more recently by Meeks and Tinaglia in their Lamination Theorem for H-Disks, see Theorem 6.1 below. A last key ingredient in the proofs of the aforementioned theorems is the following chord-arc result that allows one to relate intrinsic and extrinsic distances on an H-disk at points far from its boundary and at the same time near to points where the surface is not too flat; this chord-arc result implies that a complete simply connected H-surface must be properly embedded in $\mathbb{R}^3$. The proof of the next theorem by Meeks and Tinaglia [**142**] depends on results in [**143, 146, 148**] and the strategy of their proof follows and generalizes the proof of a similar chord-arc estimate for 0-disks by Colding and Minicozzi in [**38**].

We will denote by d_Σ, $B_\Sigma(p,r)$ respectively the intrinsic distance function and the open intrinsic ball of radius $r>0$ centered at a point p in a Riemannian surface Σ.

Theorem 1.8 (Chord-arc property for H-disks). *There exists a $C>1$ so that the following holds. Suppose that $\Sigma\subset\mathbb{R}^3$ is an H-disk, $\vec{0}\in\Sigma$ and $R>r_0>0$. If $B_\Sigma(\vec{0},CR)\subset\Sigma-\partial\Sigma$ and $\sup_{B_\Sigma(\vec{0},(1-\frac{\sqrt{2}}{2})r_0)}|A_\Sigma|>r_0^{-1}$, then*

$$\frac{1}{3}d_\Sigma(x,\vec{0})\leq\frac{1}{2}\|x\|+r_0,\ \text{for all } x\in B_\Sigma(\vec{0},R).$$

Our survey is organized as follows. We present the main definitions and background material in the introductory Section 2. In that section we also briefly describe geometrically, as well as analytically, some of the important classical examples of proper H-surfaces in $\mathbb{R}^3$ that we will need later on; understanding these key examples is crucial in obtaining a feeling for this subject (as in many other branches of mathematics), as well as in making important theoretical advances and asking the right questions. Before going further, the reader will probably benefit by taking a few minutes to view and identify the computer graphics images of these surfaces that appear near the end of Section 2.3, and to read the brief historical and descriptive comments related to the individual images.

In Section 3 we cover conservation laws that pair Killing fields in a Riemannian n-manifold X with elements in the homology group $H_{n-2}(M)$ of any H-hypersurface M in X. In our setting of Riemannian 3-manifolds X, these conservation laws are interpreted as scalar fluxes induced by a Killing field K across 1-cycles γ on an H-surface M, and these fluxes only depend on the homology class of the 1-cycle. These flux invariants play an important role in describing both local and global aspects of the geometry of H-surfaces in homogeneous 3-manifolds, as we will illustrate in later sections.

In Section 4 we summarize a number of results concerning proper H-surfaces in $\mathbb{R}^3$ of finite genus as seen in the light of the recent contributions of Colding and Minicozzi [**38**] and Meeks and Tinaglia [**143**] that demonstrate that complete H-surfaces in $\mathbb{R}^3$ of finite topology are proper. Also we briefly explain here the recent classification of proper 0-surfaces that are planar domains of infinite topology by Meeks, Pérez and Ros [**132**], as well as the description by these authors of the asymptotic behavior of proper 0-surfaces with finite genus and an infinite number of ends. In particular, we will explain how Theorems 1.3 and 1.4 follow from the results described in this section. At the end of Section 4 we explain in some detail the analytic construction by Meeks and Pérez [**113**] of certain proper 0-annuli with boundary $E_{a,b}$, where $(a,b)\in[0,\infty)\times\mathbb{R}$, that are models for ends of 0-annuli in $\mathbb{R}^3$ with infinite total curvature; in other words, every complete injective 0-immersion $\psi\colon\mathbb{S}^1\times[0,\infty)\to\mathbb{R}^3$ with infinite total curvature is, after a rigid motion, asymptotic to exactly one of the annuli $E_{a,b}$. These special embedded 0-annular ends $E_{a,b}$ are called *canonical ends* and their geometry is related to the distinct flux vectors of their boundary curves, after making certain geometric normalizations.

In Section 5 we cover recent results of Brendle [**18**] on his solution of the Lawson Conjecture, the classification of complete embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$, all of which are periodic and which intersect the level set spheres $\mathbb{S}^2 \times \{t\}$ in round circles by Hauswirth, Kilian and Schmidt [**73**], and the area estimate from below by $2\pi^2$ for closed embedded minimal surfaces of positive genus in $\mathbb{S}^3$ by Marques and Neves [**96, 97**], which led them to a proof of the Willmore conjecture.

In Sections 6 and 7 we study limits of sequences of H-surfaces. Depending on whether or not such a sequence has uniform local bounds for the area and/or for the second fundamental form, new objects can appear in the limit. For instance, in presence of local uniform bounds for the area and second fundamental form of the surfaces in the sequence, the classical Arzelà-Ascoli theorem implies subsequential convergence to an H-surface. When the sequence has local uniform bounds for the second fundamental form but it fails to have local uniform bounds for the area, then (weak) H-laminations appear in the limit; this last notion will be studied in Section 7. The reader not familiar with the subject of weak H-laminations should think about a geodesic γ on a Riemannian surface. If γ is complete and embedded (a one-to-one immersion), then its closure is a geodesic lamination $\mathcal{L}$ of the surface. When γ has no accumulation points, then it is proper and it is the unique leaf of $\mathcal{L}$. Otherwise, there pass complete, embedded, pairwise disjoint geodesics through the accumulation points, and these geodesics together with γ form the leaves of the geodesic lamination $\mathcal{L}$. A similar result is true for a complete H-surface of locally bounded curvature (i.e., whose norm of the second fundamental form is bounded in compact extrinsic balls) in a Riemannian 3-manifold [**139**]. However, when $H > 0$, two leaves of the resulting lamination might intersect non-transversely at some point p where the unit normal vectors to the leaves point in opposite directions, and in this case we call this structure a *weak* H-lamination; still it holds that nearby such a point p and on the mean convex side of each of the two intersecting leaves, there is a lamination structure (no intersections). In Section 7 we also cover the Stable Limit Leaf Theorem of Meeks, Pérez and Ros [**130, 131**] and the Limit Lamination Theorem for 0-surfaces of Finite Genus by Colding and Minicozzi [**39**].

In Section 8 we explain some further local and global results by Colding and Minicozzi in [**38**], where among other things they prove that complete 0-surfaces of finite topology in $\mathbb{R}^3$ are proper. We explain here the results of Meeks and Rosenberg [**139**] on generalizations of the work of Colding and Minicozzi in [**38**] to the Riemannian 3-manifold setting.

In Section 9, we examine how the theoretical results in the previous sections lead to deep global results in the classical theory in $\mathbb{R}^3$, as well as to a general understanding of the local geometry of any complete H-surface M in any homogeneously regular 3-manifold (see Definition 2.25

below for the concept of homogeneously regular 3-manifold). This local description is given in two local picture theorems by Meeks, Pérez and Ros [**122, 124**], each of which describes the local extrinsic geometry of M near points of concentrated curvature (the *Local Picture Theorem on the Scale of Curvature*) or of concentrated topology (the *Local Picture Theorem on the Scale of Topology*). In order to understand the second local picture theorem, we develop in this section the important notion of a *minimal parking garage structure* on $\mathbb{R}^3$, which is one of the possible limiting pictures in the topological setting. Crucial in these local pictures is a local result that calculates the rate of growth of the norm of the second fundamental form of an H-lamination in a punctured ball of a Riemannian 3-manifold when approaching a singularity of the lamination occurring at the center of the ball (the *Local Removable Singularity Theorem*). Global applications of the Local Removable Singularity Theorem to the classical theory are also discussed here; the most important of these applications are the *Quadratic Curvature Decay Theorem* and the *Dynamics Theorem* for proper 0-surfaces in $\mathbb{R}^3$ by Meeks, Pérez and Ros [**133**].

In Sections 10 and 11 we cover some results of Meeks and Tinaglia mentioned previously, as well as their Dynamics and Minimal Elements Theorems for complete strongly Alexandrov embedded 1-surfaces in $\mathbb{R}^3$ from [**149**]. This Minimal Elements Theorem is needed in the proofs of the curvature and radius estimates stated previously in Theorems 1.6 and 1.7.

In Section 12, we briefly discuss what are usually referred to as the *Calabi-Yau problems* for complete H-surfaces in $\mathbb{R}^3$ and in homogeneous 3-manifolds. These problems arose from questions asked by Calabi [**22**] and Yau (see page 212 in [**27**] and problem 91 in [**210**]) concerning the existence of complete, immersed 0-surfaces that are constrained to lie in a given region of $\mathbb{R}^3$, such as in a bounded domain. Various aspects of the Calabi-Yau problems constitute an active field of research with an interesting mix of positive and negative results. We include here a few recent fundamental advances on this problem that are not covered adequately in previous sections of the survey. We end this section with the fundamental existence Conjecture 12.3 on the embedded Calabi-Yau problem for complete 0-surfaces.

In Section 13 we discuss recent results on the Hopf Uniqueness Problem, as the aforementioned Theorem 1.5. Section 14 is devoted to material on the existence and geometry of CMC foliations of Riemannian n-manifolds. This section includes results by Meeks, Pérez and Ros on the classification of CMC foliations of $\mathbb{R}^3$ or $\mathbb{S}^3$ with a countable number of singularities given in Theorem 14.1 in the general setting of weak CMC foliations and their curvature estimates given in Theorem 14.2 for weak CMC foliations of Riemannian 3-manifolds, as well as an existence theorem for Riemannian metrics together with CMC foliations in compact n-dimensional manifolds X with the property that the Euler characteristic of X is zero (Meeks and Pérez [**112**]).

The final Section 15 of this survey is devoted to a discussion of some of the outstanding conjectures on the geometry of H-surfaces in locally homogeneous 3-manifolds.

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2. Basic results in theory of H-surfaces in $\mathbb{R}^3$

We will devote this section to giving a fast tour through the foundations of the theory, providing enough material for the reader to understand the results to be explained in future sections. While our exposition here emphasizes H-surfaces in the classical $\mathbb{R}^3$ setting, we will sometimes mention how the concept of H-surface generalizes to the Riemannian 3-manifold setting. In the sequel, $B(p,r)$ will denote the open ball centered at a point $p \in \mathbb{R}^3$ with radius $r > 0$.

2.1. Equivalent definitions of H-surfaces. One can define an H-surface from different points of view. The equivalences between these starting points give insight into the richness of the classical theory of H-surfaces in $\mathbb{R}^3$ and its connections with other branches of mathematics.

Throughout the paper, all surfaces will be assumed to be orientable unless otherwise stated. Consider the Gauss map $N\colon M \to \mathbb{S}^2$ of a surface $M \subset \mathbb{R}^3$. Then, the tangent space T_pM of M at $p \in M$ can be identified as a subspace of $\mathbb{R}^3$ under parallel translation with the tangent space $T_{N(p)}\mathbb{S}^2$ to the unit sphere at $N(p)$. Hence, one can view the differential $A_M(p) = -dN_p$ as an endomorphism of T_pM, called the *shape operator*. $A_M(p)$ is a symmetric linear transformation, whose orthogonal eigenvectors are called the *principal directions* of M at p, and the corresponding eigenvalues are the *principal curvatures* of M at p. Since the (possibly non-constant) mean curvature function H of M equals the arithmetic mean of such principal curvatures (or the average normal curvature), then we can write

$$A_M(p) = -dN_p = \begin{pmatrix} H+a & b \\ b & H-a \end{pmatrix} \tag{3}$$

in an orthonormal tangent basis (here H, a, b depend on p).

Note that by the Cauchy-Riemann equations, when H is identically zero, then the Gauss map of M is anticonformal when the sphere $\mathbb{S}^2$ is taken with its outward pointing normal, and it is conformal when the sphere $\mathbb{S}^2$ is taken with inward pointing normal, which is the orientation induced by stereographic projection of $\mathbb{S}^2$ from its north pole $(0,0,1)$ to $\mathbb{C}\cup\{\infty\}$, and we denote this meromorphic function by $g\colon M\to\mathbb{C}\cup\{\infty\}$.

DEFINITION 2.1. The formula $\langle A,B\rangle = \mathrm{Trace}(AB)$ endows the space of 2×2 real symmetric matrices with a positive definite inner product, with associated norm $|A| = \sqrt{\sum_{i,j} a_{ij}^2}$ if $A=(a_{ij})_{i,j}$. The norm of the second fundamental form $|A_M|(p)$ of M at the point p is the norm of the matrix given by (3), or equivalently, $|A_M|(p) = \sqrt{\lambda_1^2+\lambda_2^2}$, where λ_1,λ_2 are the principal curvatures of M at p.

DEFINITION 2.2.
(1) A surface $M\subset\mathbb{R}^3$ is *minimal* if and only if its mean curvature vanishes identically.
(2) A surface $M\subset\mathbb{R}^3$ is an *H-surface* if and only if has constant mean curvature $H\in\mathbb{R}$, which we will always assume is non-negative after appropriately orienting M.

Often, it is useful to identify a Riemannian surface M with its image under an isometric embedding. Since minimality is a local concept, the notion of minimality can be applied to an isometrically immersed surface $\psi\colon M\to\mathbb{R}^3$. Recall the well-known vector-valued formula

$$\Delta\psi = 2HN,$$

where Δ is the Riemannian Laplacian on M, and $H\colon M\to\mathbb{R}$ is the mean curvature function of M with respect to the Gauss map N. In particular, the coordinate functions of an immersed 0-surface are harmonic.

Let Ω be a subdomain with compact closure in a surface $M\subset\mathbb{R}^3$. If we perturb normally the inclusion map ψ on Ω by a compactly supported smooth function $u\in C_0^\infty(\Omega)$, then $\psi+tuN$ is again an immersion whenever $|t|<\varepsilon$, for some ε sufficiently small. The mean curvature function H of M relates to the infinitesimal variation of the area functional $A(t) = \mathrm{Area}[(\psi+tuN)(\Omega)]$ for compactly supported normal variations by means of the *first variation of area* (see for instance [**162**]):

$$A'(0) = -2\int_\Omega uH\,dA, \tag{4}$$

where dA stands for the area element of M. Formula (4) implies that compact immersed 0-surfaces are critical points of the area functional for compactly supported variations. In fact, a consequence of the *second variation of area* is that any point in a 0-surface has a neighborhood with least-area relative to its boundary. This property justifies the word “minimal” for these surfaces.

Another consequence of (4) is that when M is a compact H-surface with boundary (now $H\in[0,\infty)$ is a constant), then M is a critical point of the area functional for compactly supported variations that infinitesimally preserve the volume, i.e., for functions $u\in C_0^\infty(M)$ with $\int_M u\,dA=0$. This fact can be generalized to a Riemannian 3-manifold X and explains why a compact smooth domain W in X whose boundary surface area is critical with respect to the areas of the boundaries of nearby smooth domains with the same volume as W, must have boundary $M = \partial W$ with constant mean curvature. When such a domain W in X has least area with respect to the boundaries of all smooth compact subdomains in X with volume V, then Ω is called a *solution to the isoperimetric problem in X for the volume V.*

The above discussion establishes 0-surfaces as the 2-dimensional analog to geodesics in Riemannian geometry, and connects the theory of H-surfaces with one of the most important classical branches of mathematics: the calculus of variations. Coming back to our isometric immersion $\psi\colon M \to \mathbb{R}^3$, another well-known functional in the calculus of variations besides the area functional A is the *Dirichlet energy,*

$$E = \int_\Omega |\nabla \psi|^2 dA,$$

where again $\Omega \subset M$ is a subdomain with compact closure. These functionals are related by the inequality $E \geq 2A$, with equality if and only if ψ is conformal. This conformality condition is not restrictive, as follows from the existence of local isothermal or conformal coordinates for any 2-dimensional Riemannian manifold, modeled on domains of $\mathbb{C}$.

From a physical point of view, the mean curvature function of a homogeneous membrane (surface) separating two media is equal, up to a non-zero multiplicative constant, to the difference between the pressures at the two sides of the surface. When this pressure difference is zero, then the membrane has zero mean curvature. Therefore, soap films in space are physical realizations of the ideal concept of a 0-surface and soap bubbles are physical realizations of the ideal concept of an $(H > 0)$-surface.

We now summarize these various properties for 0- and H-surfaces.

DEFINITION 2.3. Let $\psi = (x_1, x_2, x_3)\colon M \to \mathbb{R}^3$ be an isometric immersion of a Riemannian surface into space and we identify M with its image. Then, M is *minimal*, or equivalently an immersed 0-surface, if and only if any of the following equivalent properties hold:

1. The mean curvature function of M vanishes identically.
2. The coordinate function x_i is a harmonic function on M for each i. In other words, $\Delta x_i = 0$, where Δ is the Riemannian Laplacian on M.
3. M is a critical point of the *area functional* for all compactly supported variations.
4. Every point $p \in M$ has a neighborhood D_p with least area relative to its boundary.
5. M is a critical point of the Dirichlet energy for all compactly supported variations, or equivalently if any point $p \in M$ has a neighborhood D_p with least energy relative to its boundary.
6. Every point $p \in M$ has a neighborhood D_p that is equal to the unique idealized soap film with boundary ∂D_p.

7. The stereographically projected Gauss map $g\colon M \to \mathbb{C} \cup \{\infty\}$ is meromorphic with respect to the underlying Riemann surface structure on M.

DEFINITION 2.4. Let $\psi = (x_1, x_2, x_3)\colon M \to \mathbb{R}^3$ be an injective isometric immersion of a Riemannian surface into space and we identify M with its image. Then, M is an *H-surface* for some $H \geq 0$ if and only if any of the following equivalent properties hold:

1. The mean curvature function of M is constant.
2. M is a critical point of the *area functional* for all compactly supported volume preserving normal variations.
3. Every point $p \in M$ has a neighborhood D_p that is equal to an idealized soap bubble with boundary ∂D_p, i.e., considering ∂D_p to be a wire, then D_p is realizable by a soap bubble bounding ∂D_p where the air pressure has a constant difference on its opposite sides.
4. Given a point $p \in M$, there exists a small $\varepsilon > 0$ such that the component D_p of $\mathbb{B}(p, \varepsilon) \cap M$ containing p, which is part of the oriented boundary of a component W of $\mathbb{B}(p, \varepsilon) - D_p$, satisfies the following constrained area-minimizing property. For any compact embedded oriented surface $\Sigma \subset \mathbb{B}(p, \varepsilon)$ with $\partial \Sigma = \partial D_p$ that is homologous in $\mathbb{B}(p, \varepsilon)$ to D_p relative to its boundary and which lies in the oriented boundary of a component W_Σ of $\mathbb{B}(p, \varepsilon) - \Sigma$ with the same volume as W, then the area of Σ is not less than the area of D_p.

This concludes our discussion of the equivalent definitions of H-surfaces. Returning to our background discussion, we note that Definition 2.3 and the maximum principle for harmonic functions imply that no compact immersed 0-surfaces in $\mathbb{R}^3$ without boundary exist. On the contrary, there exist many immersed closed surfaces with non-zero constant mean curvature [**78, 79, 205**] but by the next classical result this is not possible for spheres. We state the next theorem of Hopf in the 3-dimensional space form setting, where his original proof in [**72**] can be adapted.

THEOREM 2.5 (Hopf Theorem). *An immersed H-sphere in a complete, simply connected 3-dimensional manifold $\mathbb{Q}^3(c)$ of constant sectional curvature c is a round sphere.*

Recall that H-surfaces are assumed to be embedded, whereas *immersed* H-surfaces need not be. Round spheres are also the only closed H-surfaces in $\mathbb{R}^3$. This uniqueness result follows from the classical result of Alexandrov below and its proof is based on the so called *Alexandrov reflection principle,* which in turn is based on the interior and boundary maximum principles for H-surfaces given in Theorems 2.11 and 2.12 below. Motivated by the importance of the Alexandrov reflection principle, we will briefly explain Alexandrov's proof of the next theorem; this proof appears immediately after the statements of maximum principles given in Theorems 2.11 and 2.12.

THEOREM 2.6 (Alexandrov [**3**]). *Round spheres are the only closed H-surfaces in $\mathbb{R}^3$. More generally, if $\psi\colon M \to \mathbb{R}^3$ is a closed immersed H-surface that extends as the boundary of a compact 3-manifold which is immersed in $\mathbb{R}^3$, then $\psi(M)$ is a round sphere.*

In this survey we will focus on the study of *complete* H-surfaces (possibly with boundary), in the sense that all geodesics in them can be indefinitely extended up to the boundary of the surface. Note that with respect to the intrinsic Riemannian distance function between points on a surface, the property of being "geodesically complete" is equivalent to the surface being a complete metric space. A stronger global hypothesis, whose relationship with completeness is an active field of research in 0-surface theory, is presented in the following definition.

DEFINITION 2.7. A map $f\colon X \to Y$ between topological spaces is *proper* if $f^{-1}(C)$ is compact in X for any compact set $C \subset Y$. A subset $Y' \subset Y$ is called proper if the inclusion map $i\colon Y' \to Y$ is proper.

The Gaussian curvature function K of an immersed surface M in $\mathbb{R}^3$ is the product of its principal curvatures, or equivalently, the determinant of the shape operator A_M. Thus $|K|$ is the absolute value of the Jacobian of the Gauss map $N\colon M \to \mathbb{S}^2$. If M is minimal, then its principal curvatures are oppositely signed and thus, K is non-positive. Therefore, after integrating K on M (note that this integral may be $-\infty$ or a non-positive number), we obtain the same quantity as when computing the negative of the spherical area of M through its Gauss map, counting multiplicities. This quantity is called the *total curvature* of the immersed 0-surface:

$$C(M) = \int_M K\, dA = -\mathrm{Area}(N\colon M \to \mathbb{S}^2). \tag{5}$$

2.2. Weierstrass representation. Recall that the Gauss map of an immersed 0-surface M can be viewed as a meromorphic function $g\colon M \to \mathbb{C} \cup \{\infty\}$ on the underlying Riemann surface. Furthermore, the harmonicity of the third coordinate function x_3 of M lets us define (at least locally) its harmonic conjugate function x_3^*; hence, the so-called *height differential* $dh = dx_3 + i dx_3^*$ is a holomorphic differential on M. The pair (g, dh) is usually referred to as the *Weierstrass data* of the immersed 0-surface, and the 0-immersion $\psi\colon M \to \mathbb{R}^3$ can be expressed up to translation by $\psi(p_0)$, $p_0 \in M$, solely in terms of this data as

$$\psi(p) = \mathrm{Re} \int_{p_0}^{p} \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1 \right) dh. \tag{6}$$

The pair (g, dh) satisfies certain compatibility conditions, stated in assertions *i), ii)* of Theorem 2.8 below. The key point is that this procedure has the following converse, which gives a cookbook-type recipe for analytically defining any immersed 0-surface.

THEOREM 2.8 (Osserman [**165**]). *Let M be a Riemann surface, $g\colon M \to \mathbb{C} \cup \{\infty\}$ a meromorphic function and dh a holomorphic one-form on M. Assume that:*

i) The zeros of dh coincide with the poles and zeros of g, with the same order.
ii) For any closed curve $\gamma \subset M$,

$$\overline{\int_\gamma g\,dh} = \int_\gamma \frac{dh}{g}, \qquad \mathrm{Re}\int_\gamma dh = 0, \tag{7}$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$. Then, the map $\psi\colon M \to \mathbb{R}^3$ given by (6) *is a conformal 0-immersion with Weierstrass data* (g, dh).

All local geometric invariants of an immersed 0-surface M can be expressed in terms of its Weierstrass data. For instance, the first and second fundamental forms are respectively (see [**68, 167**]):

$$ds^2 = \left(\frac{1}{2}(|g| + |g|^{-1})|dh|\right)^2, \qquad II(v,v) = \mathrm{Re}\left(\frac{dg}{g}(v) \cdot dh(v)\right), \tag{8}$$

where v is a tangent vector to M, and the Gaussian curvature is

$$K = -\left(\frac{4\,|dg/g|}{(|g| + |g|^{-1})^2|dh|}\right)^2. \tag{9}$$

If (g, dh) is the Weierstrass data of an immersed 0-surface $\psi\colon M \to \mathbb{R}^3$, then for each $\lambda > 0$ the pair $(\lambda g, dh)$ satisfies condition *i)* of Theorem 2.8 and the second equation in (7). The first equation in (7) holds for this new Weierstrass data if and only if

$$\int_\gamma g\,dh = \int_\gamma \frac{dh}{g} = 0$$

for all homology classes γ in M, a condition that can be stated in terms of the notion of flux, which we now define. Given an immersed 0-surface M with Weierstrass data (g, dh), the *flux vector* along a closed curve $\gamma \subset M$ is defined as

$$F(\gamma) = \int_\gamma \mathrm{Rot}_{90^\circ}(\gamma') = \mathrm{Im}\int_\gamma \left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1\right) dh \in \mathbb{R}^3, \tag{10}$$

where Rot_{90° denotes the rotation by angle $\pi/2$ in the tangent plane of M at any point.

2.3. Some interesting examples of complete H-surfaces. Throughout the presentation of the examples in this section, we will freely use Collin's Theorem [**40**] that states that proper finite topology 0-surfaces in $\mathbb{R}^3$ with more than one end have finite total curvature and Theorem 4.1 on the properness of complete H-surfaces of finite topology; see Section 4 for further discussion of these important and deep results.

The most familiar examples of 1-surfaces in $\mathbb{R}^3$ are spheres of radius one and cylinders of radius 1/2, both of which are surfaces of revolution.

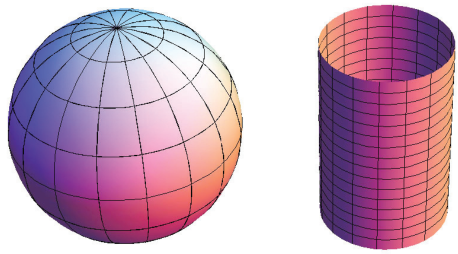

FIGURE 1. A sphere and a cylinder.

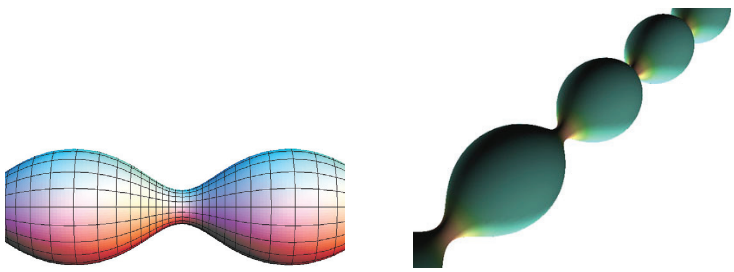

FIGURE 2. Two different Delaunay surfaces.

The Delaunay surfaces $\mathcal{D}_t$, $t \in (0, \frac{\pi}{2}]$. In 1841, Delaunay [**49**] classified the immersed 1-surfaces of revolution in $\mathbb{R}^3$. We will call the embedded ones (unduloids) *Delaunay surfaces,* see Figure 2. The $\mathcal{D}_t$, $t \in (0, \frac{\pi}{2}]$, form a one-parameter family of proper 1-surfaces of revolution that are invariant under a translation along the revolution axis. $\mathcal{D}_t$ is a cylinder for $t = \frac{\pi}{2}$, , whereas with $t \to 0$, then $\mathcal{D}_t$ converges to a chain of tangential spheres with radius 1. In fact, the parameter $t \in (0, \frac{\pi}{2}]$ can be viewed as the length of the CMC flux vector of $\mathcal{D}_t$, computed on any of its circles; see Definition 3.1 below where the CMC flux is defined.

We will next use the Weierstrass representation for introducing some of the most celebrated complete immersed 0-surfaces.

The catenoid. $M = \mathbb{C} - \{0\}$, $g(z) = z$, $dh = \frac{dz}{z}$, see Figure 3 Left. In 1741, Euler [**52**] discovered that when a catenary $x_1 = \cosh x_3$ is rotated around the x_3-axis, one obtains a surface which minimizes area among surfaces of revolution after prescribing boundary values for the generating curves. This surface was called the *alysseid* or since Plateau's time, the catenoid. In 1776, Meusnier verified that the catenoid is locally a solution of Lagrange's equation, which just means that it locally minimizes area relative to local boundaries. This surface has genus zero, two ends and total curvature -4π. Together with the plane, the catenoid is the only 0-surface of revolution (Bonnet [**14**]) and the unique complete 0-surface with genus zero, finite topology and more than one end (López and Ros [**95**]). Also, the catenoid

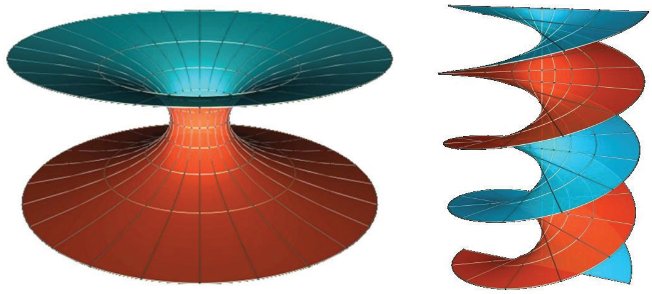

FIGURE 3. Left: The catenoid. Right: The helicoid.

is characterized as being the unique complete 0-surface with finite topology and two ends (Schoen [**186**]).

The helicoid. $M = \mathbb{C}$, $g(z) = e^z$, $dh = i\,dz$, see Figure 3 Right. This surface was first proved to be minimal by Meusnier in 1776 [**155**]. When viewed in $\mathbb{R}^3$, the helicoid has genus zero, one end and infinite total curvature. Together with the plane, the helicoid is the only ruled 0-surface (Catalan [**24**]) and the unique simply connected, complete 0-surface (Meeks and Rosenberg [**138**], see also [**11**]). The vertical helicoid can also be viewed as a genus-zero surface with two ends in a quotient of $\mathbb{R}^3$ by a vertical translation or by a screw motion. The catenoid and the helicoid are *conjugate* 0-surfaces, in the sense of the following definition.

DEFINITION 2.9. Two immersed 0-surfaces in $\mathbb{R}^3$ are said to be *conjugate* if the coordinate functions of one of them are locally the harmonic conjugates of the coordinate functions of the other one.

REMARK 2.10. There is also a notion of conjugate surface for $(H > 0)$-surfaces; see [**151**] for further discussion on the more general notion of associate surfaces to an H-surface.

Note that in the case of the helicoid and catenoid, we consider the catenoid to be defined on its universal cover $e^z : \mathbb{C} \to \mathbb{C} - \{0\}$ in order for the harmonic conjugate of x_3 to be well-defined. Equivalently, both surfaces share the Gauss map e^z and their height differentials differ by multiplication by $i = \sqrt{-1}$.

The Meeks minimal Möbius strip. $M = \mathbb{C} - \{0\}$, $g(z) = z^2\left(\frac{z+1}{z-1}\right)$, $dh = i\left(\frac{z^2-1}{z^2}\right)dz$, see Figure 4 Left. Found by Meeks [**101**], the 0-surface defined by this Weierstrass data double covers a complete, immersed 0-surface $M_1 \subset \mathbb{R}^3$ which is topologically a Möbius strip. This is the unique complete, minimally immersed surface in $\mathbb{R}^3$ of finite total curvature -6π. It contains a unique closed geodesic which is a planar circle, and also contains a line bisecting the circle.

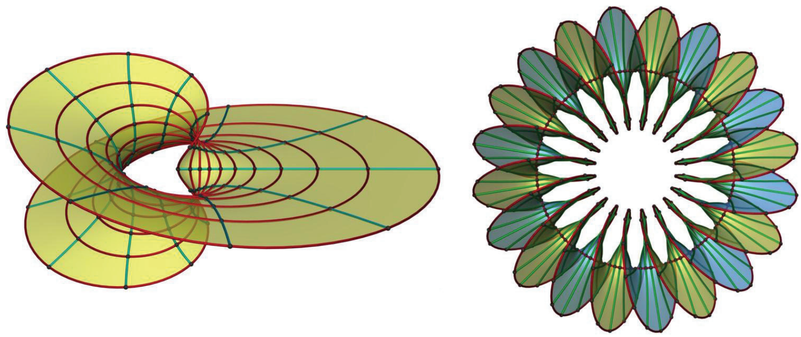

FIGURE 4. Left: The Meeks minimal Möbius strip. Right: A bent helicoid near the circle $\mathbb{S}^1$, which is viewed from above along the x_3-axis. (Images courtesy of M. Weber.)

The bent helicoids. $M = \mathbb{C} - \{0\}$, $g(z) = -z\frac{z^n+i}{iz^n+i}$, $dh = \frac{z^n+z^{-n}}{2z}dz$, see Figure 4 Right. Discovered by Meeks and Weber [**152**] and independently by Mira [**156**], these are complete, immersed 0-annuli $\widetilde{A}_n \subset \mathbb{R}^3$ with two non-embedded ends and finite total curvature; each of the surfaces $\widetilde{A}_n$ contains the unit circle $\mathbb{S}^1$ in the (x_1, x_2)-plane, and a neighborhood of $\mathbb{S}^1$ in $\widetilde{A}_n$ contains an embedded annulus A_n which approximates, for n large, a highly spinning helicoid whose usual straight axis has been periodically bent into the unit circle $\mathbb{S}^1$ (thus the name of bent helicoids). Furthermore, the A_n converge as $n \to \infty$ to the foliation of $\mathbb{R}^3$ minus the x_3-axis by vertical half-planes with boundary the x_3-axis, and with $\mathbb{S}^1$ as the singular set of C^1-convergence. The method applied by Meeks, Weber and Mira to find the bent helicoids is the classical *Björling formula* [**162**] with an orthogonal unit field along $\mathbb{S}^1$ that spins an arbitrary number n of times around the circle. This construction also makes sense when n is half an integer; in the case $n = \frac{1}{2}$, $\widetilde{A}_{1/2}$ is the double cover of the Meeks minimal Möbius strip described in the previous example. The bent helicoids A_n play an important role in proving the converse of Meeks' $C^{1,1}$-Regularity Theorem (see Meeks and Weber [**152**] and also Theorems 8.5 and 8.7 below) for the singular set of convergence in a Colding-Minicozzi limit 0-lamination.

The singly-periodic Scherk surfaces. $M = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{iz\,dz}{\prod(z\pm e^{\pm i\theta/2})}$, for fixed $\theta \in (0, \pi/2]$, see Figure 2.3 Left for the case $\theta = \pi/2$. Discovered by Scherk [**184**] in 1835, these surfaces denoted by $\mathcal{S}_\theta$ form a 1-parameter family of complete genus-zero 0-surfaces in a quotient of $\mathbb{R}^3$ by a translation, and have four annular ends. Viewed in $\mathbb{R}^3$, each surface $\mathcal{S}_\theta$ is invariant under reflection in the (x_1, x_3) and (x_2, x_3)-planes and in horizontal planes at integer heights, and can be thought of geometrically as a desingularization of two vertical planes forming an angle of θ. The special case $\mathcal{S}_{\theta=\pi/2}$ also contains pairs of orthogonal lines at planes of half-integer heights, and has implicit equation $\sin z = \sinh x \sinh y$. Together

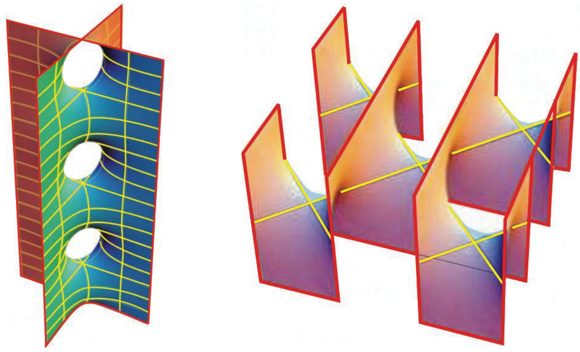

FIGURE 5. Singly-periodic Scherk surface with angle $\theta = \pi/2$ (left), and its conjugate surface, the doubly-periodic Scherk surface (right). (Images courtesy of M. Weber.)

with the plane and catenoid, the surfaces $\mathcal{S}_\theta$ are conjectured to be the only connected, complete, immersed, 0-surfaces in $\mathbb{R}^3$ whose areas in balls of radius R is less than $2\pi R^2$; see Conjecture 15.13 in Section 15 for further discussion on this open problem. This conjecture was proved by Meeks and Wolf [**153**] under the additional hypothesis that the surface have an infinite symmetry group.

The doubly-periodic Scherk surfaces. $M = (\mathbb{C} \cup \{\infty\}) - \{\pm e^{\pm i\theta/2}\}$, $g(z) = z$, $dh = \frac{z\,dz}{\prod(z \pm e^{\pm i\theta/2})}$, where $\theta \in (0, \pi/2]$ (the case $\theta = \frac{\pi}{2}$ has implicit equation $e^z \cos y = \cos x$), see Figure 2.3 Right. These surfaces, discovered by Scherk [**184**] in 1835, are the conjugate surfaces of the singly-periodic Scherk surfaces, and can be thought of geometrically as the desingularization of two families of equally spaced vertical parallel half-planes in opposite half-spaces, with the half-planes in the upper family making an angle of θ with the half-planes in the lower family. These surfaces are doubly-periodic with genus zero in their corresponding quotient $\mathbb{T}^2 \times \mathbb{R}$, and were characterized by Lazard-Holly and Meeks [**92**] as being the unique proper 0-surfaces with genus zero in any $\mathbb{T}^2 \times \mathbb{R}$. It has been conjectured by Meeks, Pérez and Ros (see Conjecture 15.16) that the singly and doubly-periodic Scherk 0-surfaces are the only complete 0-surfaces in $\mathbb{R}^3$ whose Gauss maps miss four points on $\mathbb{S}^2$. They also conjecture that the singly and doubly-periodic Scherk 0-surfaces, together with the catenoid and helicoid, are the only complete 0-surfaces of negative curvature (see Conjecture 15.17).

The Riemann minimal examples. These surfaces come in a one-parameter family defined in terms of a parameter $\lambda > 0$. Let $M_\lambda = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \mid w^2 = z(z - \lambda)(\lambda z + 1)\} - \{(0, 0), (\infty, \infty)\}$, $g(z, w) = z$, $dh = A_\lambda \frac{dz}{w}$, for each $\lambda > 0$, where A_λ is a non-zero complex number satisfying $A_\lambda^2 \in \mathbb{R}$, see Figure 6. Discovered by Riemann (and posthumously

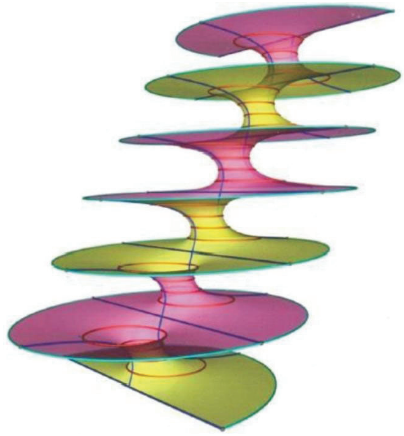

FIGURE 6. A Riemann minimal example. (Image courtesy of M. Weber.)

published, Hattendorf and Riemann [**174, 175**]), these examples are invariant under reflection in the (x_1, x_3)-plane and by a translation T_λ. The induced surfaces M_λ/T_λ in the quotient spaces $\mathbb{R}^3/T_\lambda$ have genus one and two planar ends, see [**132**] for a more precise description. The Riemann minimal examples have the amazing property that every horizontal plane intersects each of these surfaces in a circle or in a line. The conjugate minimal surface of the Riemann minimal example for a given $\lambda > 0$ is the Riemann minimal example for the parameter value $1/\lambda$ (the case $\lambda = 1$ gives the only self-conjugate surface in the family). Meeks, Pérez and Ros [**132**] showed that these surfaces are the only proper 0-surfaces in $\mathbb{R}^3$ of genus zero and infinite topology. Assuming that Conjecture 4.3 below holds, then these surfaces are the only complete 0-surfaces in $\mathbb{R}^3$ of genus zero and infinite topology. Also see [**119**] for a complete outline of the proof of uniqueness of the Riemann minimal examples and historical comments on Riemann's original proof of the classification of his examples.

2.4. Classical maximum principles. One of the consequences of the fact that H-surfaces can be viewed locally as solutions of a partial differential equation is that they satisfy certain maximum principles. We will state them for H-surfaces in $\mathbb{R}^3$, but they also hold when the ambient space is any Riemannian 3-manifold.

THEOREM 2.11 (Interior Maximum Principle [**59**]). *For $i = 1, 2$, let M_i be a connected H_i-surface in $\mathbb{R}^3$, and p an interior point to both surfaces. Suppose that M_2 lies on the mean convex side of M_1 near p $M_1 = M_2$ in a neighborhood of p.*

THEOREM 2.12 (Boundary Maximum Principle [**59**]). *For $i = 1, 2$, let M_i be a connected H_i-surface with boundary in $\mathbb{R}^3$, and p a boundary point of both surfaces. Suppose that M_2 lies on the mean convex side of M_1 near*

p, and that the surfaces are locally tangent graphs over the same half disk in $T_pM_1 = T_pM_2$ with tangent boundaries and with the same normal at p. If $H_1 \geq H_2$, then $M_1 = M_2$ in a neighborhood of p.

Before continuing with the exposition and as we announced just before the statement of Theorem 2.6, we pause to give a proof of the following classical result as an application of the previous two theorems.

THEOREM 2.13 (Alexandrov [**3**]). *Round spheres are the only closed H-surfaces in $\mathbb{R}^3$.*

PROOF. Let M be a closed H-surface and let W be the smooth compact domain in $\mathbb{R}^3$ with boundary M. We now explain how to use the Alexandrov reflection principle to prove that for the family of horizontal planes $\{P(t) = \{x_3 = t\}\}_{t\in\mathbb{R}}$, there exists a $t_M \in \mathbb{R}$ such that $P(t_M)$ that is a plane of reflectional symmetry for M and furthermore $M - P(t_M)$ consists of two components, each of which is a graph over a bounded component of $P(t_M) - M$. Assuming this symmetry result we can deduce that for any unit length vector a, M has a plane of reflective symmetry with normal vector a and, after a translation and by the compactness of M, M is invariant under the action of the orthogonal group $O(3)$, which implies that M is a round sphere.

For each $t \in \mathbb{R}$, let $R_t \colon \mathbb{R}^3 \to \mathbb{R}^3$ be reflection in the plane $P(t)$ and consider the closed lower half-space $P(t)^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \leq t\}$ determined by $P(t)$. Consider the smallest t_1 such that $P(t_1)$ intersects M. Then, there exists a small $\varepsilon > 0$ such that the following hold:

1. $P(t_1 + \varepsilon)^- \cap M$ is a graph over its possibly disconnected projection to $P(t_1 + \varepsilon)$.
2. $R_{t_1+\varepsilon}(P(t_1 + \varepsilon)^- \cap M) \subset W$.

Define $\varepsilon_1 = \max\{\varepsilon' \in [\varepsilon, \infty) \mid \text{ items 1 and 2 hold for } \varepsilon'\}$, which exists by compactness of M.

CLAIM: *The plane $P(t_1 + \varepsilon_1)$ is a plane of Alexandrov symmetry for M.*

Observe that the above claim proves the desired symmetry result for M stated in the first paragraph of this proof, in other words, $t_M = t_1 + \varepsilon_1$. Theorem 2.12 implies that the claim holds provided that the plane $P(t_1+\varepsilon)$ is orthogonal to M at some point p. So assume now that the plane $P(t_1+\varepsilon)$ is nowhere orthogonal to M. We claim that Theorem 2.11 implies

$$R_{t_1+\varepsilon_1}(P(t_1 + \varepsilon_1)^- \cap M) - P(t_1 + \varepsilon_1) \subset \operatorname{Int}(W). \tag{11}$$

Otherwise, since at a point of intersection of $R_{t_1+\varepsilon_1}(P(t_1 + \varepsilon_1)^- \cap M) - P(t_1 + \varepsilon_1)$ with $M = \partial W$, these surfaces have the same normals, then the interior maximum principle shows that $R_{t_1+\varepsilon_1}(P(t_1+\varepsilon_1)^- \cap M) \subset M$, which would imply that $P(t_1+\varepsilon_1)$ is a plane of symmetry and hence perpendicular to M at every point of $M \cap P(t_1 + \varepsilon_1)$, which is contrary to our hypothesis.

Now (11) and the compactness of M ensure that for $\delta > 0$ sufficiently small,

$$R_{t_1+\varepsilon_1+\delta}(P(t_1+\varepsilon_1+\delta)^- \cap M) - P(t_1+\varepsilon_1+\delta) \subset \mathrm{Int}(W),$$

from which we conclude that there exists a $\delta' \in (0,\delta)$ such that $\varepsilon_1+\delta'$ satisfies items 1 and 2, which contradicts the definition of ε_1. This contradiction completes the proof. □

Another beautiful application of Theorem 2.11 is the following result by Hoffman and Meeks.

THEOREM 2.14 (Strong Half-space Theorem [**69**]). *Let $f\colon M \to \mathbb{R}^3$ be a properly immersed, possibly branched, non-planar 0-surface without boundary. Then, M cannot be contained in a half-space. More generally, if $M_1, M_2 \subset \mathbb{R}^3$ are the images of two properly immersed 0-surfaces in $\mathbb{R}^3$, one of which is non-planar, then these surfaces intersect.*

Here the adjective "strong" refers to the second statement of Theorem 2.14. This second statement follows from the first statement by a previous result of Meeks, Simon and Yau [**141**], where they proved that given two proper, possibly branched 0-surfaces M_1, M_2 in $\mathbb{R}^3$ that are disjoint, there exists a proper, stable orientable 0-surface Σ in the region of W of $\mathbb{R}^3$ between the images of M_1, M_2 (see Definition 2.22 for the notion of stability). Since Σ is stable, then it is a plane by Theorem 2.24 below. The original proof by Hoffman and Meeks of the first statement of Theorem 2.14 only uses the interior maximum principle and a clever argument with catenoids as barriers. Since this argument is simple and has become a standard and useful technique for other applications, we will also include it here for the sake of completeness.

For the following arguments, please refer to Figure 7. The first step is to find a smallest open half-space that contains the surface M, which can be assumed to be $\{z < 0\}$. As M is proper, no points of $\{z = 0\}$ are accumulation points of M. Applying this property to $\vec{0} = (0,0,0)$, one finds a ball $\mathbb{B}(r)$ centered at $\vec{0}$ of radius $r > 0$ which is disjoint from M. Given $a \in (0,r)$, the vertical catenoid $C_a = \{x^2+y^2 = a^2\cosh^2(z/a)\}$ has waist circle contained in $\mathbb{B}(r)$ and thus, the upper half-catenoid $C_a^+ = C_a \cap \{x_3 \geq 0\}$ can be lowered some small height $\varepsilon > 0$ so that $C := C_a^+ - \varepsilon(0,0,1)$ is contained in $\mathbb{B}(r) \cup \{x_3 \geq 0\}$. Next one considers the 1-parameter family of half-catenoids $\{C(t) \mid t > 0\}$ obtained after applying to C a homothety of ratio $t > 0$ centered at the center $O = (0,0,-\varepsilon)$ of C. As $C(t)$ converges as $t \to 0$ to the punctured plane $\{x_3 = -\varepsilon\} - \{O\}$, then M must have points above $C(T)$ for some $T > 0$ sufficiently small. As $C(1)$ is disjoint from M, then there exists the infimum t_1 of the set $\{t > 0 \mid M \cap C(t) = \emptyset\}$. As $M \subset \{x_3 < 0\}$ and the end of $C(t_1)$ is catenoidal with positive logarithmic growth, then M and $C(t_1)$ intersect at a common interior point p (note that $\partial C(t_1) \subset \mathbb{B}(r)$) where M lies at one side of $C(t_1)$ near p, which contradicts the interior maximum principle.

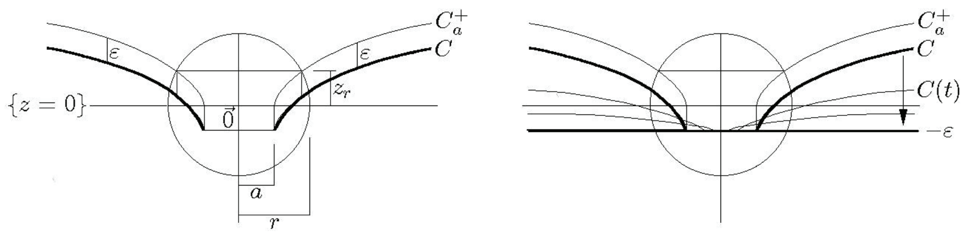

FIGURE 7. The $C(t)$, $t \in (0, 1]$, are homothetic shrinkings of C.

More generally, one has the following result of Meeks and Rosenberg based on earlier partial results in [**31, 69, 88, 135, 193**].

THEOREM 2.15 (Maximum Principle at Infinity [**140**]). *Let $M_1, M_2 \subset N$ be disjoint, connected, properly immersed 0-surfaces with (possibly empty) boundary in a complete flat 3-manifold N.*

1. *If $\partial M_1 \neq \emptyset$ or $\partial M_2 \neq \emptyset$, then after possibly reindexing, the distance between M_1 and M_2 (as subsets of N) is equal to $\inf\{d_N(p, q) \mid p \in \partial M_1,\, q \in M_2\}$, where d_N denotes Riemannian distance in N.*
2. *If $\partial M_1 = \partial M_2 = \emptyset$, then M_1 and M_2 are flat.*

We now come to a deep application of the general maximum principle at infinity. The next corollary appears in [**140**] and a slightly weaker variant of it can be found in Soret [**193**] when $H = 0$. Actually the results in [**140**] describe a slightly weaker version of the last statement in the corollary when $H > 0$, but the stronger statement given below is easily proven with the same methods.

COROLLARY 2.16 (Regular Neighborhood Theorem). *Suppose $M \subset N$ is a proper non-flat 0-surface in a complete flat 3-manifold N, with bounded second fundamental form and let $1/R$ be the supremum of the absolute values of the principal curvatures of M. Let $N_R(M)$ be the open subset of the normal bundle of M given by the normal vectors of length strictly less that R. Then, the corresponding exponential map* $\exp\colon N_R(M) \to N$ *is a smooth embedding. In particular:*

1. *M is properly embedded.*
2. *M has an open, embedded tubular neighborhood of radius R.*
3. *There exists a constant $C > 0$, depending only on R such that for all balls $B \subset N$ of radius 1, the area of $M \cap B$ is at most C times the volume of B.*

Furthermore, under the same hypotheses on M except that it is an $(H > 0)$-surface, and letting $N_R^+(M) \subset N_R(M)$ be the subset of normal vectors that have a non-negative inner product with the mean curvature vector of M, then the restriction $\exp\colon N_R^+(M) \to N$ *is a smooth embedding. In particular, property 2 holds on the mean convex side of M, and thus properties 1 and 3 also hold for M.*

DEFINITION 2.17. Let N be a smooth Riemannian n-manifold.

(1) We call a compact, immersed H-hypersurface $f\colon M \to N$ *Alexandrov embedded* if there exists an immersion $F\colon W \to N$ of a compact, mean convex n-manifold W with $\partial W = M$, such that $F|_M = f$.
(2) We call a proper, immersed H-hypersurface $f\colon M \to N$ *strongly Alexandrov embedded* if there exists a proper immersion $F\colon W \to N$ of a complete, mean convex n-manifold W with $\partial W = M$, such that F is injective on the interior of W and $F|_M = f$.

We next include a result which is analogous to Corollary 2.16 and that holds in the n-dimensional setting for certain $(H > 0)$-hypersurfaces.

THEOREM 2.18 (One-sided Regular Neighborhood, Meeks, Tinaglia [**150**]). *Suppose N is a complete n-manifold with absolute sectional curvature bounded by a constant $S_0 > 0$. Let M be a strongly Alexandrov embedded hypersurface with constant mean curvature $H_0 > 0$ and norm of its second fundamental form $|A_M| \leq A_0$ for some $A_0 > 0$. Then, the following statements hold.*

1. *There exists a positive number $\tau \in (0, \pi/S_0)$, depending on A_0, H_0, S_0, such that M has a regular neighborhood $\exp(N^+_\tau(M))$ of width τ on its mean convex side, where we are using the notation of Corollary 2.16.*
2. *There exists $C > 0$ depending on A_0, H_0, S_0, such that the $(n-1)$-dimensional volume of M in balls of radius 1 in N is less than C.*

2.5. Second variation of area, index of stability, Jacobi functions and curvature estimates of stable H-surfaces. Let $\psi\colon M \to N$ be an isometric immersion of a surface in a Riemannian 3-manifold N. Assume that $\psi(M)$ is two-sided, i.e. there exists a globally defined unit normal vector field η on M. Given a compact smooth domain (possibly with boundary) $\Omega \subset M$, we will consider variations of Ω given by differentiable maps $\Psi\colon (-\varepsilon, \varepsilon) \times \Omega \to N$, $\varepsilon > 0$, such that $\Psi(0, p) = \psi(p)$ and $\Psi(t, p) = \psi(p)$ for $|t| < \varepsilon$ and $p \in M - \Omega$. The *variational vector field* for such a variation Ψ is $\frac{\partial \Psi}{\partial t}\big|_{t=0}$ and its normal component is $u = \langle \frac{\partial \Psi}{\partial t}\big|_{t=0}, \eta \rangle$. Note that, for small t, the map $\psi_t = \Psi_{|t \times \Omega}$ is an immersion. Hence we can associate to Ψ the *area function* Area(t) =Area(ψ_t) and the *volume function* Vol(t) given by

$$\mathrm{Vol}(t) = \int_{[0,t]\times\Omega} \mathrm{Jac}(\Psi)\, dV,$$

where dV is the volume element in N. The function Vol(t) measures the signed volume enclosed between $\psi_0 = \psi$ and ψ_t.

The first variation formula for the area and volume are

$$\frac{d}{dt}\bigg|_{t=0} \mathrm{Area}(t) = -2\int_M H u\, dA, \qquad \frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(t) = -\int_M u\, dA, \tag{12}$$

where dA is the area element on M for the induced metric by ψ. The equations in (12) imply that M is a critical point of the functional Area $-$

$2c\,\mathrm{Vol}$ (here $c \in \mathbb{R}$) if and only if it has constant mean curvature $H = c$. In this case, we can consider the *Jacobi operator* on M,

$$L = \Delta + |A_M|^2 + \mathrm{Ric}(\eta), \tag{13}$$

where $\mathrm{Ric}(\eta)$ is the Ricci curvature of N along the unit normal vector field of the immersion. For an H-surface M, the second variation formula of the functional $\mathrm{Area} - 2H\,\mathrm{Vol}$ is given by (see e.g. [**8, 162**])

(14)
$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_{t=0} [\mathrm{Area}(t) - 2H\ \mathrm{Vol}(t)] &= -\int_M uLu\,dA \\ &= \int_M \left[|\nabla u|^2 - (|A_M|^2 + \mathrm{Ric}(\eta))u^2\right] dA.\end{aligned}$$

Formula (14) can be viewed as the bilinear form $Q(u,u)$ associated to the linear elliptic L^2-selfadjoint operator given by the Jacobi operator L defined in (13).

Remark 2.19. For a normal variation ψ_t of a surface $\psi\colon M \to N$ with associated normal variational vector field $u\eta$, $L(u)(p)$ is equal to $-2H'(t)|_{t=0}$ at p, where $H(t)(p)$ is the mean curvature of the immersed surface $\psi_t(M)$ at the point $\psi_t(p)$.

Definition 2.20. A C^2-function $u\colon M \to \mathbb{R}$ satisfying $Lu = 0$ on M is called a *Jacobi function.* We will let $\mathcal{J}(M)$ denote the linear space of Jacobi functions on M.

Classical elliptic theory implies that given a subdomain $\Omega \subset M$ with compact closure, the Dirichlet problem for the Jacobi operator in Ω has an infinite discrete spectrum $\{\lambda_k\}_{k\in\mathbb{N}\cup\{0\}}$ of eigenvalues with $\lambda_k \nearrow +\infty$ as k goes to infinity, and each eigenspace is a finite dimensional linear subspace of $C^\infty(\Omega) \cap H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the usual Sobolev space of L^2-functions with L^2 weak partial derivatives and trace zero.

Definition 2.21. Let $\Omega \subset M$ be a subdomain with compact closure. The *index of stability* of Ω is the number of negative eigenvalues of the Dirichlet problem associated to L in Ω. The *nullity* of Ω is the dimension of $\mathcal{J}(\Omega) \cap H_0^1(\Omega)$. Ω is called *stable* if its index of stability is zero, and *strictly stable* if both its index and nullity are zero.

When $N = \mathbb{R}^3$, (13) reduces to $L = \Delta - 2K$ (K denotes Gaussian curvature). In this case, since the Gauss map of an H-graph defined on a domain in a plane Π has image set contained in an open half-sphere, the inner product of the unit normal vector with the unit normal to Π provides a positive Jacobi function, from where we conclude that any H-graph is stable.

Coming back to the general case of a two-sided H-surface $\psi\colon M \to N$ in a Riemannian 3-manifold N, stability also makes sense in the non-compact setting for M, as we next explain.

DEFINITION 2.22. An H-surface $\psi\colon M \to N$ in a Riemannian 3-manifold N is called *stable* if any subdomain $\Omega \subset M$ with compact closure is stable in the sense of Definition 2.21. Stability is equivalent to the existence of a positive Jacobi function on M (Proposition 1 in Fischer-Colbrie [**54**]). M is said to have *finite index* if outside of a compact subset it is stable. The *index of stability* of M is the supremum of the indices of stability of subdomains with compact closure in M.

For H-surfaces, it is natural to consider a weaker notion of stability, associated to the isoperimetric problem.

DEFINITION 2.23. We say that an H-surface $\psi\colon M \to N$ in a Riemannian 3-manifold N is *weakly stable* if

$$\int_M \left[|\nabla u|^2 - (|A_M|^2 + \mathrm{Ric}(\eta))u^2\right] dA \geq 0,$$

for every $f \in C_0^\infty(M)$ with $\int_M f\, dA = 0$. Sometimes this notion is referred to as *volume preserving stable* in the literature.

The Gauss equation allows us to write the Jacobi operator of an H-surface in several interesting forms.

$$L = \Delta - 2K + 4H^2 + \mathrm{Ric}(e_1) + \mathrm{Ric}(e_2) \tag{15}$$

$$= \Delta - K + 2H^2 + \frac{1}{2}|A_M|^2 + \frac{1}{2}S \tag{16}$$

$$= \Delta - K + 3H^2 + \frac{1}{2}S + (H^2 - \det(A_M)), \tag{17}$$

where K is the Gaussian curvature of M, e_1, e_2 is an orthonormal basis of the tangent plane of $\psi\colon M \to N$ and S denotes the scalar curvature of N. Note that we take the scalar curvature function S at a point $p \in N$ to be six times the average sectional curvature of N at p.

By definition, stable surfaces have index zero. The following theorem explains how restrictive is the property of stability for complete H-surfaces in $\mathbb{R}^3$. In the case $H = 0$, the first statement in it was proved independently by Fischer-Colbrie and Schoen [**55**], do Carmo and Peng [**51**], and Pogorelov [**172**] for orientable surfaces. Later, Ros [**180**] proved that a complete, non-orientable 0-surface in $\mathbb{R}^3$ is never stable. The second statement has important applications to the study of regularity properties of H-laminations in $\mathbb{R}^3$ punctured at the origin. A short elementary proof of the next result is given in Lemma 6.4 of [**130**]. The case $H = 0$ of the second statement in the following result was also obtained by Colding and Minicozzi (Lemma A.26 in [**39**]).

THEOREM 2.24. *If $M \subset \mathbb{R}^3$ is a complete, stable immersed H-surface, then M is a plane. More generally, if $M \subset \mathbb{R}^3 - \{\vec{0}\}$ is a stable H-surface which is complete outside the origin (in the sense that every divergent path in M of finite length has as limit point the origin), then M is a plane.*

A crucial fact in H-surface theory is that stable, immersed H-surfaces with boundary in homogeneously regular 3-manifolds (see Definition 2.25 below) have curvature estimates up to their boundary. These curvature estimates were first obtained by Schoen for two-sided 0-surfaces and later improved by Ros to the one-sided 0-case, and are a simple consequence of Theorem 2.24 after a rescaling argument.

DEFINITION 2.25. A Riemannian 3-manifold N is *homogeneously regular* if there exists an $\varepsilon > 0$ such that ε-balls in N are uniformly close to ε-balls in $\mathbb{R}^3$ in the C^2-norm. In particular, if N is compact, then N is homogeneously regular.

THEOREM 2.26 (Schoen [**185**], Ros [**180**]). *Let N be a homogeneously regular 3-manifold. Then, there exists a universal constant $c > 0$ such that for any stable immersed H-surface M in N,*

$$|A_M(p)|\, d_N(p, \partial M)^2 \leq c \quad \textit{for all } p \in M,$$

where d_N denotes distance in N and ∂M is the boundary of M.

Rescaling arguments and results of López and Ros [**94**] for complete immersed 0-surfaces in $\mathbb{R}^3$ with index of stability 1 demonstrate that given a homogeneously regular 3-manifold N, there exist similar curvature estimates for two-sided H-surfaces that are weakly stable in the sense of Definition 2.23.

We also note that Rosenberg, Souam and Toubiana [**182**] have obtained the following version of Theorem 2.26 valid for H-surfaces in the two-sided case when the ambient 3-manifold has a bound on its sectional curvature.

THEOREM 2.27 (Rosenberg, Souam and Toubiana [**182**]). *Let N be a 3-manifold with a bound k_0 on its absolute sectional curvature. There exists a universal constant $c > 0$ (depending on k_0) such that for any stable, two-sided immersed H-surface M in N,*

$$|A_M(p)|\, d_N(p, \partial M)^2 \leq c \quad \textit{for all } p \in M.$$

If we weaken the stability hypothesis in Theorem 2.24 to finite index of stability and we allow compact boundary, then completeness and orientability also lead to a well-known family of immersed 0-surfaces.

THEOREM 2.28 (Fischer-Colbrie [**54**]). *Let $M \subset \mathbb{R}^3$ be a complete, orientable immersed 0-surface in $\mathbb{R}^3$, with (possibly empty) compact boundary. Then, M has finite index of stability if and only if it has finite total curvature. In this case, the index and nullity of M coincide with the index and nullity of the meromorphic extension of its Gauss map to the compactification $\overline{M}$ obtained from M after attaching its ends.*

In order to make sense of the last statement in the above theorem, recall Huber's [**74**] parabolicity result that implies that if a complete Riemannian surface with compact boundary has finite total curvature, then it is conformally a compact Riemann surface, and, as shown by Osserman [**165**], a simple application of Picard's theorem implies the Gauss map extends holomorphically across the the punctures to the conformal compactification.

3. The flux of a Killing field

We next describe the notion of the flux of a 1-cycle on an H-surface; see for instance [**86, 87, 191**] for further discussion of this invariant. This generalizes the previous definition of flux $F(\gamma)$ of a 1-cycle γ on an immersed 0-surface given in equation (10) to the H-surface setting.

DEFINITION 3.1 (CMC Flux). Let γ be a piecewise-smooth 1-cycle in an immersed H-surface $M \subset \mathbb{R}^3$. The flux vector of M along γ is

$$F(\gamma) = \int_\gamma (H\gamma + N) \times \gamma', \tag{18}$$

where N is the unit normal to M and γ' is the velocity vector of γ (compare with (10)).

In the case of a properly immersed 0-surface M in $\mathbb{R}^3$, one can associate for any $t \in \mathbb{R}$ its *scalar vertical flux* $V_M(t)$ *across the plane* $\{x_3 = t\}$, which is the possibly improper integral

$$V_M(t) = \int_{\partial(M \cap \{x_3 \leq t\})} |\nabla x_3| \quad \in (0, \infty],$$

where ∇x_3 denotes the intrinsic gradient of the third coordinate function of M.

THEOREM 3.2 (Scalar vertical flux, Meeks [**103**]). *Let M be a properly immersed 0-surface in $\mathbb{R}^3$. Then, $V_M(t)$ does not depend on $t \in \mathbb{R}$. Hence, without ambiguity we define $V_M \in (0, \infty]$ as the flux of ∇x_3 across any horizontal plane and we call V_M the* scalar vertical flux of M.

We next give a sketch of proof of Theorem 3.2, partly to motivate some other important theoretical results and techniques in the subject. We first recall the notion of a parabolic Riemannian manifold M with boundary, and refer the reader to Section 7 of the book [**118**] for further details.

DEFINITION 3.3. Let (M^n, g) be an n-dimensional Riemannian manifold with non-empty boundary. M is *parabolic* if every bounded harmonic function on M is determined by its boundary values.

In dimension $n = 2$, the property of a Riemannian surface with boundary to be parabolic is a conformal one, and any proper smooth subdomain of a parabolic manifold is also parabolic. One way to show that an n-dimensional Riemannian manifold (M^n, g) is parabolic is to prove that there exists a proper, positive superharmonic function on it [**61**].

Collin, Kusner, Meeks and Rosenberg [**42**] constructed ambient functions on certain proper non-compact regions in $\mathbb{R}^3$ with the property that they restrict to any minimal surface to be superharmonic; they called such functions *universal superharmonic functions*. Using the universal superharmonic function $f(x_1, x_2, x_3) = -x_3^2 + \ln(\sqrt{x_1^2 + x_2^2})$ defined on $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \geq 1\}$, they proved that the intersection of a properly immersed 0-surface with boundary in $\mathbb{R}^3$ and contained in a half-space is parabolic. In particular, if M is a properly immersed 0-surface in $\mathbb{R}^3$, then for any real numbers $t_1 < t_2$, the subdomain

$$M[t_1, t_2] = M \cap \{(x_1, x_2, x_3) \mid t_1 \leq x_3 \leq t_2\}$$

is a parabolic surface with boundary contained in the union of the planes $\{x_3 = t_i\}$, $i = 1, 2$; note that x_3 is a bounded harmonic function h on the parabolic Riemannian manifold $X = M[t_1, t_2]$ with $\partial X \subset h^{-1}(\{t_1, t_2\})$. In this more general setting, Meeks proved that the scalar flux of ∇h across $h^{-1}(t_1)$ is the same as the flux across $h^{-1}(t_2)$, which then proves Theorem 3.2 (see [**103**] and also see the proof of Proposition 4.16 in [**124**] for similar calculations). This completes our sketch of the proof of Theorem 3.2.

There is a related notion of flux that generalizes the formula (18) and works in the n-dimensional Riemannian manifold setting. The proof of the next theorem is straightforward and follows from two applications of the Divergence Theorem; see the proof below or the similar calculations in the proof of the conservation laws in Theorem 4.1 in [**85**].

Theorem 3.4 (CMC Flux Formula). *Let (X, g) be an n-dimensional orientable Riemannian manifold, $M \subset X$ be an orientable hypersurface of constant mean curvature and K be a Killing field. Suppose that $\Sigma, \Sigma' \subset X$ are $(n-1)$-chains with boundaries $\partial\Sigma = \Gamma \subset M$, $\partial\Sigma' = \Gamma' \subset M$, such that the $(n-2)$-cycles Γ, Γ' are homologous in M (i.e., there exists an $(n-1)$-chain $M(\Gamma, \Gamma') \subset M$ with boundary $\partial M(\Gamma, \Gamma') = \Gamma - \Gamma'$) and $\Sigma - \Sigma' + M(\Gamma, \Gamma')$ is a boundary in X (i.e., there exists an n-chain $\Omega \subset X$ with boundary $\partial\Omega = \Sigma - \Sigma' + M(\Gamma, \Gamma')$). Consider the pairing:*

$$\text{(19)} \qquad \mathrm{Flux}(\Gamma, \Sigma, K) = \int_\Gamma g(\eta_\Gamma, K) + (n-1)H \int_\Sigma g(N_\Sigma, K) \in \mathbb{R},$$

where $H \in \mathbb{R}$ is the constant value of the mean curvature of M with respect to the outward pointing normal vector to Ω, N_Σ is the unit normal field to Σ that is outward pointing on Ω, and η_Γ is the unit normal field to Γ in TM that is outward pointing on $M(\Gamma, \Gamma')$. Then, $\mathrm{Flux}(\Gamma, \Sigma, K) = \mathrm{Flux}(\Gamma', \Sigma', K)$.

In particular, if the n-th homology group $H_n(X)$ of X vanishes, then $\mathrm{Flux}(\Gamma, \Sigma, K)$ *depends only on the homology class of* $[\Gamma] \in H_{n-1}(X)$ *and on the Killing field K.*

Proof. As K is a Killing vector field, then the bilinear map $(u, v) \in TX \times TX \mapsto g(\overline{\nabla}_u K, v)$ is skew-symmetric, where $\overline{\nabla}$ stands for the metric connection of X. This implies that the divergence $\mathrm{div}_X(K)$ of K in X vanishes identically and that the divergence on M of the tangent part K^T of K to M is given by $\mathrm{div}_M(K^T) = (n-1)Hg(K, N_M)$, where N_M is the unit normal vector field of M for which H is the mean curvature (in particular, N_M is outward pointing on Ω along $M(\Gamma, \Gamma'))$.

Applying the divergence theorem to K in Ω, one obtains

$$(20)\quad 0 = \int_\Omega \mathrm{div}_X(K) = \int_\Sigma g(K, N_\Sigma) - \int_{\Sigma'} g(K, N_{\Sigma'}) + \int_{M(\Gamma,\Gamma')} g(K, N_M).$$

Analogously, the divergence theorem applied to K^T in $M(\Gamma, \Gamma')$ gives

$$(21)\qquad (n-1)H \int_{M(\Gamma,\Gamma')} g(K, N_M) = \int_\Gamma g(K, \eta_\Gamma) - \int_{\Gamma'} g(K, \eta_{\Gamma'}).$$

Plugging (21) into (20) we deduce that $\mathrm{Flux}(\Gamma, \Sigma, K) = \mathrm{Flux}(\Gamma', \Sigma', K)$, as desired. $\square$

Remark 3.5. Let ∂_{x_i}, $i = 1, 2, 3$ denote the usual constant Killing fields in $\mathbb{R}^3$ endowed with its usual flat metric, let $\gamma \subset M$ be a 1-cycle on an immersed H-surface. Then, the i-th component F_i of the flux vector F defined in (18) is equal to $\mathrm{Flux}(\gamma, M, \partial_{x_i})$, and so, it only depends on the homology class of γ in M. In the $\mathbb{R}^3$-setting, there are other Killing fields generated by one-parameter groups of rotations around a line and the corresponding fluxes obtained from these additional Killing fields give rise to other invariants for 1-cycles on an immersed H-surface (torque or momentum).

4. Classification results for H-surfaces of finite genus in $\mathbb{R}^3$

In this section we will first review some of the main results in the classical theory of complete H-surfaces in $\mathbb{R}^3$ in the context of recent results by Colding and Minicozzi [**38**], Meeks, Pérez and Ros [**123, 132**] and Meeks and Tinaglia [**143**]. After this review, we will present the classification of the asymptotic behavior of annular ends of 0-surfaces in $\mathbb{R}^3$ given by Meeks and Pérez [**113**].

4.1. Classification results. The next theorem demonstrates that classification questions for complete H-surfaces in $\mathbb{R}^3$ are equivalent to the similar classification questions for proper H-surfaces under an appropriate constraint on the global or the local topological properties of the surface. As stated, Theorem 4.1 below depends on results in several different papers. The first one of these is the proof by Colding and Minicozzi [**38**] that complete 0-surfaces of finite topology in $\mathbb{R}^3$ are proper. Using some of the techniques in [**38**], Meeks and Rosenberg [**139**] proved that the closure of a complete 0-surface with positive injectivity radius in a Riemannian 3-manifold has the structure a 0-lamination (leaves are minimal surfaces) and they used this lamination closure property to prove that a complete 0-surface in $\mathbb{R}^3$ with positive injectivity radius is proper. Recently, Meeks and Tinaglia [**143**] have been able to generalize the results in both of these previous papers to the $H > 0$ setting. Summarizing the results into one statement, we have the next fundamental theorem.

THEOREM 4.1. *A complete H-surface in $\mathbb{R}^3$ of finite topology or positive injectivity radius is proper.*

REMARK 4.2. The properness conclusion in Theorem 4.1 also holds if the H-surface has compact boundary and finite topology or if the surface has compact boundary with injectivity radius function bounded away from zero outside of some small neighborhood of its boundary. In particular, annular ends of a complete H-surface in $\mathbb{R}^3$ are proper.

In fact, by the curvature estimates in Theorem 1.6 for $(H > 0)$-disks, it can be seen that a complete $(H > 0)$-surface has bounded second fundamental form if and only if it has positive injectivity radius. We will discuss these results of Meeks and Tinaglia [**143**] in Section 11. A fundamental open problem concerning classical H-surfaces is the following one.

CONJECTURE 4.3 (Meeks, Pérez, Ros, Tinaglia). *A complete H-surface in $\mathbb{R}^3$ of finite genus is proper. More generally, for every such surface M, there exists $C_M > 0$ such that for any ball $\mathbb{B}(p,R)$ in $\mathbb{R}^3$ with radius $R \geq 1$,* $\mathrm{Area}(M \cap \mathbb{B}(p,R)) \leq C_M\, R^3$.

We remark that Meeks, Pérez and Ros [**123**] have obtained the following partial result on the above conjecture.

THEOREM 4.4. *A complete 0-surface in $\mathbb{R}^3$ of finite genus is proper if and only if it has a countable number of ends.*

A fundamental problem in classical surface theory is to describe the behavior of a proper non-compact H-surface $M \subset \mathbb{R}^3$ outside large compact sets in space. This problem is well-understood if M is minimal with finite total curvature, because in this case, each of the ends of M is asymptotic to an end of a plane or a catenoid. In [**40**], Collin proved that a proper 0-surface with at least two ends and finite topology must have finite total curvature; hence by the properness of finite topology H-surfaces in $\mathbb{R}^3$, we have the next fundamental result.

THEOREM 4.5. *A complete 0-surface in $\mathbb{R}^3$ of finite topology and at least two ends has finite total curvature. In particular, each of its ends is asymptotic to the end of a plane or a catenoid.*

The next Theorem 4.6 by Bernstein and Breiner in [**10**] states that if M is a 0-surface with finite topology but infinite total curvature (thus M has exactly one end by Theorem 4.5), then M is asymptotic to a helicoid; this result is based on some of the techniques that Meeks and Rosenberg used in the proof of the uniqueness of the helicoid. The proof of the next theorem was found independently by Meeks and Pérez [**113**] (see Section 4.2) who considered the asymptotic behaviors of annular ends in the more general context where the 0-surface M has compact boundary.

THEOREM 4.6. *A complete non-flat* 0*-surface in* $\mathbb{R}^3$ *of finite topology and one end is asymptotic to a helicoid. If the surface also has genus zero, then it is a helicoid.*

In [**144**], Meeks and Tinaglia describe proper 1-surfaces M_k in $\mathbb{R}^3$ which are doubly-periodic (invariant by two independent translations) and contained in an open slab of width $\frac{1}{2^{k+1}}$. After stacking these slabs with their surfaces on top of each other, one obtains a complete, injectively immersed disconnected surface M_∞ of constant mean curvature 1 that is properly embedded in the slab $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid -1 < x_3 < 1\}$ but M_∞ is not properly embedded in $\mathbb{R}^3$.

In the classical setting, Meeks [**102**] proved that a proper $(H > 0)$-annular end is contained in a solid half-cylinder in $\mathbb{R}^3$, and then used this result to prove that there do not exist any proper $(H > 0)$-surfaces in $\mathbb{R}^3$ with finite topology and just one end. Based in part on Meeks' results, Korevaar, Kusner and Solomon [**86**] then proved that a proper $(H > 0)$-annular end in $\mathbb{R}^3$ is asymptotic to a Delaunay surface, and that a proper $(H > 0)$-surface of finite topology and two ends is a Delaunay surface. By Theorem 4.1, we have the following result.

THEOREM 4.7. *Let* M *be a complete* $(H > 0)$*-surface in* $\mathbb{R}^3$*.*

(1) Each annular end of M *is asymptotic to the end of a Delaunay surface.*
(2) If M *has finite topology, then it has at least two ends.*
(3) If M *has finite topology and two ends, then it is a Delaunay surface.*

The first deep classification result for complete 0-surfaces with finite topology in $\mathbb{R}^3$ is the following one due to Schoen [**186**], who proved the following theorem as an application of the Alexandrov reflection technique.

THEOREM 4.8. *The catenoid is the unique complete, immersed 0-surface in* $\mathbb{R}^3$ *with finite total curvature and two embedded ends.*

After Schoen's result, López and Ros [**95**] used a deformation argument based on the Weierstrass representation to prove the following classification theorem.

THEOREM 4.9. *The only complete 0-surfaces in* $\mathbb{R}^3$ *with finite total curvature and genus zero are the plane and the catenoid.*

We next summarize some of the above classification results; in particular, Theorem 1.3 follows from the next theorem.

THEOREM 4.10. *Let* M *be a complete* H*-surface of finite topology in* $\mathbb{R}^3$*. Then:*

1. M *is proper and has bounded second fundamental form.*
2. *Each annular end of* M *is asymptotic to the end of a Delaunay surface, a plane, a catenoid or a helicoid.*

3. *If M simply connected, then it is a plane, a sphere, a catenoid or a helicoid.*
4. *If M has two ends, then it is a catenoid or a Delaunay surface.*
5. *If M has genus zero and it is a 0-surface, then it is a plane, a catenoid or a helicoid.*
6. *If $H > 0$, then M has at least two ends.*

We next explain some of the elements in the proof of Theorem 1.4, which classifies the proper 0-surfaces in $\mathbb{R}^3$ with genus zero; the case of finite topology is covered by Theorem 4.10 above and does not need the hypothesis of properness but only the weaker one of completeness. Unfortunately it is not known at the present moment if every complete H-surface of finite genus in $\mathbb{R}^3$ is proper; see Conjecture 4.3 and Theorem 4.4 for a related discussion. So for this reason, in the next theorem we will assume that the surface is proper. The results summarized in the next theorem can be found in [**128**], where it is shown that a proper finite genus 0-surface with infinite topology in $\mathbb{R}^3$ must have two limit ends, and in [**132**]; The space of ends $\mathcal{E}(M)$ of a non-compact connected manifold M has the following natural Hausdorff topology. For each proper domain $\Omega \subset M$ with compact boundary, we define the basis open set $B(\Omega) \subset \mathcal{E}(M)$ to be those equivalence classes in $\mathcal{E}(M)$ which have representatives contained in Ω. With this topology, $\mathcal{E}(M)$ is a totally disconnected compact space. Any isolated point $e \in \mathcal{E}(M)$ is called a *simple end* of M. If $e \in \mathcal{E}(M)$ is not a simple end (equivalently, if it is a limit point of $\mathcal{E}(M)$), then e is called a *limit end* of M. In the case that M is a proper 0-surface in $\mathbb{R}^3$ with more than one end, then Frohman and Meeks [**56**] showed that $\mathcal{E}(M)$ can be equipped with a linear ordering by the relative heights of the ends over the (x_1, x_2)-plane (after a rotation in $\mathbb{R}^3$). One defines the *top end* e_T of M as the unique maximal element in $\mathcal{E}(M)$ for this linear ordering. Analogously, the *bottom end* e_B of M is the unique minimal element in $\mathcal{E}(M)$. If $e \in \mathcal{E}(M)$ is neither the top nor the bottom end of M, then it is called a *middle end* of M.

THEOREM 4.11. *Let $M \subset \mathbb{R}^3$ proper 0-surface with infinite topology.*
1. *If M has genus zero, then M is one of the Riemann minimal examples.*
2. *M has finite genus greater than zero, then M has two limit ends and each of its middle ends is planar. Furthermore, after a homothety and rigid motion, the following properties hold.*
   - 2.a. *M is conformally diffeomorphic to a compact Riemann surface of genus g minus a countable closed subset $\mathcal{E}_M = \{e_n\}_{n\in\mathbb{Z}} \cup \{e_\infty, e_{-\infty}\} \subset \overline{M}$, where $\lim_{n\to-\infty} e_n = e_{-\infty}$ and $\lim_{n\to\infty} e_n = e_\infty$.*
   - 2.b. *There exists a Riemann minimal example R_t so that the top end of M converges exponentially to the top end of a translated image of $\mathcal{R}_t$ in the following sense: there exists a vector $v^+ \in \mathbb{R}^3$ and representatives $\mathcal{R}_t^+$ and M^+ of the top ends of $\mathcal{R}_t$ and M respectively, such that M^+ can be expressed as the graph over $\mathcal{R}_t^+ + v^+$ given by a smooth function f defined on the half-cylinder $\overline{\mathcal{R}_t^+}$ obtained after*

attaching to $\mathcal{R}_t^+$ its planar ends, such that f decays exponentially as the height $x_3 \to \infty$ on $\mathcal{R}_t^+$. In the same way, the bottom end of M is exponentially asymptotic to $\mathcal{R}_t^- + v^-$ for a certain translation vector $v^- \in \mathbb{R}^3$.

4.2. Embedded 0-annular ends with infinite total curvature. In this section we will describe the asymptotic behavior, conformal structure and analytic representation of an annular end of any complete 0-surface M in $\mathbb{R}^3$ with compact boundary and finite topology (hence proper by Remark 4.2). For detailed proofs of the results in this section, see Meeks and Pérez [**113**].

Take two numbers $a \in [0, \infty)$, $b \in \mathbb{R}$. Next we outline how to construct examples $E_{a,b} \subset \mathbb{R}^3$ of complete 0-annuli with compact boundary, conformally parameterized in $D(\infty, R) = \{z \in \mathbb{C} \mid |z| \geq R\}$, for some $R > 0$, so that their flux vector along their boundary is $(a, 0, -b)$ and their total Gaussian curvature is infinite. These annuli $E_{\alpha,b}$ will serve as models for the asymptotic geometry of every complete embedded minimal end with infinite total curvature and compact boundary. To define $E_{a,b}$, we will use the Weierstrass representation (g, dh) where g is the Gauss map and dh the height differential. First note that after an isometry in $\mathbb{R}^3$ and a possible change of orientation, we can assume that $b \geq 0$. We consider three separate cases.

(C1) If $a = b = 0$, then define $g(z) = e^{iz}$, $dh = dz$, which produces the end of a vertical helicoid.

(C2) If $a \neq 0$ and $b \geq 0$ (i.e., the flux vector is not vertical), we choose

$$g(z) = t\, e^{iz} \frac{z - A}{z}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \qquad z \in D(\infty, R), \tag{22}$$

where $B = \frac{b}{2\pi}$, and the parameters $t > 0$ and $A \in \mathbb{C} - \{0\}$ are to be determined (here $R > |A|$). Note that with this choice of B, the imaginary part of $\int_{\{|z|=R\}} dh$ is $-b$ because we use the orientation of $\{|z| = R\}$ as the boundary of $D(\infty, R)$.

(C3) In the case of vertical flux, i.e., $a = 0$ and $b > 0$, we take

$$g(z) = e^{iz} \frac{z - A}{z - \overline{A}}, \qquad dh = \left(1 + \frac{B}{z}\right) dz, \quad z \in D(\infty, R), \tag{23}$$

where $B = \frac{b}{2\pi}$ and $A \in \mathbb{C} - \{0\}$ is to be determined (again $R > |A|$).

In each of the three cases above, g can rewritten as $g(z) = t\, e^{iz + f(z)}$ where $f(z)$ is a well-defined holomorphic function in $D(\infty, R)$ with $f(\infty) = 0$. In particular, the differential $\frac{dg}{g}$ extends meromorphically through the puncture at ∞. The same extendability holds for dh. These properties will be collected in the next notion, that was first introduced by Rosenberg [**181**] and later studied by Hauswirth, Pérez and Romon [**67**].

Definition 4.12. *A complete immersed* 0*-surface* $M \subset \mathbb{R}^3$ *with Weierstrass data* (g, dh) *is of finite type if* M *is conformally diffeomorphic to a finitely punctured, compact Riemann surface* $\overline{M}$ *and after a possible rotation, both* dg/g*,* dh *extend meromorphically to* $\overline{M}$*.*

Coming back to our discussion about the annular minimal ends $E_{a,b}$, to determine the parameters $t > 0$, $A \in \mathbb{C} - \{0\}$ that appear in cases (C2), (C3) above, one studies the period problem for (g, dh). The only period to be killed is the first equation in (7) along $\{|z| = R\}$, which can be explicitly computed in terms of t, A. An intermediate value argument gives that given $B = \frac{b}{2\pi}$, there exist parameters $t > 0$, $A \in \mathbb{C} - \{0\}$ so that the Weierstrass data given by (22), (23) solve this corresponding period problem. At the same time, one can calculate the flux vector F of the resulting 0-immersion along its boundary and prove that its horizontal component covers all possible values. This defines for each $a, b \in [0, \infty)$ a complete immersed 0-annulus $E_{a,b}$ with compact boundary, infinite total curvature and flux vector $(a, 0, -b)$. Embeddedness of $E_{a,b}$ will be discussed below. With the notation above, we will call the end $E_{a,b}$ a *canonical end* (in spite of the name "canonical end", we note that the choice of $E_{a,b}$ depends on the explicit parameters t, A in equations (22) and (23)).

Remark 4.13. In case (C3) above, it is easy to check that the conformal map $z \overset{\Phi}{\mapsto} \overline{z}$ in the parameter domain $D(\infty, R)$ of $E_{0,b}$ satisfies $g \circ \Phi = 1/\overline{g}$, $\Phi^* dh = \overline{dh}$. Hence, after translating the surface so that the image of the point $R \in D(\infty, R)$ lies on the x_3-axis, we deduce that Φ produces an isometry of $E_{0,b}$ which extends to a 180-rotation of $\mathbb{R}^3$ around the x_3-axis; in particular, $E_{0,b} \cap (x_3\text{-axis})$ contains two infinite rays.

To understand the geometry of the canonical end $E_{a,b}$ and in particular prove that it is embedded if R is taken large enough, it is worth analyzing its multi-valued graph structure, which is the purpose of Theorem 4.15 below. Before stating this result, we need some notation.

Definition 4.14. In polar coordinates (ρ, θ) on $\mathbb{R}^2 - \{0\}$ with $\rho > 0$ and $\theta \in \mathbb{R}$, a *k-valued graph on an annulus of inner radius r and outer radius R,* is a single-valued graph of a real-valued function $u(\rho, \theta)$ defined over

$$S_{r,R}^{-k,k} = \{(\rho, \theta) \mid r \le \rho \le R,\ |\theta| \le k\pi\}, \tag{24}$$

k being a positive integer (see Figure 8). The *separation* between consecutive sheets is

$$w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta) \in \mathbb{R}. \tag{25}$$

The surface $\Sigma = \{(\rho \cos\theta, \rho \sin\theta, u(\rho, \theta)) \mid (\rho, \theta) \in S_{r,R}^{-k,k}\}$ is clearly embedded if and only if $w > 0$ (or $w < 0$). The multi-valued graph u is said to be an H-multi-valued graph if it is an H-surface.

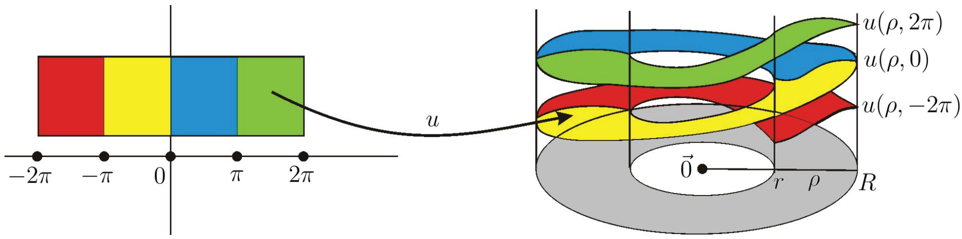

FIGURE 8. A 2-valued graph with positive separation.

Note that the separation function $w(\rho, \theta)$ used in Theorem 4.15 below refers to the vertical separation between the two disjoint multi-valued graphs Σ_1, Σ_2 appearing in the next result (versus the separation used in Definition 4.14, which measured the vertical distance between two consecutive sheets of the *same* multi-valued graph). We also use the notation $\widetilde{D}(\infty, R) = \{(\rho, \theta) \mid \rho \geq R, \theta \in \mathbb{R}\}$ and $C(R) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq R^2\}$.

THEOREM 4.15 (Asymptotic behavior of $E_{a,b}$). *Given $a \geq 0$ and $b \in \mathbb{R}$, the canonical end $E = E_{a,b}$ satisfies the following properties.*

1. *There exists $R = R_E > 0$ large such that $E_{a,b} - C(R)$ consists of two disjoint multi-valued graphs Σ_1, Σ_2 over $D(\infty, R)$ of smooth functions $u_1, u_2 \colon \widetilde{D}(\infty, R) \to \mathbb{R}$ such that their gradients satisfy $\nabla u_i(\rho, \theta) \to 0$ as $\rho \to \infty$ and the separation function $w(\rho, \theta) = u_1(\rho, \theta) - u_2(\rho, \theta)$ between both multi-valued graphs converges to π as $\rho + |\theta| \to \infty$. Furthermore for θ fixed*[1] *and $i = 1, 2$,*

$$\lim_{\rho \to \infty} \frac{u_i(\rho, \theta)}{\log(\log \rho)} = \frac{b}{2\pi}. \tag{26}$$

2. *The translated surfaces $E_{a,b} + (0, 0, -2\pi n - \frac{b}{2\pi} \log n)$ (resp. $E_{a,b} + (0, 0, 2\pi n - \frac{b}{2\pi} \log n)$) converge as $n \to \infty$ to a vertical helicoid H_T (resp. H_B) such that*

$$H_B = H_T + (0, a/2, 0). \tag{27}$$

The last equality together with item 1 imply that for different values of (a, b), the related canonical ends $E_{a,b}$ are not asymptotic after a rigid motion and homothety. See Figure 9 for a description of how the flux vector $(a, 0, -b)$ of $E_{a,b}$ influences its geometry.

The proof of Theorem 4.15 is based on a careful analysis of the horizontal and vertical projections of $E = E_{a,b}$ in terms of the Weierstrass representation. In fact, the explicit expressions of g, dh in equations (22), (23) is not

[1]This condition expresses the intersection of $E - C(R_E)$ with a vertical half-plane bounded by the x_3-axis, of polar angle θ, see Figure 9.

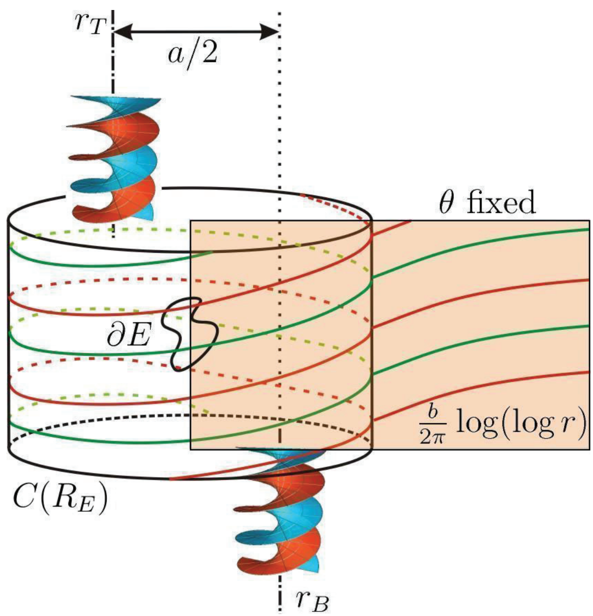

FIGURE 9. A schematic image of the embedded canonical end $E = E_{a,b}$ with flux vector $(a, 0, -b)$ (see Theorem 4.15). The half-lines r_T, r_B refer to the axes of the vertical helicoids H_T, H_B.

used, but only that those choices of g, dh have the common structure

$$g(z) = e^{iz+f(z)}, \quad dh = \left(1 + \frac{\lambda}{z-\mu}\right) dz, \tag{28}$$

where $\lambda \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $f\colon D(\infty, R) \cup \{\infty\} \to \mathbb{C}$ is a holomorphic function such that $f(\infty) = 0$ (the multiplicative constant t appearing in equation (22) can be absorbed by μ in (28) after an appropriate change of variables in the parameter domain).

With the canonical examples at hand, we are now ready to state the main result of this section.

THEOREM 4.16. *Let $E \subset \mathbb{R}^3$ be a complete 0-annulus with infinite total curvature and compact boundary. Then, E is conformally diffeomorphic to a punctured disk, its Gaussian curvature function is bounded, and after replacing E by a subend and applying a suitable homothety and rigid motion, we have:*

1. *The Weierstrass data of E is of the form (28) defined on $D(\infty, R)$ for some $R > 0$, where f is a holomorphic function in $D(\infty, R)$ with $f(\infty) = 0$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}$. In particular, dh extends meromorphically across infinity with a double pole.*
2. *E is asymptotic to the canonical end $E_{a,b}$ determined by the equality $F = (a, 0, -b)$, where F is the flux vector of E along its boundary. In*

particular, E is asymptotic to the end of a helicoid if and only if it has zero flux.

REMARK 4.17. The first statement of Theorem 4.6 is now a direct consequence of Theorem 4.16, since the divergence theorem insures that the flux vector of a one-ended 0-surface along a loop that winds around its end vanishes.

5. Constant mean curvature surfaces in $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$

As mentioned in the introduction, this survey's primarily focus is on past and recent work in the theory of minimal and constant mean curvature surface that has been done by the authors. However, in this section we will discuss some key selected results in the field which are not directly related to our work. These results are related to constant mean curvature surfaces in $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{R}$.

5.1. Brendle's proof of the Lawson Conjecture. We begin by talking about Brendle's proof of the Lawson Conjecture. See [**17**] for the actual paper and [**18**] for Brendle's complete survey of this problem and related questions. Let $\mathbb{S}^3$ denote the unit sphere in $\mathbb{R}^4$. The Clifford Torus is the torus defined by the following set of points

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2}\}. \tag{29}$$

It is a minimally embedded torus with zero intrinsic Gaussian curvature and its principal curvatures are 1 and -1.

In 1970, Lawson proposed the following conjecture, see [**91**].

CONJECTURE 5.1 (Lawson Conjecture). *Let Σ be an embedded minimal torus in $\mathbb{S}^3$, then Σ is congruent to the Clifford Torus.*

While, as we pointed out in Section 2, there are no closed minimal surfaces in $\mathbb{R}^3$, in $\mathbb{S}^3$ many examples of closed minimal surfaces have been constructed. The simplest example is the equator, namely

$$\{(x_1, x_2, x_3, x_4) \in \mathbb{S}^3 \subset \mathbb{R}^4 : x_4 = 0\}.$$

The second simplest embedded example is perhaps the Clifford Torus. Indeed, constructing closed embedded examples which are different from the equator and the Clifford Torus turned out to be a non-trivial task. In [**90**] Lawson proved that given any positive genus g, there exists at least one compact embedded minimal surface of genus g in $\mathbb{S}^3$. In fact, he also showed that when the genus is not a prime number, then there are at least two such (non-congruent) surfaces. Note that the conjecture is false if the surface is not embedded, see [**89**]. After these embedded examples, more examples were constructed (see for instance [**80, 81**]).

The first example of a classification result for minimal surfaces in $\mathbb{S}^3$ was given by Almgren. In [**5**], Almgren proved the following theorem whose proof is based on the fact that the Hopf differential of a CMC surface in $\mathbb{S}^3$ is holomorphic.

THEOREM 5.2. *Up to rigid motions of* $\mathbb{S}^3$, *the equator is the only minimal surface in* $\mathbb{S}^3$ *of genus zero.*

This is the analogous of the Hopf Theorem in $\mathbb{R}^3$ and in fact the proof uses similar arguments. Lawson Conjecture is the equivalent of Theorem 5.2 but instead for the case of tori. In [**178**] Ros was able to prove the Lawson Conjecture assuming some additional symmetries of Σ. We refer the interested reader to [**18**] and the references therein for a better discussion on previous partial results.

We now give a sketch of Brendle's proof of the Lawson Conjecture. A key ingredient in Brendle's proof is the following theorem also due to Lawson [**90**].

THEOREM 5.3. *A surface of genus one minimally immersed in* $\mathbb{S}^3$ *has no umbilical points.*

Another key result is a Simons-type identity [**189**]; namely, if Σ is a minimal surface in $\mathbb{S}^3$, then

$$\Delta_\Sigma(|A|) - \frac{|\nabla |A||^2}{|A|} + (|A|^2 - 2)|A| = 0, \tag{30}$$

where $|A|$ is the norm of the second fundamental form of Σ.

Finally, the proof by Brendle of the Lawson Conjecture relies on applying a maximum principle type argument to a function depending on two points. Results of this type were first developed by Huisken in [**75**] and then extended by Andrews in [**6**]. In [**75**], among other things, Huisken used these techniques to give a new proof of Grayson's theorem [**60**], that says that under the curve shortening flow, any embedded curve shrinks to a point in finite time and asymptotically becomes a circle. In [**6**] Andrews applied these ideas in the mean curvature flow setting.

Let $F\colon \Sigma \to \mathbb{S}^3$ be a minimal immersion of a genus-one surface Σ into $\mathbb{S}^3$ and let ν denote a unit normal vector field. Since by Theorem 5.3 Σ has no umbilical points, then $\inf_\Sigma |A| > 0$ and the following quantity is finite:

$$\omega = \sup_{x,y\in\Sigma,\, x\neq y} \sqrt{2}\frac{|\langle \nu(x), F(y)\rangle|}{|A|(x)(1-\langle F(x),F(y)\rangle)}.$$

There are then two cases to consider, $\omega \leq 1$ and $\omega > 1$. If $\omega \leq 1$ then one has that

$$\frac{|A|(x)}{\sqrt{2}}(1-\langle F(x),F(y)\rangle) - \langle \nu(x), F(y)\rangle \geq 0.$$

Using this, a calculation gives that the second fundamental form of F is parallel and therefore the principal curvatures are constant. Since by the Gauss equation,

$$K = 1 + k_1 k_2$$

where K denotes the intrinsic Gaussian curvature and k_1, k_2 the principal curvatures of Σ, it follows that K is constant and therefore Σ is flat. By a result of Lawson, this implies that Σ is congruent to the Clifford Torus.

If $\omega > 1$, then Brendle considers the function

$$Z(x,y) = \frac{\omega}{\sqrt{2}}|A|(x)(1 - \langle F(x), F(y)\rangle) + \langle \nu(x), F(y)\rangle.$$

Note that this function is non-negative and by possibly replacing ν by $-\nu$, there exist $\overline{x}, \overline{y} \in \Sigma$, $\overline{x} \neq \overline{y}$, such that $Z(\overline{x}, \overline{y}) = 0$. Since the function Z attains its global minimum at $(\overline{x}, \overline{y})$, then $Z_x(\overline{x}, \overline{y}) = Z_y(\overline{x}, \overline{y}) = 0$ and the Hessian of Z at $(\overline{x}, \overline{y})$ is non-negative. Let

$$\Delta := \{x \in \Sigma : \text{ there exists a point } y \in \Sigma \setminus \{x\} \text{ with } Z(x,y) = 0\}.$$

The set Δ is non-empty. Reasoning as before, if $x \in \Delta$ and $y \in \Sigma \setminus \{x\}$ satisfies $Z(x,y) = 0$, then $Z_x(x,y) = Z_y(x,y) = 0$ and $(\nabla^2 Z)(x,y) \geq 0$. Using equation (30), Brendle proves that Z is a subsolution of a degenerate elliptic equation in Δ. An application of Bony's strict maximum principle [**15**] then gives that Δ is open. Finally, Brendle shows that for any $x \in \Delta$, $\nabla|A|(x) = 0$. Since Ω is open, an analytic continuation type argument gives that for any $x \in \Sigma$, $\nabla|A|(x) = 0$. Just like in the case $\omega \leq 1$, this implies that Σ is flat and thus, by a result of Lawson, congruent to the Clifford Torus.

5.2. Marques and Neves' proof of the Willmore Conjecture. In this section, we give an idea of Marques and Neves' proof of the Willmore Conjecture. See [**96**] for the actual paper and [**97**] for Marques and Neves' complete survey of this problem and related questions. Let $\Sigma \subset \mathbb{S}^3$ be a closed surface. The Willmore energy of Σ is

$$\mathcal{W}(\Sigma) := \int_\Sigma (1 + H^2).$$

The Willmore energy of the Clifford Torus is $2\pi^2$. The Willmore conjecture [**207**] states that the Clifford Torus minimizes the Willmore energy among tori, and thus, in this sense, is the best torus. Namely,

CONJECTURE 5.4. *Let $\Sigma \subset \mathbb{S}^3$ be a closed surface of genus one. Then*

$$\mathcal{W}(\Sigma) \geq 2\pi^2.$$

An important feature of the Willmore energy is that it is conformally invariant. Namely, given $v \in \mathbb{B}^4$ (unit ball in $\mathbb{R}^4$), consider the centered dilation of $\mathbb{S}^3$ that fixes $v/|v|$ and $-v/|v|$, namely

$$F_v \colon \mathbb{S}^3 \to \mathbb{S}^3, \quad F_v(x) = \frac{1 - |v|^2}{|x - v|^2}(x - v) - v. \tag{31}$$

Then for any $v \in \mathbb{B}^4$, $\mathcal{W}(F_v(\Sigma)) = \mathcal{W}(\Sigma)$.

In their seminal paper [**96**], Marques and Neves proved the Willmore conjecture. Before their very elegant proof, several results and techniques have been used to understand the Willmore Conjecture, see for instance [**9, 19, 20, 25, 29, 176, 179, 187, 188, 199, 204, 208**]. We refer the interested reader to Section 2 of [**97**] and the references therein for a more comprehensive and detailed discussion.

We now give a sketch of their proof. A key tool in their proof are techniques which come from the min-max principle which is used to find unstable critical points of a given functional. See Section 2.5 for a discussion on stability and index of a minimal surface. In [**4**], Almgren developed a min-max theory for the area functional. Let $\mathcal{Z}_2(\mathbb{S}^3)$ denote the space of integral 2-currents with zero boundary and let $I^k = [0,1]^k$ denote the k-dimensional cube. Given a continuous function $\Phi\colon I^k \to \mathcal{Z}_2(\mathbb{S}^3)$, let $[\Phi]$ denote the set of all continuous functions from I^k to $\mathcal{Z}_2(\mathbb{S}^3)$ that are homotopic to Φ through homotopies that fix the functions on ∂I^k. Let

$$\mathbf{L}([\Phi]) := \inf_{\Psi\in[\Phi]} \sup_{x\in I^k} \mathrm{Area}(\Psi(x)).$$

In [**171**] Pitts proved the following theorem.

THEOREM 5.5. *If* $\mathbf{L}([\Phi]) > \sup_{x\in I^k} \mathrm{Area}(\Phi(x))$ *then there exists a disjoint collection of smooth, closed, embedded minimal surfaces* $\Sigma_1,\dots,\Sigma_N$ *in* $\mathbb{S}^3$ *such that*

$$\mathbf{L}([\Phi]) = \sum_{i=1}^N m_i \,\mathrm{Area}(\Sigma_i),$$

for some positive integers multiplicities $m_1,\dots,m_N$.

Let F_v be the conformal map defined in equation (31). Given an embedded surface $S = \partial\Omega$, where Ω is a region of $\mathbb{S}^3$, and using the ambient distance, let

$$S_t := \partial\{x \in \mathbb{S}^3 : d(x,\Omega) \le t\} \quad \text{if } t \in [0,\pi]$$

and

$$S_t := \partial\{x \in \mathbb{S}^3 : d(x,\mathbb{S}^3 \setminus \Omega) \le -t\} \quad \text{if } t \in [-\pi,0].$$

Given an embedded compact surface $\Sigma \subset \mathbb{S}^3$, Marques and Neves define a five parameters deformation of Σ, $\{\Sigma_{(v,t)}\}_{(v,t)\in\mathbb{B}^4\times[-\pi,\pi]}$, called the canonical family, where

$$\Sigma_{(v,t)} := (F_v(\Sigma))_t \in \mathcal{Z}_2(\mathbb{S}^3)$$

Thanks to a result of Ros [**179**] that was inspirational to their approach and the fact that the Willmore energy is conformally invariant, it follows that

$$\mathrm{Area}(\Sigma_{(v,t)}) \le \mathcal{W}(F_v(\Sigma)) = \mathcal{W}(\Sigma), \text{ for all } (v,t) \in \mathbb{B}^4 \times [-\pi,\pi].$$

The idea is to prove that the min-max principle applied to the homotopy class of the canonical family of a closed surface of genus one produces the Clifford Torus. If that is the case, then

$$2\pi^2 = \mathrm{Area}(\text{Clifford Torus}) \le \sup_{(v,t)\in\mathbb{B}^4\times[-\pi,\pi]} \mathrm{Area}(\Sigma_{(v,t)}) \le \mathcal{W}(\Sigma)$$

However, the canonical family so defined is discontinuous on $\partial\mathbb{B}^4 = \mathbb{S}^3$ but Marques and Neves were able to reparameterize the canonical family of Σ by a continuous map $\Phi_\Sigma \colon I^5 \to \mathcal{Z}_2(\mathbb{S}^3)$ such that the image $\Phi_\Sigma(I^5)$ is equal to the closure of the canonical family in $\mathcal{Z}_2(\mathbb{S}^3)$ and Φ_Σ satisfies the following properties:

(A) $\sup_{x\in I^5} \text{Area}(\Phi_\Sigma(x)) \leq \mathcal{W}(\Sigma)$.
(B) $\Phi_\Sigma(x,0) = \Phi_\Sigma(x,1) = 0$ for all $x \in I^4$.
(C) For any $x \in \partial I^4$ there exists $Q(x) \in \mathbb{S}^3$ such that $\Phi_\Sigma(x,t)$ is a sphere of radius πt centered at $Q(x)$ for any $t \in I$.
(D) The degree of $Q \colon \mathbb{S}^3 \to \mathbb{S}^3$ is equal to the genus of Σ and hence it is non-zero if the genus is not zero.

The definition of Q can be found in the papers [**96, 97**]. Without going into the details, when the genus of Σ is not zero, Property (D) guarantees that $\mathbf{L}([\Phi_\Sigma]) > 4\pi$, where $\mathbf{L}$ and $[\Phi_\Sigma]$ are defined in the previous discussion about the min-max principle. Given that, Marques and Neves apply the min-max argument to prove that there exists a closed minimal surface $\widehat{\Sigma}$ with $\text{Area}(\widehat{\Sigma}) = \mathbf{L}([\Phi_\Sigma])$. If $\text{Area}(\widehat{\Sigma}) \geq 8\pi$ then

$$\mathcal{W}(\Sigma) \geq \text{Area}(\widehat{\Sigma}) \geq 8\pi > 2\pi^2$$

and the conjecture holds. Thus $\text{Area}(\widehat{\Sigma}) < 8\pi$ which gives that Σ is connected because the area of a closed minimal surface in $\mathbb{S}^3$ is at least 4π.

By the nature of the deformation, it is natural to expect that the index of $\widehat{\Sigma}$ is 5. If this is the case then using a theorem by Urbano [**204**] gives that $\widehat{\Sigma}$ must be a Clifford torus. It is important to notice that Almgren-Pitts Theory does not give an estimate on the Morse index of $\widehat{\Sigma}$. Indeed, Marques and Neves were able to prove that the index of $\widehat{\Sigma}$ is 5 and therefore $\widehat{\Sigma}$ is a Clifford Torus. Therefore, for any closed surface Σ in $\mathbb{S}^3$ of genus at least one, we have that

$$\mathcal{W}(\Sigma) \geq 2\pi^2.$$

The fact that when equality holds then Σ is a Clifford Torus requires extra work and we refer the interested reader to Marques and Neves' papers for the argument.

5.3. Minimal surfaces in $\mathbb{S}^2\times\mathbb{R}$ foliated by circles: the classification theorem of Hauswirth, Kilian and Schmidt. In [**137**] Meeks and Rosenberg studied the geometry of complete minimal annuli in Riemannian manifolds that can be expressed as the Riemannian product $M\times\mathbb{R}$ of a closed Riemannian surface M with $\mathbb{R}$. In the case that M is a sphere with a metric of positive Gaussian curvature, they proved that any complete embedded minimal annulus Σ in $M\times\mathbb{R}$ is properly embedded with bounded second fundamental form and Σ intersects each level set sphere $M\times\{t\}, t\in\mathbb{R}$, in a simple closed curve. In this case they also showed that the moduli space of such minimal annuli with a fixed vertical flux is compact, which is a useful property since it implies that any sequence of vertical translations of Σ has a convergent subsequence. In particular, if $M = \mathbb{S}^2$ is the sphere of constant Gaussian curvature 1, then $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ is properly embedded and intersects each level set sphere in a round circle. They also described in that paper a 1-parameter family $\mathcal{A}$ of periodic minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$ and each surface in this family intersects each level set sphere of $\mathbb{S}^2 \times \mathbb{R}$ in a circle; this family first described by Hauswirth in [**65**] is similar to the Riemann family of properly embedded minimal planar domains in $\mathbb{R}^3$.

In fact Hauswirth [**65**] was able to construct a Jacobi function similar to the Shiffman function, an indispensable tool used by Meeks, Pérez and Ros in their proof of Theorem 1.4, the proof of which depended upon methods in the theory of integrable systems. Also using methods from the theory of integrable systems, Hauswirth, Kilian and Schmidt [**66**] recently proved the following classification result for complete embedded minimal annuli in $\mathbb{S}^2 \times \mathbb{R}$. For some related classification results for strongly Alexandrov embedded minimal annuli in the 3-sphere see Kilian and Schmidt [**83**].

THEOREM 5.6 (Hauswirth, Kilian, Schmidt [**66**]). *Every complete embedded minimal annulus in $\mathbb{S}^2 \times \mathbb{R}$ lies in the family $\mathcal{A}$. In particular, each such minimal annulus Σ intersects every level set sphere in $\mathbb{S}^2 \times \mathbb{R}$ in a circle, Σ is invariant under a reflection in a vertical totally geodesic annulus and Σ is periodic under a vertical translation that is the composition of two rotational symmetries around circles that are great circles in level set spheres.*

6. Limits of H-surfaces without local area or curvature bounds

Two central problems in the classical theory of H-surfaces are to understand the possible geometries or shapes of those H-surfaces in $\mathbb{R}^3$ that have finite genus, as well as the structure of limits of sequences of H-surfaces with fixed finite genus. The classical theory deals with these limits when the sequence has uniform local area and curvature bounds, since in this case one can reduce the problem of taking limits of H-surfaces to taking limits of H-graphs (for this reduction, one uses the local curvature bound in order to express the surfaces as local graphs of uniform size, and the local area bound to constrain locally the number of such graphs to a fixed finite number). In this graphical framework, the existence and properties of limits is given by the classical Arzelà-Ascoli theorem, see e.g., [**169**]. Hence, we will concentrate here on the case where we do not have such estimates.

The starting point consists of analyzing local structure of a sequence of compact 0-surfaces Σ_n with fixed finite genus in a fixed extrinsic ball in $\mathbb{R}^3$, which is an issue first tackled by Colding and Minicozzi in a series of papers where they study the structure of a sequence of compact 0-surfaces Σ_n with fixed genus but without area or curvature bounds in balls $\mathbb{B}(R_n) = \mathbb{B}(\vec{0}, R_n) \subset \mathbb{R}^3$, whose radii R_n either remain bounded or tend to infinity as $n \to \infty$ and with boundaries $\partial\Sigma_n \subset \partial\mathbb{B}_n$, $n \in \mathbb{N}$. These two possibilities on R_n lead to very different situations for the limit object of (a subsequence of) the Σ_n, as we will explain below. Generalizations these results to the $(H > 0)$-setting by Meeks and Tinaglia will also be discussed.

6.1. Colding-Minicozzi theory for 0-surfaces and generalizations by Meeks and Tinaglia to H-surfaces. As we indicated above, a main goal of the Colding-Minicozzi theory, as adapted by Meeks and Tinaglia, is to understand the limit objects for a sequence of compact H_n-surfaces Σ_n with fixed genus but not *a priori* area or curvature bounds, each one with compact boundary contained in the boundary sphere of $\mathbb{B}(R_n)$. Typically, one finds *weak H-laminations* (see Definition 7.1 for the concept of weak H-lamination) possibly with singularities as limits of subsequences of the Σ_n. Nevertheless, we will see in Theorems 6.1 and 7.7, that the behavior of the limit lamination changes dramatically depending on whether R_n diverges to ∞ or stays bounded, among others, in the following two aspects:

(I) The limit lamination might develop removable (case $R_n \to \infty$) or essential singularities (case R_n bounded).
(II) The leaves of the lamination are proper (case $R_n \to \infty$) or might fail to have this property (case R_n bounded).

These two phenomena connect with major open problems in the current state of the theory of H-surfaces:

(I)' Finding removable singularity results for weak H-laminations (or equivalently, finding extension theorems for weak H-laminations defined outside of a small set). In this line, we can mention the work by Meeks, Pérez and Ros [**121, 133**], see also Sections 9 and 14 below.
(II)' Finding conditions under which a complete H-surface must be proper (*embedded Calabi-Yau problem*), which we will treat further in Sections 8 and 11.

Coming back to the Colding-Minicozzi theory, the most important goal is to understand the shape of a 0-disk Σ in $\mathbb{R}^3$ depending on its Gaussian curvature. Roughly speaking, only two models are possible: if the curvature of Σ is everywhere small, then Σ is a graph (with the plane as a model); and if the Gaussian curvature of Σ is large at some point, then Σ consists of two multi-valued graphs pieced together (this is called a *double staircase,* whose model is the helicoid; see [**197, 198**] for related generalizations). Multi-valued graphs $\Sigma_g \subset \Sigma$ are subsets such that every point in Σ_g has a neighborhood which is a graph over its projection over the plane $\{z = 0\}$ (up to a rotation), but the global projection from Σ to $\{z = 0\}$ fails to be one-to-one; for a precise definition of a k-valued graph, see Definition 4.14. For instance, a vertical helicoid can be thought as two ∞-valued graphs joined along the vertical axis.

The study of H-multi-valued graphs, i.e., multi-valued graphs with constant mean curvature H, relies heavily on PDE techniques, some of which aspects we will comment next. Since the third component of the

Gauss map on a H-multi-valued graph Σ as in Definition 4.14 is a positive Jacobi function, then Σ is stable, and thus it has curvature estimates away from its boundary by Theorem 2.26. Also in this case the separation w given by equation (25) is a difference between two solutions of the same mean curvature H equation, thus w satisfies a second order elliptic, partial differential equation in divergence form:

$$\operatorname{div}(\mathcal{A}\nabla w) = 0, \tag{32}$$

where $\mathcal{A}$ is a smooth map valued in the space of real symmetric, positive definite 2×2 matrices. Furthermore, the eigenvalues λ_i of $\mathcal{A}$ satisfy

$$0 < \mu \leq \lambda_i \leq 1/\mu, \tag{33}$$

where the constant μ only depends on an upper bound for the gradient $|\nabla u|$ (in particular, (32) resembles the Laplace equation if $|\nabla u|$ is extremely small). A consequence of (32)-(33) is that w satisfies a Harnack-type inequality:

$$\sup_{\Omega'} w \leq C \inf_{\Omega'} w$$

whenever Ω' has compact closure in the domain of w for some constant $C > 0$ depending solely on an upper bound for $|\nabla u|$ and on the distance from Ω' to the boundary of the domain of w.

With these preliminaries at hand, the statement of the so-called *Limit Lamination Theorem for 0-Disks* (Theorem 0.1 of [**37**]) is easy to understand (see also [**13**] for related results on the topology and geometry of leaves of a lamination obtained as limit of a sequence of 0-disks); to make this statement optimal in this 0-disk setting, we will make use of the result by Meeks [**105**] (see Theorem 8.5 below) that the singular set $\mathcal{S}$ in Theorem 6.1 is a vertical line instead of a Lipschitz curve parameterized by its height, as given in [**37**]. It is worth mentioning that the proof of Theorem 8.5 uses the uniqueness of the helicoid among properly embedded non-flat 0-surfaces in $\mathbb{R}^3$, whose proof in turn depends on the original statement of the Limit Lamination Theorem for 0-Disks with a Lipschitz curve as singular set $\mathcal{S}$. Recently, Meeks and Tinaglia [**146**] generalized the Limit Lamination Theorem for Disks to the case that the surfaces are H-disks. Hence, we state this theorem in the general H-setting.

Theorem 6.1 (Limit Lamination Theorem for H-Disks). *Let $\Sigma_n \subset \mathbb{B}(R_n)$ be a sequence of H_n-disks with $\vec{0} \in \Sigma_n$, $\partial\Sigma_n \subset \partial\mathbb{B}(R_n)$, $R_n \to \infty$ and $\sup |A_{\Sigma_n}(\vec{0})| \to \infty$. Let $\mathcal{S}$ denote the x_3-axis. Then, there exists a subsequence Σ_n (denoted in the same way) and numbers $\widehat{R}_n \to \infty$ such that up to a rotation of $\mathbb{R}^3$ fixing $\vec{0}$:*

1. *Each $\Sigma_n \cap \mathbb{B}(\widehat{R}_n)$ consists of exactly two multi-valued graphs away from $\mathcal{S}$, which spiral together.*
2. *For each $\alpha \in (0,1)$, the surfaces $\Sigma_n - \mathcal{S}$ converge in the C^α-topology to the foliation $\mathcal{F} = \{x_3 = t\}_{t\in\mathbb{R}}$ of $\mathbb{R}^3$ by horizontal planes.*

3. *If* $\mathcal{S}(t) = (0, 0, t)$ *for every* $t \in \mathbb{R}$*, then given* $t \in \mathbb{R}$ *and* $r > 0$*,*

$$\sup_{\Sigma_n \cap \mathbb{B}(\mathcal{S}(t), r)} |A_{\Sigma_n}| \to \infty \quad \textit{as } n \to \infty.$$

Items 1 and 2 in the statement of Theorem 6.1 mean that for every compact subset $C \subset \mathbb{R}^3 - \mathcal{S}$ and for every $n \in \mathbb{N}$ sufficiently large depending on C, the surface $\Sigma_n \cap C$ consists of multi-valued graphs over a portion of $\{x_3 = 0\}$, and the sequence $\{\Sigma_n \cap C\}_n$ converges to $\mathcal{F} \cap C$ as graphs in the C^α-topology. Item 3 deals with the behavior of the sequence along the singular set of convergence.

The basic example to visualize Theorem 6.1 is a sequence of rescaled helicoids $\Sigma_n = \lambda_n H = \{\lambda_n x \mid x \in H\}$, where H is a fixed vertical helicoid with axis the x_3-axis and $\lambda_n > 0$, $\lambda_n \searrow 0$. The Gaussian curvature of $\{\Sigma_n\}_n$ blows up along the x_3-axis and the Σ_n converge away from the axis to the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes. The x_3-axis $\mathcal{S}$ is the singular set of C^1-convergence of the Σ_n to $\mathcal{F}$; i.e., the Σ_n do not converge C^1 to the leaves of $\mathcal{F}$ along the x_3-axis. Finally, each leaf L of $\mathcal{F}$ extends smoothly across $L \cap \mathcal{S}$; (S consists of removable singularities of $\mathcal{F}$).

The proof of Theorem 6.1 in the $H = 0$ setting is involved and runs along several highly demanding papers [**33, 34, 35, 36, 37**]. Since a through sketch of this proof is provided in Chapter 4 of [**118**], we do not provide any further details here. However, before going on to the next section of this survey, it is worthwhile to highlight one of the crucial ingredients of this proof when $H = 0$, namely the scale invariant *1-sided curvature estimate for 0-disks* by Colding, Minicozzi [**37**]. We remark that Meeks and Tinaglia [**148**] have proved a companion *1-sided curvature estimate for H-disks* that is not scale invariant, and that will be stated in Theorem 11.10.

THEOREM 6.2 (One-Sided Curvature Estimate for 0-Disks). *There exists an* $\varepsilon > 0$ *such that the following holds. Given* $r > 0$ *and a 0-disk* $\Sigma \subset \mathbb{B}(2r) \cap \{x_3 > 0\}$ *with* $\partial\Sigma \subset \partial\mathbb{B}(2r)$*, then for any component* Σ' *of* $\Sigma \cap \mathbb{B}(r)$ *which intersects* $\mathbb{B}(\varepsilon r)$*,*

$$r \sup_{\Sigma'} |A_\Sigma| \leq 1. \tag{34}$$

(See Figure 10).

7. Weak H-laminations, the Stable Limit Leaf Theorem and the Limit Lamination Theorem for Finite Genus

We have mentioned the importance of understanding limits of sequences of H-surfaces with fixed (or bounded) genus but no *a priori* curvature or area bounds in a 3-manifold N, a situation of which Theorem 6.1 is a particular case. This result shows that one must consider limit objects other than H-surfaces, such as minimal foliations or more generally, H-laminations of N.

In this section we start by recalling the classical notion of lamination and discuss some results on the regularity of these objects when the leaves

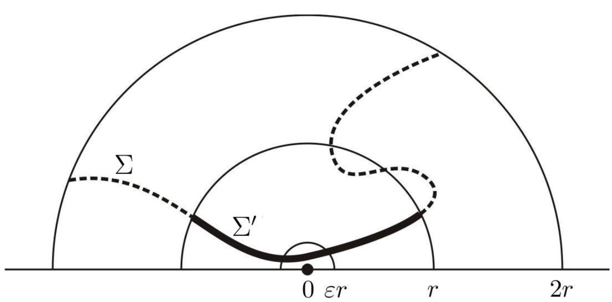

FIGURE 10. The one-sided curvature estimate.

have constant mean curvature. Then we will enlarge the class to admit *weak laminations* by allowing certain tangential intersections between the leaves. These weak laminations and foliations will be studied in subsequent sections.

DEFINITION 7.1. A *codimension-one lamination* of a Riemannian 3-manifold N is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. $\mathcal{L}$ is a closed subset of N;
2. $\mathcal{A} = \{\varphi_\beta \colon \mathbb{D} \times (0,1) \to U_\beta\}_\beta$ is an atlas of topological coordinate charts of N (here $\mathbb{D}$ is the open unit disk in $\mathbb{R}^2$, $(0,1)$ is the open unit interval in $\mathbb{R}$ and U_β is an open subset of N); note that although N is assumed to be smooth, we only require that the regularity of the atlas (i.e., that of its change of coordinates) is of class C^0; in other words, $\mathcal{A}$ is an atlas with respect to the topological structure of N.
3. For each β, there exists a closed subset C_β of $(0,1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

We will simply denote laminations by $\mathcal{L}$, omitting the charts φ_β in $\mathcal{A}$ unless explicitly necessary. A lamination $\mathcal{L}$ is said to be a *foliation of* N if $\mathcal{L} = N$. Every lamination $\mathcal{L}$ decomposes into a collection of disjoint, connected topological surfaces (locally given by $\varphi_\beta(\mathbb{D} \times \{t\})$, $t \in C_\beta$, with the notation above), called the *leaves* of $\mathcal{L}$. Observe that if $\Delta \subset \mathcal{L}$ is any collection of leaves of $\mathcal{L}$, then the closure of the union of these leaves has the structure of a lamination within $\mathcal{L}$, which we will call a *sublamination.*

A codimension-one lamination $\mathcal{L}$ of N is called a *CMC lamination* if each of its leaves is smooth and has constant mean curvature, possibly varying from leaf to leaf. Given $H \in \mathbb{R}$, an *H-lamination* of N is a CMC lamination all whose leaves have the same mean curvature H. If $H = 0$, the H-lamination will also be called a *minimal lamination.*

Since the leaves of a lamination $\mathcal{L}$ are disjoint, it makes sense to consider the second fundamental form $A_\mathcal{L}$ as being defined on the union of the leaves. A natural question to ask is whether or not the norm $|A_\mathcal{L}|$ of the second fundamental form of a (minimal, H- or CMC) lamination $\mathcal{L}$ in a

Riemannian 3-manifold is locally bounded. Concerning this question, we make the following observations.

(O.1) If $\mathcal{L}$ is a minimal lamination, then Theorem 6.2 implies that $|A_{\mathcal{L}}|$ is locally bounded (to prove this, one only needs to deal with limit leaves, where the one-sided curvature estimates apply).

(O.2) As a consequence of recent work of Meeks and Tinaglia [**143, 145, 148**] on curvature estimates for $(H > 0)$-disks (Theorem 11.10 below), a CMC lamination $\mathcal{L}$ of a Riemannian 3-manifold N has $|A_{\mathcal{L}}|$ locally bounded.

Given a sequence of CMC laminations $\mathcal{L}_n$ of a Riemannian 3-manifold N with uniformly bounded second fundamental form on compact subsets of N, a simple application of the uniform graph lemma for surfaces with constant mean curvature (see Colding and Minicozzi [**32**] or Pérez and Ros [**169**] from where this well-known result can be deduced) and of the Arzelà-Ascoli Theorem, gives that there exists a limit object of (a subsequence of) the $\mathcal{L}_n$, which in general fails to be a CMC lamination since two "leaves" of this limit object could intersect tangentially with mean curvature vectors pointing in opposite directions; nevertheless, if $\mathcal{L}_n$ is a 0-lamination for every n, then the maximum principle ensures that the limit object is indeed a 0-lamination, see Proposition B1 in [**37**]. Still, in the general case of CMC laminations, such a limit object always satisfies the conditions in the next definition.

DEFINITION 7.2. A (codimension-one) *weak CMC lamination* $\mathcal{L}$ of a Riemannian 3-manifold N is a collection $\{L_\alpha\}_{\alpha\in I}$ of (not necessarily injectively) immersed constant mean curvature surfaces, called the *leaves* of $\mathcal{L}$, satisfying the following three properties.

1. $\bigcup_{\alpha\in I} L_\alpha$ is a closed subset of N.
2. If $p \in N$ is a point where either two leaves of $\mathcal{L}$ intersect or a leaf of $\mathcal{L}$ intersects itself, then each of these local surfaces at p lies at one side of the other (this cannot happen if both of the intersecting leaves have the same signed mean curvature as graphs over their common tangent space at p, by the maximum principle).
3. The function $|A_{\mathcal{L}}|\colon \mathcal{L} \to [0,\infty)$ given by

$$|A_{\mathcal{L}}|(p) = \sup\{|A_L|(p) \mid L \text{ is a leaf of } \mathcal{L} \text{ with } p \in L\}. \tag{35}$$

   is uniformly bounded on compact sets of N.

Furthermore:

- If $N = \bigcup_\alpha L_\alpha$, then we call $\mathcal{L}$ a *weak CMC foliation* of N.
- If the leaves of $\mathcal{L}$ have the same constant mean curvature H, then we call $\mathcal{L}$ a *weak H-lamination* of N (or H-foliation, if additionally $N = \bigcup_\alpha L_\alpha$).

The following proposition follows immediately from the definition of a weak H-lamination and from Theorem 2.12.

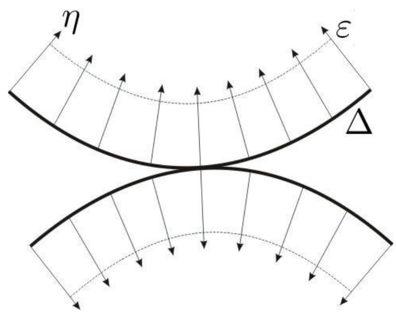

FIGURE 11. The leaves of a weak H-lamination with $H \neq 0$ can intersect each other or themselves, but only tangentially with opposite mean curvature vectors. Nevertheless, on the mean convex side of these locally intersecting leaves, there is a lamination structure.

PROPOSITION 7.3. *Let $\mathcal{L}$ be a weak H-lamination of a 3-manifold N. Then $\mathcal{L}$ has a local H-lamination structure on the mean convex side of each leaf. More precisely, given a leaf L_α of $\mathcal{L}$ and a small disk $\Delta \subset L_\alpha$, there exists an $\varepsilon > 0$ such that if (q,t) denotes the normal coordinates for* $\exp_q(t\eta_q)$ *(here* exp *is the exponential map of N and η is the unit normal vector field to L_α pointing to the mean convex side of L_α), then the exponential map* exp *is an injective submersion in $U(\Delta,\varepsilon) := \{(q,t) \mid q \in \mathrm{Int}(\Delta),\, t \in (-\varepsilon,\varepsilon)\}$, and the inverse image* $\exp^{-1}(\mathcal{L}) \cap \{q \in \mathrm{Int}(\Delta),\, t \in [0,\varepsilon)\}$ *is an H-lamination of $U(\Delta,\varepsilon)$ in the pulled back metric, see Figure 11.*

DEFINITION 7.4. Let M be a complete, embedded surface in a Riemannian 3-manifold N. A point $p \in N$ is a *limit point* of M if there exists a sequence $\{p_n\}_n \subset M$ which diverges to infinity in M with respect to the intrinsic Riemannian topology on M but converges in N to p as $n \to \infty$. Let $\lim(M)$ denote the set of all limit points of M in N; we call this set the *limit set of M*. In particular, $\lim(M)$ is a closed subset of N and $\overline{M} - M \subset \lim(M)$, where $\overline{M}$ denotes the closure of M.

The above notion of limit point can be extended to the case of a lamination $\mathcal{L}$ of N as follows: A point $p \in \mathcal{L}$ is a *limit point* if there exists a coordinate chart $\varphi_\beta \colon \mathbb{D} \times (0,1) \to U_\beta$ as in Definition 7.1 such that $p \in U_\beta$ and $\varphi_\beta^{-1}(p) = (x,t)$ with t belonging to the accumulation set of C_β. The notion of limit point can be also extended to the case of a weak H-lamination of N, by using that such an weak H-lamination has a local lamination structure at the mean convex side of any of its points, given by Proposition 7.3.

It is not difficult to show that if p is a limit point of a lamination $\mathcal{L}$ (resp. of a weak H-lamination), then the leaf L of $\mathcal{L}$ passing through p consists entirely of limit points of $\mathcal{L}$; in this case, L is called a *limit leaf* of $\mathcal{L}$.

The following result, called the *Stable Limit Leaf Theorem,* concerns the behavior of limit leaves for a weak H-lamination.

THEOREM 7.5 (Meeks, Pérez, Ros [**130, 131**]). *Any limit leaf L of a codimension-one weak H-lamination of a Riemannian manifold is stable for the Jacobi operator defined in equation (13). More strongly, every two-sided cover of such a limit leaf L is stable. Therefore, the collection of stable leaves of a weak H-lamination $\mathcal{L}$ has the structure of a sublamination containing all the limit leaves of $\mathcal{L}$.*

We next return to discuss more aspects related to the Limit Lamination Theorem for H-Disks (Theorem 6.1). The limit object in that result is an example of a limiting *parking garage structure* on $\mathbb{R}^3$ with one column, see the next to last paragraph before Theorem 8.5 for a description of the notion of minimal parking garage structure. We will find again a limiting parking garage structure in Theorem 7.7 below, but with two columns instead of one. In a parking garage structure one can travel quickly up and down the horizontal levels of the limiting surfaces only along the (helicoidal[2]) columns, in much the same way that some parking garages are configured for traffic flow; hence, the name parking garage structure. We will study these structures in Section 8.1.

Theorem 6.1 deals with limits of sequences of H-disks, but it is also useful when studying more general situations, as for instance, locally simply connected sequences of H-surfaces, a notion which we now define.

DEFINITION 7.6. Suppose that $\{M_n\}_n$ is a sequence of H_n-surfaces (possibly with boundary) in an open set U of $\mathbb{R}^3$. If for any $p \in U$ there exists a number $r(p) > 0$ such that $\mathbb{B}(p, r(p)) \subset U$ and for n sufficiently large, M_n intersects $\mathbb{B}(p, r(p))$ in compact disks whose boundaries lie on $\partial\mathbb{B}(p, r(p))$, then we say that $\{M_n\}_n$ is *locally simply connected in U.* If $\{M_n\}_n$ is a locally simply connected sequence in $U = \mathbb{R}^3$ and the positive number $r(p)$ can be chosen independently of $p \in \mathbb{R}^3$, then we say that $\{M_n\}_n$ is *uniformly locally simply connected.*

There is a subtle difference between our definition of uniformly locally simply connected and that of Colding and Minicozzi [**39**], which may lead to some confusion. Colding and Minicozzi define a sequence $\{M_n\}_n$ to be uniformly locally simply connected in an open set $U \subset \mathbb{R}^3$ if for any compact set $K \subset U$, the number $r(p)$ in Definition 7.6 can be chosen independently of $p \in K$. It is not difficult to check that this concept coincides with our definition of a locally simply connected sequence in U.

The Limit Lamination Theorem for H-Disks (Theorem 6.1) admits a generalization to a locally simply connected sequence of non-simply connected H_n-planar domains passing through the origin and having unbounded curvature at the origin, which we now explain since it will be useful for our goal of classifying minimal planar domains. Instead of the scaled-down limit

[2]See Remark 8.6.

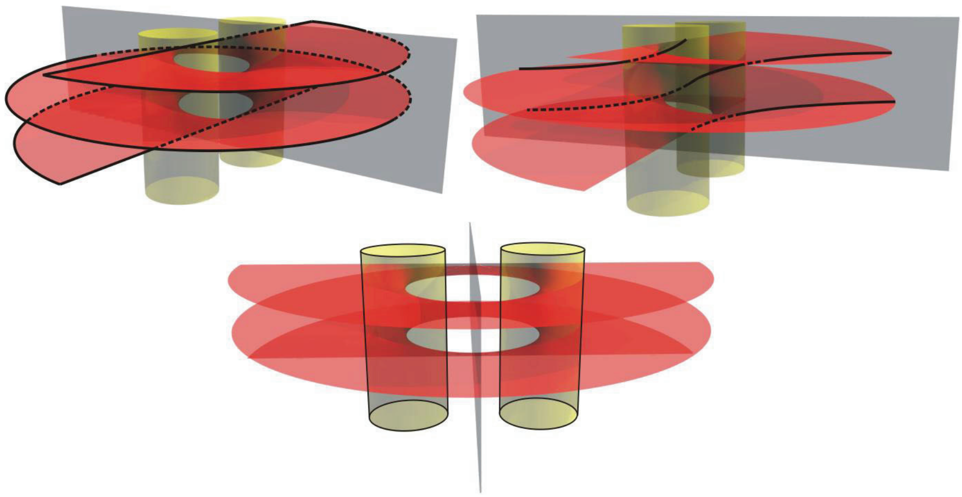

FIGURE 12. Three views of the same Riemann minimal example, with large horizontal flux and two oppositely handed vertical helicoids forming inside solid almost vertical cylinders, one at each side of the vertical plane of symmetry.

of the helicoid, the basic example in this case is an appropriate scaled-down limit of Riemann minimal examples $\mathcal{R}_t$, $t > 0$. To understand this limit, normalize each Riemann minimal example $\mathcal{R}_t$ so that $\mathcal{R}_t$ is symmetric by reflection in the (x_1, x_3)-plane Π and the *flux* $F(t)$ of $\mathcal{R}_t$, which is the flux vector along any compact horizontal section $\mathcal{R}_t \cap \{x_3 = \text{constant}\}$, has third component equal to one. The fact that $\mathcal{R}_t$ is invariant by reflection in Π forces $F(t)$ to be contained in Π for each $t > 0$. Furthermore, $t > 0 \mapsto F = F(t)$ is a bijection whose inverse map $t = t(F)$ parameterizes the whole family of Riemann minimal examples, with F running from horizontal to vertical (with monotonically increasing slope function). When F tends to vertical, then it can be proved that $\mathcal{R}_{t(F)}$ converges to a vertical catenoid with waist circle of radius $\frac{1}{2\pi}$. When F tends to horizontal, then one can shrink $\mathcal{R}_{t(F)}$ so that F tends to $(4, 0, 0)$, and in that case the $\mathcal{R}_{t(F)}$ converge to the foliation of $\mathbb{R}^3$ by horizontal planes, outside of the two vertical lines $\{(0, \pm 1, x_3) \mid x_3 \in \mathbb{R}\}$, along which the shrunk surface $\mathcal{R}_{t(F)}$ with F very horizontal approximates two oppositely handed, highly sheeted, scaled-down vertical helicoids, see Figures 12 and 13.

With this basic family of examples in mind, we state the following result by Colding and Minicozzi. We refer the reader to the paper [**147**] by Meeks and Tinaglia for theorems that describe the limiting object of a locally simply connected sequence of H_n-surfaces of fixed finite genus in $\mathbb{R}^3$ with boundaries diverging to infinity; in particular, the reader might compare the statement of the next theorem with the similar statement of Theorem 1.4 in [**147**].

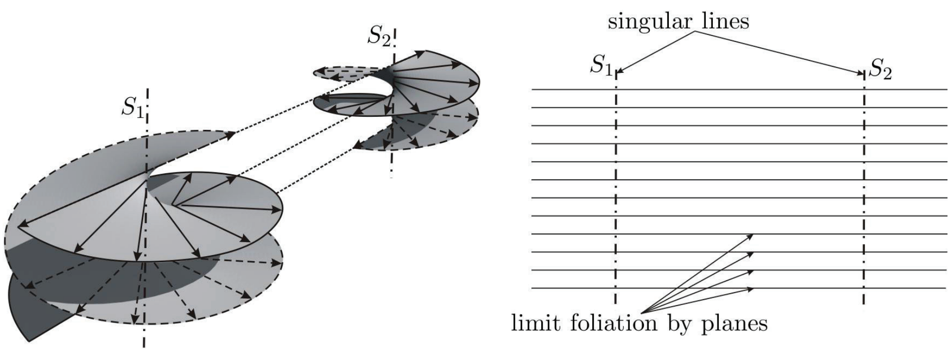

FIGURE 13. Left: Two oppositely handed double spiral staircases. Right: The limit foliation by parallel planes and the singular set of convergence $S_1 \cup S_2$.

THEOREM 7.7 (Limit Lamination Theorem for 0-surfaces of Finite Genus [**39**]). *Let $\Sigma_n \subset \mathbb{B}(R_n)$ be a locally simply connected sequence of 0-surfaces of finite genus g, with $\partial\Sigma_n \subset \partial\mathbb{B}(R_n)$, $R_n \to \infty$, such that $\Sigma_n \cap \mathbb{B}(2)$ contains a component which is not a disk for any n. If $\sup |A_{\Sigma_n \cap \mathbb{B}(1)}| \to \infty$ as $n \to \infty$, then there exists a subsequence of the Σ_n (denoted in the same way) and two vertical lines S_1, S_2, such that after a rotation in $\mathbb{R}^3$, then following properties hold.*

(1) Away from $S_1 \cup S_2$, each Σ_n consists of exactly two multi-valued graphs spiraling together.
(2) For each $\alpha \in (0,1)$, the surfaces $\Sigma_n - (S_1 \cup S_2)$ converge in the C^α-topology to the foliation $\mathcal{F}$ of $\mathbb{R}^3$ by horizontal planes.
(3) Along S_1 and S_2 the norm of the second fundamental form of the Σ_n blows up as $n \to \infty$.
(4) The pair of multi-valued graphs appearing in item 1 inside Σ_n for n large, form double spiral staircases with opposite handedness at S_1 and S_2. Thus, circling only S_1 or only S_2 results in going either up or down, while a path circling both S_1 and S_2 closes up, see Figure 13.

Theorem 7.7 gives rise to a second example of a limiting parking garage structure on $\mathbb{R}^3$ (we obtained the first example in Theorem 6.1 above), now with two columns which are $(+,-)$-handed[3], just like in the case of the Riemann minimal examples $\mathcal{R}_t$ discussed before the statement of Theorem 7.7. We refer the reader to Section 8.1 for more details about parking garage structures on $\mathbb{R}^3$, and to Theorem 9.5 for a generalization to the case where there is no bound on the genus of the surfaces Σ_n.

[3]Here, $+$, (resp. $-$) means that the corresponding forming helicoid or multi-valued graph is right-handed.

8. Properness results for 0-surfaces

In previous sections we have seen that H-laminations constitute a key tool in the understanding of the global behavior of H-surfaces in $\mathbb{R}^3$. In this section, we will present some results about the existence and structure of 0-laminations, which have deep consequences in various aspects of 0-surface theory, including the general 3-manifold setting. For example, we will give a natural condition under which the closure of a complete 0-surface in a Riemannian 3-manifold has the structure of a 0-lamination. We will deduce from this analysis, among other things, that certain complete 0-surfaces are proper, a result which is used to prove the uniqueness of the helicoid in the complete setting (Theorem 1.3) by reducing it to the corresponding characterization assuming properness (Theorem 4.6).

In the Introduction we mentioned the result, proved in [**138**] by Meeks and Rosenberg, that the closure of a complete 0-surface M with locally bounded second fundamental form in a Riemannian 3-manifold N, has the structure of a 0-lamination (they stated this result in [**138**] in the particular case $N = \mathbb{R}^3$, but their proof extends to the general case for N as mentioned in [**139**]). The same authors have demonstrated that this result still holds true if we substitute the locally bounded curvature assumption by a weaker hypothesis, namely that for every $p \in N$, there exists a neighborhood D_p of p in N where the injectivity radius function I_M of M restricted to $M \cap D_p$ is bounded away from zero.

Definition 8.1. Let Σ be a complete Riemannian manifold. The *injectivity radius function* $I_\Sigma \colon \Sigma \to (0, \infty]$ is defined at a point $p \in \Sigma$, to be the supremum of the radii $r > 0$ of disks $D(\vec{0}, r) \subset T_p\Sigma$, such that the exponential map $\exp_p \colon T_p\Sigma \to \Sigma$, restricts to $D(\vec{0}, r)$ as a diffeomorphism onto its image. The *injectivity radius of* Σ is the infimum of I_Σ.

The next theorem was proved by Meeks and Rosenberg [**139**].

Theorem 8.2 (Minimal Lamination Closure Theorem). *Let M be a complete 0-surface of positive injectivity radius in a Riemannian 3-manifold N (not necessarily complete). Then, the closure $\overline{M}$ of M in N has the structure of a 0-lamination $\mathcal{L}$, some of whose leaves are the connected components of M. Furthermore:*

1. *If N is homogeneously regular, then there exist $C, \varepsilon > 0$ depending on N and on the injectivity radius of M, such that the norm of the second fundamental form of M in the ε-neighborhood of any limit leaf of $\overline{M}$ is less than C (recall that limit leaves were introduced in Definition 7.4).*
2. *If M is connected, then exactly one of the following three statements holds for the set* $\lim(M) \subset \mathcal{L}$ *of limit points of M:*
   - *2.a. M is properly embedded in N, and* $\lim(M) = \emptyset$.
   - *2.b.* $\lim(M)$ *is non-empty and disjoint from M, and M is properly embedded in the open set* $N - \lim(M)$.
   - *2.c.* $\lim(M) = \mathcal{L}$ *and $\mathcal{L}$ contains an uncountable number of leaves.*

In the particular case $N = \mathbb{R}^3$, more can be said. Suppose $M \subset \mathbb{R}^3$ is a connected, complete 0-surface with positive injectivity radius. By Theorem 8.2, the closure of M has the structure of a 0-lamination of $\mathbb{R}^3$. If item 2.a in Theorem 8.2 does not hold for M, then the sublamination of limit points $\lim(M) \subset \overline{M}$ contains some leaf L. By Theorem 7.5 L is stable, hence L is a plane by Theorem 2.24. Now Theorem 8.2 insures that M has bounded curvature in some ε-neighborhood of the plane L, which contradicts Lemma 1.3 in [**138**]. This contradiction proves the following result.

Corollary 8.3 (Meeks, Rosenberg [**139**]). *Every connected, complete 0-surface in $\mathbb{R}^3$ with positive injectivity radius is properly embedded.*

Suppose M is a complete 0-surface of finite topology in $\mathbb{R}^3$. If the injectivity radius of M is zero, then there exists a divergent sequence of embedded geodesic loops $\gamma_n \subset M$ (i.e., closed geodesics with at most one corner) with lengths going to zero. Since M has finite topology, we may assume the γ_n are all contained in a fixed annular end E of M. By the Gauss-Bonnet formula, each γ_n is homotopically non-trivial, and so, the cycles $\gamma_n \cup \gamma_1$, $n \geq 2$, bound compact annular subdomains in E, whose union is a subend of E. However, the Gauss-Bonnet formula implies that the total Gaussian curvature of this union is finite (greater than -4π). Hence, E is asymptotic to an end of a plane or of a half-catenoid, which is absurd. This argument proves that the following result holds.

Corollary 8.4 (Colding-Minicozzi [**38**]). *A complete 0-surface of finite topology in $\mathbb{R}^3$ is properly embedded.*

8.1. Regularity of the singular sets of convergence to 0-laminations. An important technique which is used when dealing with a sequence of 0-surfaces M_n is to rescale each surface in the sequence to obtain a well-defined limit after rescaling, from where one deduces information about the original sequence. An important case in these rescaling processes is that of blowing up a sequence of H-surfaces *on the scale of curvature* (for details, see Theorem 1.1 in [**122**], and also see the proofs of Theorem 15 in [**115**] or of Corollary 2.2 in [**106**]). When the surfaces in the sequence are complete and embedded in $\mathbb{R}^3$, this blowing-up process produces a limit which is a proper non-flat 0-surface with bounded second fundamental form, whose genus and rank of homology groups are bounded above by the ones of the M_n. For example, if each M_n is a planar domain, then the same property holds for the limit surface.

Recall that we defined in Section 6.1 the concept of a locally simply connected sequence of proper 0-surfaces in $\mathbb{R}^3$. This concept can be easily generalized to a sequence of proper 0-surfaces in a Riemannian 3-manifold[4] N. For useful applications of the notion of locally simply connected sequence, it is essential to consider sequences of proper 0-surfaces which *a priori* may

[4]To do this, simply exchange the Euclidean balls $\mathbb{B}(p, r(p))$ in Definition 7.6 by extrinsic balls $B_N(p, r(p))$ relative to the Riemannian distance function on N.

not satisfy the locally simply connected condition, and then modify them to produce a new sequence which satisfies that condition. We accomplish this by considering a blow-up argument on a scale which, in general, is different from blowing up on the scale of curvature. We call this procedure *blowing up on the scale of topology*. This scale was defined and used in [**127, 128**] to prove that any proper 0-surface in $\mathbb{R}^3$ of finite genus and infinitely many ends has bounded curvature and is recurrent. We now explain the elements of this new blow-up procedure, which is also the basis for the proof of Theorem 9.5 below in the general 3-manifold setting.

Suppose $\{M_n\}_n$ is a sequence of non-simply connected, proper 0-surfaces in $\mathbb{R}^3$ which is not uniformly locally simply connected. Note that the Gaussian curvature of the collection M_n is not uniformly bounded, and so, one could blow up these surfaces on the scale of curvature to obtain a proper, non-flat 0-surface which may or may not be simply connected. Also note that, after choosing a subsequence, there exist points $p_n \in \mathbb{R}^3$ such that by defining

$$r_n(p_n) = \sup\{r > 0 \mid \mathbb{B}(p_n, r) \text{ intersects } M_n \text{ in disks}\},$$

then $r_n(p_n) \to 0$ as $n \to \infty$. Let $\widetilde{p}_n$ be a point in $\mathbb{B}(p_n, 1) = \{x \in \mathbb{R}^3 \mid \|x - p_n\| < 1\}$ where the function $x \mapsto d(x, \partial\mathbb{B}(p_n, 1))/r_n(x)$ attains its maximum (here d denotes extrinsic distance). Then, the translated and rescaled surfaces $\widetilde{M}_n = \frac{1}{r_n(\widetilde{p}_n)}(M_n - \widetilde{p}_n)$ intersect for all n the closed unit ball $\overline{\mathbb{B}}(\vec{0}, 1)$ in at least one component which is not simply connected, and for n large they intersect any ball of radius less than $1/2$ in simply connected components, in particular the sequence $\{\widetilde{M}_n\}_n$ is uniformly locally simply connected (see the proof of Lemma 8 in [**127**] for details).

For the sake of clarity, we now illustrate this blow-up procedure on certain sequences of Riemann minimal examples, defined in Section 2.3. Each of these surfaces is foliated by circles and straight lines in horizontal planes, with the (x_1, x_3)-plane as a plane of reflective symmetry. After a translation and a homothety, we can also assume that these surfaces are normalized so that any ball of radius less than 1 intersects these surfaces in compact disks, and the closed unit ball $\overline{\mathbb{B}}(\vec{0}, 1)$ intersects every Riemann example in at least one component which is not a disk. Under this normalization, any sequence of Riemann minimal examples is uniformly locally simply connected. The flux of each Riemann minimal example along a compact horizontal section has horizontal and vertical components which are not zero; hence it makes sense to consider the ratio V of the norm of its horizontal component over the vertical one.

As explained before Theorem 7.7, V parameterizes the family $\{\mathcal{R}(V)\}_V$ of Riemann minimal examples, with $V \in (0, \infty)$. When $V \to 0$, the surfaces $\mathcal{R}(V)$ converge smoothly to the vertical catenoid centered at the origin with waist circle of radius 1. But for our current purposes, we are more interested in the limit object of $\mathcal{R}(V)$ as $V \to \infty$. In this case, the Riemann minimal examples $\mathcal{R}(V)$ converge smoothly to a foliation of $\mathbb{R}^3$ by horizontal planes away from the two vertical lines passing through $(0,-1,0),(0,1,0)$. Given a horizontal slab $S \subset \mathbb{R}^3$ of finite width, the description of $\mathcal{R}(V) \cap S$ for V large is as follows, see Figure 12.

(a) If C_1, C_2 are disjoint vertical cylinders in S with axes $S \cap [\{(0,-1)\} \times \mathbb{R}]$, $S \cap [\{(0,1)\} \times \mathbb{R}]$, then $\mathcal{R}(V) \cap C_i$ is arbitrarily close to the intersection of S with a highly sheeted vertical helicoid with axis the axis of C_i, $i = 1,2$. Furthermore, the fact that the (x_1,x_3)-plane is a plane of reflective symmetry of $\mathcal{R}(V)$ implies that these limit helicoids are oppositely handed.
(b) In $S - (C_1 \cup C_2)$, the surface $\mathcal{R}(V)$ consists of two almost flat, almost horizontal multi-valued graphs, with number of sheets increasing to ∞ as $V \to \infty$.
(c) If C is a vertical cylinder in S containing $C_1 \cup C_2$, then $\mathcal{R}(V)$ intersects $S - C$ in a finite number $n(V)$ of univalent graphs, each one representing a planar end of $\mathcal{R}(V)$. Furthermore, $n(V) \to \infty$ as $V \to \infty$.

This description shows an example of a particular case of what we call a *parking garage structure* on $\mathbb{R}^3$ for the limit of a sequence of 0-surfaces (we mentioned this notion before Definition 7.6). Roughly speaking, a parking garage surface with n columns is a smooth embedded surface in $\mathbb{R}^3$ (not necessarily minimal), which in any fixed finite width horizontal slab $S \subset \mathbb{R}^3$, can be decomposed into 2 disjoint, almost flat horizontal multi-valued graphs over the exterior of n disjoint disks $D_1, \dots, D_n$ in the (x_1,x_2)-plane, together n topological strips each one contained in one of the solid cylinders $S \cap (D_i \times \mathbb{R})$ (these are the *columns* of the parking garage structure), such that each strip lies in a small regular neighborhood of the intersection of a vertical helicoid H_i with $S \cap (D_i \times \mathbb{R})$. One can associate to each column a $+$ or $-$ sign, depending on the handedness of the corresponding helicoid H_i. Note that a vertical helicoid is the basic example of a parking garage surface with 1 column, and the Riemann minimal examples $\mathcal{R}(V)$ with $V \to \infty$ have the structure of a parking garage structure with two oppositely handed columns. Other limiting parking garage structures with varying numbers of columns (including the case where $n = \infty$) and associated signs can be found in Traizet and Weber [**203**] and in Meeks, Pérez and Ros [**124**].

There are interesting cases where the locally simply connected condition guarantees the convergence of a sequence of 0-surfaces in $\mathbb{R}^3$ to a limiting parking garage structure. Typically, one proves that the sequence converges (up to a subsequence and a rotation) to a foliation of $\mathbb{R}^3$ by horizontal planes, with singular set of convergence consisting of a locally finite set of Lipschitz curves parameterized by heights. In fact, these Lipschitz curves are vertical lines by Theorem 8.5 below, and locally around the lines the surfaces in the sequence can be arbitrarily approximated by highly sheeted vertical helicoids, as follows from the uniqueness of the helicoid (Theorem 4.6) after a blowing-up process on the scale of curvature. By work of Meeks and

Tinaglia [**145**], the next theorem can be generalized to a locally simply connected sequence of H_n-surfaces in a Riemannian 3-manifold.

THEOREM 8.5 ($C^{1,1}$-regularity of $S(\mathcal{L})$, Meeks [**105**]). *Suppose $\{M_n\}_n$ is a locally simply connected sequence of proper 0-surfaces in a Riemannian 3-manifold N, that converges C^α, $\alpha \in (0,1)$, to a 0-lamination $\mathcal{L}$ of N, outside a locally finite collection of Lipschitz curves $S(\mathcal{L}) \subset N$ transverse to $\mathcal{L}$, along which the Gaussian curvatures of the M_n blow up and the convergence fails to be C^α. Then, $\mathcal{L}$ is a 0-foliation in a neighborhood of $S(\mathcal{L})$, and $S(\mathcal{L})$ consists of $C^{1,1}$-curves orthogonal to the leaves of $\mathcal{L}$.*

Next we give an idea of the proof of Theorem 8.5. First note that the nature of this theorem is local, hence it suffices to consider the case of a sequence of proper 0-disks M_n in the unit ball $\mathbb{B}(1) = \mathbb{B}(\vec{0}, 1)$ of $\mathbb{R}^3$ (the general case follows from adapting the arguments to a Riemannian 3-manifold). After passing to a subsequence, one can also assume that the surfaces M_n converge to a $C^{0,1}$-minimal foliation[5] $\mathcal{L}$ of $\mathbb{B}(1)$ and the convergence is C^α, $\alpha \in (0,1)$, outside of a transverse Lipschitz curve $S(\mathcal{L})$ that passes through the origin. Since unit normal vector field $N_{\mathcal{L}}$ to $\mathcal{L}$ is Lipschitz (Solomon [**192**]), then the integral curves of $N_{\mathcal{L}}$ are of class $C^{1,1}$. Then, the proof consists of demonstrating that $S(\mathcal{L})$ is the integral curve of $N_{\mathcal{L}}$ passing through the origin. To do this, one first proves that $S(\mathcal{L})$ is of class C^1, hence it admits a continuous tangent field T, and then one shows that T is orthogonal to the leaves of $\mathcal{L}$. These properties rely on a local analysis of the singular set $S(\mathcal{L})$ as a limit of minimizing geodesics γ_n in M_n that join pairs of appropriately chosen *points of almost maximal curvature,* (in a sense similar to the points p_n in Theorem 9.4 below), together with the fact that the minimizing geodesics γ_n converge C^1 as $n \to \infty$ to the integral curve of $N_{\mathcal{L}}$ passing through the origin. Crucial in this proof is the uniqueness of the helicoid (Theorem 4.6), since it gives the local picture of the 0-disks M_n near the origin as being closely approximated by portions of a highly sheeted helicoid near its axis.

REMARK 8.6. The local structure of the surfaces M_n for n large near a point in $S(\mathcal{L})$ as an approximation by a highly sheeted helicoid is the reason for the parenthetical comment of the columns being helicoidal for the surfaces limiting to a parking garage structure, see the paragraph just after Theorem 7.5.

Meeks and Weber [**152**] have shown that the $C^{1,1}$-regularity of $S(\mathcal{L})$ given by Theorem 8.5 is the best possible result. They do this by proving the following theorem, which is based on the *bent helicoids* described in Section 2.3, also see Figure 4 Right.

[5] Any codimension-one minimal foliation is of class $C^{0,1}$ and its unit normal vector field is $C^{0,1}$ as well, see Solomon [**192**].

THEOREM 8.7 (Meeks, Weber [**152**]). *Let Γ be a properly embedded $C^{1,1}$-curve in an open set U of $\mathbb{R}^3$. Then, Γ is the singular set of convergence for some Colding-Minicozzi type limit foliation of some neighborhood of Γ in U.*

9. Local pictures, local removable singularities and dynamics

An important application of the Local Removable Singularity Theorem 9.1 below is a characterization of all complete 0-surfaces in $\mathbb{R}^3$ of quadratic decay of Gaussian curvature (Theorem 9.6 below).

Given a 3-manifold N and a point $p \in N$, we will denote by d_N the distance function in N to p and $B_N(p, r)$ the metric ball of center p and radius $r > 0$. For a lamination $\mathcal{L}$ of N, we will denote by $|A_{\mathcal{L}}|$ the norm of the second fundamental form function on the leaves of $\mathcal{L}$. Meeks, Pérez and Ros [**121**] obtained the following remarkable local removable singularity result in any Riemannian 3-manifold N for certain possibly singular weak H-laminations.

THEOREM 9.1 (Local Removable Singularity Theorem). *A weak H-lamination $\mathcal{L}$ of a punctured ball $B_N(p, r)-\{p\}$ in a Riemannian 3-manifold N extends to a weak H-lamination of $B_N(p, r)$ if and only if there exists a positive constant c such that $|A_{\mathcal{L}}|d_N < c$ in some subball.*

REMARK 9.2. There is a natural setting where the curvature estimate $|A_{\mathcal{L}}|d_N < c$ described in the above theorem holds; namely, if the weak H-lamination $\mathcal{L}$ is a sublamination of a CMC foliation of a punctured ball $B_N(p, r) - \{p\}$. This will be discussed further in Section 14.

Since stable immersed H-surfaces have local curvature estimates which satisfy the hypothesis of Theorem 9.1 and every limit leaf of a H-lamination is stable (Theorem 7.5), we obtain the next extension result for the sublamination of limit leaves of any H-lamination in a punctured Riemannian 3-manifold.

COROLLARY 9.3. *Suppose that N is a Riemannian 3-manifold, which is not necessarily complete. If $W \subset N$ is a closed countable subset*[6] *and $\mathcal{L}$ is a weak H-lamination of $N - W$, then:*

1. *The sublamination of $\mathcal{L}$ consisting of the closure of any collection of its stable leaves extends across W to a weak H-lamination $\mathcal{L}_1$ of N. Furthermore, each leaf of $\mathcal{L}_1$ is stable.*
2. *The sublamination of $\mathcal{L}$ consisting of its limit leaves extends across W to a weak H-lamination of N.*
3. *If $\mathcal{L}$ is a minimal foliation of $N - W$, then $\mathcal{L}$ extends across W to a minimal foliation of N (this result will be generalized in Section 14 to the case of a CMC foliation of $N - W$, provided that there is a bound on the mean curvature of the leaves of the foliation).*

[6]An argument based on the classical Baire's theorem allows us to pass from the isolated singularity setting of Theorem 9.1 to a closed countable set of singularities, see [**133**].

Recall that Corollary 2.16 ensured that every complete H-surface in $\mathbb{R}^3$ with bounded Gaussian curvature is properly embedded. The next theorem by Meeks, Pérez and Ros [**122**] shows that any complete H-surface in $\mathbb{R}^3$ which is not properly embedded, has natural limits under dilations, which are properly embedded 0-surfaces. By *dilation,* we mean the composition of a homothety and a translation.

Theorem 9.4 (Local Picture on the Scale of Curvature). *Suppose M is a complete H-surface with unbounded second fundamental form in a homogeneously regular 3-manifold N. Then, there exists a sequence of points $p_n \in M$ and positive numbers $\varepsilon_n \to 0$, such that the following statements hold.*

1. *For all n, the component M_n of $B_N(p_n, \varepsilon_n) \cap M$ that contains p_n is compact, with boundary $\partial M_n \subset \partial B_N(p_n, \varepsilon_n)$.*
2. *Let $\lambda_n = \sqrt{|A_{M_n}|(p_n)}$ (where as usual, $|A_{\widehat{M}}|$ stands for the norm of the second fundamental form of a surface $\widehat{M}$). Then, $\frac{|A_{M_n}|}{\lambda_n} \leq 1 + \frac{1}{n}$ on M_n, with $\lim_{n\to\infty} \varepsilon_n \lambda_n = \infty$.*
3. *The rescaling of the metric balls $B_N(p_n, \varepsilon_n)$ by factor λ_n converge uniformly to $\mathbb{R}^3$ with its usual metric (so that we identify p_n with $\vec{0}$ for all n), and there exists a properly embedded 0-surface M_∞ in $\mathbb{R}^3$ with $\vec{0} \in M_\infty$, $|A_{M_\infty}| \leq 1$ on M_∞ and $|A_{M_\infty}|(\vec{0}) = 1$, such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.*

The above theorem gives a *local picture* or description of the local geometry of an H-surface M in an extrinsic neighborhood of a point $p_n \in M$ of concentrated curvature. The points $p_n \in M$ appearing in Theorem 9.4 are called *blow-up points on the scale of curvature.*

Now assume that for any positive ε, the intrinsic ε-balls of such an H-surface M are not always disks. Then, the curvature of M certainly blows up at some points in these non-simply connected intrinsic ε-balls as $\varepsilon \to 0$. Thus, one could blow up M on the scale of curvature as in Theorem 9.4 but this process would create a simply connected non-flat limit, hence a helicoid. Now imagine that we want to avoid this helicoidal blow-up limit, and note that the injectivity radius of M is zero, i.e., there exists an intrinsically divergent sequence of points $p_n \in M$ where the injectivity radius function of M tends to zero; If we choose these points p_n carefully and blow up M around the p_n in a similar way as we did in the above theorem, but exchanging the former ratio of rescaling (which was the norm of the second fundamental form at p_n) by the inverse of the injectivity radius at these points, then we will obtain a sequence of H_n-surfaces, $H_n \to 0$, which are non-simply connected in a fixed ball of space (after identifying again p_n with $\vec{0} \in \mathbb{R}^n$), and it is natural to ask about possible limits of such a blow-up sequence. This is the purpose of the next result.

For a complete Riemannian manifold M, we will let $I_M\colon M \to (0,\infty]$ be the injectivity radius function of M, and given a subdomain $\Omega \subset M$, $I_\Omega = (I_M)|_\Omega$ will stand for the restriction of I_M to Ω. Recall that the infimum of I_M is called the *injectivity radius* of M.

The next theorem by Meeks, Pérez and Ros appears in [**124**].

THEOREM 9.5 (Local Picture on the Scale of Topology). *There exists a smooth decreasing function $\delta\colon (0,\infty) \to (0,\infty)$ with $\lim_{r\to\infty} r\delta(r) = \infty$ such that the following statements hold. Suppose M is a complete 0-surface with injectivity radius zero in a homogeneously regular 3-manifold N. Then, there exists a sequence of points $p_n \in M$ (called "points of almost-minimal injectivity radius") and positive numbers $\varepsilon_n = n\, I_M(p_n) \to 0$ such that:*

1: For all n, the closure M_n of the component of $M \cap B_N(p_n, \varepsilon_n)$ that contains p_n is a compact surface with boundary in $\partial B_N(p_n, \varepsilon_n)$. Furthermore, M_n is contained in the intrinsic open ball $B_M(p_n, \frac{r_n}{2} I_M(p_n))$, where $r_n > 0$ satisfies $r_n\delta(r_n) = n$.

2: Let $\lambda_n = 1/I_M(p_n)$. Then, $\lambda_n I_{M_n} \geq 1 - \frac{1}{n}$ on M_n.

3: The rescaling of the metric balls $B_N(p_n, \varepsilon_n)$ by factor λ_n converge uniformly as $n \to \infty$ to $\mathbb{R}^3$ with its usual metric (as in Theorem 9.4, we identify p_n with $\vec{0}$ for all n).

Furthermore, exactly one of the following three possibilities occurs.

4: The surfaces $\lambda_n M_n$ have uniformly bounded Gaussian curvature on compact subsets[7] *of $\mathbb{R}^3$ and there exists a connected, properly embedded 0-surface $M_\infty \subset \mathbb{R}^3$ with $\vec{0} \in M_\infty$, $I_{M_\infty} \geq 1$ and $I_{M_\infty}(\vec{0}) = 1$, such that for any $k \in \mathbb{N}$, the surfaces $\lambda_n M_n$ converge C^k on compact subsets of $\mathbb{R}^3$ to M_∞ with multiplicity one as $n \to \infty$.*

5: After a rotation in $\mathbb{R}^3$, the surfaces $\lambda_n M_n$ converge to a minimal parking garage structure on $\mathbb{R}^3$, consisting of a foliation $\mathcal{L}$ of $\mathbb{R}^3$ by horizontal planes, with columns forming a locally finite set $S(\mathcal{L})$ of vertical straight lines (the set $S(\mathcal{L})$ is the singular set of convergence of $\lambda_n M_n$ to $\mathcal{L}$), and:

(5.1) $S(\mathcal{L})$ contains a line l_1 which passes through the closed ball of radius 1 centered at the origin, and another line l_2 at distance one from l_1, and all of the lines in $S(\mathcal{L})$ have distance at least one from each other.

(5.2) There exist oriented, homotopically non-trivial simple closed curves $\gamma_n \subset \lambda_n M_n$ with lengths converging to 2, which converge to the line segment γ that joins $(l_1 \cup l_2) \cap \{x_3 = 0\}$ and such that the integrals of the unit conormal vector of $\lambda_n M_n$ along

[7] As $M_n \subset B_N(p_n, \varepsilon_n)$, the convergence $\{\lambda_n B_N(p_n, \varepsilon_n)\}_n \to \mathbb{R}^3$ explained in item *3* allows us to view the rescaled surface $\lambda_n M_n$ as a subset of $\mathbb{R}^3$. The uniformly bounded property for the Gaussian curvature of the induced metric on $M_n \subset N$ rescaled by λ_n on compact subsets of $\mathbb{R}^3$ now makes sense.

γ_n in the induced exponential $\mathbb{R}^3$-coordinates of $\lambda_n B_N(p_n, \varepsilon_n)$ converge to a horizontal vector of length 2 orthogonal to γ.

(5.3) If there exists a bound on the genus of the surfaces $\lambda_n M_n$, then:

(a) $S(\mathcal{L})$ consists of just two lines l_1, l_2 and the associated limiting double multigraphs in $\lambda_n M_n$ are oppositely handed.

(b) Given $R > 0$, for $n \in \mathbb{N}$ sufficiently large depending on R, the surface $(\lambda_n M_n) \cap B_{\lambda_n N}(p, R/\lambda_n)$ has genus zero.

6: There exists a non-empty, closed set $\mathcal{S} \subset \mathbb{R}^3$ and a 0-lamination $\mathcal{L}$ of $\mathbb{R}^3 - \mathcal{S}$ such that the surfaces $(\lambda_n M_n) - \mathcal{S}$ converge to $\mathcal{L}$ outside of some singular set of convergence $S(\mathcal{L}) \subset \mathcal{L}$, and $\mathcal{L}$ has at least one non-flat leaf. Furthermore, if we let $\Delta(\mathcal{L}) = \mathcal{S} \cup S(\mathcal{L})$, then, after a rotation of $\mathbb{R}^3$:

(6.1) Let $\mathcal{P}$ be the sublamination of flat leaves in $\mathcal{L}$. Then, $\mathcal{P} \neq \emptyset$ and the closure of every such flat leaf is a horizontal plane. Furthermore, if $L \in \mathcal{P}$, then the plane $\overline{L}$ intersects $\Delta(\mathcal{L})$ in a set containing at least two points, each of which are at least distance 1 from each other in $\overline{L}$, and either $\overline{L} \cap \Delta(\mathcal{L}) \subset \mathcal{S}$ or $\overline{L} \cap \Delta(\mathcal{L}) \subset S(\mathcal{L})$.

(6.2) $\Delta(\mathcal{L})$ is a closed set of $\mathbb{R}^3$ which is contained in the union of planes $\bigcup_{L\in\mathcal{P}} \overline{L}$. Furthermore, every plane in $\mathbb{R}^3$ intersects $\mathcal{L}$.

(6.3) There exists $R_0 > 0$ such that the sequence of surfaces $\left\{M_n \cap B_M(p_n, \frac{R_0}{\lambda_n})\right\}_n$ does not have bounded genus.

(6.4) There exist oriented closed geodesics $\gamma_n \subset \lambda_n M_n$ with uniformly bounded lengths which converge to a line segment γ in the closure of some flat leaf in $\mathcal{P}$, which joins two points of $\Delta(\mathcal{L})$, and such that the integrals of $\lambda_n M_n$ along γ_n in the induced exponential $\mathbb{R}^3$-coordinates of $\lambda_n B_N(p_n, \varepsilon_n)$ converge to a horizontal vector orthogonal to γ with length $2\,\mathrm{Length}(\gamma)$.

The results in the series of papers [**37, 38, 39**] by Colding and Minicozzi and the minimal lamination closure theorem by Meeks and Rosenberg [**139**] play important roles in deriving the above compactness result. The first two authors conjecture that item 6 in Theorem 9.5 does not actually occur.

The local picture theorems on the scales of curvature and topology deal with limits of a sequence of blow-up rescalings for a complete 0-surface. Next we will study an interesting function which is invariant by rescalings, namely the Gaussian curvature of a surface in $\mathbb{R}^3$ times the squared distance to a given point. A complete Riemannian surface M is said to have *intrinsic quadratic curvature decay constant $C > 0$ with respect to a point $p \in M$*, if the absolute curvature function $|K_M|$ of M satisfies

$$|K_M(q)| \leq \frac{C}{d_M(p,q)^2} \qquad \text{for all } q \in M - \{p\},$$

where d_M denotes the Riemannian distance function. Note that if such a Riemannian surface M is a complete surface in $\mathbb{R}^3$ with $p = \vec{0} \in M$, then it also has extrinsic quadratic decay constant C with respect to the radial distance R to $\vec{0}$, i.e., $|K_M|R^2 \leq C$ on M. For this reason, when we say that a 0-surface in $\mathbb{R}^3$ has *quadratic decay of curvature*, we will always refer to curvature decay with respect to the extrinsic distance R to $\vec{0}$, independently of whether or not M passes through or limits to $\vec{0}$.

THEOREM 9.6 (Quadratic Curvature Decay Theorem, Meeks, Pérez, Ros [**133**]). *Let $M \subset \mathbb{R}^3 - \{\vec{0}\}$ be a 0-surface with compact boundary (possibly empty), which is complete outside the origin $\vec{0}$, i.e., all divergent paths of finite length on M limit to $\vec{0}$. Then, M has quadratic decay of curvature if and only if its closure in $\mathbb{R}^3$ has finite total curvature. In particular, every complete 0-surface $M \subset \mathbb{R}^3$ with compact boundary and quadratic decay of curvature is properly embedded in $\mathbb{R}^3$. Furthermore, if C is the maximum of the logarithmic growths of the ends of M, then*

$$\lim_{R\to\infty} \sup_{M-\mathbb{B}(R)} |K_M|R^4 = C^2,$$

where $\mathbb{B}(R)$ denotes the extrinsic ball of radius R centered at $\vec{0}$.

REMARK 9.7. The above Quadratic Curvature Decay Theorem is a crucial tool in understanding the asymptotic behavior of all properly embedded minimal surfaces in $\mathbb{R}^3$ via rescaling arguments. This application is called the *Dynamics Theorem for Properly Embedded Minimal Surfaces* by Meeks, Pérez and Ros [**122**], which we will not explain in this article; instead, we will discuss in the next section a closely related dynamics type theorem for certain $(H > 0)$- surfaces in $\mathbb{R}^3$.

10. The Dynamics Theorem for H-surfaces in $\mathbb{R}^3$

We now apply some of the theoretical results in the previous sections to analyze some aspects of the asymptotic behavior of a given proper H-surface M in $\mathbb{R}^3$. To understand this asymptotic behavior, we consider two separate cases.

In the case that M has bounded second fundamental form, the answer to this problem consists of classifying the space $\mathbb{T}(M)$ of limits of sequences of the form $\{M - p_n\}_n$, where $p_n \in M$, $|p_n| \to \infty$ (equivalently $\{p_n\}_n$ is a divergent sequence in M, note that M is proper as follows from Corollary 2.16 and Theorem 2.18). Observe that $\{M - p_n\}_n$ has area estimates in balls of any fixed radius by Corollary 2.16 and Theorem 2.18; hence, every divergent sequence of translations of M has a subsequence that converges on compact subsets of $\mathbb{R}^3$ to a possibly immersed H-surface. In fact, the surfaces in $\mathbb{T}(M)$ are possibly disconnected, strongly Alexandrov embedded H-surfaces.

In the case that M is proper but does not have bounded second fundamental form, a more natural way to understand its asymptotic behavior is to consider all proper non-flat surfaces in $\mathbb{R}^3$ that can obtained as a limit of a sequence of the form $\{\lambda_n(M-p_n)\}_n$, where $p_n \in M$ is a sequence of points that diverges in $\mathbb{R}^3$ and $\lambda_n = |A_M|(p_n)$ is unbounded (as usual, $|A_M|$ is the norm of the second fundamental form of M). This set of limits by divergent dilations of M was studied by Meeks, Pérez and Ros assuming that M is a 0-surface (this is their *Dynamics Theorem* [**122**], which is an application of their Quadratic Curvature Decay Theorem 9.6); also see Chapter 11 in [**118**] for further discussion in this minimal case when $|A_M|$ is not bounded.

The material covered here is based on [**149**] by Meeks and Tinaglia, which was motivated by the earlier work in [**122**] and we refer the reader to [**149**] for further details. We will focus our attention here on some of the less technical results in [**149**] and the basic techniques developed there.

DEFINITION 10.1. Suppose that $M \subset \mathbb{R}^3$ is a complete, non-compact, connected H-surface with compact boundary (possibly empty) and with bounded second fundamental form.

1. For any divergent sequence of points $p_n \in M$, a subsequence of the translated surfaces $M - p_n$ converges to a properly immersed H-surface which bounds a smooth open subdomain on its mean convex side. Let $\mathbb{T}(M)$ denote the collection of all such limit surfaces.
2. We say that M is *chord-arc* if there exists a constant $C > 0$ such that for all $p, q \in M$ with $\|p - q\| \geq 1$, we have $d_M(p, q) \leq C\|p - q\|$. Note that if M is chord-arc with constant C and $p, q \in M$ with $\|p - q\| < 1$, then $d_M(p, q) \leq 5C$ by the following argument. Let $z \in M \cap \partial B(p, 2)$; applying the chord-arc property of M to p, z and to z, q we obtain $d_M(p, z) \leq 2C$ and $d_M(z, q) \leq C|q - z| \leq 3C$. Hence, the triangle inequality gives $d_M(p, q) \leq 5C$.

In order to study $\mathbb{T}(M)$ it is convenient for technical reasons to study a closely related space $\mathcal{T}(M)$ that can be thought of a subset of $\mathbb{T}(M)$.

DEFINITION 10.2. Suppose $M \subset \mathbb{R}^3$ is a non-compact, strongly Alexandrov embedded H-surface with bounded second fundamental form.

1. We define the set $\mathcal{T}(M)$ of all connected, strongly Alexandrov embedded H-surfaces $\Sigma \subset \mathbb{R}^3$, which are obtained in the following way.

   There exists a divergent sequence of points $p_n \in M$ such that the translated surfaces $M - p_n$ converge C^2 on compact sets of $\mathbb{R}^3$ to a strongly Alexandrov embedded H-surface Σ', and Σ is a connected component of Σ' passing through the origin. Actually we consider the immersed surfaces in $\mathcal{T}(M)$ to be *pointed* in the sense that if such a surface is not embedded at the origin, then we consider the surface to represent two different elements in $\mathcal{T}(M)$ depending on a choice of one of the two preimages of the origin.
2. $\Delta \subset \mathcal{T}(M)$ is called *$\mathcal{T}$-invariant*, if $\Sigma \in \Delta$ implies $\mathcal{T}(\Sigma) \subset \Delta$.
3. A non-empty subset $\Delta \subset \mathcal{T}(M)$ is called a *minimal* $\mathcal{T}$-invariant set, if it is $\mathcal{T}$-invariant and contains no smaller non-empty $\mathcal{T}$-invariant sets.

4. If $\Sigma \in \mathcal{T}(M)$ and Σ lies in a minimal $\mathcal{T}$-invariant set of $\mathcal{T}(M)$, then Σ is called a *minimal element* of $\mathcal{T}(M)$.

$\mathcal{T}(M)$ has a natural compact metric space topology, that we now describe. Suppose that $\Sigma \in \mathcal{T}(M)$ is embedded at the origin. In this case, there exists an $\varepsilon > 0$ depending only on the bound of the second fundamental form of M, so that there exists a disk $D(\Sigma) \subset \Sigma \cap \overline{\mathbb{B}}(\varepsilon)$ with $\partial D(\Sigma) \subset \partial\overline{\mathbb{B}}(\varepsilon)$, $\vec{0} \in D(\Sigma)$ and such that $D(\Sigma)$ is a graph with gradient at most 1 over its projection to the tangent plane $T_{\vec{0}}D(\Sigma) \subset \mathbb{R}^3$. Given another such $\Sigma' \in \mathcal{T}(M)$, define

$$d_{\mathcal{T}(M)}(\Sigma, \Sigma') = d_{\mathcal{H}}(D(\Sigma), D(\Sigma')), \tag{36}$$

where $d_{\mathcal{H}}$ is the Hausdorff distance. If $\vec{0}$ is not a point where Σ is embedded, then since we consider Σ to represent one of two different pointed surfaces in $\mathcal{T}(M)$, we choose $D(\Sigma)$ to be the disk in $\Sigma \cap \mathbb{B}(\varepsilon)$ corresponding to the chosen base point. With this modification, the above metric is well-defined on $\mathcal{T}(M)$. Using the curvature and local area estimates of elements in $\mathcal{T}(M)$, it is straightforward to prove that $\mathcal{T}(M)$ is sequentially compact and therefore it has a compact metric space structure.

Given a surface $\Sigma \in \mathcal{T}(M)$, it can be shown that $\mathcal{T}(\Sigma)$ is a subset of $\mathcal{T}(M)$. In particular, we can consider $\mathcal{T}$ to represent a function:

$$\mathcal{T} \colon \mathcal{T}(M) \to \mathcal{P}(\mathcal{T}(M)),$$

where $\mathcal{P}(\mathcal{T}(M))$ denotes the power set of $\mathcal{T}(M)$. Using the natural compact metric space structure on $\mathcal{T}(M)$, we can obtain classical dynamics type results on $\mathcal{T}(M)$ with respect to the mapping $\mathcal{T}$. These dynamics results include the existence of non-empty minimal $\mathcal{T}$-invariant sets in $\mathcal{T}(M)$.

THEOREM 10.3 (CMC Dynamics Theorem). *Let $M \subset \mathbb{R}^3$ be a connected, non-compact, strongly Alexandrov embedded $(H > 0)$-surface with bounded second fundamental form. Then the following statements hold:*

1. *$\mathcal{T}(M)$ is non-empty and $\mathcal{T}$-invariant.*
2. *$\mathcal{T}(M)$ has a natural compact topological space structure given by the metric $d_{\mathcal{T}(M)}$ defined in (36).*
3. *Every non-empty $\mathcal{T}$-invariant set of $\mathcal{T}(M)$ contains a non-empty minimal $\mathcal{T}$-invariant set. In particular, since $\mathcal{T}(M)$ is itself a non-empty $\mathcal{T}$-invariant set, then $\mathcal{T}(M)$ contains non-empty minimal invariant elements.*

DEFINITION 10.4. For any point p in a surface $M \subset \mathbb{R}^3$, we denote by $M(p, R)$ the closure of the connected component of $M \cap \mathbb{B}(p, R)$ which contains p.

If M is a surface satisfying the hypotheses of Theorem 10.3 and M is not embedded at p having two immersed components $M(p, R)$, $M'(p, R)$ that correspond to two pointed immersions, then in what follows we will consider both of these components separately.

As an application of the above Dynamics Theorem, we have the following result on the geometry of minimal elements of $\mathcal{T}(M)$.

THEOREM 10.5 (Minimal Element Theorem). *Let $M \subset \mathbb{R}^3$ be a complete, non-compact, $(H > 0)$-surface with possibly empty compact boundary and bounded second fundamental form. Then, the following statements hold.*

1. *If $\Sigma \in \mathcal{T}(M)$ is a minimal element, then either every surface in $\mathbb{T}(\Sigma)$ is the translation of a fixed Delaunay surface, or every surface in $\mathbb{T}(\Sigma)$ has one end. In particular, every surface in $\mathbb{T}(\Sigma)$ is connected and, after ignoring base points, $\mathcal{T}(\Sigma) = \mathbb{T}(\Sigma)$.*
2. *Minimal elements of $\mathcal{T}(M)$ are chord-arc, in the sense of Definition 10.1.*
3. *Suppose Σ is a minimal element of $\mathcal{T}(M)$. Then, the following statements are equivalent.*
   a. *Σ is a Delaunay surface.*
   b. *$\lim_{R\to\infty} \underline{A}(R) = 0$, where $\underline{A}(R) = \inf_{R_1\geq R}(\inf_{p\in\Sigma}(\mathrm{Area}[\Sigma(p,R_1)] \cdot R_1^{-2}))$.*
   c. *$\lim_{R\to\infty} \underline{G}(R) = 0$, where $\underline{G}(R) = \inf_{R_1\geq R}(\inf_{p\in\Sigma}(\mathrm{Genus}[\Sigma(p,R_1)] \cdot R_1^{-2}))$.*

11. The curvature and radius estimates of Meeks-Tinaglia

A longstanding problem in classical surface theory is to classify the complete, simply connected H-surfaces in $\mathbb{R}^3$. In the case the surface is simply connected and compact, this classification follows by work of either Hopf [**72**] in 1951 or of Alexandrov [**3**] in 1956, who gave different proofs that a round sphere is the only possibility. In [**143**], Meeks and Tinaglia have recently proved that a complete, simply connected $(H > 0)$-surface is compact.

THEOREM 11.1. *Complete simply connected $(H > 0)$-surfaces in $\mathbb{R}^3$ are compact, and thus are round spheres.*

The two main ingredients in the proof of Theorem 11.1 in [**143**] are the radius estimates Theorem 1.7 and the curvature estimates Theorem 1.6. For the reader's convenience, we restate them here.

THEOREM 11.2 (Radius Estimates, Meeks-Tinaglia [**143**]). *There exists an $\mathcal{R} \geq \pi$ such that any H-disk in $\mathbb{R}^3$ with $H > 0$ has radius less than $\mathcal{R}/H$.*

THEOREM 11.3 (Curvature Estimates, Meeks-Tinaglia [**143**]). *Given δ, $\mathcal{H} > 0$, there exists a $K(\delta,\mathcal{H}) \geq \sqrt{2}\mathcal{H}$ such that any H-disk M in $\mathbb{R}^3$ with $H \geq \mathcal{H}$ satisfies*

$$\sup_{\{p\in M \,|\, d_M(p,\partial M)\geq\delta\}} |A_M| \leq K(\delta,\mathcal{H}),$$

where d_M is the intrinsic distance function of M.

Since every point on an H-surface M of positive injectivity radius r_0 is the center of a geodesic H-disk in M of radius r_0, the curvature estimate in Theorem 11.3 has the following immediate consequence.

COROLLARY 11.4. *If M is a complete $(H > 0)$-surface with positive injectivity radius r_0, then*

$$\sup_M |A_M| \leq K(r_0, H).$$

As complete $(H > 0)$-surfaces of bounded curvature are properly embedded in $\mathbb{R}^3$ by Theorem 2.18, then Corollary 11.4 implies the next result.

COROLLARY 11.5. *A complete $(H > 0)$-surface with positive injectivity radius is properly embedded in $\mathbb{R}^3$.*

Since there exists an $\varepsilon > 0$ such that for any $C > 0$, every complete immersed surface Σ in $\mathbb{R}^3$ with $\sup_\Sigma |A_\Sigma| < C$ has injectivity radius greater than ε/C, then Corollary 11.4 also demonstrates that a necessary and sufficient condition for an $(H > 0)$-surface in $\mathbb{R}^3$ to have bounded curvature is that it has positive injectivity radius.

COROLLARY 11.6. *A complete $(H > 0)$-surface has positive injectivity radius if and only if it has bounded curvature.*

We now give an outline of Meeks and Tinaglia's approach to proving Theorems 11.2 and 11.3.

Step 1: *Prove analogous curvature estimates for $(H > 0)$-disks in terms of extrinsic rather than intrinsic distances of points to the boundaries of their disks.*

The proofs of this extrinsic version of Theorem 11.3 is by contradiction and relies on an accurate geometric description of a 1-disk near interior points where the norm of the second fundamental form becomes arbitrarily large. This geometric description is inspired by the pioneering work of Colding and Minicozzi in the minimal case [**34, 35, 37**].

The extrinsic curvature estimates just alluded to are the following ones.

LEMMA 11.7 (First Extrinsic Curvature Estimate). *Given $\delta > 0$ and $H \in (0, \frac{1}{2\delta})$, there exists $K_0(\delta, H) > 0$ such that for any H-disk $M \subset \mathbb{R}^3$,*

$$\sup_{\{p \in M \mid d_{\mathbb{R}^3}(p, \partial M) \geq \delta\}} |A_M| \leq K_0(\delta, H),$$

The arguments in the proof of the above lemma deal only with the component Δ of the intersection $\mathbb{B}(p, \delta) \cap M$ that contains p. Since the convex hull property does not hold for $(H > 0)$-disks, in principle the topology of the planar domain Δ might be arbitrarily complicated, in the sense that the number k of boundary components might not have a universal upper bound. However, this potential problem is solved by proving, for any $k \in \mathbb{N}$, an extrinsic curvature estimate $K_0(\delta, H, k)$ valid when Δ has at most k boundary components, and then by demonstrating the following result on the existence of an upper bound N_0 on the number of boundary components of Δ:

Proposition 11.8 (Proposition 3.1 in [**143**]). *There exists $N_0 \in \mathbb{N}$ such that for any $R \leq \frac{1}{2}$ and $H \in [0,1]$, if M is a compact H-disk with $\partial M \subset \mathbb{R}^3 - \mathbb{B}(R)$ and M is transverse to $\partial\mathbb{B}(R)$, then each component of $M \cap \mathbb{B}(R)$ has at most N_0 boundary components.*

Since Δ is a subset of a disk, then every 1-cycle on it has zero flux. Hence, Lemma 11.7 follows from Proposition 11.8 and the next extrinsic curvature estimate.

Lemma 11.9 (Second Extrinsic Curvature Estimate). *Given $\delta > 0$ and $H \in (0, \frac{1}{2\delta})$, there exists $K_0(\delta, H, k) > 0$ such that any H-planar domain Δ with zero flux and at most k boundary components satisfies:*

$$\sup_{\{p\in\Delta \,|\, d_{\mathbb{R}^3}(p,\partial\Delta)\geq\delta\}} |A_\Delta| \leq K_1(\delta, H, k),$$

For details on the following outline of the proof of Lemma 11.9, see [**143**]. Arguing by contradiction, assume that the lemma fails. One can easily reduce the proof of Lemma 11.9 to the following situation. There exists a sequence $\{\Delta(j)\}_j$ of 1-planar domains with zero flux satisfying the following properties for each $j \in \mathbb{N}$:

1. $\Delta(j) \subset \mathbb{B}(\delta)$ and $\partial\Delta(j) \subset \partial\mathbb{B}(\delta)$.
2. $\vec{0}$ is a point of almost-maximal curvature on $\Delta(j)$ with $|A_{\Delta(j)}|(\vec{0}) > j$, in the sense that there exists a sequence of positive numbers $\delta_j \to 0$ with $\delta_j \cdot |A_{\Delta(j)}|(\vec{0}) \to \infty$ and $\lim_{j\to\infty} \max\{|A_{\Delta(j)}|(q) \mid q \in \mathbb{B}(\delta_j)\} \cdot |A_{\Delta(j)}|(\vec{0}) = 1$.

The proof of the Local Picture Theorem on the Scale of Curvature (Theorem 9.4) implies, after replacing by a subsequence, that the homothetically scaled surfaces

$$\Sigma(j) = |A_{\Delta(j)}|(\vec{0}) \cdot \Delta(j)$$

converge to a non-flat, proper 0-planar domain $\Sigma_\infty \subset \mathbb{R}^3$ with zero flux. By the classification Theorem 1.4 of such 0-planar domains, Σ_∞ must be helicoid that we will assume has a vertical axis passing through the origin. What this means geometrically is that on the scale of curvature, vertical helicoids are forming around the point $\vec{0} \in \Delta(j)$ as $j \to \infty$.

Hence, after replacing by a subsequence, for any $n \in \mathbb{N}$, there exists an integer $J(n)$ such that for $j > J(n)$, the following statements hold. On the scale of curvature and near the origin, there exists in $\Delta(j)$ a pair $\widehat{G}_j^{up}, \widehat{G}_j^{down}$ of n-valued graphs which correspond to "large" n-valued graphs in the scaled almost-helicoids and that have "small" gradients for their n-valued graphing functions; here the superscripts "up" and "down" refer to the direction of their mean curvature vectors. The most difficult part in demonstrating Step 1 is to prove that, for n sufficiently large, the n-valued graphs $\widehat{G}_j^{up}, \widehat{G}_j^{down}$ contain 2-valued subgraphs that extend to 2-valued graphs G_j^{up}, G_j^{down} in $\Delta(j)$ on a scale proportional to δ. Furthermore, it is shown that the 2-valued graph G_j^{up} can be chosen to contain a sheet that lies between the two sheets of G_j^{down}. Crucial in the proof of this extension results is the work of Colding and Minicozzi [**34**] on the extension of multi-valued graphs inside of certain 0-disks in $\mathbb{R}^3$. In our situation, one applies their results to stable 0-disks $E(n)$ that are shown to exist in the complement of $\Delta(j)$ in the small ball $\mathbb{B}(\delta)$; see [**143**] for details. Finally one obtains a contradiction by showing that one can choose the multigraphs G_j^{up}, G_j^{down}, so that as $j \to \infty$, they collapse to a 1-graph over an annulus in the(x_1, x_2)-plane, which is impossible since G_j^{up}, G_j^{down} have oppositely signed mean curvatures.

Step 2: *Relate the existence of an extrinsic curvature estimate in Step 1 to the existence of extrinsic radius estimates for H-disks.*

Arguing again by contradiction, we may assume that $E(n)$ is a sequence of 1-disks with $\vec{0} \in E(n)$ and $\partial E(n) \subset \mathbb{R}^3 - \mathbb{B}(n+1)$. By Step 1, the 1-planar domains $E(n) \cap \mathbb{B}(n)$ have bounded norm of their second fundamental forms. By rather standard arguments like those used in the proof of the Dynamics Theorem 10.3, a subsequence of the $E(n) \cap \mathbb{B}(n)$ converges to a strongly Alexandrov embedded 1-surface M in $\mathbb{R}^3$ with zero flux. But item 3 in the Minimal Element Theorem 10.5 implies that there exists a Delaunay surface which is a limit of a sequence of translations of subdomains in M. This is contradiction, since a Delaunay surface has non-zero flux.

Step 3: *Prove the following one-sided curvature estimate for H-disks.*

Theorem 11.10 (One-sided curvature estimate for H-disks, Meeks, Tinaglia [**148**]). *There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $M \subset \mathbb{R}^3$ be an H-disk such that $M \cap \mathbb{B}(R) \cap \{x_3 = 0\} = \emptyset$ and $\partial M \cap \mathbb{B}(R) \cap \{x_3 > 0\} = \emptyset$. Then,*

$$\sup_{x \in M \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\}} |A_M|(x) \leq \frac{C}{R}. \tag{37}$$

In particular, if $M \cap \mathbb{B}(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $H \leq \frac{C}{R}$.

A key technical result that is needed in the proof of the above one-sided curvature estimate is the existence of the extrinsic curvature estimates in Step 1. Perhaps even more important in the proof of Theorem 11.10 are the extension results for multi-valued graphs in the surfaces $\Delta(j)$ described in the sketch of that proof of Step 1, and some rather technical results on 0-laminations of $\mathbb{R}^3$ with a finite number of singularities. Some of the tools used in the proof of Step 3 include the one-sided curvature estimates for 0-disks in Theorem 6.2, the Stable Limit Leaf Theorem 7.5, the Local Removable Singularity Theorem 9.1 and the Local Picture Theorem 9.5 on the Scale of Topology.

Step 4: *Relate the existence of an extrinsic curvature estimate in Step 1 to the existence of an intrinsic curvature estimate via the following weak chord-arc result for H-disks.*

Recall from Definition 10.4 that given a point p in a surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap \mathbb{B}(p, R)$ passing through p.

THEOREM 11.11 (Weak Chord Arc Estimate, Theorem 1.2 in [**142**]). *There exists a $\delta_1 \in (0, \frac{1}{2})$ such that the following holds. Given an H-disk in $\Sigma \subset \mathbb{R}^3$ and an intrinsic closed ball $\overline{B}_\Sigma(x, R)$ which is contained in $\Sigma - \partial\Sigma$, we have*

1. $\Sigma(x, \delta_1 R)$ is a disk with $\partial\Sigma(\vec{0}, \delta_1 R) \subset \partial\mathbb{B}(\delta_1 R)$.
2. $\Sigma(x, \delta_1 R) \subset B_\Sigma(x, \frac{R}{2})$.

Theorem 1.6 is a straightforward consequence of the Theorem 11.11. Once one has obtained the 1-sided curvature estimate in Step 3, the strategy of the proof of Theorem 11.11 is similar to the strategy of the proof of Proposition 1.1 in [**38**] by Colding and Minicozzi.

Step 5: *Relate the existence of an intrinsic curvature estimate in Step 4 to the existence of radius estimates for H-disks.* This final step of the proof of Theorem 11.3 is similar to that of Step 2, which completes our sketch of the proofs of Theorems 11.2 and 11.3.

In [**143**], Meeks and Tinaglia also obtain curvature estimates for $(H > 0)$-annuli. However, while these estimates are analogous to the curvature estimates in Theorem 11.3 for H-disks, they necessarily must depend on the length of the flux vector of the generator of the first homology group of the given annulus. An immediate consequence of these curvature estimates for H-annuli is the next Theorem 11.12 on the properness of complete H-surfaces of finite topology. The next theorem is what allows us to obtain the properness and curvature estimates for classical finite topology $(H > 0)$-surfaces described in Section 4.

THEOREM 11.12. *A complete H-surface with smooth compact boundary (possibly empty) and finite topology has bounded curvature and is properly embedded in $\mathbb{R}^3$.*

The theory developed in this manuscript also provides key tools for understanding the geometry of $(H > 0)$-disks in a Riemannian 3-manifold, especially in the case that the manifold is complete and locally homogeneous. These generalizations and applications of the work presented here will appear in [**145**], and we mention two of them here and refer the reader to [**144**] for details.

First of all, one has the next generalization of Theorem 11.3.

THEOREM 11.13 (Curvature Estimates, Meeks-Tinaglia [**145**]). *Let X be a homogeneous 3-manifold. Given δ, $\mathcal{H} > 0$, there exists $K(\delta, \mathcal{H}, X) \geq \sqrt{2}\mathcal{H}$ such that any H-disk M in X with $H \geq \mathcal{H}$ satisfies*

$$\sup_{\{p \in M \,|\, d_M(p, \partial M) \geq \delta\}} |A_M| \leq K(\delta, \mathcal{H}, X).$$

The main difficulty in generalizing the curvature estimate in Theorem 11.3 to the setting of homogeneous 3-manifolds is that one does not have a corresponding result like Proposition 11.8. But this problem can be solved by applying and adapting the manifold techniques applied by Meeks and Rosenberg [**139**], in their proof of the Minimal Lamination Closure Theorem 8.2. One obtains a modified version of Theorem 11.3 in the setting of homogeneously regular 3-manifolds for any compact $(H > 0)$-surface whose injectivity radius function is bounded away from zero outside of a small neighborhood of its boundary and in the small neighborhood this function is equal to the distance to the boundary. This approach gives the following generalization of Corollary 11.6.

THEOREM 11.14. *A complete $(H > 0)$-surface in a homogeneously regular 3-manifold has positive injectivity radius if and only if it has bounded second fundamental form.*

As an application of these results, one can prove the following theorem.

THEOREM 11.15 (Meeks, Tinaglia [**144**]). *Let $H \geq 1$. Then, any complete H-surface of finite topology in a complete hyperbolic 3-manifold is proper.*

This result is sharp when the complete hyperbolic 3-manifold is $\mathbb{H}^3$:

THEOREM 11.16 (Coskunuzer, Meeks, Tinaglia [**44**]). *For every $H \in [0, 1)$, there exists a complete, stable simply-connected H-surface in $\mathbb{H}^3$ that is not proper.*

See also [**43**] and [**177**] for examples of non-properly embedded complete simply-connected 0-surfaces in $\mathbb{H}^3$ and $\mathbb{H}^2 \times \mathbb{R}$ respectively.

12. Calabi-Yau problems

The Calabi-Yau problems or conjectures refer to a series of questions concerning the non-existence of a complete, 0-immersion $f\colon M \to \mathbb{R}^3$ whose image $f(M)$ is constrained to lie in a particular region of $\mathbb{R}^3$ (see [**21**], page 212 in [**27**], problem 91 in [**210**] and page 360 in [**211**]). Calabi's original conjecture states that a complete non-flat minimal surface cannot be contained either in the unit ball $\mathbb{B}(1)$ or in a slab. The first important negative result on the Calabi-Yau problem was given by Jorge and Xavier [**76**], who proved the existence of a complete 0-surface contained in an open slab of $\mathbb{R}^3$. In 1996, Nadirashvili [**160**] constructed a complete minimal disk in $\mathbb{B}(1)$; such a minimal disk cannot be embedded by the Colding-Minicozzi Theorem 4.1. A clever refinement of the ideas used by Nadirashvili, allowed Morales [**158**] to construct a conformal 0-immersion of the open unit disk that is proper in $\mathbb{R}^3$. These same techniques were then applied by Martín and Morales [**100**] to prove that if $\mathcal{D} \subset \mathbb{R}^3$ is either a smooth open bounded domain or a possibly non-smooth open convex domain, then there exists a complete, properly immersed 0-disk in $\mathcal{D}$; again, this disk cannot be embedded by Theorem 4.1.

In fact, embeddedness creates a dichotomy in results concerning the Calabi-Yau questions, as we have already seen in Theorems 4.1 and 8.2.

In contrast to the existence results described in the previous paragraph, Martín and Meeks have shown that there exist many bounded non-smooth domains in $\mathbb{R}^3$ which do not admit any complete, properly immersed surfaces with bounded absolute mean curvature function and at least one annular end. This generalized their previous joint work with Nadirashvili [**99**] in the minimal setting.

Theorem 12.1 (Martín and Meeks [**98**])). *Given any bounded domain $\mathcal{D}' \subset \mathbb{R}^3$, there exists a proper family $\mathcal{F}$ of horizontal simple closed curves in $\mathcal{D}'$ such that the bounded domain $\mathcal{D} = \mathcal{D}' - \bigcup \mathcal{F}$ does not admit any complete, connected properly immersed surfaces with compact (possibly empty) boundary, an annular end and bounded absolute mean curvature function.*

Ferrer, Martín and Meeks have given the following general result on the classical Calabi-Yau problem.

Theorem 12.2 (Ferrer, Martín and Meeks [**53**]). *Let M be an open, connected orientable surface and let $\mathcal{D}$ be a domain in $\mathbb{R}^3$ which is either convex or bounded and smooth. Then, there exists a complete, proper minimal immersion $f\colon M \to \mathcal{D}$.*

The following conjecture and the earlier stated Conjecture 4.3 are the two most important problems related to the properness of complete H-surfaces in $\mathbb{R}^3$. In relation to the following conjecture for complete 0-surfaces, one can ask whether there exists a complete, bounded non-compact $(H > 0)$-surface in $\mathbb{R}^3$. The next problem is largely motivated and suggested by the work of Martín, Meeks, Nadirashvili, Pérez and Ros.

Conjecture 12.3 (Embedded Calabi-Yau Conjectures).

1. *A necessary and sufficient condition for a connected, open topological surface M to admit a complete bounded minimal embedding in $\mathbb{R}^3$ is that every end of M has infinite genus.*
2. *A necessary and sufficient condition for a connected, open topological surface M to admit a proper minimal embedding in every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that M is orientable and every end of M has infinite genus.*
3. *A necessary and sufficient condition for a connected, non-orientable open topological surface M to admit a proper minimal embedding in some bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that every end of M has infinite genus.*

13. The Hopf Uniqueness Problem

There are two highly influential results on the classification and geometric description of H-spheres in homogeneous 3-manifolds, see [**1, 2, 72**]:

THEOREM 13.1 (Hopf Theorem). *An immersed H-sphere in a complete, simply connected 3-dimensional manifold $\mathbb{Q}^3(c)$ of constant sectional curvature c is a round sphere.*

THEOREM 13.2 (Abresch-Rosenberg Theorem). *An immersed H-sphere in a simply connected homogeneous 3-manifold with a four-dimensional isometry group is a rotationally symmetric immersed sphere.*

When the ambient space is an arbitrary homogeneous 3-manifold X, the type of description of immersed H-spheres given by the above theorems is no longer possible, due to the lack of continuous families of ambient rotations in X. Because of this, one natural way to describe immersed H-spheres in this general setting is to parameterize explicitly the moduli space of these spheres up to ambient isometries, and to determine their most important geometric properties.

In this section we describe a theoretical framework for studying immersed H-surfaces in any simply connected homogeneous 3-manifold X that is not diffeomorphic to $\mathbb{S}^2\times\mathbb{R}$; in $\mathbb{S}^2\times\mathbb{R}$, there is a unique immersed H-sphere for each value of the mean curvature $H \in [0,\infty)$, and each such immersed H-sphere is embedded as consequence of Theorem 13.2.

The common framework for every simply connected homogeneous 3-manifold X not diffeomorphic to $\mathbb{S}^2 \times \mathbb{R}$ is that such a X is isometric to a metric Lie group, i.e., to a 3-dimensional Lie group equipped with a left invariant metric. For background material on the classification and geometry of 3-dimensional metric Lie groups, the reader can consult the introductory textbook-style article [**117**] by the first two authors, and for further details on the proofs outlined in this section, we refer the reader to the papers [**108, 109, 110**] by Meeks, Mira, Pérez and Ros.

We wish to explain here how this general theory leads to the classification and geometric study of immersed H-spheres when X is compact. Specifically, Theorem 13.3 below gives a classification of immersed H-spheres in any homogeneous 3-manifold diffeomorphic to $\mathbb{S}^3$, and determines the essential properties of such spheres with respect to their existence, uniqueness, moduli space, symmetries, embeddedness and stability. Since we will refer to smooth families of oriented H-spheres parameterized by the values H of their constant mean curvature, in this section we will allow H to be any real number.

THEOREM 13.3 (Meeks, Mira, Pérez, Ros [**109**]). *Let X be a compact, simply connected homogeneous 3-manifold. Then:*

1. *For every $H \in \mathbb{R}$, there exists an immersed oriented sphere S_H in X of constant mean curvature H.*
2. *Up to ambient isometry, S_H is the unique immersed sphere in X with constant mean curvature H.*
3. *There exists a well-defined point in X called the* center of symmetry *of S_H such that the isometries of X that fix this point also leave S_H invariant.*

4. S_H is Alexandrov embedded, in the sense that the immersion $f\colon S_H \looparrowright X$ of S_H in X can be extended to an isometric immersion $F\colon B \to X$ of a Riemannian 3-ball such that $\partial B = S_H$ is mean convex.
5. S_H has index one and nullity three for the Jacobi operator.

Moreover, let $\mathcal{M}_X$ be the set of oriented immersed H-spheres in X whose center of symmetry is a given point $e \in X$. Then, $\mathcal{M}_X$ is an analytic family $\{S(t) \mid t \in \mathbb{R}\}$ parameterized by the mean curvature value t of $S(t)$.

Every compact, simply connected homogeneous 3-manifold is isometric to the Lie group SU(2) given by (1), endowed with a left invariant metric. There exists a 3-dimensional family of such homogeneous manifolds, which includes the 3-spheres $\mathbb{S}^3(c)$ of constant sectional curvature $c > 0$ and the two-dimensional family of rotationally symmetric *Berger spheres*, each of which has a four-dimensional isometry group. Apart from these two more symmetric families, any other left invariant metric on SU(2) has a 3-dimensional isometry group, with the isotropy group of every point being isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Item 3 in Theorem 13.3 provides the natural generalization of the theorems by Hopf and Abresch-Rosenberg to this more general context, since it implies that any immersed H-sphere S_H in such a space inherits all the ambient isometries fixing some point; in particular, S_H is round in $\mathbb{S}^3(c)$ and rotationally symmetric in the Berger spheres.

Items 1 and 2 together with the last statement of Theorem 13.3 provide an explicit description of the moduli space of immersed H-spheres in any compact, simply connected homogeneous 3-manifold X. Items 4 and 5 in Theorem 13.3 describe general embeddedness and stability type properties of immersed H-spheres in X which are essentially sharp, as we explain next. In $\mathbb{S}^3(c)$, immersed H-spheres are round, embedded and weakly stable (see Definition 2.23 for the notion of weak stability). However, for a general homogeneous X diffeomorphic to $\mathbb{S}^3$, immersed H-spheres need not be embedded (Torralbo [**200**] for certain ambient Berger spheres) or weakly stable (Torralbo and Urbano [**201**] for certain ambient Berger spheres, see also Souam [**194**]), and they are not geodesic spheres if X is not isometric to some $\mathbb{S}^3(c)$. Nonetheless, item 4 in Theorem 13.3 shows that any immersed H-sphere in a general X is Alexandrov embedded, a weaker notion of embeddedness, while item 5 describes the index and the dimension of the kernel of the Jacobi operator of an immersed H-sphere.

Just as in the classical case of $\mathbb{R}^3$, the left invariant Gauss map of an oriented surface Σ in a metric Lie group X (not necessarily compact) takes values in the unit sphere of the Lie algebra of X and contains essential information on the geometry of the surface, especially when Σ is an immersed H-surface.

DEFINITION 13.4. Given an oriented immersed surface $f\colon \Sigma \looparrowright X$ with unit normal vector field $N\colon \Sigma \to TX$ (here TX refers to the tangent bundle of X), we define the *left invariant Gauss map* of the immersed surface to be the map $G\colon \Sigma \to \mathbb{S}^2 \subset T_eX$ that assigns to each $p \in \Sigma$ the unit tangent vector to X at the identity element e given by $(dl_{f(p)})_e(G(p)) = N_p$.

An additional property of the immersed H-spheres in X that is not listed in the statement of Theorem 13.3 is that, after identifying X with the Lie group SU(2) endowed with a left invariant metric, the left invariant Gauss map of every immersed H-sphere in X is a diffeomorphism to $\mathbb{S}^2$; this diffeomorphism property is crucial in the proof of Theorem 13.3 and follows from the next more general result.

THEOREM 13.5 (Theorem 4.1 in [**109**]). *Any index-one H-sphere S_H in a 3-dimensional, simply-connected metric Lie group X satisfies:*

(1) The left invariant Gauss map of S_H is an orientation-preserving diffeomorphism to $\mathbb{S}^2$.
(2) S_H is unique up to left translations among H-spheres in X.
(3) S_H lies inside a real-analytic family $\{S_{H'} \mid H' \in (H-\varepsilon, H+\varepsilon)\}$ of index-one spheres in X for some $\varepsilon > 0$, where $S_{H'}$ has constant mean curvature of value H'.

As an application of Theorem 13.3, Meeks, Mira, Pérez and Ros provide a more detailed description of the special geometry of immersed 0-spheres in a general compact X.

THEOREM 13.6 (Theorem 7.1 in [**109**]). *For X as in Theorem 13.3, the unique (up to left translations) immersed 0-sphere S_0 in X is embedded. Furthermore, since the stabilizer of any point in a left invariant metric on* SU(2) *contains $\mathbb{Z}_2 \times \mathbb{Z}_2$ and S_0 is also invariant under the antipodal map $A \mapsto -A$, then by item 3 of Theorem 13.3 the related group of ambient isometries $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ leaves S_0 invariant; in fact when the isometry group of X is 3-dimensional, then G is the subgroup of ambient isometries of X that leaves S_0 invariant.*

Since it is well-known [**190**] that for every Riemannian metric on $\mathbb{S}^3$, there exists an embedded minimal sphere, then one could apply the uniqueness statement in item 2 of Theorem 13.3 to give an alternative proof that S_0 is embedded.

The theorems of Hopf and Abresch-Rosenberg rely on the existence of a holomorphic quadratic differential for immersed H-surfaces in homogeneous 3-manifolds with isometry group of dimension at least four. This approach using holomorphic quadratic differentials does not seem to work when the isometry group of the homogeneous 3-manifold has dimension three. Instead, the approach to proving Theorem 13.3 is inspired by two recent works on immersed H-spheres in the Thurston geometry Sol_3, i.e., in the solvable Lie group Sol_3 equipped with its standard left invariant metric. One of these works is the local parameterization by Daniel and Mira [**48**] of the space $\mathcal{M}^1_{\mathrm{Sol}_3}$ of index-one, immersed H-spheres in Sol_3 equipped with its standard metric, via the left invariant Gauss map and the uniqueness of such spheres.

The other one is Meeks' [**107**] area estimates for the subfamily of spheres in $\mathcal{M}^1_{\mathrm{Sol}_3}$ whose mean curvatures are bounded from below by any fixed positive constant; these two results lead to a complete description of the immersed H-spheres in Sol_3 endowed with its standard metric. However, the proof of Theorem 13.3 for a general compact, simply connected homogeneous 3-manifold X requires the development of new techniques and theory, which are needed to prove that the left invariant Gauss map of an index-one immersed H-sphere in X is a diffeomorphism, that immersed H-spheres in X are Alexandrov embedded and have a center of symmetry, and that there exist a priori area estimates for the family of index-one immersed H-spheres in X.

Here is a brief outline of the proof of Theorem 13.3. One first identifies the compact, simply connected homogenous 3-manifold X isometrically with SU(2) endowed with a left invariant metric. Next, one shows that any index-one immersed H-sphere S_H in X has the property that any other immersed sphere of the same constant mean curvature H in X is a left translation of S_H. The next step in the proof is to show that the set $\mathcal{H}$ of values $H \in \mathbb{R}$ for which there exists an index-one immersed H-sphere in X is non-empty, open and closed in $\mathbb{R}$ (hence, $\mathcal{H} = \mathbb{R}$). That $\mathcal{H}$ is non-empty follows from the existence of isoperimetric spheres in X of small volume. Openness of $\mathcal{H}$ follows from an application of the implicit function theorem, an argument that also proves that the space of index-one immersed H-spheres in X modulo left translations is an analytic one-dimensional manifold. By elliptic theory, closedness of $\mathcal{H}$ can be reduced to obtaining a priori area and curvature estimates for index-one, immersed H-spheres with any fixed upper bound on their mean curvatures. The existence of these curvature estimates is obtained by a rescaling argument. The most delicate part of the proof of Theorem 13.3 is obtaining a priori area estimates; for this, one first shows that the non-existence of an upper bound on the areas of all immersed H-spheres in X implies the existence of a complete, stable, constant mean curvature surface in X that can be seen to be the lift via a certain fibration $\Pi\colon X \to \mathbb{S}^2$ of an immersed curve in $\mathbb{S}^2$, and then one proves that such a surface cannot be stable to obtain a contradiction. This contradiction completes the proof of the fact that index-one immersed H-spheres exist for all values of H, and so, they are the unique immersed H-spheres in X. The Alexandrov embeddedness of immersed H-spheres follows from a deformation argument, using the smoothness of the family of immersed H-spheres in X and the maximum principle for H-surfaces in Theorem 2.11. Finally, the existence of a center of symmetry for any immersed H-sphere in X is deduced from the Alexandrov embeddedness and the uniqueness up to left translations of the sphere.

We next describe some key results and definitions that are essential in pushing forward and generalizing the arguments for classifying immersed H-spheres described above in the compact case to the setting where the metric Lie group is diffeomorphic to $\mathbb{R}^3$.

DEFINITION 13.7. Let Y be a complete homogeneous 3-manifold.

(1) The *critical mean curvature* $H(Y)$ of Y is defined as

$$H(Y) = \inf\{\max|H_M| : M \text{ is an immersed closed surface in } Y\},$$

where $\max|H_M|$ denotes the maximum of the absolute mean curvature function H_M.

(2) The *Cheeger constant* $\mathrm{Ch}(Y)$ of Y is defined as

$$\mathrm{Ch}(Y) = \inf_{\substack{K\subset Y \\ \text{compact}}} \frac{\mathrm{Area}(\partial K)}{\mathrm{Volume}(K)}.$$

The strategy of Meeks, Mira, Pérez and Ros in [**108**] to generalize Theorem 13.3 to the case where X is diffeomorphic to $\mathbb{R}^3$ is to obtain a result similar to Theorem 13.3 except that in this case, index-one H-spheres in X exist precisely for the values $H \in (H(X), \infty)$; note that the definition of $H(X)$ only permits immersed H-spheres to occur in X if $H \geq H(X)$ and the case of $H = H(X)$ is also easily ruled out. The expected proof of this generalization of Theorem 13.3 follows the same general reasoning as the outline given above. At the present moment, the main difficulty in completing the proof of this final result on the Hopf Uniqueness Problem is to obtain the following area estimates for immersed H-spheres in X:

($\star$) *For any $\varepsilon > 0$, there exists $A(X, \varepsilon) > 0$ such that every index-one H-sphere in X with $H \in (H(X) + \varepsilon, \infty)$ has area less than $A(X, \varepsilon)$.*

Meeks, Mira, Pérez and Ros in [**108**] are presently writing up the proof of these area estimates in an essentially case-by-case study of the possible metric Lie groups X that can occur when X is diffeomorphic to $\mathbb{R}^3$.

We end this section with some comments about the geometry of solutions of the isoperimetric problem in metric Lie groups diffeomorphic to $\mathbb{R}^3$ and the relationship between the two constants $H(X), \mathrm{Ch}(X)$ in Definition 13.7. In this subject there are still many important open problems concerning H-surfaces in simply connected homogeneous 3-manifolds, and we refer the interested reader to the last section of [**117**] for a long list of them; however, we mention some of our favorite ones below related to the isoperimetric problem in metric Lie groups diffeomorphic to $\mathbb{R}^3$.

CONJECTURE 13.8 (Isoperimetric Domains Conjecture). *Let X denote a metric Lie group diffeomorphic to $\mathbb{R}^3$. Then:*

1. *Isoperimetric domains in X are topological balls. More generally, closed Alexandrov embedded H-surfaces in X are spheres.*
2. *Immersed H-spheres in X are embedded, and the balls that they bound are isoperimetric domains.*
3. *For each fixed volume V_0, solutions to the isoperimetric problem in X for volume V_0 are unique up to left translations in X.*

There seems to be no direct method for computing the critical mean curvature $H(X)$ of a metric Lie group X diffeomorphic to $\mathbb{R}^3$, whereas when X is not isomorphic to the universal cover $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the special linear group $\mathrm{SL}(2,\mathbb{R})$, it is straightforward to compute the more familiar Cheeger constant of X directly from its metric Lie algebra, see [**117**] for this computation. Since the validity of Conjecture 13.8 would imply that H-spheres in X are the boundaries of isoperimetric domains, then it is perhaps not too surprising that one has the following result.

THEOREM 13.9 (Meeks, Mira, Pérez, Ros [**110**]). *If Y is a simply connected homogeneous 3-manifold, then $2\,H(Y) = \mathrm{Ch}(Y)$. Furthermore, if Δ_n is a sequence of isoperimetric domains in X with diverging volumes, then, as $n \to \infty$, the mean curvatures of their boundary surfaces converge to $H(X)$ and the radii R_n of Δ_n converge to infinity.*

REMARK 13.10. Another theoretical tool developed by Meeks, Mira, Pérez and Ros in [**109**] is a conformal PDE that the stereographic projection g of the left invariant Gauss map of an immersed H-surface in a simply connected 3-dimensional Lie group X must satisfy; this PDE depends on the value of H and invariants of its metric Lie algebra of X. Conversely, it follows from the representation Theorem 3.7 in [**109**] that any function $g\colon M \to \mathbb{C} \cup \{\infty\}$ on a simply connected Riemann surface satisfying this PDE, can be integrated to obtain a conformal H-immersion of M into X with g as its stereographically projected left invariant Gauss map.

14. CMC foliations

This section is devoted to results on the existence and geometry of CMC foliations of Riemannian n-manifolds.

14.1. The classification of singular CMC foliations of $\mathbb{R}^3$. The following classification theorem is stated for weak CMC foliations of $\mathbb{R}^3$, which are similar to weak H-laminations in that the leaves can intersect non-transversely, and where two such leaves intersect at a point, then locally they lie on one side of the other one near this point; for the definition of the more general notion of a weak CMC lamination, see Definition 7.2. Critical to its proof are the existence of curvature estimates given in Theorem 14.2 for weak CMC foliations of any Riemannian 3-manifold with bounded absolute sectional curvature. The following result generalizes the classical theorem of Meeks [**102**] that the only CMC foliations of $\mathbb{R}^3$ are foliations by parallel planes.

THEOREM 14.1 (Meeks, Pérez, Ros [**121**]). *Suppose that $\mathcal{F}$ is a weak CMC foliation of $\mathbb{R}^3$ with a closed countable set $\mathcal{S}$ of singularities (these are the points where the weak CMC structure of $\mathcal{F}$ cannot be extended). Then, each leaf of $\mathcal{F}$ is contained in either a plane or a round sphere, and $\mathcal{S}$ contains at most 2 points. Furthermore if $\mathcal{S}$ is empty, then $\mathcal{F}$ is a foliation by planes.*

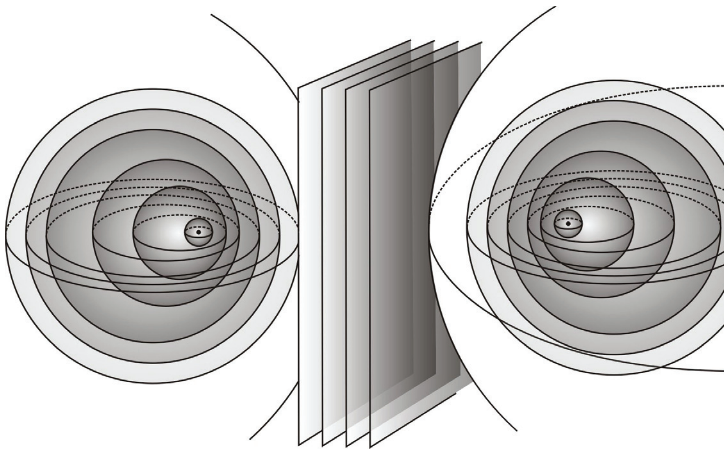

FIGURE 14. A foliation of $\mathbb{R}^3$ by spheres and planes with two singularities.

The simplest examples of weak CMC foliations of $\mathbb{R}^3$ with a closed countable set of singularities are families of parallel planes or concentric spheres around a given point. A slightly more complicated example appears when considering a family of pairwise disjoint planes and spheres as in Figure 14, where the set $\mathcal{S}$ consists of two points.

In the case of the unit 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ with its constant 1 sectional curvature, we obtain a similar result:

The leaves of every weak CMC foliation of $\mathbb{S}^3$ with a closed countable set $\mathcal{S}$ of singularities are contained in round spheres, and $\mathcal{S}$ consists of 1 or 2 points.

We note that in the statement of the above theorem, we made no assumption on the regularity of the foliation $\mathcal{F}$. However, the proofs require that $\mathcal{F}$ has bounded second fundamental form on compact sets of $N = \mathbb{R}^3$ or $\mathbb{S}^3$ minus the singular set $\mathcal{S}$. This bounded curvature assumption always holds for a topological CMC foliation by recent work of Meeks and Tinaglia [**143, 145**] on curvature estimates for embedded, non-zero constant mean curvature disks and a related 1-sided curvature estimate for embedded surfaces of any constant mean curvature (see Theorem 11.10 in the $\mathbb{R}^3$-setting and observation (O.2) in Section 7); in the case that all of the leaves of the lamination of a 3-manifold are minimal, this 1-sided curvature estimate was given earlier by Colding and Minicozzi [**37**] which also holds in the 3-manifold setting.

Consider a foliation $\mathcal{F}$ of a Riemannian 3-manifold N with leaves having constant absolute mean curvature, with this constant possibly depending on the given leaf. After possibly passing to a four-sheeted cover, we can assume X is oriented and that all leaves of $\mathcal{F}$ are oriented consistently, in the sense that there exists a continuous, nowhere zero vector field in X which is transversal to the leaves of $\mathcal{F}$. In this situation, the mean curvature function of the leaves of $\mathcal{F}$ is well-defined and so $\mathcal{F}$ is a CMC foliation. Therefore, when analyzing the structure of such a CMC foliation $\mathcal{F}$, it is natural to consider for each $H \in \mathbb{R}$, the subset $\mathcal{F}(H)$ of $\mathcal{F}$ of those leaves that have mean curvature H. Such a subset $\mathcal{F}(H)$ is closed since the mean curvature function is continuous on $\mathcal{F}$; $\mathcal{F}(H)$ is an example of an H-lamination. A cornerstone in proving Theorem 14.1 is to analyze the structure of an H-lamination $\mathcal{L}$ (or more generally, a weak H-lamination, see Definition 7.2) of a punctured ball in a Riemannian 3-manifold, in a small neighborhood of the puncture. This local problem can be viewed as a desingularization problem, see Theorem 9.1.

Besides Theorem 9.1, a second key ingredient is needed in the proof of Theorem 14.1: a universal scale-invariant curvature estimate valid for any weak CMC foliation of a compact Riemannian 3-manifold with boundary, solely in terms of an upper bound for its sectional curvature. The next result is inspired by previous curvature estimates described in Section 2.5 for stable constant mean curvature surfaces.

THEOREM 14.2 (Curvature Estimates for CMC foliations, [**121**]). *There exists a constant $C > 0$ such that the following statement holds. Given $\Lambda \geq 0$, a compact Riemannian 3-manifold X with boundary whose absolute sectional curvature is at most Λ, a weak CMC foliation $\mathcal{F}$ of X and a point $p \in \mathrm{Int(X)}$, we have*

$$|A_{\mathcal{F}}|(p) \leq \frac{C}{\min\{d_X(p, \partial X), \frac{\pi}{\sqrt{\Lambda}}\}},$$

¡ where $|A_{\mathcal{F}}|\colon X \to [0, \infty)$ is the function that assigns to each $p \in X$ the supremum of the norms of the second fundamental forms of leaves of $\mathcal{F}$ passing through p, and d_X is the Riemannian distance in X.

If $\mathcal{F}$ were a non-flat weak CMC foliation of $\mathbb{R}^3$, then the norms of the second fundamental forms of foliations obtained by scaling $\mathcal{F}$ by $\frac{1}{n}$, $n \in \mathbb{N}$, are not uniformly bounded which contradicts the conclusions of Theorem 14.2. This contradiction proves that the only CMC foliations of $\mathbb{R}^3$ are foliations by parallel planes.

The above curvature estimate is also an essential tool for analyzing the structure of a weak CMC foliation of a small geodesic Riemannian 3-ball punctured at its center. Among other things, in [**121**] Meeks, Pérez and Ros proved that if the mean curvatures of the leaves of such a weak CMC foliation are bounded in a neighborhood of the puncture, then the weak CMC foliation extends across the puncture to a weak CMC foliation of the ball. Theorem 14.1 and a blow-up argument lead to a model for the structure of a weak CMC foliation of a punctured ball in any Riemannian 3-manifold. From here, one can deduce that a compact, orientable Riemannian 3-manifold not diffeomorphic to the 3-sphere $\mathbb{S}^3$ does not admit any weak

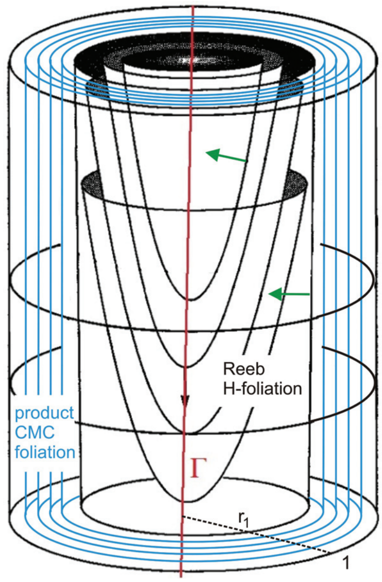

FIGURE 15. This figure depicts the structure of a generalized Reeb foliation on $D \times \mathbb{R}$ with leaves of constant mean curvature H.

(transversely oriented) CMC foliation with a non-empty countable closed set of singularities; see [**111**] for this and other related results.

14.2. CMC foliations of closed n-manifolds. By the next theorem by Meeks and Pérez, the vanishing of the Euler characteristic of a closed n-manifold X is equivalent to the existence of a CMC foliation of X with respect to some Riemannian metric. In the case X is orientable, this theorem was proved by Oshikiri [**164**]; we emphasize that the proof of Theorem 14.3 below by Meeks and Pérez in [**112**] does not use Oshikiri's results. Furthermore, when $n \geq 3$ the CMC foliations $\mathcal{F}$ that we construct on X with vanishing Euler characteristic satisfy that there are a finite number of components of the complement of the sublamination of minimal leaves in $\mathcal{F}$ such that each of these foliated components is diffeomorphic to the product of an open $(n-1)$-disk D and a circle $\mathbb{S}^1$, with isometry group containing $SO(n-1) \times \mathbb{S}^1$; furthermore, the universal cover $D \times \mathbb{R}$ of each such component together with its lifted foliation and metric are equivalent to a rather explicit CMC foliation $\mathcal{F}_n$ on $D \times \mathbb{R}$ with a product metric g_n, such that this structure is invariant under the action of $SO(n-1) \times \mathbb{R}$ and depends only on the dimension n; see Figure 15.

Recall that by definition, a CMC foliation is necessarily smooth.

THEOREM 14.3 (Existence Theorem for CMC Foliations). *A closed n-manifold admits a CMC foliation for some Riemannian metric if and only* *if its Euler characteristic is zero. When $n \geq 2$, the CMC foliation can be taken to be non-minimal.*

Since closed (topological) 3-manifolds admit smooth structures and the Euler characteristic of any closed manifold of odd dimension is zero, the previous theorem has the following corollary.

COROLLARY 14.4. *Every closed topological 3-manifold admits a smooth structure together with a Riemannian metric and a non-minimal CMC foliation.*

The proof of Theorem 14.3 is motivated by two seminal works. The first one, due to Thurston (Theorem 1(a) in [**196**]), shows that a necessary and sufficient condition for a smooth closed n-manifold X to admit a smooth, codimension-one foliation $\mathcal{F}$ is for its Euler characteristic to vanish; for our applications, $\mathcal{F}$ can be chosen to be transversely oriented. The second one is the result by Sullivan (Corollary 3 in [**195**]) that given such a pair $(X, \mathcal{F})$ where $\mathcal{F}$ is orientable (this means that the subbundle of the tangent bundle to X which is tangent to the foliation is orientable), then X admits a smooth Riemannian metric g_X for which $\mathcal{F}$ is a minimal foliation (this is called $\mathcal{F}$ is *geometrically taut*) if and only if for every compact leaf L of $\mathcal{F}$ there exists a closed transversal that intersects L (called $\mathcal{F}$ is *homologically taut*); in the proof of Theorem 14.3, it is needed the generalization of the implication 'homologically taut $\Rightarrow$ geometrically taut' without Sullivan's hypothesis that the foliation $\mathcal{F}$ be orientable.

In the case when $n = 2$, Theorem 14.3 follows by giving explicit examples. Consider the curve $\alpha = \{(t, 3 + \cos t) \mid t \in \mathbb{R}\}$ in the (x_1, x_2)-plane and let C in $\mathbb{R}^3$ be the surface obtained by revolving α around the x_1-axis. Let $\mathcal{F}$ be the foliation of C by circles contained in planes orthogonal to the x_1-axis, whose leaves have constant geodesic curvature, see Figure 16. $\mathcal{F}$ is transversely oriented by the normal vectors to the circles in C that have positive inner product in $\mathbb{R}^3$ with ∂_{x_1}. Since the map $R(x_1, x_2, x_3) = (2\pi + x_1, x_2, -x_3)$ preserves the transverse orientation of the CMC foliation, then $\mathcal{F}$ descends to a CMC foliation of the Klein bottle C/R or to the torus $C/(R^2)$. By classification of closed surfaces, a closed surface with Euler characteristic zero must be a torus or a Klein bottle. Thus, Theorem 14.3 trivially holds when $n = 2$.

So assume $n \geq 3$ and we will give a sketch of the proof of Theorem 14.3 in this case. One first studies the existence of codimension-one, $(SO(n-1)\times\mathbb{R})$-invariant CMC foliations $\mathcal{R}_{n-1}$ of the Riemannian product of the real number line $\mathbb{R}$ with the closed unit $(n-1)$-disk $\overline{\mathbb{D}}(1) \subset \mathbb{R}^{n-1}$ with respect to a certain $SO(n-1)$-invariant metric, see Figure 15. The leaves of this foliation $\mathcal{R}_{n-1}$ are of one of two types: those leaves that intersect $\mathbb{D}(r_1) \times \mathbb{R}$ (here $\mathbb{D}(r_1) = \{x \in \mathbb{R}^{n-1} \mid |x| < r_1\}$ and $0 < r_1 < 1$) are rotationally symmetric hypersurfaces which are graphical over $\mathbb{D}(r_1) \times \{0\}$ and asymptotic to the vertical $(n-1)$-cylinder $\mathbb{S}^{n-2}(r_1) \times \mathbb{R}$; the remaining leaves of $\mathcal{R}_{n-1}$ are the vertical cylinders $\mathbb{S}^{n-2}(r) \times \mathbb{R}$, $r \in [r_1, 1]$. All leaves of $\mathcal{R}_{n-1}$ in $\mathbb{D}^{n-1}(r_1) \times \mathbb{R}$

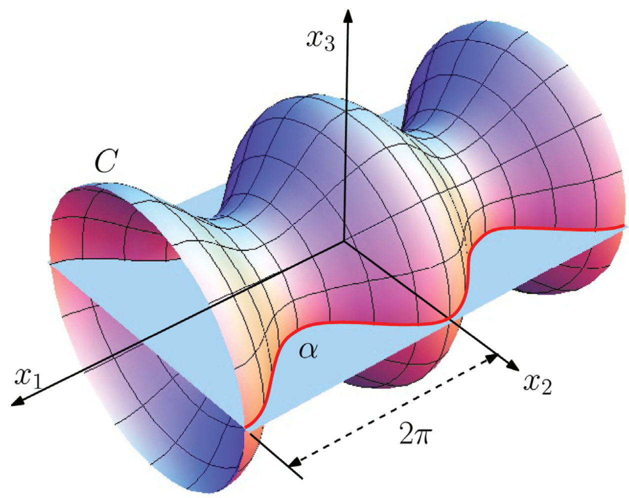

FIGURE 16. The circles in C foliate the surface by curves of constant geodesic curvature. The symmetry R is the composition of the reflection in the (x_1, x_2)-plane (depicted in the figure) with the translation by the vector $(2\pi, 0, 0)$.

are vertical translates of a single such leaf (in particular, they all have the same constant mean curvature, equal to the constant value of the mean curvature of $\mathbb{S}^{n-2}(r_1)\times\mathbb{R}$), while the (constant) mean curvature values of the cylinders $\mathbb{S}^{n-2}(r) \times \mathbb{R}$, $r \in [r_1, 1]$, vary from leaf to leaf. This foliation $\mathcal{R}_{n-1}$ gives rise under the quotient action of $\mathbb{Z} \subset \mathbb{R}$ to what we call an *enlarged foliated Reeb component* $\mathcal{R}_{n-1}/\mathbb{Z}$, that is diffeomorphic to $\overline{\mathbb{D}}(1) \times \mathbb{S}^1$.

The sufficient implication in Theorem 14.3 follows directly from the Poincaré-Hopf index theorem. As for the necessary implication, the results in [**196**] imply that a smooth, closed n-manifold X with Euler characteristic zero admits a smooth, transversely oriented foliation $\mathcal{F}'$ of codimension one. After a simple modification of $\mathcal{F}'$ along some smooth simple closed curve Γ transverse to the foliation by the classical technique of turbularization, $\mathcal{F}'$ can be assumed to have at least one non-compact leaf. Recall that in this process one modifies the previous foliation in a small tubular neighborhood of Γ and one ends up with a new foliation where we have introduced what we called in the previous paragraph a generalized Reeb component centered along Γ. Then one proves the existence of a finite collection $\Delta = \{\gamma_1, \ldots, \gamma_k\}$ of pairwise disjoint, compact embedded arcs in X that are transverse to the leaves of $\mathcal{F}'$ and such that every compact leaf of the foliation intersects at least one of these arcs; this existence result follows from work of Haefliger [**64**] on the compactness of the set of compact leaves of any codimension-one foliation of X. The next step consists of modifying $\mathcal{F}'$ using again turbularization by introducing pairs of enlarged Reeb components, one pair for each $\gamma_i \in \Delta$. These modifications give rise to a new transversely oriented foliation $\mathcal{F}$. By a careful application of a generalization of Sullivan's theorem to the case of non-orientable codimension-one foliations, one can check that in the complement of the sewn in generalized Reeb components in $\mathcal{F}$, the resulting manifold $\widehat{X}$ with boundary admits a metric so that all of the leaves of the restricted foliation are minimal and in a neighborhood of $\partial\widehat{X}$ where the foliation is a product foliation, the metric is also a product metric. Then one proceeds by extending this minimal metric on $\widehat{X}$ to X and so that the regions modified by turbularization now have the rotationally invariant metrics mentioned in the previous paragraph. Crucial in obtaining this smooth metric on X so that all of the leaves of $\mathcal{F}$ have constant mean curvature, is the application of the classical Theorem 2 in Moser [**159**]. This smooth "gluing" result of Moser is closely related to his following well-known classical result: If g_1, g_2 are two metrics with respective volume forms dV_1, dV_2 on a closed orientable Riemannian n-manifold Y with the same total volume, then there exists an orientation preserving diffeomorphism $f\colon Y \to Y$ such that $f^*(dV_1) = dV_2$; furthermore, f can be chosen isotopic to the identity.

We finish this article mentioning another result from [**112**], which is the Structure Theorem 14.5 given below on the geometry and topology of non-minimal CMC foliations of a closed n-manifold. Before stating this theorem, we fix some notation for a CMC foliation $\mathcal{F}$ of a (connected) closed Riemannian n-manifold X:

- $N_{\mathcal{F}}$ denotes the unit normal vector field to $\mathcal{F}$ whose direction coincides with the given transverse orientation.
- $H_{\mathcal{F}}\colon X \to \mathbb{R}$ stands for the *mean curvature function* of $\mathcal{F}$ with respect to $N_{\mathcal{F}}$.
- $H_{\mathcal{F}}(X) = [\min H_{\mathcal{F}}, \max H_{\mathcal{F}}]$ is the image of $H_{\mathcal{F}}$.
- $\mathcal{C}_{\mathcal{F}}$ denotes the union of the compact leaves in $\mathcal{F}$, which is a compact subset of X by the aforementioned result of Haefliger [**64**].

THEOREM 14.5 (Structure Theorem for CMC Foliations). *Let (X, g) be a closed, connected Riemannian n-manifold which admits a non-minimal CMC foliation $\mathcal{F}$. Then:*

1. *$\int_X H_{\mathcal{F}}\, dV = 0$ and so, $H_{\mathcal{F}}$ changes sign (here dV denotes the volume element with respect to g).*
2. *For H a regular value of $H_{\mathcal{F}}$, $H_{\mathcal{F}}^{-1}(H)$ consists of a finite number of compact leaves of $\mathcal{F}$ contained in* $\mathrm{Int}(\mathcal{C}_{\mathcal{F}})$.
3. *$X - \mathcal{C}_{\mathcal{F}}$ consists of a countable number of open components and the leaves in each such component Δ have the same mean curvature as the finite positive number of compact leaves in $\partial\Delta$; furthermore, every leaf in the closure of $X - \mathcal{C}_{\mathcal{F}}$ is stable. In particular, except for a countable subset of $H_{\mathcal{F}}(X)$, every leaf of $\mathcal{F}$ with mean curvature H is compact, and for every $H \in H_{\mathcal{F}}(X)$, there exists at least one compact leaf of $\mathcal{F}$ with mean curvature H.*

4. a. *Suppose that L is a leaf of $\mathcal{F}$ that contains a regular point of $H_{\mathcal{F}}$. Then L is compact, it consists entirely of regular points of $H_{\mathcal{F}}$ and lies in* $\mathrm{Int}(\mathcal{C}_{\mathcal{F}})$. *Furthermore, L has index zero if and only if the function $g(\nabla H_{\mathcal{F}}, N_{\mathcal{F}}) = N_{\mathcal{F}}(H_{\mathcal{F}})$ is negative along L, and if the index of L is zero, then it also has nullity zero.*
   b. *Suppose that L is a leaf of $\mathcal{F}$ that is disjoint from the regular points of $H_{\mathcal{F}}$. Then the index of L is zero, and if L is a limit leaf*[8] *of the CMC lamination of X consisting of the compact leaves of $\mathcal{F}$, then L is compact with nullity one.*
5. *Any leaf of $\mathcal{F}$ with mean curvature equal to* $\min H_{\mathcal{F}}$ *or* $\max H_{\mathcal{F}}$ *is stable and such a leaf can be chosen to be compact with nullity one.*

15. Outstanding problems and conjectures

In this last section, we present many of the fundamental conjectures in minimal and constant mean curvature surface theory. In the statement of most of these conjectures we have listed the principal researchers to whom the given conjecture might be attributed and/or those individuals who have made important progress in its solution.

15.1. Conjectures in the classical case of $\mathbb{R}^3$. The classical Euclidean isoperimetric inequality states that the inequality $4\pi A \leq L^2$ holds for the area A of a compact subdomain of $\mathbb{R}^2$ with boundary length L, with equality if and only if Ω is a round disk. The same inequality is known to hold for compact minimal surfaces with boundary in $\mathbb{R}^3$ with at most two boundary components (Reid [**173**], Osserman and Schiffer [**168**], Li, Schoen and Yau [**93**], see also Osserman's survey paper [**166**]).

Conjecture 15.1 (Isoperimetric Inequality Conjecture). *Every connected, compact minimal surface Ω with boundary in $\mathbb{R}^3$ satisfies*

$$4\pi A \leq L^2, \tag{38}$$

where A is the area of Ω and L is the length of its boundary. Furthermore, equality holds if and only if Ω is a planar round disk.

More generally, if Ω is a connected, compact surface with boundary in a Hadamard 3-manifold N with sectional curvature at most $-a^2$, and the absolute mean curvature function of Ω is at most $|a| \geq 0$, then equation (38) *is satisfied and equality holds if and only if Ω is a planar round disk in a flat totally umbilic simply-connected hypersurface of constant mean curvature $|a|$.*

Gulliver and Lawson [**63**] proved that if Σ is an orientable, stable minimal surface with compact boundary that is properly embedded in the punctured unit ball $\mathbb{B} - \{\vec{0}\}$ of $\mathbb{R}^3$, then its closure is a compact, embedded minimal surface. If Σ is not stable, then the corresponding result is not known. Meeks, Pérez and Ros [**133, 128**] proved that a properly embedded

[8]See Definition 7.4 for the definition of a limit leaf of a codimension-one lamination.

minimal surface M in $\mathbb{B} - \{\vec{0}\}$ with $\partial M \subset \mathbb{S}^2$ extends across the origin if and only if the function $K|R|^2$ is bounded on M, where K is the Gaussian curvature function of M and $R^2 = x_1^2 + x_2^2 + x_3^2$ (Theorem 2.26 implies that if M is stable, then $K|R|^2$ is bounded). In fact, this removable singularity result holds true if we replace $\mathbb{R}^3$ by an arbitrary Riemannian 3-manifold (Theorem 9.1). The following conjecture can be proven to hold for any minimal surface of finite topology (in fact, with finite genus, see Corollary 2.4 in [**126**]).

Conjecture 15.2 (Isolated Singularities Conjecture, Gulliver-Lawson). *The closure of a properly embedded minimal surface with compact boundary in the punctured ball $\mathbb{B} - \{\vec{0}\}$ is a compact, embedded minimal surface.*

The most ambitious conjecture about removable singularities for minimal surfaces is the following one, which deals with laminations instead of with surfaces.

Conjecture 15.3 (Fundamental Singularity Conjecture, Meeks-Pérez-Ros). *If $A \subset \mathbb{R}^3$ is a closed set with zero 1-dimensional Hausdorff measure and $\mathcal{L}$ is a minimal lamination of $\mathbb{R}^3 - A$, then $\mathcal{L}$ extends to a minimal lamination of $\mathbb{R}^3$.*

In Section 9, we saw how the Local Removable Singularity Theorem 9.1 is a cornerstone for the proof of the Quadratic Curvature Decay Theorem 9.6 and the Dynamics Theorem in [**122**], which illustrates the usefulness of removable singularities results.

In the discussion of the conjectures that follow, it is helpful to fix some notation for certain classes of complete embedded minimal surfaces in $\mathbb{R}^3$.

- Let $\mathcal{C}$ be the space of connected, $\mathcal{C}$omplete, embedded minimal surfaces.
- Let $\mathcal{P} \subset \mathcal{C}$ be the subspace of $\mathcal{P}$roperly embedded surfaces.
- Let $\mathcal{M} \subset \mathcal{P}$ be the subspace of surfaces with $\mathcal{M}$ore than one end.

In what follows we will freely use the properness result of complete 0-surfaces of finite topology given in Corollary 8.4 and Collin's Theorem [**40**] that properly embedded minimal surfaces of finite topology with more than 1 end have finite total curvature.

Conjecture 15.4 (Finite Topology Conjecture I, Hoffman-Meeks). *An orientable surface M of finite topology with genus g and r ends, $r \neq 0, 2$, occurs as a topological type of a surface in $\mathcal{C}$ if and only if $r \leq g + 2$.*

The main theorem in [**120**] insures that for each positive genus g, there exists an upper bound $e(g)$ on the number of ends of an $M \in \mathcal{M}$ with finite topology and genus g. Hence, the non-existence implication in Conjecture 15.4 will be proved if one can show that $e(g)$ can be taken as $g + 2$. Concerning the case $r = 2$, the classification result of Schoen [**186**] implies that the only examples in $\mathcal{M}$ with finite topology and two ends are catenoids.

On the other hand, Theorem 1.3 characterizes the helicoid among complete, embedded, non-flat minimal surfaces in $\mathbb{R}^3$ with genus zero and one end. Concerning one-ended surfaces in $\mathcal{C}$ with finite positive genus, first note that all these surfaces are proper by Theorem 4.1. Furthermore, every example $M \in \mathcal{P}$ of finite positive genus and one end has a special analytic representation on a once punctured compact Riemann surface, as follows from the works of Bernstein and Breiner [**10**] and Meeks and Pérez [**113**], see Theorems 4.6 and 4.16. In fact, these authors showed that any such minimal surface has *finite type*[9] and is asymptotic to a helicoid.

All these facts motivate the next conjecture, which appeared in print for the first time in the paper [**138**] by Meeks and Rosenberg, although several versions of it as questions were around a long time before appearing in [**138**]. The finite type condition and work of Colding and Minicozzi were applied by Hoffman, Traizet and White [**70, 71**] to prove of the existence implication of the next conjecture. A step in the proof of the uniqueness statement of the next conjecture in the case of genus one is the result of Bernstein and Breiner [**12**] that states that every genus 1 helicoid has an axis of rotational symmetry; here uniqueness means up to the composition of an ambient isometry and a homothety.

Conjecture 15.5 (Finite Topology Conjecture II, Meeks-Rosenberg). *For every non-negative integer g, there exists a unique non-planar $M \in \mathcal{C}$ with genus g and one end.*

The Finite Topology Conjectures I and II together propose the precise topological conditions under which a non-compact orientable surface of finite topology can be properly minimally embedded in $\mathbb{R}^3$. What about the case where the non-compact orientable surface M has infinite topology? In this case, either M has infinite genus or M has an infinite number of ends. Results of Collin, Kusner, Meeks and Rosenberg imply such an M must have at most two limit ends. Meeks, Pérez and Ros proved in [**128**] that such an M cannot have one limit end and finite genus. The claim is that these restrictions are the only ones.

Conjecture 15.6 (Infinite Topology Conjecture, Meeks). *A non-compact, orientable surface of infinite topology occurs as a topological type of a surface in $\mathcal{P}$ if and only if it has at most one or two limit ends, and when it has one limit end, then its limit end has infinite genus.*

Traizet [**202**] constructed a properly embedded minimal surface with infinite genus and one limit end, all whose simple ends are annuli and whose Gaussian curvature function is unbounded. In a closely related paper, Morabito and Traizet [**157**] constructed a properly embedded minimal surface with two limit ends, one of which has genus zero and the other with infinite genus, such that all of its middle ends are annuli. These results represent progress on Conjecture 15.6.

[9]See Definition 4.12 for the concept of minimal surface of finite type.

If $M \in \mathcal{C}$ has finite topology, then M has finite total curvature or is asymptotic to a helicoid by Theorems 4.1 and 4.6. It follows that for any such surface M, there exists a constant $C_M > 0$ such that the injectivity radius function $I_M : M \to (0, \infty]$ satisfies

$$I_M(p) \geq C_M \|p\|, \quad p \in M.$$

Work of Meeks, Pérez and Ros in [**124, 133**] indicates that this linear growth property of the injectivity radius function should characterize the examples in $\mathcal{C}$ with finite topology, in a similar manner that the inequality $K_M(p)\|p\|^2 \leq C_M$ characterizes finite total curvature for a surface $M \in \mathcal{C}$ (Theorem 9.6, here K_M denotes the Gaussian curvature function of M).

CONJECTURE 15.7 (Injectivity Radius Growth Conjecture, Meeks-Pérez-Ros). *A surface $M \in \mathcal{C}$ has finite topology if and only if its injectivity radius function grows at least linearly with respect to the extrinsic distance from the origin.*

The results in [**124, 133**] and the earlier described Theorems 6.1 and 7.7 also motivated several conjectures concerning the limits of locally simply connected sequences of minimal surfaces in $\mathbb{R}^3$, like the following one, which in the case that M is allowed to have compact boundary represents the necessary implication in the embedded Calabi-Yau Conjecture 15.10 below.

CONJECTURE 15.8 (Finite Genus Properness Conjecture, Meeks-Pérez-Ros). *If $M \in \mathcal{C}$ and M has finite genus, then $M \in \mathcal{P}$.*

In [**123**], Meeks, Pérez and Ros proved Conjecture 15.8 under the additional hypothesis that M has a countable number of ends (this assumption is necessary for M to be proper in $\mathbb{R}^3$ by work in [**42**]).

Conjecture 15.8 can be shown to follow from the next beautiful structure conjecture, which we include here in spite of the fact that it is stated when the ambient space is a general Riemannian manifold.

CONJECTURE 15.9 (Finite Genus Conjecture in 3-manifolds, Meeks-Pérez-Ros). *Suppose M is a connected, complete, embedded minimal surface with empty boundary and finite genus in a Riemannian 3-manifold N. Let $\overline{M} = M \cup \lim(M)$, where $\lim(M)$ is the set of limit points*[10] *of M. Then, one of the following possibilities holds.*

1. *$\overline{M}$ has the structure of a minimal lamination of N.*
2. *$\overline{M}$ fails to have a minimal lamination structure, $\lim(M)$ is a non-empty minimal lamination of N consisting of stable leaves and M is properly embedded in $N - \lim(M)$.*

The next conjecture is a more ambitious version of the previously stated Conjecture 12.3.

CONJECTURE 15.10 (Embedded Calabi-Yau Conjectures, Martín, Meeks, Nadirashvili, Pérez, Ros).

[10] See the paragraph just before Theorem 8.2 for the definition of $\lim(M)$.

1. *There exists an $M \in \mathcal{C}$ contained in a bounded domain in $\mathbb{R}^3$. In particular, $\mathcal{P} \neq \mathcal{C}$.*
2. *There exists an $M \in \mathcal{C}$ whose closure $\overline{M}$ has the structure of a minimal lamination of a slab, with M as a leaf and with two planes as limit leaves.*
3. *A necessary and sufficient condition for a connected, open topological surface M to admit a complete bounded minimal embedding in $\mathbb{R}^3$ is that every end of M has infinite genus.*
4. *A necessary and sufficient condition for a connected, open topological surface M to admit a proper minimal embedding in every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that M is orientable and every end of M has infinite genus.*
5. *A necessary and sufficient condition for a connected, non-orientable open topological surface M to admit a proper minimal embedding in some bounded domain $\mathcal{D} \subset \mathbb{R}^3$ as a complete surface is that every end of M has infinite genus.*

We now discuss two conjectures related to the underlying conformal structure of a minimal surface.

CONJECTURE 15.11 (Liouville Conjecture, Meeks-Sullivan). *If $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a positive harmonic function, then h is constant.*

The above conjecture is closely related to work in [**42, 129, 140**]. We also remark that Neel [**161**] proved that if a surface $M \in \mathcal{P}$ has bounded Gaussian curvature, then M does not admit non-constant bounded harmonic functions. A related conjecture is the following one:

CONJECTURE 15.12 ((Multiple-End Recurrency Conjecture, Meeks). *If $M \in \mathcal{M}$, then M is recurrent for Brownian motion.*

Assuming that one can prove the last conjecture, the proof of the Liouville Conjecture would reduce to the case where $M \in \mathcal{P}$ has infinite genus and one end. Note that in this setting, a surface could satisfy Conjecture 15.11 while at the same time being transient. For example, by work of Meeks, Pérez and Ros [**129**] every doubly or triply-periodic minimal surface with finite topology quotient satisfies the Liouville Conjecture, and these minimal surfaces are never recurrent. On the other hand, every doubly or triply-periodic minimal surface has exactly one end (Callahan, Hoffman and Meeks [**23**]), which implies that the assumption in Conjecture 15.12 that $M \in \mathcal{M}$, not merely $M \in \mathcal{P}$, is a necessary one. It should be also noted that the previous two conjectures need the hypothesis of global embeddedness, since there exist properly immersed minimal surfaces with two embedded ends and which admit bounded non-constant harmonic functions [**42**].

CONJECTURE 15.13 (Scherk Uniqueness Conjecture, Meeks-Wolf). *If M is a connected, properly immersed minimal surface in $\mathbb{R}^3$ and* $\mathrm{Area}(M \cap \mathbb{B}(R)) \leq 2\pi R^2$ *holds in extrinsic balls $\mathbb{B}(R)$ of radius R, then M is a plane, a catenoid or one of the singly-periodic Scherk minimal surfaces.*

By the Monotonicity Formula (see e.g., [**32**]), any connected, properly immersed minimal surface in $\mathbb{R}^3$ with

$$\lim_{R\to\infty} R^{-2}\text{Area}(M\cap\mathbb{B}(R))\leq 2\pi,$$

is actually embedded. A related conjecture on the uniqueness of the doubly-periodic Scherk minimal surfaces was solved by Lazard-Holly and Meeks [**92**]; they proved that if $M\in\mathcal{P}$ is doubly-periodic and its quotient surface has genus zero, then M is one of the doubly-periodic Scherk minimal surfaces. The basic approach used in [**92**] was adapted later on by Meeks and Wolf [**153**] to prove that Conjecture 15.13 holds under the assumption that the surface is singly-periodic. We recall that Meeks and Wolf's proof uses that the Unique Limit Tangent Cone Conjecture below holds in their periodic setting; this approach suggests that a good way to solve the general Conjecture 15.13 is first to prove Conjecture 15.14 on the uniqueness of the limit tangent cone of M, from which it follows (unpublished work of Meeks and Ros) that M has two Alexandrov-type planes of symmetry. Once M is known to have these planes of symmetry, one can describe the Weierstrass representation of M, which Meeks and Wolf (unpublished) claim would be sufficient to complete the proof of the conjecture.

Conjecture 15.14 (Unique Limit Tangent Cone at Infinity Conjecture, Meeks). *If $M\in\mathcal{P}$ is not a plane and has extrinsic quadratic area growth, then $\lim_{t\to\infty}\frac{1}{t}M$ exists and is a minimal, possibly non-smooth cone over a finite balanced configuration of geodesic arcs in the unit sphere, with common ends points and integer multiplicities. Furthermore, if M has area not greater than $2\pi R^2$ in extrinsic balls of radius R, then the limit tangent cone of M is either the union of two planes or consists of a single plane with multiplicity two passing through the origin.*

By unpublished work of Meeks and Wolf, the above conjecture is closely tied to the validity of the next classical one.

Conjecture 15.15 (Unique Limit Tangent Cone at Punctures Conjecture). *Let $f\colon M\to\mathbb{B}-\{\vec{0}\}$ be a proper immersion of a surface with compact boundary in the punctured unit ball, such that $f(\partial M)\subset\partial\mathbb{B}$ and whose mean curvature function is bounded. Then, $f(M)$ has a* unique *limit tangent cone at the origin under homothetic expansions.*

A classical result of Fujimoto [**58**] establishes that the Gauss map of any orientable, complete, non-flat, immersed 0-surface in $\mathbb{R}^3$ cannot exclude more than 4 points, which improved the earlier result of Xavier [**209**] that the Gauss map of such a surface cannot miss more than 6 points. If one assumes that a surface $M\in\mathcal{C}$ is periodic with finite topology quotient, then Meeks, Pérez and Ros solved the first item in the next conjecture [**125**]. Also see Kawakami, Kobayashi and Miyaoka [**82**] for related results on this problem, including some partial results on the conjecture of Osserman that states that the Gauss map of an orientable, complete, non-flat, immersed 0-surface with finite total curvature in $\mathbb{R}^3$ cannot miss 3 points of $\mathbb{S}^2$.

CONJECTURE 15.16 (Four Point Conjecture, Meeks, Pérez, Ros). *Suppose $M \in \mathcal{C}$. If the Gauss map of M omits 4 points on $\mathbb{S}^2$, then M is a singly or doubly-periodic Scherk minimal surface.*

We next deal with the question of when a surface $M \in \mathcal{C}$ has strictly negative Gaussian curvature. Suppose again that a surface $M \in \mathcal{C}$ has finite topology, and so, M either has finite total curvature or is a helicoid with handles. It is straightforward to check that such a surface has negative curvature if and only if it is a catenoid or a helicoid (note that if $g\colon M \to \mathbb{C} \cup \{\infty\}$ is the stereographically projected Gauss map of M, then after a suitable rotation of M in $\mathbb{R}^3$, the meromorphic differential $\frac{dg}{g}$ vanishes exactly at the zeros of the Gaussian curvature of M; from here one deduces easily that if M has finite topology and strictly negative Gaussian curvature, then the genus of M is zero). More generally, if we allow a surface $M \in \mathcal{C}$ to be invariant under a proper discontinuous group G of isometries of $\mathbb{R}^3$, with M/G having finite topology, then M/G is properly embedded in $\mathbb{R}^3/G$ by an elementary application of the Minimal Lamination Closure Theorem (see Proposition 1.3 in [**170**]). Hence, in this case M/G has finite total curvature by a result of Meeks and Rosenberg [**134, 136**]. Suppose additionally that M/G has negative curvature, and we will discuss which surfaces are possible. If the ends of M/G are helicoidal or planar, then a similar argument using $\frac{dg}{g}$ gives that M has genus zero, and so, it is a helicoid. If M/G is doubly-periodic, then M is a Scherk minimal surface, see [**125**]. In the case M/G is singly-periodic, then M must have Scherk-type ends but we still do not know if the surface must be a Scherk singly-periodic minimal surface. These considerations motivate the following conjecture.

CONJECTURE 15.17 (Negative Curvature Conjecture, Meeks, Pérez, Ros). *If $M \in \mathcal{C}$ has negative curvature, then M is a catenoid, a helicoid or one of the singly or doubly-periodic Scherk minimal surfaces.*

We end this section of conjectures about H-surfaces in $\mathbb{R}^3$ by reminding the reader the already stated Conjecture 4.3 about properness of complete H-surfaces in $\mathbb{R}^3$ with finite genus.

15.2. Open problems in homogeneous 3-manifolds. In all of the conjectures below, X will denote a simply-connected, 3-dimensional metric Lie group.

CONJECTURE 15.18 (Isoperimetric Domains Conjecture). *Let X denote a metric Lie group diffeomorphic to $\mathbb{R}^3$. Then:*

1. *Isoperimetric domains (resp. surfaces) in X are topological balls (resp. spheres).*
2. *Immersed H-spheres in X are embedded, and the balls that they bound are isoperimetric domains.*

3. *For each fixed volume V_0, solutions to the isoperimetric problem in X for volume V_0 are unique up to left translations in X.*

In reference to the following open problems and conjectures, the reader should note that Meeks, Mira, Pérez and Ros are in the final stages of completing paper [**108**] that solves some of them; this work should give complete solutions to Conjectures 15.19 and 15.21 below. Their claimed results would also demonstrate that every H-sphere in X has index one (see the first statement of Conjecture 15.20). In [**110**], in the case that X is diffeomorphic to $\mathbb{R}^3$, it is shown that as the volumes of isoperimetric domains in X go to infinity, their radii[11] go to infinity and the mean curvatures of their boundaries converge to the critical mean curvature $H(X)$ of X (introduced in Definition 13.7). We expect that by the time this survey is published, [**108**] will be available and consequentially, some parts of this section on open problems should be updated by the reader to include these new results.

Conjecture 15.19 (Hopf Uniqueness Conjecture, Meeks-Mira-Pérez-Ros). *For every $H \geq 0$, any two H-spheres immersed in X differ by a left translation of X.*

It is easy to see that the index of an H-sphere S_H immersed in X is at least one; indeed, if $\{F_1, F_2, F_3\}$ denotes a basis of right invariant vector fields of X (that are Killing vector fields for the left invariant metric of X), then the functions $u_i = \langle F_i, N\rangle$, $i = 1, 2, 3$, are Jacobi functions on S_H (see Definition 2.20, here N is a unit normal vector field to S_H). Since right invariant vector fields on X are identically zero or never zero and spheres do not admit a nowhere zero tangent vector field, then the functions u_1, u_2, u_3 are linearly independent. Hence, 0 is an eigenvalue of the Jacobi operator of S_H of multiplicity at least three. As the first eigenvalue is simple, then 0 is not the first eigenvalue of the Jacobi operator and thus, the index of S_H is at least one. Moreover, if the index of S_H is exactly one, then it follows from Theorem 3.4 in Cheng [**26**] (see also [**48, 183**]) that the nullity of S_H is three. Finally, recall that every weakly stable compact H-surface has index at most one (two eigenfunctions associated to different negative eigenvalues are L^2-orthogonal, and thus produce a linear combination with zero mean, that contradicts weak stability). Therefore, a weakly stable H-sphere in X has index one and nullity three. The next conjecture claims that this index-nullity property does not need the hypothesis on weak stability, and that weak stability holds whenever X is non-compact.

Conjecture 15.20 (Index-one Conjecture, Meeks-Mira-Pérez-Ros). *Every H-sphere in X has index one. Furthermore, when X is diffeomorphic to $\mathbb{R}^3$, then every H-sphere in X is weakly stable.*

Note that by Theorem 13.3, the first statement in Conjecture 15.20 holds in the case X is SU(2) with a left invariant metric. Also note that

[11]The radius of a compact Riemannian manifold M with boundary is the maximum distance of points in M to its boundary.

the hypothesis that X is diffeomorphic to $\mathbb{R}^3$ in the second statement of Conjecture 15.20 is necessary since the second statement fails to hold in certain Berger spheres, see Torralbo and Urbano [**201**]. By Theorem 4.1 in [**109**], the validity of the first statement in Conjecture 15.20 implies Conjecture 15.19 holds as well.

Hopf [**72**] proved that the moduli space of non-congruent H-spheres in $\mathbb{R}^3$ is the interval $(0, \infty)$ (parameterized by their mean curvatures H) and all of these H-spheres are embedded and stable, hence of index one; these results and arguments of Hopf readily extend to the case of $\mathbb{H}^3$ with the interval being $(1, \infty)$ and $\mathbb{S}^3$ with interval $[0, \infty)$, both $\mathbb{H}^3$ and $\mathbb{S}^3$ endowed with their standard metrics; see Chern [**28**]. By Theorem 13.3, if X is a metric Lie group diffeomorphic to $\mathbb{S}^3$, then the moduli space of non-congruent H-spheres in X is the interval $[0, \infty)$, again parameterized by their mean curvatures H. However, Torralbo [**200**] proved that some H-spheres fail to be embedded in certain Berger spheres. These results motivate the next two conjectures.

Conjecture 15.21 (Hopf Moduli Space Conjecture, Meeks-Mira-Pérez-Ros). *When X is diffeomorphic to $\mathbb{R}^3$, then the moduli space of non-congruent H-spheres in X is the interval $(H(X), \infty)$, which is parameterized by their mean curvatures H. In particular, every H-sphere in X is Alexandrov embedded and $H(X)$ is the infimum of the mean curvatures of H-spheres in X.*

The results of Abresch and Rosenberg [**1, 2**] and previous classification results for rotationally symmetric H-spheres demonstrate that Conjecture 15.21 holds when X is some $\mathbb{E}(\kappa, \tau)$-space (see e.g., [**45**] for a description of these spaces). Work of Daniel and Mira [**48**] and of Meeks [**107**] imply that Conjectures 15.18, 15.19, 15.20 and 15.21 hold for Sol_3 with its standard metric.

The next conjecture is known to hold in the flat $\mathbb{R}^3$ as proved by Alexandrov [**3**] and subsequently extended to $\mathbb{H}^3$ and to a hemisphere of $\mathbb{S}^3$.

Conjecture 15.22 (Alexandrov Uniqueness Conjecture). *If X is diffeomorphic to $\mathbb{R}^3$, then the only compact, Alexandrov embedded H-surfaces in X are topologically spheres.*

In the case there exist two orthogonal foliations of X by planes of reflectional symmetry, as is the case of Sol_3 with its standard metric, then using the Alexandrov reflection method, the last conjecture is known to hold; see [**48**] for details in the special case of the standard metric on Sol_3.

If Conjecture 15.22 holds, then the unique compact H-surfaces which bound regions in X are constant mean curvature spheres. In particular, one would have the validity of items 1 and 3 of Conjecture 15.18.

Although we do not state it as a conjecture, it is generally believed that for any value of $H > H(X)$ and $g \in \mathbb{N}$, there exist compact, genus-g, immersed, non-Alexandrov embedded H-surfaces in X, as is the case in classical $\mathbb{R}^3$ setting (Wente [**205**] and Kapouleas [**78**]).

CONJECTURE 15.23 (Stability Conjecture for SU(2), Meeks-Pérez-Ros). *If X is diffeomorphic to $\mathbb{S}^3$, then X contains no strongly stable (the 2-sided cover admits a positive Jacobi function) complete H-surfaces.*

Conjecture 15.23 holds when the metric Lie group X is in one of the following two cases:

- X is a Berger sphere with non-negative scalar curvature (see item (5) of Corollary 9.6 in Meeks, Pérez and Ros [**130**]).
- X is SU(2) endowed with a left invariant metric of positive scalar curvature (by item (1) of Theorem 2.13 in [**130**], a complete stable H-surface Σ in X must be compact, in fact must be topologically a two-sphere or a projective plane; hence one could find a right invariant Killing field on X which is not tangent to Σ at some point of Σ, thereby inducing a Jacobi function which changes sign on Σ, a contradiction).

It is also proved in [**130**] that if Y is a 3-sphere with a Riemannian metric (not necessarily a left invariant metric) such that it admits no stable complete minimal surfaces, then for each integer $g \in \mathbb{N} \cup \{0\}$, the space of compact embedded minimal surfaces of genus g in Y is compact, a result which is known to hold for Riemannian metrics on $\mathbb{S}^3$ with positive Ricci curvature (Choi and Schoen [**30**]).

CONJECTURE 15.24 (Stability Conjecture, Meeks-Mira-Pérez-Ros). *Suppose X is diffeomorphic to $\mathbb{R}^3$. Then*

$$H(X) = \sup\{\textit{mean curvatures of complete stable } H\textit{-surfaces in } X\}. \tag{39}$$

Regarding Conjecture 15.24, define $\widehat{H}(X)$ to be the supremum in the right-hand-side of (39). By Theorem 1.5 in [**110**], there exists a properly embedded, complete stable $H(X)$-surface in X that is part of an $H(X)$-foliation of X. Thus, $H(X) \leq \widehat{H}(X)$.

Remember from the discussion after Definition 13.7 that the main difficulty in completing the proof of the generalization of Theorem 13.3 to the case that X is diffeomorphic to $\mathbb{R}^3$, is to obtain the area estimates $(\star)$ for index-one H-spheres in X. We next explain why the validity of Conjecture 15.24 would imply that the area estimates $(\star)$ hold. To see this, consider a sequence of index-one spheres S_{H_n} immersed in X with $H_n \searrow H_\infty \geq 0$ and with areas diverging to infinity. In [**109**] it is proved that one can produce an appropriate limit of left translations of S_{H_n} which is a stable H_∞-surface in X. Therefore, $H_\infty \leq \widehat{H}(X)$. As by definition $H(X) \leq H_n$ for all $n \in \mathbb{N}$, then $H(X) \leq H_\infty$. As $H(X) = \widehat{H}(X)$ because we are assuming the validity of Conjecture 15.24, then we conclude that

$H_\infty = H(X)$, which proves the area estimates ($\star$). This argument also shows that the validity of Conjecture 15.24 would imply that both Conjecture 15.19 and the first statement in Conjecture 15.20 hold.

CONJECTURE 15.25 (CMC Product Foliation Conjecture, Meeks-Mira-Pérez-Ros).

(1) If X is diffeomorphic to $\mathbb{R}^3$, then given $p \in X$ there exists a smooth CMC product foliation of $X - \{p\}$ by spheres.
(2) Let $\mathcal{F}$ be a CMC foliation of X, i.e., a foliation all whose leaves have constant mean curvature (possibly varying from leaf to leaf). Then $\mathcal{F}$ is a product foliation by topological planes with absolute mean curvature function bounded from above by $H(X)$.

Since spheres of radius R in $\mathbb{R}^3$ or in $\mathbb{H}^3$ have constant mean curvature, item (1) of the above conjecture holds in these spaces. In fact it can be shown that the conjecture holds if the isometry group of X is at least four-dimensional.

Regarding item (2) of Conjecture 15.25, we remark that the existence of a CMC foliation in X implies that X is diffeomorphic to $\mathbb{R}^3$. To see this, we argue by contradiction: suppose that $\mathcal{F}$ is a CMC foliation of a metric Lie group diffeomorphic to $\mathbb{S}^3$. Novikov [**163**] proved that any foliation of $\mathbb{S}^3$ by surfaces has a Reeb component C, which is topologically a solid doughnut with a boundary torus leaf ∂C and the other leaves of $\mathcal{F}$ in C all have ∂C as their limits sets. Hence, all of leaves of $\mathcal{F}$ in C have the same mean curvature as ∂C. By the Stable Limit Leaf Theorem for H-laminations, ∂C is stable. But an embedded compact, two-sided H-surface in SU(2) is never stable, since some right invariant Killing field induces a Jacobi function which changes sign on the surface.

Suppose for the moment that item (1) in Conjecture 15.25 holds and we will point out some important consequences. Suppose $\mathcal{F}$ is a smooth CMC product foliation of $X - \{p\}$ by spheres, p being a point in X. Parameterize the space of leaves of $\mathcal{F}$ by their mean curvature; this can be done by the maximum principle for H-surfaces, which shows that the spheres in $\mathcal{F}$ decrease their positive mean curvatures at the same time that the volume of the enclosed balls by these spheres increases. Thus, the mean curvature parameter for the leaves of $\mathcal{F}$ decreases from ∞ (at p) to some value $H_0 \geq 0$. We claim that

$H_0 = H(X)$ and every compact H-surface in X satisfies $H > H(X)$.

To see the claim, we argue by contradiction. Suppose that there exists an immersed closed surface M in X such that the maximum value of the absolute mean curvature function of M is less than or equal to H_0. Since M is compact, then M is contained in the ball enclosed by some leaf Σ of $\mathcal{F}$. By left translating M until touching Σ at a first time, we obtain a contradiction to the usual comparison principle for the mean curvature, which finishes the proof of the claim. With this property in mind, we now list some consequences of item (1) in Conjecture 15.25.

(1) All leaves of $\mathcal{F}$ have index one. This is because the leaves of $\mathcal{F}$ bounding balls of small volume have this property and as the volume increases, the multiplicity of zero as an eigenvalue of the Jacobi operator of the corresponding boundary sphere cannot exceed three by Cheng's theorem [**26**].
(2) All leaves of $\mathcal{F}$ are weakly stable. To see this, note that every function ϕ in the nullity of a leaf Σ of $\mathcal{F}$ is induced by a right invariant Killing field on X (this is explained in the paragraph just before Conjecture 15.20), and hence, $\int_\Sigma \phi = 0$ by the Divergence Theorem applied to the ball enclosed by Σ. In this situation, Koiso [**84**] proved that the weak stability of Σ is characterized by the non-negativity of the integral $\int_\Sigma u$, where u is any smooth function on Σ such that $Lu = 1$ on Σ (see also Souam [**194**]). Since the leaves of $\mathcal{F}$ can be parameterized by their mean curvatures, the corresponding normal part u of the associated variational field satisfies $u > 0$ on Σ, $Lu = 1$ and $\int_\Sigma u > 0$. Therefore, Σ is weakly stable.
(3) The leaves of $\mathcal{F}$ are the unique H-spheres in X (up to left translations), by Theorem 13.5.

If additionally the Alexandrov Uniqueness Conjecture 15.22 holds, then the constant mean curvature spheres in $\mathcal{F}$ are the unique (up to left translations) compact H-surfaces in X which bound regions. As explained in the second paragraph just after Conjecture 15.22, this would also imply that the leaves of $\mathcal{F}$ are the unique (up to left translations) solutions to the isoperimetric problem in X.

The next conjecture is motivated by the isoperimetric inequality of White [**206**].

Conjecture 15.26 (Isoperimetric Inequality Conjecture, Meeks-Mira-Pérez-Ros). *Suppose that X is diffeomorphic to $\mathbb{R}^3$. Given any $\varepsilon > 0$ and $L_0 > 0$, there exists $C(\varepsilon, L_0)$ such that for any compact immersed surface Σ in X with connected boundary of length at most L_0 and which is minimal or has absolute mean curvature function bounded from above by* $\mathrm{Ch}(X) - \varepsilon$, *then*

$$\mathrm{Area}(\Sigma) \leq C(\varepsilon, L_0).$$

The next conjecture exemplifies another aspect of the special role that the critical mean curvature $H(X)$ of X might play in the geometry of H-surfaces in X.

Conjecture 15.27 (Stability Conjecture, Meeks-Mira-Pérez-Ros). *A complete, stable H-surface Σ in X with $H = H(X)$ is an entire graph with respect to some Killing field V, i.e., every integral curve of V intersects exactly once to Σ (transversely). In particular, if $H(X) = 0$, then any* *complete, stable minimal surface* Σ *in* X *is a leaf of a minimal foliation of* X *and so* Σ *is actually homologically area-minimizing in* X*.*

The previous conjecture is closely related to the next conjecture, which in turn is closely tied to recent work of Daniel, Meeks and Rosenberg [**46, 47**] on halfspace-type theorems in simply-connected, 3-dimensional metric semidirect products.

CONJECTURE 15.28 (Strong-Halfspace Conjecture in Nil_3, Daniel-Meeks-Rosenberg). *A complete, stable minimal surface in* Nil_3 *is either an entire graph with respect to the Riemannian submersion* $\Pi\colon \mathrm{Nil}_3 \to \mathbb{R}^2$ *or a vertical plane* $\Pi^{-1}(l)$*, where* l *is a line in* $\mathbb{R}^2$*. In particular, by the results in* [**47**]*, any two properly immersed disjoint minimal surfaces in* Nil_3 *are parallel vertical planes or they are entire graphs* F_1, F_2 *over* $\mathbb{R}^2$*, where* F_2 *is a vertical translation of* F_1*.*

CONJECTURE 15.29 (Positive Injectivity Radius, Meeks-Pérez-Tinaglia). *A complete embedded* H*-surface of finite topology in* X *has positive injectivity radius.*

Conjecture 15.29 is motivated by the partial result of Meeks and Pérez [**114**] that the injectivity radius of a complete, embedded *minimal* surface of finite topology in a homogeneous 3-manifold is positive (hence Conjecture 15.29 holds for $H = 0$). A related result of Meeks and Peréz [**114**] when $H = 0$ and of Meeks and Tinaglia (unpublished) when $H > 0$, is that if Y is a complete locally homogeneous 3-manifold with positive injectivity radius and Σ is a complete embedded H-surface in Y with finite topology, then the injectivity radius function of Σ is bounded away from zero on compact domains in Y. Meeks and Tinaglia (unpublished) have also shown that Conjecture 15.29 holds for complete embedded H-surfaces of finite topology in metric Lie groups X with four or six-dimensional isometry group.

CONJECTURE 15.30 (Bounded Curvature Conjecture, Meeks-Pérez-Tinaglia). *A complete embedded* H*-surface of finite topology in* X *with* $H > 0$ *has bounded second fundamental form.*

The previous two conjectures are related as follows. Curvature estimates of Meeks and Tinaglia [**145**] for embedded H-disks imply that every complete embedded H-surface with $H > 0$ in a homogeneously regular 3-manifold has bounded second fundamental form if and only if it has positive injectivity radius.

CONJECTURE 15.31 (Calabi-Yau Properness Problem, Meeks-Pérez-Tinaglia). *A complete, connected, embedded* H*-surface of positive injectivity radius in* X *with* $H \geq H(X)$ *is always proper.*

In the classical setting of $X = \mathbb{R}^3$, where $H(X) = 0$, Conjecture 15.31 was proved by Meeks and Rosenberg [**139**] for the case $H = 0$. This result was based on work of Colding and Minicozzi [**38**] who demonstrated that complete embedded minimal surfaces in $\mathbb{R}^3$ with finite topology are proper, thereby proving what is usually referred to as the *classical embedded Calabi-Yau problem* for finite topology minimal surfaces. Recently, Meeks and Tinaglia [**143**] proved Conjecture 15.31 in the case $X = \mathbb{R}^3$ and $H > 0$, which completes the proof of the conjecture in the classical setting.

As we have already mentioned, Meeks and Pérez [**114**] have shown that every complete embedded minimal surface M of finite topology in X has positive injectivity radius; hence M would be proper whenever $H(X) = 0$ and Conjecture 15.31 holds for X. Meeks and Tinaglia [**144**] have shown that any complete embedded H-surface M in a complete 3-manifold Y with constant sectional curvature -1 is proper provided that $H \geq 1$ and M has injectivity radius function bounded away from zero on compact domains of in Y; they also proved that any complete, embedded, finite topology H-surface in such a Y has bounded second fundamental form. In particular, for $X = \mathbb{H}^3$ with its usual metric, an annular end of any complete, embedded, finite topology H-surface in X with $H \geq H(X) = 1$ is asymptotic to an annulus of revolution by the classical results of Korevaar, Kusner, Meeks and Solomon [**85**] when $H > 1$ and of Collin, Hauswirth and Rosenberg [**41**] when $H = 1$.

The next conjecture is motivated by the classical results of Meeks and Yau [**154**] and of Frohman and Meeks [**57**] on the topological uniqueness of minimal surfaces in $\mathbb{R}^3$ and partial unpublished results by Meeks.

Conjecture 15.32 (Topological Uniqueness Conjecture, Meeks). *If M_1, M_2 are two diffeomorphic, connected, complete embedded H-surfaces of finite topology in X with $H = H(X)$, then there exists a diffeomorphism $f\colon X \to X$ such that $f(M_1) = M_2$.*

We recall that Lawson [**91**] proved a beautiful unknottedness result for minimal surfaces in $\mathbb{S}^3$ equipped with a metric of positive Ricci curvature. He demonstrated that whenever M_1, M_2 are compact, embedded, diffeomorphic minimal surfaces in such a Riemannian 3-sphere, then M_1 and M_2 are ambiently isotopic. His result was generalized by Meeks, Simon and Yau [**141**] to the case of metrics of non-negative scalar curvature on $\mathbb{S}^3$. Meeks and Pérez proved the above conjecture in the case that X is diffeomorphic to $\mathbb{S}^3$; see Corollary 4.19 in [**117**].

The next conjecture is motivated by the classical case of $X = \mathbb{R}^3$, where it was proved by Meeks [**102**], and in the case of $X = \mathbb{H}^3$ with its standard constant -1 curvature metric, where it was proved by Meeks and Tinaglia [**144**].

Conjecture 15.33 (One-end / Two-ends Conjecture, Meeks-Tinaglia). *Suppose that M is a connected, non-compact, properly embedded H-surface of finite topology in X with $H > H(X)$. Then:*

(1) M has more than one end.
(2) If M has two ends, then M is an annulus.

The previous conjecture also motivates the next one.

Conjecture 15.34 (Topological Existence Conjecture, Meeks). *Suppose X is diffeomorphic to $\mathbb{R}^3$. Then for every $H > H(X)$, X admits connected properly embedded H-surfaces of every possible orientable topology, except for connected finite genus surfaces with one end or connected finite positive genus surfaces with 2 ends which it never admits.*

Conjecture 15.34 is probably known in the classical settings of $X = \mathbb{R}^3$ and $\mathbb{H}^3$ but the authors do not have a reference of this result for either of these two ambient spaces. For the non-existence results alluded to in this conjecture in these classical settings see [**85, 86, 102, 144**]. The existence part of the conjecture should follow from gluing constructions applied to collections of non-transversely intersecting embedded H-spheres appropriately placed in X, as in the constructions of Kapouleas [**77**] in the case of $X = \mathbb{R}^3$.

We end our discussion of open problems in X with the following generalization of the classical properly embedded Calabi-Yau problem in $\mathbb{R}^3$; see item 3 of Conjecture 15.10. Variations of this conjecture can be attributed to many people but in the formulation below, it is primarily due to Martín, Meeks, Nadirashvili, Pérez and Ros and their related work.

Conjecture 15.35 (Embedded Calabi-Yau Problem). *Suppose X is diffeomorphic to $\mathbb{R}^3$ and Σ is a connected, non-compact surface. A necessary and sufficient condition for Σ to be diffeomorphic to some complete, embedded bounded minimal surface in X is that every end of Σ has infinite genus.*

In the case of $X = \mathbb{R}^3$ with its usual metric, the non-existence implication in the last conjecture was proved by Colding and Minicozzi [**38**] for complete embedded minimal surfaces with an annular end; also see the related more general results of Meeks and Rosenberg [**139**] and of Meeks, Peréz and Ros [**123**].

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