Topology of the space of cycles and existence of minimal varieties

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Abstract. In this article we survey what is known about the existence of minimal varieties of dimension $l \geq 2$ in compact Riemannian manifolds. We describe how min-max methods can be used in conjunction with the nontrivial topology of the space of cycles. In the final section, we propose some open questions in the subject.

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1. Introduction

Let (M^{n+1}, g) be a closed Riemannian manifold of dimension $(n+1)$. We are interested in studying the existence of minimal varieties in M, of a given dimension $1 \leq l \leq n$, in connection with nontrivial topology of the space of l-dimensional cycles.

The case $l = 1$ goes back to the work of Birkhoff [**6**], who used min-max methods and the notion of sweepouts to construct a smooth closed geodesic in every Riemannian two-sphere. This answered affirmatively a question posed by Poincaré [**28**]. Later Lusternik and Schnirelmann [**21**] proved the existence of at least three simple closed geodesics in Riemannian two-spheres,

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and the existence of infinitely many immersed ones follows from the works of Franks [**14**] and Bangert [**5**]. We refer the reader to the article of Hingston [**18**] for more on the $l = 1$ case.

We define the space of cycles by means of Geometric Measure Theory. We can take coefficients in $\mathbb{Z}$ (oriented cycles) or in $\mathbb{Z}_2$ (unoriented cycles). The choice of integer coefficients leads to the notion of integral current, while the choice of $\mathbb{Z}_2$ coefficients leads to the notion of modulo 2 flat chains. For reasons that will become clear later we focus in the $\mathbb{Z}_2$ case.

We suppose M is isometrically embedded in some Euclidean space $\mathbb{R}^N$ and consider the space $\mathcal{I}_l(M;\mathbb{Z}_2)$ of l-dimensional modulo 2 flat chains in $\mathbb{R}^N$ with support contained in M (see [**12**, 4.2.26] for more details). A typical element of this space is a formal finite sum $c = \sum_{i=1}^m S_i$ of compact (unoriented) C^1 submanifolds S_i contained in M. There is a boundary operator ∂, taking values in the space of mod 2 $(l-1)$-chains, that in the above case gives $\partial c = \sum_{i=1}^m \partial S_i$. Of course $\partial \circ \partial = 0$. Each element $T \in \mathcal{I}_l(M;\mathbb{Z}_2)$ has a mass $M(T)$, that measures the l-dimensional area, and the natural topology is induced by the flat distance:

$$\mathcal{F}(T_1, T_2) = \inf\{M(S) + M(U) : T_1 - T_2 = S + \partial U, S \in \mathcal{I}_l, U \in \mathcal{I}_{l+1}\}.$$

Our space of l-cycles is then defined to be

$$\mathcal{Z}_l(M;\mathbb{Z}_2) = \{T \in \mathcal{I}_l(M;\mathbb{Z}_2) : \partial T = 0\},$$

endowed with the flat topology.

A fundamental result ([**2**]) is the following isoperimetric inequality: there exist positive constants $\nu \leq 1$ and c, depending only on M, such that for any cycle $T \in \mathcal{Z}_l(M;\mathbb{Z}_2)$ with mass $M(T) < \nu$, we can find $U \in \mathcal{I}_{l+1}(M, \mathbb{Z}_2)$ such that $\partial U = T$ and

$$M(U) \leq cM(T)^{\frac{l+1}{l}}.$$

In particular, it follows from the definition of the flat distance that if T_1 and T_2 are cycles with $\mathcal{F}(T_1, T_2) < \nu$ then there exists an $(l+1)$-chain U such that $\partial U = T_1 - T_2$ and $M(U) \leq \rho \mathcal{F}(T_1, T_2)$ for some $\rho \geq 1$ depending only on M. Intuitively speaking, two l-cycles are very close in the flat topology if together they are the boundary of a chain with very small $(l+1)$-area.

The first step in the direction of a Morse theory was done by Almgren in 1960 [**2**]. He computed all homotopy groups of the space of cycles. He considered coefficients in $\mathbb{Z}$ but the arguments extend immediately to the case of coefficients modulo 2. Notice first that any cycle $T \in \mathcal{Z}_l(M;\mathbb{Z}_2)$ induces a well-defined element $\Lambda_0(T) = [T] \in H_l(M, \mathbb{Z}_2)$. By representing elements of $H_l(M, \mathbb{Z}_2)$ by polyhedral chains it follows that we have an isomorphism $\Lambda_0 : \pi_0(\mathcal{Z}_l(M;\mathbb{Z}_2)) \to H_l(M, \mathbb{Z}_2)$. Almgren proved:

THEOREM 1.1 (Almgren's Isomorphism Theorem, [**2**]). *For any $k \geq 1$, there exists a canonical isomorphism*

$$\Lambda_k : \pi_k(\mathcal{Z}_l(M;\mathbb{Z}_2), 0) \to H_{k+l}(M, \mathbb{Z}_2).$$

Almgren's Isomorphism Theorem implies the space of l-dimensional mod 2 cycles has nontrivial topology:

COROLLARY 1.2. $\pi_{n+1-l}(\mathcal{Z}_l(M, \mathbb{Z}_2)) = \mathbb{Z}_2$.

This follows immediately from $H_{n+1}(M^{n+1}, \mathbb{Z}_2) = \mathbb{Z}_2$, and suggests that critical points for the area functional can be found by Morse theory.

In Section 2, we describe the isomorphism of Almgren. In Section 3, we discuss the Almgren-Pitts min-max theory for the area functional and the basic existence results. In Section 4, we consider the case of hypercycles and define the notion of k-sweepout. We discuss work of Gromov and Guth on the sublinear growth of the corresponding sequence of min-max values $\{\omega_k\}$. In Section 5, we describe our proof of Yau's conjecture (about existence of infinitely many minimal surfaces in dimension three) for manifolds with positive Ricci curvature. In Section 6, we propose a number of open questions related to min-max theory.

2. Almgren's Isomorphism Theorem

Let us describe the isomorphism first when $k = 1$. Let $\phi : [0,1] \to \mathcal{Z}_l(M; \mathbb{Z}_2)$ be continuous in the flat topology with $\phi(0) = \phi(1) = 0$. For each $j \in \mathbb{N}$, $I(1,j)$ denotes the cube complex on $I^1 = [0,1]$ whose 1-cells and 0-cells (those are also called vertices) are, respectively,

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \ldots, [1 - 3^{-j}, 1] \quad \text{and} \quad [0], [3^{-j}], \ldots, [1 - 3^{-j}], [1].$$

Given $0 < \varepsilon < \nu/(4\rho)$ very small, if j is sufficiently large we have that

$$\mathcal{F}(\phi(x), \phi(y)) < \varepsilon$$

for every $x, y \in [i \cdot 3^{-j}, (i+1) \cdot 3^{-j}]$, $i = 0, \ldots, 3^j - 1$. In particular there exist $U_i \in \mathcal{I}_{l+1}(M; \mathbb{Z}_2)$ such that

$$\partial U_i = \phi((i+1) \cdot 3^{-j}) - \phi(i \cdot 3^{-j})$$

and $M(U_i) \leq \rho\varepsilon$. Note that $\partial(\sum_i U_i) = 0$, hence we can define

$$\Lambda_1([\phi]) = \Lambda_0(\sum_i U_i) \in H_{l+1}(M, \mathbb{Z}_2).$$

We need to check that this is well-defined. Notice that if U_i' are different choices in $\mathcal{I}_{l+1}(M; \mathbb{Z}_2)$ with $\partial U_i' = \phi((i+1) \cdot 3^{-j}) - \phi(i \cdot 3^{-j})$ and $M(U_i') \leq \rho\varepsilon$, we have that $\partial(U_i - U_i') = 0$ and $M(U_i - U_i') \leq 2\rho\varepsilon < \nu$. Hence $U_i - U_i' = \partial A_i$ for some $(l+2)$-chain A_i. This implies $\left[\sum_i U_i\right] = \left[\sum_i U_i'\right] \in H_{l+1}(M, \mathbb{Z}_2)$. Hence the construction does not depend on the choices of the U_i. In order to check that it does not depend on the subdivision, we subdivide the interval $[i \cdot 3^{-j}, (i+1) \cdot 3^{-j}]$ into three equal parts and choose $U_{i,1}, U_{i,2}, U_{i,3} \in \mathcal{I}_{l+1}(M; \mathbb{Z}_2)$ with $\partial U_{i,q} = \phi(i \cdot 3^{-j} + q \cdot 3^{-j-1}) - \phi(i \cdot 3^{-j} + (q-1) \cdot 3^{-j-1})$ and $M(U_{i,q}) \leq \rho\varepsilon$. We have that $\partial(U_i - (U_{i,1} + U_{i,2} + U_{i,3})) = 0$ and $M(U_i - (U_{i,1} + U_{i,2} + U_{i,3})) \leq 4\rho\varepsilon < \nu$. Hence there exists some $(l+2)$-chain B_i such that $\partial B_i = U_i - (U_{i,1} + U_{i,2} + U_{i,3})$. This implies, as before, that $\left[\sum_i U_i\right] = \left[\sum_i (U_{i,1} + U_{i,2} + U_{i,3})\right] \in H_{l+1}(M, \mathbb{Z}_2)$.

The construction of Λ_k is analogous. We denote by $I(k,j)$ the cell complex on the unit cube I^k:

$$I(k,j) = I(1,j) \otimes \ldots \otimes I(1,j) \quad (k \text{ times}).$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_k$ is a q-cell of $I(k,j)$ if and only if α_i is a cell of $I(1,j)$ for each i, and $\sum_{i=1}^k \dim(\alpha_i) = q$. We often abuse notation by identifying a q-cell α with its support: $\alpha_1 \times \cdots \times \alpha_k \subset I^k$.

Given $\phi : I^k \to \mathcal{Z}_l(M;\mathbb{Z}_2)$ continuous in the flat topology with $\phi(\partial I^k) = 0$, we can consider subdivisions $I(k,j)$ for large j so that $\mathcal{F}(\phi(x),\phi(y))$ is sufficiently small whenever x,y belong to the same k-cell of $I(k,j)$. This allows us to construct a chain map

$$\hat{\phi} : I(k,j) \to \mathcal{I}_*(M;\mathbb{Z}_2)$$

with the properties:

- $\hat{\phi}(x) = \phi(x)$ if $x \in I(k,j)_0$,
- $\hat{\phi}(\partial\sigma) = \partial\hat{\phi}(\sigma)$ for every cell $\sigma \in I(k,j)$,
- $M(\hat{\phi}(\sigma))$ is small for every cell $\sigma \in I(k,j)$ with $\dim(\sigma) \geq 1$.

Then

$$\Lambda_k([\phi]) = \Lambda_0\big(\sum_{\sigma \in I(k,j)_k} \hat{\phi}(\sigma)\big) \in H_{k+l}(M,\mathbb{Z}_2).$$

The idea to prove surjectivity of Λ_k is to perform a slicing argument. If A denotes a small tubular neighborhood of M in $\mathbb{R}^N$, the nearest-point projection $\pi : A \to M$ is a strong deformation retraction. We can assume, without loss of generality, that $A \subset [0,\lambda]^N \subset \mathbb{R}^N$ for some $\lambda > 0$, and represent a given element of $H_{k+l}(M,\mathbb{Z}_2)$ by a polyhedral chain T in $A \subset [0,\lambda]^N$. The desired map ϕ in $\pi_k(\mathcal{Z}_l(M;\mathbb{Z}_2),0)$ can be constructed inductively by first setting

$$\psi(t_1,0,\ldots,0) = \partial\big(T \cap \{x_1 < \lambda t_1\}\big),$$
$$\psi(t_1,t_2,0,\ldots,0) = \partial\big(\psi(t_1,0,\ldots,0) \cap \{x_2 < \lambda t_2\}\big),$$
$$\cdots,$$
$$\psi(t_1,\ldots,t_k) = \partial\big(\psi(t_1,t_2,\ldots,t_{k-1},0) \cap \{x_k < \lambda t_k\}\big),$$

after perturbing slightly T so that no face is parallel to a coordinate vector of $\mathbb{R}^N$, and defining $\phi = \pi_\# \circ \psi$.

The proof of injectivity is more intricate. It uses the interpolation technique of Almgren ([**2**]). The basic result is:

THEOREM 2.1. *Let $\hat{\phi} : I(k,0) \to \mathcal{I}_*(M;\mathbb{Z}_2)$ be a chain map of degree l, i.e., $\hat{\phi}(x) \in I_l(M,\mathbb{Z}_2)$ for every $x \in I(k,0)_0$. There exists a map $\phi : I^k \to \mathcal{I}_*(M;\mathbb{Z}_2)$, continuous in the flat topology, such that*

(i) *$\phi(x) = \hat{\phi}(x)$ for all $x \in I(k,0)_0$;*

(ii) *for every $\alpha \in I(k,0)_p$, $\phi_{|\alpha}$ depends only on the values assumed by $\hat{\phi}$ on the subfaces of α;*

(iii) *there exists a constant* $C = C(M,k) > 0$ *such that*

$$\sup\{\mathbf{M}(\phi(x)) : x \in I^k\} \leq C \sup_{\alpha \in I(k,0)} \{\mathbf{M}(\hat{\phi}(\alpha))\},$$

$$\sup\{\mathcal{F}(\phi(x), \phi(y)) : x, y \in I^k\} \leq C \sup_{\alpha \in I(k,0), \dim(\alpha) > 0} \{\mathbf{M}(\hat{\phi}(\alpha))\}.$$

We call ϕ the Almgren extension $\mathrm{Alm}(\hat{\phi})$ of $\hat{\phi}$. Given any $\psi : I^k \to \mathcal{Z}_l(M, \mathbb{Z}_2)$, continuous in the flat topology, it follows from the isoperimetric inequality mentioned earlier that if j is sufficiently large, the restriction $\psi : I(k,j)_0 \to \mathcal{Z}_l(M, \mathbb{Z}_2)$ can be extended to a chain map $\hat{\psi}$. The previous theorem implies there is an Almgren extension $\mathrm{Alm}(\hat{\psi}) : I^k \to \mathcal{Z}_l(M, \mathbb{Z}_2)$. The idea to prove injectivity of Λ_k is to show first that if $\psi : I^k \to \mathcal{Z}_l(M, \mathbb{Z}_2)$ is such that $\psi(\partial I^k) = \{0\}$, then ψ is homotopic to $\mathrm{Alm}(\hat{\psi})$ relative to ∂I^k (assuming j sufficiently large). The second step is to prove that if $\Lambda_k([\psi]) = 0$, then $\mathrm{Alm}(\hat{\psi})$ is homotopically trivial. This uses again the Interpolation Theorem 2.1. Notice that the interpolation technique (improved to the mass topology) was an important technical ingredient in the authors proof of the Willmore conjecture [**23**].

3. Min-max theory

In 1965, Almgren devised a min-max theory for the area functional. For a given nontrivial homotopy class Π of the space $\mathcal{Z}_l(M, \mathbb{Z}_2)$, he defined the min-max number

$$L(\Pi) = \inf_{\phi \in \Pi} \sup_{x \in I^k} M(\phi(x)).$$

We say $L(\Pi)$ is the *width* of Π.

Almgren [**3**] proved the following theorem:

Theorem 3.1. *If* Π *is nontrivial, then* $L(\Pi) > 0$. *Moreover, there exists an* l*-dimensional minimal variety (stationary integral varifold of dimension* l*)* V *such that*

$$M(V) = L(\Pi).$$

Note that the mass functional is only lower semicontinuous in the flat topology. This is not a problem in minimization schemes, like in the Plateau problem, but causes serious difficulties if one wants to produce unstable critical points. Almgren introduced the notion of varifolds to deal with this problem. The point is that even though the families are taken to be continuous in the flat topology, the final convergence of a min-max sequence to the minimal variety is in the sense of varifolds so there is no cancellation of mass. The flat cycle/varifold duality is one of the reasons the theory is highly nontrivial.

The following example is quite degenerate but still instructive. Let M be the product manifold $S^n(1) \times S^1(r)$, where r is very large, and let us consider one-parameter families of hypercycles. This means we will apply the min-max theory to the nontrivial element Π of $\pi_1(\mathcal{Z}(M,\mathbb{Z}_2),\{0\})=\mathbb{Z}_2$. A connected closed minimal hypersurface in M must be either one of the leaves $S^n(1)\times\{\theta\}$ or it crosses every leaf, but in this last case it will have very large area by the monotonicity formula. One can argue that an optimal sweepout (a family whose supremum of the areas equals $L(\Pi)$) must be of the form $\phi(t)=\Sigma(0)+\Sigma(t)$, where $\Sigma(s)=S^n(1)\times\{2\pi rs\}$, $s\in[0,1]$. If $t_i\to 0$, the sequence $\phi(t_i)$ is a min-max sequence that converges to 0 in the flat topology and to $2|\Sigma(0)|$ as varifolds. To make this rigorous one has to exclude the possibility of getting a leaf with multiplicity one as the min-max surface. This can be done by noticing that a leaf is not homologous to zero and using results of Zhou [**34**].

By putting together Corollary 1.2 and Theorem 3.1, we get

Corollary 3.2. *There exists at least one l-dimensional minimal variety in M^{n+1} for every $1\le l\le n$.*

Such minimal varieties are smooth almost everywhere, by Allard's regularity theory [**1**]. When the codimension is one Pitts improved the regularity significantly. He used the curvature estimates of Schoen-Simon-Yau [**30**] and of Schoen-Simon [**29**] in high dimensions to prove the following result:

Theorem 3.3 (Pitts, [**27**]). *Suppose $(n+1)\ge 3$. In codimension one ($l=n$), the min-max minimal variety V can be chosen as*

$$V=n_1\cdot\Sigma_1+\cdots+n_p\cdot\Sigma_p$$

where $n_i\in\mathbb{Z}_+$ and $\{\Sigma_i\}$ is a disjoint collection of closed minimal hypersurfaces that are smooth embedded outside a set of codimension 7. In particular, they are smooth if $(n+1)\le 7$.

Remarks: In the case of codimension one, this result can be proven by performing min-max with sweepouts that enjoy better regularity properties. We refer the reader to the papers of Smith [**31**], Colding-De Lellis [**9**] and De Lellis-Tasnady [**11**] for more details. Pitts Theorem is not true when $(n+1)=2$, as the min-max variety can be a geodesic network (stationary graph). In Montezuma [**26**], a modified min-max theory is devised to produce minimal hypersurfaces with intersecting properties.

4. The space of hypercycles

Let us go back to the topology of the space of cycles in the case of hypersurfaces (codimension one). The Isomorphism Theorem implies:

- $\pi_1(\mathcal{Z}_n(M^{n+1},\mathbb{Z}_2),\{0\})=\mathbb{Z}_2$,
- for $k\ge 2$,

$$\pi_k(\mathcal{Z}_n(M^{n+1},\mathbb{Z}_2),\{0\})=H_{k+n}(M^{n+1},\mathbb{Z}_2)=0.$$

These are precisely the homotopy groups of the infinite dimensional real projective space $\mathbb{RP}^\infty$. In fact, one can define a weak homotopy equivalence between $\mathcal{Z}_n(M^{n+1},\mathbb{Z}_2)$ and $\mathbb{RP}^\infty$ in the following way. Let $f:M\to\mathbb{R}$ be a

Morse function, with $f(M) = [0,1]$. Then the sweepout $t \in [0,1] \mapsto \Sigma(t) = \partial(\{x \in M : f(x) < t\})$ generates π_1 of the space of hypercycles. We can extend this map to $\mathbb{RP}^\infty$: let

$$\Phi : \mathbb{RP}^\infty \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$$

be defined by

$$\Phi(a = [a_0 : a_1 : \cdots : a_k : 0 : \cdots : 0 : \cdots]) = \partial(\{x \in M : p_a(f(x)) < 0\}),$$

where $p_a(t) = a_0 + a_1 t + \cdots + a_k t^k$. This map is well-defined and continuous in the flat topology because we are considering coefficients in $\mathbb{Z}_2$.

In general, we consider the following class of maps:

Definition: Let X be a finite-dimensional simplicial complex. A continuous map $\Psi : X \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is called a *k-sweepout* if the induced homomorphism in homology

$$\Psi_* : H_k(X, \mathbb{Z}_2) \to H_k(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$$

is surjective.

This has an equivalent characterization in terms of cohomology. Let $\bar{\lambda} \in H^1(\mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2), \mathbb{Z}_2) = \mathbb{Z}_2$ be the nontrivial element. The map $\Psi : X \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ will be a k-sweepout when the pullback class

$$\lambda = \Psi^*(\bar{\lambda}) \in H^1(X, \mathbb{Z}_2)$$

satisfies

$$\lambda^k \neq 0 \in H^k(X, \mathbb{Z}_2),$$

where $\lambda^k = \lambda \cup \cdots \cup \lambda$ denotes the k-th cup power of λ.

If $\Phi : \mathbb{RP}^\infty \to \mathcal{Z}_n(M^{n+1}, \mathbb{Z}_2)$ is the map defined above, the restriction $\Phi_{|\mathbb{RP}^k}$ is a k-sweepout for every k. Since $\Phi(\mathbb{RP}^1)$ is a one-parameter sweepout of M, i.e., generates π_1 of the space of hypercycles, for any standard $\mathbb{RP}^1 \subset \mathbb{RP}^k$, we conclude that $\lambda = \Phi^*(\bar{\lambda})$ is the generator σ of the first cohomology $H^1(\mathbb{RP}^k, \mathbb{Z}_2) = \mathbb{Z}_2$. Since the cohomology ring of $\mathbb{RP}^k$ is the truncated (in degree k) polynomial ring in σ with $\mathbb{Z}_2$ coefficients, we get that $\lambda^k = \sigma^k \neq 0 \in H^k(\mathbb{RP}^k, \mathbb{Z}_2) = \mathbb{Z}_2$.

By applying the Almgren-Pitts min-max theory to the class $\mathcal{P}_k$ of k-sweepouts, one gets a sequence of numbers that have been investigated earlier by Gromov and Guth:

$$\omega_k(M) = \inf_{\psi \in \mathcal{P}_k} \sup_{x \in \mathrm{dmn}(\psi)} M(\psi(x)),$$

where $\mathrm{dmn}(\psi)$ denotes the domain of ψ.

The following result was proven by Gromov in [**15**, Section 4.2.B], and by Guth in [**17**] via an elegant bend–and–cancel argument. Guth proved this result when the ambient space is a Euclidean ball, but his arguments carry over to the setting of Riemannian manifolds (see [**24**]).

THEOREM 4.1. *There exists a constant $C = C(M) > 0$ so that*

$$\omega_k(M) \leq Ck^{\frac{1}{n+1}}$$

for every $k \in \mathbb{N}$.

Remark: The lower bound $\omega_k(M) \geq C'k^{\frac{1}{n+1}}$ also holds for some constant $C' = C'(M) > 0$ (Gromov [**16**], Guth [**17**]).

In order to have some intuition about the sublinear growth of ω_k, one can consider the following example. We take M to be the standard sphere S^{n+1} and $\mathcal{H}_d$ to be the space of harmonic polynomials in $\mathbb{R}^{n+2}$ of degree less than or equal to d. This space can be identified with a Euclidean space of dimension $D(d) + 1$. Because we are taking coefficients modulo 2, so orientation does not matter, the following map is well-defined

$$\Phi([p]) = \{x \in S^{n+1} : p(x) = 0\},$$

where $[p] \in \mathbb{RP}^{D(d)}$ is the image of the polynomial p under the standard projection $\pi : \mathbb{R}^{D(d)+1} \to \mathbb{RP}^{D(d)}$. Because $M(\Phi([p])) \leq C(n)d$ and $D(d) \approx d^{n+1}$ as $d \to \infty$, we conclude that the estimate of Theorem 4.1 is sharp.

5. Yau's conjecture

In 1982, Yau [**33**] (first problem in the Minimal Surfaces section) proposed the following conjecture:

Conjecture: *Any compact three-manifold admits infinitely many smooth closed immersed minimal surfaces.*

In [**24**], we were able to settle this conjecture for manifolds of positive Ricci curvature. We prove the existence of infinitely many smooth closed embedded minimal hypersurfaces in manifolds with positive Ricci curvature of dimension $(n + 1) \leq 7$. For general manifolds we prove the existence of at least $(n + 1)$ minimal hypersurfaces. The statement can be extended to higher dimensions if we allow singular sets of codimension 7, by using Schoen-Simon regularity theory:

THEOREM 5.1. *Let (M^{n+1}, g) be a compact Riemannian manifold, $(n + 1) \geq 3$. Then either*

(i) *there exists a disjoint collection $\{\Sigma_1, \ldots, \Sigma_{n+1}\}$ of $(n+1)$ connected closed embedded minimal hypersurfaces, smooth outside sets of codimension 7,*
(ii) *or there exist infinitely many connected closed embedded minimal hypersurfaces, smooth outside sets of codimension 7.*

The proof combines Lusternik-Schnirelmann theory [**21**], that we carry to the Almgren-Pitts setting, with counting arguments. The idea is to apply min-max theory to the class of k-sweepouts, for every k, and investigate whether the obtained minimal hypersurfaces are really distinct from each other. If $\omega_k = \omega_{k+1}$ for some k, we can use a topological argument based on the cohomological definion of k-sweepouts and prove that there must be infinitely many minimal surfaces. If the sequence $\{\omega_k\}_k$ is strictly increasing, we get that for some fixed constant $c > 0$,

$$\#\{\omega_j : \omega_j \leq c \cdot m\} \geq m^{n+1}$$

for every m. The basic problem is that one could be getting the same hypersurface with higher and higher multiplicities. In fact, if there exists a collection $\{\Sigma_1, \ldots, \Sigma_{n+1}\}$ like in alternative (i) above, the min-max hypersurfaces achieving $\{\omega_k\}_k$ could be all of the form $m_{1,k}\Sigma_1 + \cdots + m_{n+1,k}\Sigma_{n+1}$, $m_{i,j} \in \mathbb{Z}_+$. This is compatible with Theorem 4.1.

If the Ricci curvature is positive, we need to rule out alternative (i) above. Frankel's theorem [**13**] implies that any two smooth closed minimal hypersurfaces must intersect each other. To deal with the case when there are singularities, we can use the next result. This is because the Almgren-Pitts theory always produces a min-max minimal variety that is almost minimizing in annuli (see definition in [**27**], [**24**]).

Theorem 5.2. *Let (M^{n+1}, g) be a compact Riemannian manifold with positive Ricci curvature. Then any stationary integral n-varifold V in M that is almost minimizing in small annuli must be of the form $V = k \cdot \Sigma$, where $k \in \mathbb{Z}_+$ and Σ is a connected closed smooth embedded minimal hypersurface outside a set of codimension 7.*

Proof. It follows from the partial regularity theory of Schoen and Simon that the support of V is a closed embedded minimal hypersurface Σ, smooth outside a set of codimension 7 (Section 7 of [**29**]). Let Σ_1, Σ_2 be connected components of Σ. We are going to prove that necessarily $\Sigma_1 = \Sigma_2$.

If not, we have $\Sigma_1 \cap \Sigma_2 = \emptyset$. If $T \subset M$ is a closed subset with n-dimensional Hausdorff measure $\mathcal{H}^n(T) < \infty$, we denote by $\mathrm{reg}(T)$ the regular set of T (as a hypersurface), i.e., the set of points $p \in T$ such that for some neighborhood U of p the intersection $T \cap U$ is a smooth embedded hypersurface. The singular set is then defined by $\mathrm{sing}(T) = T \setminus \mathrm{reg}(T)$. We can choose $p_i \in \Sigma_i$ so that $d(p_1, p_2) = d(\Sigma_1, \Sigma_2) = d > 0$. Let $\gamma : [0, d] \to M$ be the minimizing geodesic with $\gamma(0) = p_1$ and $\gamma(d) = p_2$. If $p_1 \in \mathrm{reg}(\Sigma_1)$ and $p_2 \in \mathrm{reg}(\Sigma_2)$, the proof proceeds as in the nonsingular case. By averaging the second variation formula of the energy over parallel variations that are orthogonal to γ', one finds an energy decreasing variation γ_t with $\gamma_t(0) \in \Sigma_1$ and $\gamma_t(d) \in \Sigma_2$, $t \in (-\varepsilon, \varepsilon)$. This contradicts the choice of p_i.

On the other hand, if $p_i \in \mathrm{sing}(\Sigma_i)$ for some $i \in \{1, 2\}$ then the tangent cone of Σ_i at p_i must be a hyperplane with some positive integer multiplicity. This is because $\Sigma_1 \cap B_\varepsilon(\gamma_1(\varepsilon)) = \emptyset$ and $\Sigma_2 \cap B_\varepsilon(\gamma_2(d - \varepsilon)) = \emptyset$, forcing the tangent cones at p_1 and p_2 to be contained in halfspaces. But Schoen-Simon theory (see also Theorem B in [**19**]) implies the multiplicity must be 1, and by Allard p_i is a regular point. Contradiction.

Theorem A in [**19**] implies the regular set of Σ is connected. By the Constancy Theorem we conclude the multiplicity of V must be constant along $\mathrm{reg}(\Sigma)$. The result follows.

□

Hence alternative (i) of Theorem 5.1 cannot happen in positive Ricci curvature, and thus:

THEOREM 5.3. *Let (M^{n+1}, g) be a compact Riemannian manifold, $(n+1) \geq 3$. If $Ric_g > 0$, then there exist infinitely many connected closed embedded minimal hypersurfaces, smooth outside sets of codimension 7.*

Note that if $(n+1) \leq 7$, the min-max hypersurfaces are everywhere smooth. Hence the next result follows from Theorem 5.3:

COROLLARY 5.4. *Let (M^{n+1}, g) be a compact Riemannian manifold, $3 \leq (n+1) \geq 7$. If $Ric_g > 0$, then there exist infinitely many connected closed smooth embedded minimal hypersurfaces.*

6. Future directions

A central open problem in the min-max theory for the area functional is to relate the Morse index of the min-max minimal submanifold with the number of parameters. In general, one should expect that $\mathrm{index}(\Sigma) \leq k$, where k is the number of parameters. This is a subtle question, specially because of the phenomenon of multiplicity.

In [**34**], Zhou proves this for $k = 1$ in the case of compact manifolds M^n of positive Ricci curvature, with $3 \leq n \leq 7$. He is also able to characterize the area and the multiplicity of the min-max hypersurface. This extends results for $n = 3$ of the authors [**22**]. Recently Zhou [**35**] extended his study to arbitrary dimensions, in which case the minimal hypersurface might have singularities. Mazet and Rosenberg [**25**] and Song [**32**] applied min-max methods to study the geometry of the minimal hypersurface of least area.

The regularity of the min-max minimal submanifold in the case of codimension one follows from the works of Pitts [**27**] and Schoen-Simon [**29**]. We think the following partial regularity conjecture is natural in the general high codimension case:

CONJECTURE 6.1. *Suppose $(n+1) \geq 3$. The min-max minimal variety V can be chosen to be smooth outside a set of codimension 2.*

This is motivated by work of Almgren [**4**], that establishes the partial regularity codimension two result for area-minimizing currents of general codimension. Almgren's proof has been improved and simplified recently by De Lellis and Spadaro (see [**10**]). A proof of the above conjecture would likely require new curvature estimates for stable submanifolds.

The proofs of Theorem 5.3 and Corollary 5.4 are by contradiction ([**24**]). Therefore it should be very interesting to have a description of these minimal hypersurfaces. We propose a conjecture:

CONJECTURE 6.2. *For a generic metric g on S^{n+1}, $3 \leq (n+1) \leq 7$, there should be a list $\{\Sigma_k\}$ of smooth closed embedded minimal hypersurfaces such that:*

- $\text{index}(\Sigma_k) = k$,
- $\text{mult}(\Sigma_k) = 1$,
- $\text{area}(\Sigma_k) \geq c \cdot k^{\frac{1}{n+1}}$.

Some progress has been obtained recently by Li and Zhou in [**20**], Chodosh-Ketover-Maximo [**8**] and Carlotto [**7**] where they prove the set of Morse indices of these minimal hypersurfaces must be unbounded.

The previous conjecture should follow from:

CONJECTURE 6.3 (Multiplicity One Conjecture). *For a generic metric g on S^{n+1}, $3 \leq (n+1) \leq 7$, any unstable component of a min-max minimal hypersurface has multiplicity one.*

We believe it should be very interesting to study the existence of minimal varieties in higher codimension as well. It turns out the spaces of cycles have rich topological structure. For instance, one can look at codimension two cycles with integer coefficients in the unit sphere S^{n+1}. Almgren's Isomorphism Theorem gives that

- $\pi_1(\mathcal{Z}_{n-1}(S^{n+1}, \mathbb{Z}), \{0\}) = 0$,
- $\pi_2(\mathcal{Z}_{n-1}(S^{n+1}, \mathbb{Z}), \{0\}) = \mathbb{Z}$,
- $\pi_k(\mathcal{Z}_{n-1}(S^{n+1}, \mathbb{Z}), \{0\}) = 0$ for $k \geq 3$.

Hence the space $\mathcal{Z}_{n-1}(S^{n+1}, \mathbb{Z})$ should be weakly homotopic to the infinite dimensional complex projective space $\mathbb{CP}^\infty$.

It makes sense to perform min-max for the area functional over families that detect the $(2k)$-dimensional homology of $\mathcal{Z}_{n-1}(S^{n+1}, \mathbb{Z})$. The methods of Guth [**17**] to prove sublinear growth of the min-max values require the coefficients to be in $\mathbb{Z}_2$. It should be interesting to study the case of integer coefficients. For instance, in the case of $\mathcal{Z}_1(S^3, \mathbb{Z})$, one can consider multiparameter families of zero sets of complex polynomials in $\mathbb{C}^2$ to justify sublinear growth (Example 1 of Appendix 2 in [**17**]). We point out that, in general, for $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$, the space of l-dimensional cycles $\mathcal{Z}_l(S^{n+1}, G)$ should be weakly homotopic to the Eilenberg-MacLane space $K(G, n+1-l)$.

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