Preface

General relativity (GR) was born in 1915, when Albert Einstein formulated his field equations. They give, to the physical theory of gravitation, a geometric structure, where gravitation acts through curvature. *Mathematical general relativity* finds solutions to those equations and studies their properties. It has been very fruitful, answering crucial questions both within physics and within some fascinating realms of mathematics, and advancing research into important and previously unexplored regions of mathematics and physics. Its objects of study are manifolds equipped with a Lorentzian metric satisfying the Einstein equations. When other fields are present, the Einstein equations are coupled to the corresponding matter equations. Examples of resulting systems include the Einstein-Maxwell, Einstein-Euler and Einstein-Yang-Mills equations. These systems of nonlinear partial differential equations (PDE) present many problems of hyperbolic, parabolic, and elliptic nature. Nowadays, the Einstein equations are among the most important and challenging research topics in geometric analysis and PDE theory.

The Einstein equations, on the one hand, describe the Universe as a whole; and on the other hand, smaller regions like galaxies, clusters of stars, binary stars, or black holes. These latter examples form so-called isolated gravitating systems. They have in common that, mathematically, they are described by asymptotically flat spacetimes—that is to say, manifolds that decay to flat Minkowski space outside of spatially compact regions.

This volume comprises articles by leading experts covering the most important aspects of GR research from the past 100 years. Each article deals in some manner with the topics explained below.

Since the beginnings of GR—which aims at discovering the universe's secrets—geometry, analysis and physics have interacted fruitfully in pursuit of that aim. It is precisely those interactions which make mathematical GR such a prolific theory. The first exact solution to Einstein's equations was derived in 1916 by Karl Schwarzschild. On all fronts—in cosmological and non-cosmological cases, in theory and in observation—new results bore fruits. A breakthrough for general relativity theory came with the confirmation—as an outcome of Arthur Eddington's expedition of 1919—that light bends. Cosmological solutions by Albert Einstein 1917, by Willem de Sitter 1917, and by Alexander Friedmann in 1922, stirred discussions of reconciling redshifts of astronomical objects. Vesto Slipher observed nebulae, nowadays called galaxies, and from 1912 had been finding redshifts, for which there was no explanation at the time. The solution came with Georges

Lemaître, who in 1927 deduced that the universe must be expanding. His linear velocity-distance relation was confirmed by Edwin Hubble 1929. Whereas many big steps were taken by the pioneers in the 1920s and 1930s, a rigorous formulation of the mathematical initial value problem was missing. It wasn't until the 1950s and 1960s, with the works by Yvonne Choquet-Bruhat and Robert Geroch, that the Cauchy problem in GR was rigorously formulated. Based on these results, mathematical GR set out to solve burning problems in the field. Among the most spectacular solutions have been those to the global nonlinear stability of Minkowksi spacetime, by Demetrios Christodoulou and Sergiu Klainerman (1993); to the formation of black holes, by Demetrios Christodoulou (2008); and to the positive mass theorem, by Richard Schoen and Shing-Tung Yau (1979), and by Edward Witten (1981). An elegant proof of black hole formation due to concentration of matter in a small region was given by Richard Schoen and Shing-Tung Yau (1983). Various stability results have been established since then. A current big question is the stability of the Kerr solution, the latter describing a black hole with angular momentum. When black holes collide, GR predicts that gravitational waves are sent out which propagate at the speed of light. They have not yet been detected directly. Current and future experiments aim at detecting these waves. Not only should these experiments lead to yet another confirmation of GR, but it will open a new gate to astronomical regions that have so far been opaque to telescopes. Recently, there have been predictions of a specific feature of gravitational radiation, the so-called memory effect displacing test masses permanently. In order to understand gravitational radiation and related phenomena, we have to investigate the null asymptotics of physically relevant spacetimes. Various methods have been developed that have yielded new insights about null asymptotics. The latter leads to another special topic in GR, namely energy. As the gravitational field can be transformed away at a point, there is no generic way of defining energy as in Newtonian physics. For certain systems, one can define total energy and/or quasi-local energies, the latter being integrals over a sphere-like region. However, the general picture has yet to be revealed. There are today many definitions of quasilocal mass. Among them, most promising ones go back to Melissa Liu and Shing-Tung Yau (2003) as well as to Shing-Tung Yau and Mu-Tao Wang (2009). In the 1980s Roger Penrose listed the search for a definition of a quasilocal mass as a priority in classical GR. Its importance is tightly related to the Penrose inequality, which is a conjecture proposed by Roger Penrose in the 1970s bounding the total mass of a spacetime from below in terms of the area of suitable surfaces representing black holes. A full understanding of quasilocal energy should also lead to new progress in the study of the Cauchy problem for the Einstein equations. Further, the weak and strong cosmic censorship conjectures put forth by Roger Penrose have engaged the GR community in discussions about the nature of singularities. The weak hypothesis says that singularities in isolated gravitating systems do not affect distant observers; that is to say, every singularity is hidden inside a black hole; whereas the strong version claims that the fate of all observers is uniquely predictable by the initial data. The latter hypothesis, not requiring asymptotic flatness, is also valid for cosmological situations. Both conjectures can be shown to hold under certain assumptions, but there exist very specific violations as well. It remains open in what sense they may hold generically. We also mention the incompleteness (often called singularity) theorems by Stephen Hawking and Roger Penrose. One of the big challenges for the future is to combine GR with quantum theory. Very little is known about how to unify these theories, though suggestions have been put forth. It is yet too early to say what directions this field of research will take. Moreover, since the revolutionary astronomical observations of 1998 which showed that our universe is expanding at an accelerated rate, cosmological research has gained fresh momentum. Besides the current model of a Friedmann-Lemaître-Robertson-Walker (FLRW) Universe—where the cosmological term added to the Einstein equations is related to dark energy driving the expansion—other models have been proposed. Support or elimination of such models will depend on future experiments.

In the past 100 years, mathematical general relativity has enjoyed many breakthroughs, and more exciting questions have emerged to be solved in the future.

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