An alternative proposal for the anomalous acceleration

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Abstract. The purpose of this article is to give an overview of an alternative proposal for the anomalous acceleration of nearby galaxies as discovered in the 1998 supernova observations. The most common model introduces a cosmological constant into the Einstein equations to account for dark energy which is driving the accelerated expansion of the Universe. The authors give an alternative explanation within Einstein's original equations without the cosmological term, and without dark energy. The new model explains the anomalous acceleration due to a local under-density created by a self-similar wave from the radiation epoch that triggers an instability in the standard model when the pressure drops to zero.

1. Background

In 1998 supernova observations by astronomers led to the discovery of the Anomalous Acceleration (AA) of nearby galaxies. A best fit of the data to the two parameter family of Friedmann Spacetimes with curvature k and cosmological constant Λ led to the best fit model being a critical $k = 0$ Friedmann space-time with $\Omega_\Lambda \approx .7$. Cosmologists have thus hypothesized a repulsive anti-gravitational force coming from a cosmic vacuum energy, Dark Energy, accounting for approximately seventy percent of the energy density of the universe. By this interpretation, Dark Energy is represented by the addition of an extra term in the form of the cosmological constant to the right hand side of Einstein's equations of General Relativity (GR). This is the only way to preserve the uniform Friedmann space-time in the presence of the observed accelerated expansion, and hence the only way to preserve the Cosmological Principle, that the earth is not in a special place in the universe. Dark Energy has never been observed.

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2. Alternative explanation for the anomalous acceleration

The authors looked for an alternative explanation of the AA *wholly within* Einstein's original equations and without the cosmological constant, and *without Dark Energy*, (c.f. [**21**]).

Our Proposal: The AA is due to a local under-density on the scale of the supernova data, created by a self-similar wave from the radiation epoch that triggers an instability in the SM when the pressure drops to zero.

The instability is characterized by a new (closed) asymptotic ansatz which we introduce for spherical under-dense perturbations of the SM when $p = 0$, [**18, 20, 21**]. We show that the resolution of the instability is to create a large region of accelerated uniform expansion on the scale of the supernova data, (one order of magnitude larger in extent than expected), that expands outward from the center of the perturbation. Local under-densities induce local velocity increases and the Cosmological Principle can only hold approximately on the scale of the perturbations. The discovery of these instabilities arose from author's previous work.

Our project began with the idea from shock wave theory that the enormous pressure $p = \rho c^2/3$, one third of the total energy density, and consequent strong nonlinearities present in the Einstein equations during the radiation epoch of the Big Bang would lead one to conjecture that perturbations from the SM during the radiation epoch should decay into simple wave forms by the end of radiation, [**24, 18, 19**]. Since simple waves are typically "noninteracting" solutions on which the equations reduce from PDE's to ODE's, (c.f.[**6, 22**]), we set out to find spherical solutions which perturb the SM during the radiation epoch and on which the Einstein equations reduce to ODE's. In [**18, 19**] we identified a *unique* one parameter family of self-similar solutions that meet these requirements, which we call a-waves[1], depending on the *acceleration* parameter a [**19**], and normalized so that $a = 1$ is the SM, (the critical $k = 0$ Friedmann space-time with $\Omega_\Lambda = 0$). Parameter values $a < 1$ produce under-dense perturbations of SM near the center. The a-waves exist during the radiation epoch, which lasts from microseconds after the Big Bang until some tens of thousands of years after the Big Bang when the pressure drops precipitously to zero.[2] Our proposal is that a-waves are the prime candidates for the local time-asymptotic behavior of perturbations of SM near the center of perturbation, by the end of the radiation epoch.

[1]These self-similar solutions were first discovered, (from a different point of view), in [**1**], and further studies, including a discussion of these solutions as a possible mechanism for creating voids between galaxies, are recorded in the survey [**2**]. Authors were unaware of these connections in [**24, 18, 19**]. Our proposal here is the first attempt to connect these waves with the AA.

[2]The pressure drops to $p \approx 0$ about one order of magnitude before the time of uncoupling of matter and radiation at about $300,000$ years after the Big Bang, [**11, 12**].

The self-similar a-waves that exist when $p = \rho c^2/3$ do not persist to $p = 0$, [**2, 20**]. Thus our problem since [**18, 19**] has been to continue these a-waves into the $p = 0$ epoch. We accomplish this by showing that initial data corresponding to small perturbations $a < 1$ of SM at the end of the radiation epoch, trigger our identified instability in the SM when the pressure drops to $p = 0$. Surprisingly, perturbations of the SM by a-waves do not evolve trivially to the later observation, as we originally conjectured in [**18, 19**], but rather, it is the non-trivial phase portrait of the *instability* they *trigger* when the pressure drops to zero, that determines the evolution of a-waves and the anomalous accelerations they induce in the central region. According to the phase portrait, the SM is a classic unstable saddle rest point, and under-dense perturbations near SM evolve to a nearby stable rest point M corresponding to flat Minkowski space. Evolution toward the stable rest point M creates a large flat region of accelerated uniform expansion one order of magnitude larger in extent than expected. Moreover, we discover that *exactly* the same range of quadratic corrections Q to redshift vs luminosity are produced during the evolution from the SM at the end of radiation, to the stable rest point M after $p = 0$, as are produced in Dark Energy theory as Ω_Λ ranges from zero to one. By numerical simulation we determine the unique wave in the family that accounts for the same present value H_0 of the Hubble constant, and the same quadratic correction Q as Dark Energy theory with $\Omega_\Lambda = .7$. The third order correction C is a prediction that distinguishes our wave theory from Dark Energy theory. Determining the consistency of this wave theory with other measurements in cosmology would require further assumptions about the space-time far from the center of the perturbation. At this stage we make no such assumptions.

3. The perturbation equations

We begin by considering metrics in (t, r)=Standard Schwarzschild Coordinate (SSC) where the gravitational metric takes the usual form

$$ds^2 = -B(t,r)dt^2 + \frac{1}{A(t,r)}dr^2 + r^2 d\Omega^2. \tag{3.1}$$

Our results however will be given in different coordinates, namely, (t, ξ) where $\xi = r/t$, so that (our convention is to let $c = 1$ when convenient),

$$\xi = \frac{r}{ct} = \frac{\text{arclength distance at fixed } t}{\text{distance of light travel since Big Bang}}.$$

Thus we interpret ξ as fractional distance to the Hubble length $c/H \approx 10^{10}$ *lightyears*, a measure of the distance across the visible universe. For example, when we neglect terms on the order of ξ^4 below, we incur errors on the order of $\xi^4 \approx .0001$ at a tenth of the way across the visible universe. We consider ourselves as observers at present time t_0 positioned at the center $r = \xi = 0$, and our results will be given in terms of small ξ.

Putting the ansatz (3.1) into the Einstein equations $G = \kappa T$ for a perfect fluid

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij},$$

assuming spherical symmetry and setting $p = 0$, (c.f. [**25**]), leads to the following equations in (t, ξ) coordinates which are equivalent to the Einstein equations:

$$\begin{aligned}
tz_t + \xi\left\{(-1 + Dw)z\right\}_\xi &= -Dwz, \\
tw_t + \xi\left(-1 + Dw\right)w_\xi &= w - D\left\{w^2 + \tfrac{1-\xi^2 w^2}{2A}\left[\tfrac{1-A}{\xi^2}\right]\right\} \\
\xi A_\xi &= (A - 1) - z \\
\tfrac{\xi D_\xi}{D} &= (A - 1) - \tfrac{\left(1-\xi^2 w^2\right)}{2} z.
\end{aligned}$$

Here $D = \sqrt{AB}$, z is a dimensionless density and w is a dimensionless velocity,

$$\begin{aligned}
z &= \tfrac{\kappa \rho r^2}{1-\left(\frac{v}{c}\right)^2}, \\
w &= v/\xi,
\end{aligned}$$

where v is the fluid velocity, and $\kappa/c^2 = 8\pi\mathcal{G}/c^4$ is Einstein's gravitational constant, [**7**].

Our new ansatz for corrections to SM closes within even powers of ξ, and is given by, [**19, 20**]:

$$z(t,\xi) = z_{SM}(\xi) + \Delta z(t,\xi) \qquad \Delta z = z_2(t)\xi^2 + z_4(t)\xi^4 \tag{3.2}$$

$$w(t,\xi) = w_{SM}(\xi) + \Delta w(t,\xi) \qquad \Delta w = w_0(t) + w_2(t)\xi^2 \tag{3.3}$$

$$A(t,\xi) = A_{SM}(\xi) + \Delta A(t,\xi) \qquad \Delta A = A_2(t)\xi^2 + A_4(t)\xi^4 \tag{3.4}$$

$$D(t,\xi) = D_{SM}(\xi) + \Delta D(t,\xi) \qquad \Delta D = D_2(t)\xi^2 \tag{3.5}$$

where $z_{SM}, w_{SM}, A_{SM}, D_{SM}$ are the expressions for the unique self-similar representation of the SM when $p = 0$, given by, [**20**],

$$z_{SM}(\xi) = \tfrac{4}{3}\xi^2 + \tfrac{40}{27}\xi^4 + O(\xi^6), \quad w_{SM}(\xi) = \tfrac{2}{3} + \tfrac{2}{9}\xi^2 + O(\xi^4), \tag{3.6}$$

$$A_{SM}(\xi) = 1 - \tfrac{4}{9}\xi^2 - \tfrac{8}{27}\xi^4 + O(\xi^6),\ D_{SM}(\xi) = 1 - \tfrac{1}{9}\xi^2 + O(\xi^4). \tag{3.7}$$

This gives

$$\begin{aligned}
z(t,\xi) &= \left(\frac{4}{3} + z_2(t)\right)\xi^2 + \left\{\frac{40}{27} + z_4(t)\right\}\xi^4 + O(\xi^6), \\
w(t,\xi) &= \left(\frac{2}{3} + w_0(t)\right) + \left\{\frac{2}{9} + w_2(t)\right\}\xi^2 + O(\xi^4).
\end{aligned}$$

We prove the equations close within the unknowns $z_2, z_4, w_0, w_2, A_2, A_4, D_2$. The asymptotic equations for these unknowns are given by the following autonomous equations:[3]

$$(3.8)\qquad z_2' = -3w_0\left(\frac{4}{3}+z_2\right),$$

$$(3.9)\qquad w_0' = -\frac{1}{6}z_2 - \frac{1}{3}w_0 - w_0^2,$$

$$(3.10)\qquad z_4' = 5\left\{\frac{2}{27}z_2 + \frac{4}{3}w_2 - \frac{1}{18}z_2^2 + z_2w_2\right\} +5w_0\left\{\frac{4}{3} - \frac{2}{9}z_2 + z_4 - \frac{1}{12}z_2^2\right\},$$

$$(3.11)\qquad w_2' = -\frac{1}{10}z_4 - \frac{4}{9}w_0 + \frac{1}{3}w_2 - \frac{1}{24}z_2^2 + \frac{1}{3}z_2w_0 +\frac{1}{3}w_0^2 - 4w_0w_2 + \frac{1}{4}w_0^2z_2,$$

with

$$(3.12)\qquad A_2 = -\frac{1}{3}z_2,\quad A_4 = -\frac{1}{5}z_4,\quad D_2 = -\frac{1}{12}z_2.$$

In particular, (3.8)-(3.11) closes within the w's and z's, and we prove that if the constraints (3.12) hold initially, then they are maintained by the equations for all time. Conditions (3.12) are not invariant under time transformations, even though the SSC metric form is invariant under arbitrary time transformations, so we can interpret (3.12), and hence the ansatz (3.2)-(3.5), as fixing the time coordinate gauge of the SSC metric. This gauge agrees with FRW co-moving time up to errors of order $O(\xi^2)$.

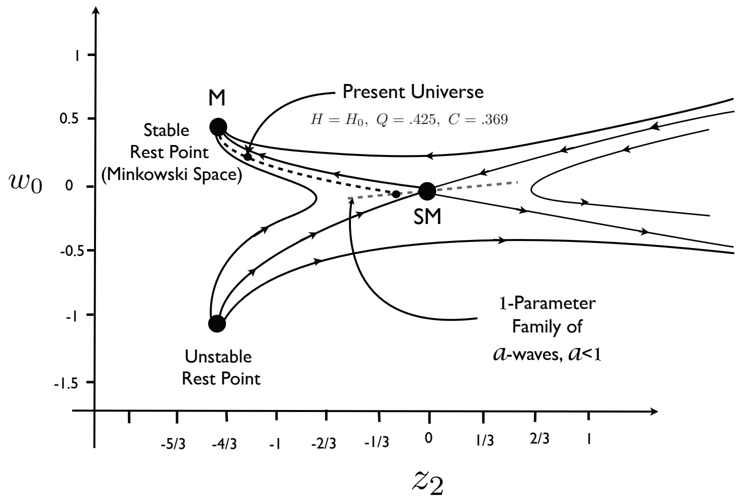
The importance of this ansatz is that, neglecting errors of order $O(\xi^4)$, corrections satisfying the ansatz describe an (approximate) uniformly expanding spacetime of density $\rho(t)$, constant at each time t, and it looks very similar to a "speeded up" Friedmann space-time. That is, since the ansatz is,

$$(3.13)\qquad z(\xi,t) = \kappa\rho(t,\xi)r^2 + O(\xi^4) = \left(\frac{4}{3}+z_2(t)\right)\xi^2 + O(\xi^4),$$

neglecting the $O(\xi^4)$ error gives $\kappa\rho = (4/3+z_2(t))/t^2$, a function of *time* alone. For the SM, $z_2 \equiv 0$ and this gives $\kappa\rho(t) = (4/3)\,t^{-2}$, which is the exact evolution of the density for the SM Friedmann spacetime with $p=0$ in co-moving coordinates, [**16**]. For the evolution of our specific under-densities in the wave theory, we show $z_2(t) \to -4/3$ as the solution tends to the stable rest point M along the eigen-direction which is the vertical w_0-axis, implying that the instability creates an accelerated drop in the density in a large uniform spacetime expanding outward from the center, (c.f. Figure 1).

[3] These asymptotic equations are written as functions of τ, where $\tau = \ln t$, $0<\tau<11$, and prime denotes $d/d\tau$. Introducing τ in place of t solves the long time simulation problem.

FIGURE 1. Phase Portrait for Central Region



Specifically, we prove that

$$\rho(t) = \frac{k}{t^3(1+\omega)},$$

where $k = k(t)$ and $\omega = \omega(t)$ change exponentially slowly during the convergence to M.

4. Analysis of the perturbation equations

The autonomous system (3.8)-(3.11) contains within it the the closed subsystem (3.8), (3.9),

$$z_2' = -3w_0\left(\frac{4}{3} + z_2\right), \tag{4.14}$$

$$w_0' = -\frac{1}{6}z_2 - \frac{1}{3}w_0 - w_0^2. \tag{4.15}$$

Remarkably, this subsystem alone determines the corrections to the SM at the order of the observed AA, accurate when errors $O(\xi^4)$ in z and $O(\xi^3)$ in $v = w\xi$ are neglected.

The phase portrait for the system (4.14)-(4.15) can easily be determined, (see Figure 1.): The unstable rest point at $(z_2, w_0) = (0,0)$ corresponds to the SM at the order of the observed AA within the central region, and clearly displays the instability of the SM. A calculation shows that the initial data from a-waves, projected into the (z_2, w_0)-plane and parameterized by a , cuts between the stable and unstable manifold of $(0,0)$, as plotted by the dotted line in the phase portrait depicted in Figure 1. This implies that a small under-density corresponding to $a < 1$ will evolve away from SM, following (approximately) the unstable manifold of SM to the stable rest point $M = (z_2, w_0) = (-4/3, 1/3)$, (c.f. Figure 1). Moreover, using (3.12)) we see that the metric components A and B are both of order $1 + O(\xi^4)$. This shows that at the stable rest point, the metric is Minkowski up to errors $O(\xi^4)$. We conclude that a small under-density created by an a-wave at the end of the radiation era causes the formation of a large under-dense region of accelerated uniform expansion moving outward from the center, in which the metric tends to flat Minkowski space.

5. Numerical results

Within the framework of the Einstein equations with cosmological constant, the best fit to the supernova data among Friedmann spacetimes with curvature parameter k and cosmological constant Λ leads to $k = 0$ and

$$\Omega_\Lambda \approx .7,$$

which leads to the conclusion that the universe consists of seventy percent Dark Energy, [**13, 14**]. Now the Hubble constant H at a given time in a given model is defined via the redshift vs luminosity relation

$$Hd_\ell = \mathrm{z} + O(\mathrm{z}^2). \tag{5.16}$$

We let H_0 denote the current measured value of the Hubble constant at present time, $H_0 = 100h_0\frac{km}{s\,mpc}$, with $h_0 \approx .68$. The redshift vs luminosity in the Dark Energy model is

$$H_0 d_\ell = \mathrm{z} + \frac{1}{4}(1+\Omega_\Lambda)^2 \mathrm{z}^2 - \frac{1}{8}\left(1 + \frac{2}{3}\Omega_\Lambda + \Omega_\Lambda^2\right)\mathrm{z}^3 + O(\mathrm{z}^4). \tag{5.17}$$

Here the quadratic correction to the SM with Dark Energy is

$$Q = .25(1+\Omega_\Lambda)^2. \tag{5.18}$$

As Ω_Λ increases from 0 to 1, Q increases through

$$.25 \le Q \le .5. \tag{5.19}$$

At the present time value $\Omega_\Lambda = .7$, (5.18) gives the value

$$Q = .425,$$

and (5.17) gives the coefficient C of z^3 when $\Omega_\Lambda = .7$ as

$$C = -0.1804.$$

On the other hand, a calculation shows that in our wave theory model,

$$H_0 d_\ell = \mathrm{z} + Q(z_2, w_0)\mathrm{z}^2 + C(z_2, z_4, w_0, w_2)\mathrm{z}^3 + O(\mathrm{z}^4),$$

where

$$Q_2(z_2, w_0) = \frac{1}{4} + \frac{24w_0 + 45w_0^2 + 3z_2}{4(2+3w_0)^2}, \tag{5.20}$$

but C is more complicated, (details omitted). From (5.20) we see directly that *again*, as the orbit the orbit (4.14), (4.15) evolves from the unstable rest point $SM = (0, 0)$, to the stable rest point $M = (-4/3, 1/3)$, Q increases from $Q(0,0) = .25$ to $Q(-4/3, 1/3) = .5$, *precisely* the same range (5.19) that DE theory produces. A numerical simulation of solutions up to *present time* t_0, the time when the Hubble constant takes its present value H_0, determines the unique value

$$a = \underline{a} = 0.99999959 = 1 - (4.3) \times 10^{-7},$$

corresponding to an a-wave that creates an under-density relative to the SM at the end for the radiation epoch, such that the subsequent $p = 0$ evolution starting from this initial data evolves to time $t = t_0$ with $H = H_0$ and $Q = .425$ *in agreement with* the values of H and Q at present time t_{DE} in the DE model $\Omega_\Lambda = .7$.

Now comparing the initial density $\rho_{wave}(t_*)$ at the center of the wave $a = \underline{a}$ to the corresponding initial density $\rho_{SM}(t_*)$ at the same time t_* at the end of radiation, gives

$$\frac{\rho_{wave}(t_*)}{\rho_{SM}(t_*)} = 1 - (7.45)\, 10^{-6} \approx 1. \tag{5.21}$$

During the $p = 0$ evolution up to present time t_0, this density ratio evolves to

$$\frac{\rho_{wave}(t_0)}{\rho_{SM}(t_0)} = 0.145, \tag{5.22}$$

a seven-fold under-density. We conclude that the wave $a = \underline{a}$, which accounts for the correct value of the Hubble constant and the quadratic correction to redshift vs luminosity at the center of wave, also quantifies the severity of the instability in the SM triggered by the perturbations (3.2)-(3.5).

6. Initial data from the radiation epoch

After the radiation epoch, the pressure drops precipitously to zero, [**11, 12**]. This happens not on a constant time surface $t = t_*$, but on a constant temperature surface $T = T_*$. Thus the initial data for the equations (3.8)-(3.11) must be computed from the restriction of our self-similar waves at the end of radiation, to the constant temperature surface $T = T_*$. We must then convert this data to a constant time surface $t = t_*$. This step is achieved by first using the Stefan-Boltzmann law

$$\rho_* c^2 = \frac{a_{SB}}{4} c T_*^4,$$

(a_{SB}=Stefan-Boltzmann constant), to relate the temperature T_* to the constant density surface $\rho = \rho_*$, and then pulling this back to the constant time surface $t = t_*$. Since we are working with asymptotic solutions, the pullback to $t = t_*$ is accomplished by use of the equations together with Taylor's Theorem. Finally, we must address the issue that the initial data from the end of radiation is given in a *different gauge* from the gauge determined by our ansatz for the $p = 0$ evolution. This stems from the fact that time since the Big Bang is different for each of the different a-waves. To address this, we define a gauge transformation to post-process the data by converting it into the gauge to which our asymptotic evolution applies.

For each value of (a,T_*), we compute the values t_0, Q and C, where t_0 is the time at which $H = H_0$, and Q and C are the quadratic and cubic corrections to redshift vs luminosity at that time, computed from our derived formulas for $Q(z_2, w_0)$ and $C(z_2, w_0, w_2)$. Our numerics show that the dependence on the starting temperature is negligible for T_* in the range $3000^o K \leq T_* \leq 9000^o K$, the range relevant to cosmology, [**11**]. Thus for the temperatures appropriate for Cosmology, t_0, Q and C are determined by a alone. The value $a = \underline{a}$ is then uniquely determined by the condition that $a < 1$ and $Q = .425$.

Our results are recorded in the following theorem. Again we let present time in a given model denote the time at which the Hubble constant H (as defined in (5.16)) reaches its present measured value $H = H_0$, this time being different in different models.

THEOREM 1. *Let $t = t_0$ denote present time since the Big Bang in the wave model and $t = t_{DE}$ present time since the Big Bang in the Dark Energy*[4] *model. Then there exists a unique value of the acceleration parameter $\underline{a} = 0.99999959 \approx 1 - 4.3 \times 10^{-7}$ corresponding to an under-density relative to the SM at the end of radiation, such that the subsequent $p = 0$ evolution starting from this initial data evolves to time $t = t_0$ with $H = H_0$ and $Q = .425$, in agreement with the values of H and Q at $t = t_{DE}$ in the Dark Energy model. The cubic correction at $t = t_0$ in the wave theory is then $C = 0.3591$, while Dark Energy theory gives $C = -0.1804$ at $t = t_{DE}$. The times are related by $t_0 = .95\, t_{DE}$.*

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