Decay of linear waves on black hole space-times

Daniel Tataru

Abstract. The aim of these notes is to describe recent and ongoing work of the author and collaborators on decay estimates for linear waves on black hole space times. For comparison purposes, our discussion will also include nontrapping asymptotically flat space-times. We will consider the scalar wave equation and perturbations thereof, as well as the corresponding Maxwell system.

1. Introduction

These notes are concerned with the long time dynamics for solutions to wave equations on asymptotically flat space times. Precisely, we are interested in several classes of decay estimates for linear waves. The motivation for studying such decay estimates, however, comes in good part from nonlinear wave equations.

While for comparison purposes we do consider nontrapping space-times, the main focus of this work is on black hole space-times, where trapping necessarily occurs. Such space-times are of interest for instance from the perspective of the black hole stability problem.

There are several types of decay estimates we consider, which will be described in detail in the sequel. The first bounds we consider are not decay bounds, but simply uniform energy bounds; these are crucial to our discussion, but not a-priori true. In terms of decay bounds, we are interested in the following:

- Local energy decay estimates. These measure the L^2 averaged energy decay in compact sets. These are invariant with respect to time translation, but also, in some sense, fit well with the asymptotically flat space-times.

2010 *Mathematics Subject Classification.* 35L05, 35Q61, 35Q76, 83C57.

Key words and phrases. Kerr spacetime, black hole stability, decay of linear waves, waves, Maxwell system, energy estimates, Strichartz estimates, Morawetz estimates.

- Strichartz estimates. These are averaged $L^p_t L^q_x$ decay bounds, which are invariant with respect to both space and time translation, and are useful in the study of nonlinear problems with H^s type data.
- Pointwise decay estimates. These correspond to problems with localized initial data; they fit better the nonlinear stability problems.

One key point in our results is that the local energy decay estimates are the fundamental ones. Indeed, two of the results we have so far assert that, under suitable asymptotic flatness assumptions, local energy decay implies both Strichartz estimates and pointwise decay. Because of this, most of these notes are devoted to explaining the local energy decay results and ideas.

Concerning the class of wave operators we work with, an important distinction we make is between the self-adjoint and nonself-adjoint cases. For the most part, our results apply to the self-adjoint case. Some of the ideas carry over to small nonself-adjoint perturbations of self-adjoint problems. However, so far there is no complete approach that applies to the general nonself-adjoint case.

A second distinction we make is between stationary and nonstationary metrics. In the stationary case, one can characterize the decay properties in terms of (i) spectral assumptions and (ii) non-trapping type assumptions. For the non-stationary case, however, we can only treat small perturbations of stationary metrics, for the most part.

Further, in the black hole case, another important feature turns out to be whether there exists a second, rotational group of symmetries, associated to a second, space-like Killing vector field. This is particularly important since a large class of relativistic stationary black hole space times necessarily have this property.

For the remainder of the introduction we define the classes of space-times we work with, and then discuss some of their geometric properties. Then we describe all types of decay estimates listed above in the context of the scalar wave equation. In the following sections of the paper we present some of the results we have so far, as well as outline some open problems. The last section is devoted to similar questions in the context of the Maxwell system.

1.1. Asymptotically flat nontrapping space-times. Here we work in $\mathbb{R}^{3+1}$, where we have a Lorentzian metric (i.e. with signature $(3,1)$):

$$g = g_{\alpha\beta} dx^\alpha dx^\beta.$$

The simplest such example is the Minkowski metric,

$$m = -dt^2 + dx^2.$$

We will assume that the time slice foliation $\{t = const\}$ is space-like, and denote by $N = \nabla t$ its time-like normal. Concerning the metric g we will always assume the following conditions:

(NT1) g is asymptotically flat,

$$g = m + O(r^{-\epsilon}), \qquad \nabla g = O(r^{-\epsilon-1}), \qquad \nabla^2 g = O(r^{-\epsilon-2}).$$

(NT2) g is nontrapping, i.e, all null geodesics escape to infinity.

In terms of the time dependence of g, for some of our results we will make no assumption, but for some we will restrict ourselves to either

(S) Stationary, where we have a time-like Killing field $X = \partial_t$, so that $\nabla_X g = 0$ or

(SV) Slowly varying, where $X = \partial_t$ is an almost Killing field, $\nabla_X g = O(\epsilon)$ with ϵ small enough.

We remark that all the assumptions here are time reversible.

1.2. Asymptotically flat black hole space-times (e.g. Schwarzschild, Kerr). The set-up here is as follows. We work in a domain

$$\mathbb{R}^{3+1} \supset \mathcal{M} = \{r > r_0\} \times \mathbb{R}.$$

As above, g is a Lorentzian metric in $\mathcal{M}$, and we have the space-like foliation $\{t = const\}$, with time-like forward normal $N = \nabla t$.

Concerning the metric g, we begin with the following conditions:

(BH1) g is asymptotically flat, as above.

(BH2) The inner cylinder $\mathcal{C} = \{r = r_0\}$ is space-like, with forward outgoing time-like normal $N_0 = -\nabla r$.

These two assumptions imply that there must be both trapped null geodesics, as well as null geodesics which exit on the inner cylinder $\mathcal{C}$. In particular we can define the black hole $\mathcal{BH}$ as the (spatially compact) region of points which cannot escape to infinity along null geodesics. Its outer boundary is called the event horizon $\mathcal{H}$. We denote by $\mathcal{T}$ the set of trapped null geodesics which stay away from the black hole. Concerning these geometric objects, we will make the following assumptions:

(BH-3) (Red Shift) $\mathcal{H}$ is smooth and non-degenerate, i.e. in suitable coordinates we have $\mathcal{H} = \{r = r_\mathcal{H}\}$, with null forward normal $L = -\nabla r$, tangent to $\mathcal{H}$, and satisfying the non-degeneracy condition

$$\nabla_L L = \sigma L, \qquad \sigma \geq c > 0.$$

(BH-4) (Hyperbolic trapping) The trapped set $\mathcal{T}$ is hyperbolic. It also has positive energy, i.e. $X = \partial_t$ is forward time-like on $\mathcal{T}$.

These two assumptions are not satisfied by all black hole space-times, but are satisfied by most relativistic space times. They insure that the two regions $\mathcal{H}$ and $\mathcal{T}$ are at least micro-locally separated, and that the red shift property holds uniformly.

As before, we will consider in greater detail the stationary case and the slowly varying case. In addition, we will further consider the case when there is an additional one dimensional group of symmetries:

(BH-5) The space $(\mathcal{M}, g)$ admits a second one parameter family of symmetries, generated by a space-like Killing vector field $\Omega = (x_1\partial_2 - x_2\partial_1)$, commuting with X.

Examples of BH space times satisfying all the assumptions above are the Schwarzschild and Kerr space-times, and indeed much of the research in this field has been devoted to the study of these special cases. Our goal here is to develop some general principles instead.

1.3. Scalar wave equations. Our main target here is the inhomogeneous wave equation:

$$\Box_g u = f, \qquad u[0] := (u(0), Nu(0)) = (u_0, u_1), \tag{1.1}$$

where

$$\Box_g = \frac{1}{\sqrt{g}} \partial_\alpha g^{\alpha\beta} \sqrt{g}\, \partial_\beta.$$

More generally, we will add a real magnetic field A and potential V, and consider the evolution

$$Pu = f, \qquad u[0] := (u(0), Nu(0)) = (u_0, u_1). \tag{1.2}$$

where

$$P = \Box_{g,A} + V$$

with

$$\Box_{g,A} = \frac{1}{\sqrt{g}} (\partial_\alpha + iA_\alpha) g^{\alpha\beta} \sqrt{g} (\partial_\beta + iA_\beta)$$

As defined, the above operator is self-adjoint. In the non-self-adjoint case we will include small nonzero imaginary components in A and V.

To the wave operator $\Box_g$ we associate the energy momentum tensor:

$$T_{\alpha\beta} = \partial_\alpha u \partial_\beta u - \frac{1}{2} g_{\alpha\beta} \partial^\nu u \partial_\nu u.$$

This is divergence free when $\Box_g u = 0$,

$$\nabla^\alpha T_{\alpha\beta} = 0,$$

and has a natural extension to the general self-adjoint case.

When X is a Killing vector field, this yields

$$\nabla^\alpha (T_{\alpha\beta} X^\beta) = 0,$$

which gives a conserved energy,

$$E(u(t)) = \int T(X, N)\, dx.$$

This is a positive definite energy for the equation (1.1) in the nontrapping case, when X is time-like. However for the more general problem (1.2), as well as in the black hole case it is merely positive definite outside a compact set.

1.4. Energy and resolvent bounds. For the scalar wave equation we define the energy norm as

$$\|u[t]\|^2_{\dot{H}^1 \times L^2} = \|\nabla_{x,t} u(t)\|^2_{L^2}.$$

DEFINITION 1.1. *We say that uniform energy estimates hold for the homogeneous scalar wave equation* (1.2) *if we have the uniform bound*

$$\|\nabla u(t)\|_{L^2} \lesssim \|\nabla u(s)\|_{L^2}, \qquad t > s. \tag{1.3}$$

We note that by the Duhamel formula one can also rewrite this as a bound for the inhomogeneous problem,

$$\|\nabla u(t)\|_{L^2} \lesssim \|\nabla u(s)\|_{L^2} + \|f\|_{L^1[s,t]L^2_x}, \qquad t > s. \tag{1.4}$$

We also remark that, in general, by local theory we only have exponential energy bounds,

$$\|\nabla u(t)\|_{L^2} \lesssim e^{C(t-s)} \|\nabla u(s)\|_{L^2}, \qquad t > s.$$

For various purposes, it is also useful to track the time evolution of the higher Sobolev norms, namely

$$\|\nabla u(t)\|_{H^k}, \qquad k > 0.$$

DEFINITION 1.2. *We say that higher Sobolev energy bounds hold for the homogeneous scalar wave equation* (1.2) *if for $k \geq 1$ we have the uniform bound*

$$\|\nabla u(t)\|_{H^k} \lesssim \|\nabla u(s)\|_{H^k} \qquad t > s. \tag{1.5}$$

For NT or stationary BH metrics and operators P, the transition between energy bounds and higher energy bounds is relatively straightforward:

THEOREM 1.3. *Let $(\mathcal{M}, g)$ be either a NT or a stationary BH space-time as above. If uniform energy estimates hold for the wave equation* (1.2) *then higher Sobolev uniform energy estimates hold as well.*

In the stationary NT case this is easily proved by commuting with ∂_t and some elliptic analysis. In the nonstationary NT case, the argument requires essentially the full high frequency part of the local energy decay proof.

In the stationary BH case, the elliptic analysis works only partially, and is supplemented by a microlocal high frequency bound near the horizon $\mathcal{H}$, which takes advantage of the red shift assumptions (BH-3) and (BH-4). These guarantee that any null geodesic on which ∂_t is not timelike spends only a limited amount of time in the exterior region and away from the horizon $\mathcal{H}$ and the null rays generated by L along $\mathcal{H}$. The rays near $\mathcal{H}$ and L, on the other hand, are red shifted, so they correspond to exponential energy decay and exponential frequency decay.

We further remark that in some sense one would hope that the BH result extends to the slowly varying nonstationary case. However, this does not appear to work unless some additional conditions are imposed near the trapped set $\mathcal{T}$.

In the stationary case, one can further interpret the energy estimates in terms of resolvent bounds. To define the resolvent take a time Fourier transform in (1.2):

$$Pu = f \longleftrightarrow P_\tau \hat{u}(\tau) = \hat{f}(\tau) \longleftrightarrow \hat{u}(\tau) = R_\tau \hat{f}(\tau)$$

where the resolvent

$$R_\tau : L^2 \to \dot{H}^1$$

is defined whenever P_τ is invertible. A-priori we have exponential bounds

$$\|\nabla u(t)\|_{L^2} \lesssim e^{Mt} \|\nabla u(0)\|_{L^2},$$

so the resolvent is well defined and holomorphic at least for $\Im \tau < -M$. In effect the operators P_τ are uniformly elliptic away from the real axis, so by Fredholm theory the resolvent is well defined as a meromorphic function away from the real axis.

The uniform energy bounds are related to the resolvent as follows:

Proposition 1.4. *Uniform energy estimates are equivalent to the resolvent bound*[1]

$$\|R_\tau\|_{L^2 \to \dot{H}^1} \lesssim |\Im \tau|^{-1}, \qquad \Im \tau < 0.$$

It is immediately apparent that an obstruction to having uniform energy bounds is the presence of negative eigenfunctions, which solve

$$P_\tau u_\tau = 0, \qquad \Im \tau < 0$$

By standard elliptic analysis, such eigenfunctions u_τ must decay rapidly (indeed exponentially) at infinity.

They correspond to exponentially growing solutions to (1.2). We remark that the assumption that P is self-adjoint does not guarantee that there are no eigenvalues outside of the continuous spectrum (i.e. the real axis). However, it does guarantee that these eigenvalues are isolated and away from the real axis, and also that they come in conjugate pairs.

A less obvious obstruction to uniform energy bounds is given by resonances on the real axis, which correspond to polynomially growing modes for the wave evolution. These are discussed in greater detail in the next section, with a special emphasis on the zero modes.

1.5. Local energy decay. The simplest local energy norm is defined at the L^2 level as

$$\|u\|_{LE} = \sup_k \|\langle r\rangle^{-\frac{1}{2}} u\|_{L^2(\mathbb{R}\times A_k)}, \qquad A_k = \{\langle x\rangle \approx 2^k\}.$$

We also define its H^1 counterpart,

$$\|u\|_{LE^1} = \|\nabla u\|_{LE} + \|\langle r\rangle^{-1} u\|_{LE}$$

[1]This applies to our problem but not in general; compare with the Hille-Yosida theorem.

as well as the dual norm

$$\|f\|_{LE^*} = \sum_{k\geq 0} \|\langle r\rangle^{\frac{1}{2}} f\|_{L^2(\mathbb{R}\times A_k)}.$$

Then the sharp formulation of local energy decay in the NT case is as follows:

Definition 1.5. *We say that local energy decay holds for the wave equation* (1.2) *in the NT case if*

$$\|u\|_{LE^1} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{LE^*+L^1L^2} + \|\nabla u(0)\|_{L^2} \tag{1.6}$$

The heuristics for local energy decay are quite simple, at least in the nontrapping case. As waves propagate with unit speed, they will spend a time of at most 2^k in the dyadic cylinder A_k. Thus their integrated local energy in A_k is controlled by their energy with an integration factor of 2^k.

Just as in the case of the uniform energy bounds, we also can look at higher local energy decay bounds. For this we define the higher local energy norms as

$$\|u\|_{LE^k} = \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{LE}$$

and similarly $LE^{k,1}$ and $LE^{*,k}$. Then we have

Definition 1.6. *We say that higher local energy decay bounds hold for the equation* (1.2) *in the NT case if*

$$\|u\|_{LE^{k,1}} + \|\nabla u\|_{L^\infty H^k} \lesssim \|f\|_{LE^{*,k}+L^1H^k} + \|\nabla u(0)\|_{H^k}.$$

The same local energy norms cannot be used ad literam in the BH case, due to trapping. Heuristically, one can have high frequency waves which stay for an arbitrarily long time near the trapped set. The redeeming feature here is the hyperbolicity of the trapped set. By the uncertainty principle, a frequency λ wave must spread spatially at least λ^{-1}. But by hyperbolicity, most of the λ^{-1} spatial ball will have spread out when propagated along the Hamilton flow for a time about $\log\lambda$ (Ehrenfest time). This hints that there is at most a $|\log\lambda|^{\frac{1}{2}}$ loss in the local energy decay bounds, occurring only on the trapped set.

This leads us to introduce a modified local energy norm $LE_{\mathcal{T}}$ which has log losses on the trapped set, and its dual $LE^*_{\mathcal{T}}$. Thus we have the inclusions

$$LE^1 \subset LE^1_{\mathcal{T}}, \qquad LE^*_{\mathcal{T}} \subset LE^*,$$

with equality away from $\mathcal{T}$. The modified local energy decay is as follows

$$\|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{LE^*_{\mathcal{T}}+L^1L^2} + \|\nabla u(0)\|_{L^2}.$$

The transition to higher local energy decay bounds follows the same rules as for the uniform energy estimates:

Theorem 1.7. *Let* $(\mathcal{M}, g)$ *be either a NT or a stationary BH space-time as above. If local energy decay estimates hold for the wave equation* (1.2) *then higher local energy decay estimates hold as well.*

Again, the result extends to slowly varying BH space times once some care is taken with the trapped set $\mathcal{T}$.

We next consider the interpretation of local energy decay in terms of resolvent bounds in the stationary case. For that we use the norms $\mathcal{LE}$, $\mathcal{LE}^1$ and $\mathcal{LE}^*$ obtained by freezing time in the definition of LE, LE^1 respectively LE^*. We will also use the subscript zero, as in $\mathcal{LE}_0$, for the closure of C_0^∞ in the corresponding space.

Proposition 1.8. *Consider either a stationary NT or BH space-time. Then local energy decay is equivalent to the uniform resolvent bound*

$$\|R_\tau f\|_{\mathcal{LE}^1_{\mathcal{T}}} \lesssim \|f\|_{\mathcal{LE}^*_{\mathcal{T}}} \quad , \qquad \Im \tau \leq 0.$$

Here one needs to carefully define the resolvent on the real line as the limit when the real line is approached from the lower half-space (this is often called the *limiting absorption principle*). Here it is natural to distinguish two cases:

(i) $\tau \neq 0$. Then $u = R_\tau f$ must satisfy the outgoing radiation condition (or the *Sommerfeld radiation condition*)

$$r^{-\frac{1}{2}}(\partial_r - i\tau)u \in L^2$$

Thus, in addition to complex eigenvalues, one is led to introduce the embedded resonances as an obstruction to local energy decay:

Definition 1.9. *A function $u \in \mathcal{LE}^1$ is an embedded resonance associated to the real time frequency $\tau \in \mathbb{R}\backslash\{0\}$ if it satisfies the outgoing radiation condition above and $P_\tau u = 0$.*

Less obviously, embedded resonances also cause uniform energy bounds to fail. It is not too difficult to show that in polar coordinates the embedded resonances must have the asymptotics

$$u(r,\omega) = v(\omega)r^{-1}e^{i\tau r} + O(r^{-2}), \qquad v \neq 0$$

(ii) *(ii)* $\tau = 0$. There the outgoing radiation condition is meaningless. These have simpler asymptotics

$$u(r,\omega) = vr^{-1} + O(r^{-2})$$

and a symbol type behavior at infinity. Unlike the case of embedded resonances, here we can also have $v = 0$, in which case these become eigenvalues. The zero resonances play a special role in our story. Their absence is equivalent to a bound of the form

$$\|u\|_{\mathcal{LE}^1} \lesssim \|P_0 u\|_{\mathcal{LE}^*} \tag{1.7}$$

which we will refer to in the sequel as the *zero energy resolvent bound.*

1.6. Strichartz estimates (averaged decay). The Strichartz estimates are averaged decay bounds, where the decay is measured using mixed norm spaces $L^p L^q = L^p_t L^q_x$, with an associated Sobolev index ρ. The range of indices depends on the decay properties of the fundamental solution for the wave equation, and thus on the dimension; in $3+1$ dimensions it is

$$2 < p \leq \infty, \qquad \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, \qquad \rho = \frac{3}{2} - \frac{1}{p} - \frac{3}{q}.$$

Then we have

DEFINITION 1.10. *Strichartz estimates hold for the equation* (1.2) *if*

$$\| |D_x|^{-\rho} \nabla u\|_{L^p L^q} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{|D_x|^{-\rho} L^{p'} L^{q'} + L^1 L^2} + \|u[0]\|_{\dot H^1 \times L^2} \tag{1.8}$$

for all admissible indices[2] (p, q).

Standard methods show that at least in the NT case it suffices to have the estimates for the homogeneous wave equation (1.2)

$$\| |D_x|^{-\rho} \nabla u\|_{L^p L^q} \lesssim \|\nabla u_0\|_{L^2} + \|u_1\|_{L^2},$$

or equivalently to have an inhomogeneous estimate for the forward inhomogeneous problem $Pu = f$ (i.e. with vanishing data at $t = -\infty$):

$$\|\nabla u\|_{L^\infty L^2} \lesssim \| |D_x|^{\rho} f\|_{L^{p'} L^{q'}}.$$

Strichartz estimates have been originally proved in the constant coefficient case, and then locally in time for variable coefficients.

1.7. Pointwise decay estimates (Price's Law). The context here is that we start with regular initial data, which is further localized in a compact set. Then the question we ask is what is the pointwise decay rate at infinity for solutions to the homogeneous wave equation.

The context we choose here is the one which agrees with most relativistic space-times, where the asymptotic flatness condition takes the more precise form:

$$g = m + O_{rad}(r^{-1}) + O(r^{-2}), \qquad V = O_{rad}(r^{-3}) + O(r^{-4}).$$

Here the $O_{rad}(r^{-1})$ corresponds to the relativistic mass term, while the $O(r^{-2})$ nonradial term is associated to the angular momentum.

For the initial data we use the $H^{m.k}$ spaces, which require m derivatives and k moments to be in L^2.

The expected decay rate for linear waves along the light cone is still t^{-1}. The interesting question is what is the decay rate inside the cone, and in particular, within a compact set. The latter case was the subject of some heuristic computations in the 70's, due to a physicist, Robert Price. His computed t^{-3} decay rate [**25**] was later referred to as Price's law. It is also interesting to fill in the cone and understand the proper decay rates in all

[2] Here and below one can use different pairs (p, q) on the left and on the right in the inequality,

directions. The correct rates, which we will still refer to as Price's law for simplicity, are as follows:

$$
\begin{aligned}
|u(t,x)| &\lesssim \frac{1}{\langle t\rangle \langle t-|x|\rangle^2} \|\nabla u(0)\|_{H^{m,k}}, \\
|\partial_t u(t,x)| &\lesssim \frac{1}{\langle t\rangle \langle t-|x|\rangle^3} \|\nabla u(0)\|_{H^{m,k}}, \\
|\partial_x u(t,x)| &\lesssim \frac{1}{\langle r\rangle \langle t-|x|\rangle^3} \|\nabla u(0)\|_{H^{m,k}}.
\end{aligned} \tag{1.9}
$$

Remark 1.11. *When true, the above decay rates are sharp due to the contribution of the leading order radial terms at spatial infinity in the metric or potential. Interestingly enough, this has nothing to do with the local properties of the metric.*

1.8. Local Energy Decay as the core decay bound. While each of the three types of decay estimates may be more useful for certain purposes, it is the local energy decay that has emerged in the work of the author and collaborators as the fundamental estimate. Precisely, one has the following heuristic relations between local energy decay and the other types of decay estimates:

Principle 1.12. *Let $(\mathcal{M}, g)$ be either a NT or a BH space-time. If uniform energy estimates and local energy decay hold for the wave equation* (1.2)*, then Strichartz estimates hold as well.*

The ideas that lead to this, as well as more precise results of this type are described in Section 2.

Principle 1.13. *Let $(\mathcal{M}, g)$ be either a NT or a BH space-time. If uniform energy estimates and local energy decay hold for the wave equation* (1.2)*, then pointwise decay bounds (Price's law) hold as well.*

Results of this type, together with the appropriate assumptions, are described in Section 3.

2. Strichartz estimates

The starting point in our description of the results concerning Strichartz estimates for the wave equation on asymptotically flat space-times is the article [**20**], where only NT space-times are considered. The first main result in [**20**] is as follows:

Theorem 2.14. *Let $(\mathcal{M}, g)$ be an asymptotically flat space-time which is a small perturbation of the Minkowski space-time. Then both local energy decay and Strichartz estimates hold in $(\mathcal{M}, g)$.*

The local energy decay part of this result is not so difficult, and is proved using exactly the same multiplier method as in the case of the Minkowski space-time. The interesting part is the assertion on Strichartz estimates.

This is in turn a consequence of a result which gives the existence of an outgoing parametrix for the wave equation. Unlike the more classical concept of a forward parametrix, an outgoing parametrix corresponds to solving the wave equation partially forward in time and partially backward, so that all waves move in an outward direction. The parametrix construction, in turn, is a special case of a more general result proved earlier by the author in [**27**].

One useful observation in [**20**] is that one can put together the local energy bounds and the Strichartz estimates to obtain the more complete bound

$$\begin{aligned} & \| |D_x|^{-\rho} \nabla u\|_{L^pL^q} + \|\nabla u\|_{L^\infty L^2} + \|u\|_{LE^1} \\ & \lesssim \|f\|_{|D_x|^{-\rho} L^{p'}L^{q'} + L^1L^2 + LE^*} + \|\nabla u(0)\|_{L^2}. \end{aligned} \tag{2.10}$$

This comes in very handy in the proof of the next result in [**20**]:

THEOREM 2.15. *Let $(\mathcal{M}, g)$ be an asymptotically flat NT space-time. Assume that uniform energy bounds and local energy decay hold in $(\mathcal{M}, g)$. Then the Strichartz estimates hold as well.*

This is essentially an immediate consequence of the estimate (2.10). To see this, suppose at first that $f \in L^1L^2 + LE^*$. One then splits the space $\mathbb{R}^{3+1}$ in two parts, namely the inside of a fixed large cylinder C_{int} and its exterior C_{ext}. Correspondingly, the solution u to (1.2) is split into two parts $u = u_{int} + u_{ext}$, using an appropriate partition of unit. By hypothesis u satisfies the local energy decay bounds. Using these to estimate the truncation errors in L^2, the problem is reduced to proving the L^pL^q Strichartz estimates separately for u_{int} and u_{ext}.

In the exterior domain C_{ext} one can view the metric as a small perturbation of the Minkowski metric, and use the bound (2.10). In the interior domain, on the other hand, one can use the local energy decay bound (which is now a global H^1 bound on u) to further localize on the unit time scale, retaining l^2 summability in this decomposition. Then it suffices to apply the local Strichartz estimates to each of these pieces.

To work with $f \in |D_x|^{-\rho} L^{p'}L^{q'}$, it suffices to produce an approximate solution v so that $Pv - f \in LE^*$. But this can be done separately in the exterior region and in unit time slices of the interior region, using Theorem 2.14 for the former and again the local Strichartz estimates for the latter.

The same method applies in the black hole setting, provided one has only hyperbolic trapping and a good result near the trapped set $\mathcal{T}$:

THEOREM 2.16. *Let $(\mathcal{M}, g)$ be an asymptotically flat BH space-time. Assume that uniform energy bounds and local energy decay hold in $(\mathcal{M}, g)$. Then the Strichartz estimates hold as well.*

The proof of this theorem still uses the local energy decay to split the space, but this time into four regions:

(i) An inner region, near and inside the event horizon.

(ii) A small neighbourhood of the trapped region.
(iii) A spatially bounded large exterior nontrapping annulus
(iv) An exterior region.

Theorem 2.14 and the bound (2.10) applies in the exterior region, while in regions (i) and (iii) one can further truncate to the unit temporal scale, as above. That leaves us with the near-trapped region. A direct bound for that was obtained in [**19**] and [**30**] in special cases (Schwarzschild/Kerr), but one can also treat the general case using (variations[3] of) the results in [**7**].

3. Pointwise decay (Price's law)

Our goal here is to work with either NT or BH asymptotically flat metrics $(\mathcal{M}, g)$ and associated magnetic potentials A and potentials V, and seek to understand the decay rates at infinity for linear waves for regular localized initial data.

A-priori there are two factors which may influence these local decay rates, namely the local behavior and the asymptotic behavior at infinity. However, these two factors turn out to play very different roles. The local behavior will be the more fundamental factor, which decides whether there is decay or not; we measure this in terms of the uniform energy bounds and local energy decay. On the other hand, once we know that there is decay, the actual decay rate will depend on the asymptotic behavior at infinity.

For this reason, we need to be more precise about the asymptotics at infinity. We describe this using vector fields as follows:

$$\begin{aligned} T &= \{\partial_t, \partial_i\}, && \text{(translations)} \\ \Omega &= \{x_i\partial_j - x_j\partial_i\}, && \text{(rotations)} \\ S &= t\partial_t + x\partial_x. && \text{(scaling)} \end{aligned}$$

We set $Z = \{T, \Omega, S\}$. Then we define the classes $S^Z(r^k)$ of functions in $\mathbb{R}^+ \times \mathbb{R}^3$ by

$$a \in S^Z(r^k) \Longleftrightarrow |Z^j a(t,x)| \leq c_j \langle r\rangle^k, \quad j \geq 0.$$

By $S^Z_{rad}(r^k)$ we denote spherically symmetric functions in $S^Z(r^k)$.

Then for the purpose of this section we consider metrics g which are of the following form:

$$g = m + g_{sr} + g_{lr},$$

where m stands for the Minkowski metric, g_{lr} is a stationary long range spherically symmetric component, with $S^Z_{rad}(r^{-1})$ coefficients, of the form

$$g_{lr} = g_{lr,tt}(r)dt^2 + g_{lr,tr}(r)dtdr + g_{lr,rr}(r)dr^2 + g_{lr,\omega\omega}(r)r^2d\omega^2$$

and g_{sr} is a short range component of the form

$$g_{sr} = g_{sr,tt}dt^2 + 2g_{sr,ti}dtdx_i + g_{sr,ij}dx_idx_j$$

[3]As the results in [**7**] are only for a corresponding Schrödinger equation with time independent coefficients, there is still some work to be done here.

with $S^Z(r^{-2})$ coefficients.

Similarly, respecting the same scaling as above, we consider magnetic potentials A of the form

$$A = A_{sr} + A_{lr}, \qquad A_{lr} \in S_{rad}(r^{-2}), \quad A_{sr} \in S(r^{-3})$$

as well as potentials V of the form

$$V = V_{sr} + V_{lr}, \qquad V_{lr} \in S_{rad}(r^{-3}), \quad V_{sr} \in S(r^{-4})$$

This definition is motivated by standard classes of relativistic metrics, e.g. Schwarzschild and Kerr. In turn, this leads to the decay rates prescribed by Price's law. We remark that choosing different polynomial decay rates for the coefficients will lead in a straightforward manner to adjusted decay rates.

Finally, our decay results are expressed relative to the distance to the Minkowski null cone $\{t = |x|\}$. This can only be done provided that there is a null cone associated to the metric g which is within $O(1)$ of the Minkowski null cone. However, in general the long range component of the metric produces a logarithmic correction to the cone. This issue can be remedied via a change of coordinates which roughly corresponds to using Regge-Wheeler coordinates in Schwarzschild/Kerr near spatial infinity, see [**28**]. This corresponds to choosing g_{lr} of the form $g_{lr} = g_{lr,\omega\omega}(r)r^2 d\omega^2$.

Now we can state our main result, which is a more accurate version of Principle 1.13 stated earlier:

THEOREM 3.17. *Let $(\mathcal{M}, g)$ be either an NT or BH asymptotically flat space time and potentials A, V so that g, A, V have size and regularity as described above. Assume that uniform energy bounds and local energy decay hold for the scalar wave equation* (1.2). *Then the pointwise decay bounds* (1.9) *hold (Price's Law).*

This result was first established by the author in [**28**] for stationary space-times, and then by the author and collaborators in the nonstationary case in [**22**]. These results follow a large body of work where partial results were obtained in various BH space-times, see e.g. [**1, 6, 11, 13**] and references therein. However, in all prior works the issues of obtaining various weighted energy estimates and that of obtaining pointwise decay rates were combined. It was only in [**28**] and [**22**] that the relation between these two concepts became clear. We also refer the reader to [**14**] for a more complete analysis in the Schwarzschild space-time.

The common feature of the two papers is the use of the above vector fields in combination with energy estimates and Klainerman-Sobolev type embeddings.

The idea of the argument in [**28**] is to use a time Fourier transform to translate the results in terms of resolvent bounds, and then use a careful analysis of the resolvent in order to get good pointwise bounds. The delicate part is that of obtaining a precise resolvent expansion near time frequency zero. This also provides a good clue as to the key importance of the zero frequency bound (1.7).

The argument in [**22**] happens all in the physical space. In this way it is more robust, and has greater potential to be useful in nonlinear stability problems. This is the proof that we outline in what follows. It is convenient to split it into several steps:

Step 1. Vector fields. A-priori, the local energy bounds apply to the linear waves and their derivatives by commuting the vector fields Z with the equation. Assuming compactly supported data, one expands this to bounds of the type

$$\|Z^\alpha u\|_{LE^1} \lesssim \|\nabla u(0)\|_{H^m}, \qquad |\alpha| \leq cm. \tag{3.11}$$

Step 2. Klainerman-Sobolev estimates. Once we have the above local energy decay estimates, one can ask about the pointwise bounds which can be derived from it. This is a weighted version of the Sobolev embeddings, and leads essentially to a bound of the form

$$|\nabla u| \lesssim \frac{1}{\langle r\rangle \langle t-r\rangle^{\frac{1}{2}}} \|\nabla u(0)\|_{H^m}. \tag{3.12}$$

Step 3. Near cone analysis. The goal of this step is to provide a better tool to obtain improved bounds near the cone, as well as to make the transition from ∇u to u in the previous bound. This is done by rewriting our equation in the form

$$\Box u = Qu,$$

where $\Box$ is the Minkowski d'Allembertian and Q is the remaining part of the operator P. Then we estimate the right hand side directly using the prior bounds and the solution u using the fundamental solution for the d'Alembertian. This leads to a decay bound for u, which is better than what one would obtain directly by integrating the previous bound on ∇u:

$$|u| \lesssim \frac{\log\langle t-r\rangle}{\langle r\rangle \langle t-r\rangle^{\frac{1}{2}}} \|\nabla u(0)\|_{H^N} \tag{3.13}$$

Step 4. Low frequency analysis. The aim of this step is to improve the above bound inside the inner region $r \leq t/2$. In this region we can use the scaling derivative S as a proxy for ∂_t, so our problem becomes essentially a zero frequency problem. Hence, using the zero resolvent bound (1.7) and appropriate Sobolev embeddings, we can improve the previous bound to

$$|u| \lesssim \frac{\log\langle t-r\rangle}{\langle t\rangle \langle t-r\rangle^{\frac{1}{2}}} \|\nabla u(0)\|_{H^N} \tag{3.14}$$

We note that in [**22**], an alternate estimate, called the stationary local energy decay was used instead of (1.7). However, the two bounds are essentially equivalent.

Step 5. Reiteration. The final bound is obtained by successively repeating Steps 3 and 4 in an alternating manner. Some additional care is required at the last application of Step 4, where the fundamental solution for the d'Alembertian is not applied directly in a pointwise fashion, but instead some cancellation is used.

4. Local energy decay

The goal of this section is to discuss local energy decay bounds in increasingly difficult situations. We begin our discussion with the Minkowski space-time, and then continue with the NT case, first stationary and then nonstationary. Finally we discuss the BH case. Some of this is still work in progress.

4.1. Local energy decay in Minkowski space-time. Here we consider the equation

$$\Box\phi = 0 \qquad \text{in } \mathbb{R}^{n+1}, \qquad \phi[0] = (\phi_0, \phi_1).$$

In the simplest form, local energy decay (also known as *Morawetz estimates*), can be stated as:

$$\|\nabla_{x,t}\phi(x,t)\|_{L^2(\mathbb{R}\times B_R)} \lesssim R^{\frac{1}{2}}\|\nabla_{x,t}\phi(x,0)\|_{L^2}.$$

These are based on the very simple heuristic that a speed 1 wave spends at most $O(R)$ time inside B_R.

Morawetz's proof uses the positive commutator method. If P and Q are self-adjoint, respectively skew-adjoint operators then

$$2\Re\langle P\phi, Q\phi\rangle = \langle [P,Q]\phi, \phi\rangle$$

Applying this with

$$P = \Box, \qquad Q = \partial_r + \frac{n-1}{2r},$$

already yields a positive commutator,

$$\|r^{-\frac{1}{2}}\nabla\!\!\!/\,\phi(x,t)\|_{L^2} + \|\phi(0,t)\|_{L^2} \lesssim \|\nabla_{x,t}\phi(x,0)\|_{L^2}, \qquad n = 3$$

but not exactly what is needed. The desired result is obtained in a similar manner by replacing the operator ∂_r with $a(r/R)\partial_r$ for a suitably chosen increasing function a.

4.2. Local energy decay in the nontrapping case. The simplest result in this direction is Theorem 2.14, which deals with small perturbations of the Minkowski space-time. Small perturbations have two advantages, namely (i) they are nontrapping, and (ii) the low frequency bounds are stable with respect to small metric perturbations.

Moving up to general NT space-times, we retain the first property above but not the second. Thus the difficulties come from low frequencies. In effect one could prove a high frequency bound which allows for lower order errors, and asserts that

$$\|u\|_{LE^1} + \|\nabla u\|_{L^\infty L^2} \lesssim \|f\|_{LE^*+L^1L^2} + \|\nabla u(0)\|_{L^2} + \|u\|_{L^2(\mathbb{R}\times B(0,R))}. \tag{4.15}$$

This shows that any obstruction to local energy decay is localized and compact. To continue our discussion, we consider first the stationary case. As described earlier, two enemies of local energy decay are

(i) negative eigenfunctions $u \in L^2$ with $P_\tau u = 0$, which correspond to time frequencies $\Im\tau < 0$ and to exponentially increasing solutions.

(ii) resonances $u \in \mathcal{LE}$ with $P_\tau u = 0$ and the outgoing radiation condition, and $\tau \in \mathbb{R}$. These correspond to polynomially increasing solutions.

One key element here is the following classical result, due originally to Kato and Agmon; see [**16**] and the references therein.

Theorem 4.18. *Assume that P is asymptotically flat and* self-adjoint. *Then there are no nonzero resonances for P embedded in the continuous spectrum.*

We emphasize here the self-adjointness requirement, without which the result is no longer true. However, it is stable with respect to small perturbations, even nonself-adjoint.

Given the above discussion, the following result in [**21**] is natural:

Theorem 4.19. *Assume that no negative eigenfunctions and zero resonances exist for a P in* (1.2) *in a stationary symmetric NT context. Then local energy decay holds.*

A similar result was previously proved for the nontrapping Schrödinger flow in the asymptotically flat setting in [**18**]. One key intermediate result in the proof of the above theorem is the two point local energy bound

$$\|u\|_{LE^1(0,T)} + \|\nabla u\|_{L^\infty(0,T;L^2)} \lesssim \|f\|_{LE^*+L^1L^2(0,T)} + \|\nabla u(0)\|_{L^2} + \|\nabla u(T)\|_{L^2} \tag{4.16}$$

which holds under the sole assumption that there are no zero resonances. This concludes our discussion of the self-adjoint stationary case, though the results in [**21**] go beyond the above theorem and also apply to operators P which have eigenvalues τ with $\Im\tau < 0$.

The remaining part of this section is devoted to the time dependent case. Suppose first that we have a continuous family of asymptotically flat NT operators P_h (and thus, also a continuous family of metrics). Can we continue the above spectral properties of P along this family ? The next stability result in [**21**] describes the only way this could happen:

Theorem 4.20. *Bifurcations to negative eigenfunctions for asymptotically flat NT symmetric operators P can occur only via zero resonances.*

As an extension of these ideas we are able in [**21**] to expand the range of our results to slowly varying operators:

THEOREM 4.21. *Assume that $P(t)$ is a slowly varying and almost self-adjoint NT asymptotically flat operator so that no negative eigenfunctions and zero resonances exist for $P(t)$ in* (1.2). *Then local energy decay holds for the linear wave equation* (1.2).

A brief outline of the proof of this theorem is as follows:

Step 1: Establish a two point local energy decay bound (4.16). This is done in three stages, partitioning the time frequencies into low, medium and high. The high frequency bounds are obtained by standard microlocal propagation arguments. The medium frequencies, instead, require Carleman estimates for the near part. Finally, the low frequencies are dealt with perturbatively, starting from the zero resolvent bound. The slowly varying property allows us to harmlessly absorb the errors in gluing the three regions. Interestingly enough, this step does not use the nonexistence of negative eigenvalues.

Step 2: Almost energy conservation, which is another consequence of the self-adjointness and the slowly varying property, is combined with the two point local energy decay in order to distinguish two disjoint energy growth modes, bounded versus exponentially growing energy (at a rate bounded away from zero). Notably, this step does not require a positive definite energy except near infinity.

Step 3. By the slowly varying property, the absence of negative eigenfunctions shows that only very small exponential growth rates are allowed. Then, by the previous step, energy actually has to stay bounded. Then the two point local energy decay turns into full local energy decay.

4.3. Local Energy Decay in black hole space-times. The geometry of BH space-times is more complicated than the NT space-times. Here we can identify three distinct regions, along with their main features:

(i) Exterior region $r \gg 1$.
Feature: asymptotically flat, $g = m + O(r^{-1})$.

(ii) Trapped set $\mathcal{T}$.
Features: (a) hyperbolic trapping,
(b) separate from horizon, and
(c) $\tau \neq 0$ on the trapped set (i.e. ∂_t energy positive there).

(iii) The event horizon $\mathcal{H}$.
Features: smooth, nondegenerate red shift and convexity.

The challenges we encounter are two-fold:

- Understand the coupling of three regions at high frequency, and the energy propagation in the high frequency limit.
- The separation between the three regions is blurred at medium and low frequencies, in particular region (ii) disappears there.

In the high frequency limit, null geodesics (light rays) can come from (a) infinity, (b) trapped set or (c) the event horizon. On the other hand, forward in time they can go to (i) the black hole (in finite time), (ii) the event horizon (in infinite time) (iii) the trapped set and (iv) to infinity. Our condition (BH-4) is critical here, as it disallows the scenario (b) $\to$ (ii), and thus prevents reiterations between the horizon and the trapped set.

At low frequencies, the difficulty is to seamlessly connect infinity with the event horizon. This works in a straightforward manner in the spherically symmetric case. Otherwise, the assumption (BH-5) seems to serve as an adequate replacement.

In what follows we provide a brief outline of ongoing work, joint with Jacob Sterbenz. This is still work in progress, so the reader should take it with a grain of salt.

As in the NT case, we begin our discussion with the self-adjoint stationary case. Our first result is as follows:

THEOREM 4.22. *Consider the wave equation* (1.2) *in the BH, stationary, self-adjoint case. Assume that there are no eigenvalues in* $\Im\tau < 0$, *and no resonances on* $\Im\tau = 0$. *Then local energy decay holds. The converse is also true.*

As in the NT case, the first step in the proof is to establish the high frequency local energy bound similar to (1.6), namely

$$\|u\|_{LE^1_{\mathcal{T}}} + \|\nabla u\|_{L^\infty L^2} \lesssim \|\Box u\|_{LE^*_{\mathcal{T}}+L^1L^2} + \|\nabla u(0)\|_{L^2} + \|u\|_{L^2_{loc}}. \tag{4.17}$$

This is done using microlocal analysis tools, and shows that we have complete resolvent bounds outside a compact set. We can also characterize eigenvalues and resonances:

PROPOSITION 4.23. *a) Eigenvalues and resonances can only occur in a compact subset of* $\{\Im\tau \le 0\}$.

b) Eigenvalues in $\Im\tau < 0$ *are smooth and decay exponentially at infinity.*

c) Resonances in $\Im\tau = 0$ *are smooth and decay like* $r^{-1}e^{i\tau r}$ *at infinity.*

From here on, a compactness argument combined with the absence of negative eigenvalues yields a resolvent extension to the full lower half-space, with a bound of the form

$$\|R_\tau\|_{L^2\to\dot H^1} \lesssim c(|\Im\tau|)$$

for some decreasing function $c(y)$ which equals $\dfrac{1}{y}$ for large y. In particular this gives sub-exponential decay bounds in the wave flow (1.2).

This then allows for a second compactness argument as we approach the real line, which gives a local energy resolvent bound,

$$\|R_\tau\|_{\mathcal{LE}^*\to\mathcal{LE}^1} \lesssim 1.$$

In turn this implies the desired local energy decay.

The above result is not entirely satisfactory because it only provides a qualitative bound, which is not very useful for the transition to the slowly varying case. This is also related to our lack of a result on the absence of embedded resonances. As a mechanism to remedy the above difficulty, we are proposing to use the second Killing field given in the assumption (BH-5). This implies that the null generator L extends to a Killing vector field which is time-like near the horizon. Then we first seek to prove the following result on absence of embedded resonances:

THEOREM 4.24. *For stationary BH space-times as above which also satisfy the assumption (BH-5) and associated self-adjoint operators P, there are no nonzero resonances for P embedded in the continuous spectrum.*

As a consequence of this, we aim to directly obtain the following improvement of the Theorem 4.22,

THEOREM 4.25. *Consider the wave equation* (1.2) *in the BH, stationary, self-adjoint case, where the assumption (BH-5) is also satisfied. Assume that there are no eigenvalues in $\Im\tau < 0$, and no zero resonances. Then local energy decay holds.*

Our goal is also to provide an alternate proof of this result, which yields a quantitative constant. This proof follows the same steps as in the proof of Theorem 4.21, using the two point local energy decay bound as an intermediate step. One difference here is in the treatment of medium frequencies; instead of a single Carleman weight coming from infinity, here we use a pair of weights, one coming from infinity and the other from the event horizon. This is where the additional Killing vector field is used.

Once this is achieved, the (qualitative) subexponential decay coming from the nonexistence of negative eigenfunctions yields a quantitative uniform energy bound. This in turn leads to local energy decay.

The last step in our analysis is to switch to slowly varying metrics. The results are summarized as follows:

THEOREM 4.26. *For continuous families of BH space-times as above, negative eigenvalues can only bifurcate via a zero resonance.*

THEOREM 4.27. *a) The local energy decay result above is stable with respect to small* stationary *perturbations.*

b) The local energy decay result above extends to slowly varying metrics (with a suitable extra assumption on the trapped set).

We observe that as a corollary of this result, one can get local energy decay for Kerr with large a by continuity only by knowing that no zero resonances exist in Kerr.

We also observe that the trapped set dynamics are a-priori unstable with respect to small nonstationary non-decaying perturbations.

5. Maxwell fields

In this section we describe recent and current work on decay estimates for electromagnetic waves, in either NT or BH space-time. This is the next natural step after the scalar wave equations. For local energy decay we have comparatively fewer results. For the Strichartz estimates there is no difference whatsoever, as we can rewrite the Maxwell system as a coupled system of wave equations. Finally, our results for pointwise decay are on par with those for the scalar wave equation.

5.1. Electromagnetic waves. The Maxwell system is a first order system for the electromagnetic field F, which is a two form on our Lorentzian space time $(\mathcal{M}, g)$. Using differential forms, the homogeneous Maxwell system can be written as

$$dF = 0, \qquad d * F = 0, \qquad F(0) = F_0. \tag{5.18}$$

In $3+1$ dimensions this is a six dimensional system with two compatibility conditions. In particular the initial condition cannot be chosen arbitrarily. From the above system one can also derive a coupled wave system for F, namely

$$\Box_g F = 0$$

where $\Box_g$ is the Hodge d'Alembertian.

The above equations can also be written using covariant differentiation:

$$\nabla^\alpha F_{\alpha\beta} = 0, \qquad \nabla_{[\gamma} F_{\alpha\beta]} = 0 \tag{5.19}$$

or using electromagnetic potential A, $F = dA$, with gauge condition $\nabla^\alpha A_\alpha = 0$, as

$$\nabla^\alpha \nabla_\alpha A_\beta = 0$$

though we will not take advantage of that here.

We are also interested in the inhomogeneous system

$$dF = G_1, \qquad d * F = G_2, \qquad F(0) = F_0, \tag{5.20}$$

Here the source terms are subject to compatibility conditions $dG_1 = 0$, $dG_2 = 0$, and F_0 also needs to satisfy two compatibility conditions. The physical set-up corresponds to $G_1 = 0$; however, for mathematical purposes it is also interesting to allow a nonzero G_1.

Similarly to the wave equation, we have an energy-momentum tensor

$$T_{\alpha\beta} = g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} + \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta},$$

which is divergence free in the homogeneous case,

$$\nabla^\alpha T_{\alpha\beta} = 0.$$

If X is a Killing field then

$$\nabla^\alpha (T_{\alpha\beta} X^\beta) = 0$$

and one obtains a conserved energy,

$$E_X(F) = \int_{\Sigma_t} *i_X T = \int_{\Sigma_t} T(X, N) dV_\Sigma.$$

Again, this is positive definite if X is space-like and Σ is space-like (e.g. in the stationary NT case). Then

$$E_X(F) \approx \|F\|^2_{L^2(\Sigma_t)}.$$

A special role is played in the Maxwell case by the electric and magnetic charges. Given a compact two dimensional surface S, we define the electric charge inside S by

$$Q^* = \int_S F^*$$

and the magnetic charge inside S as

$$Q = \int_S F$$

In the black hole case is natural to take S which includes the black hole inside. Then these are conserved quantities for the homogeneous Maxwell problem. However, in the inhomogeneous case the charges are nonconstant, and are associated to non-decaying solutions for the inhomogeneous equation. In the BH case, we even have non-decaying charged solutions for the homogeneous equation. As an example, in the Schwarzschild space-time we have the Hodge dual stationary solutions

$$F_0 = \frac{Q}{4\pi} d\omega_{\mathbb{S}^2}, \qquad *F_0 = \frac{Q^*}{4\pi} r^{-2} dr \wedge dt.$$

There is also a straightforward modification for Kerr, with a similar decay rate at infinity. The Minkowski space-time, and any NT space-time for that matter, does not admit such solutions.

5.2. Energy estimates. For the Maxwell system we define the energy norm as $\|F(t)\|_{L^2}$, and the higher energy norm as $\|F(t)\|_{H^k}$. For NT or stationary BH metrics, the transition between energy bounds and higher energy bounds is relatively straightforward, and follows the model of the scalar wave equation:

THEOREM 5.28. *Let $(\mathcal{M}, g)$ be either a NT or a stationary BH space-time as above. If uniform energy estimates hold for the Maxwell system* (5.18) *then higher Sobolev uniform energy estimates hold as well.*

In the stationary case, one can further interpret the energy estimates in terms of resolvent bounds. To define the resolvent take a time Fourier transform in (5.20) to obtain the stationary system:

$$d^0_\tau \hat{F}(\tau) = \hat{G}_1(\tau), \qquad d^0_\tau * \hat{F}(\tau) = \hat{G}_2$$

where the functions $\hat{G}_i(\tau)$ are subject to the compatibility conditions

$$d^0_\tau \hat{G}_i(\tau) = 0.$$

Here we define the operator d^0_τ as follows:

$$d^0_\tau F = dF - dt \wedge (\mathcal{L}_{\partial_t} - i\tau)F. \tag{5.21}$$

Then the resolvent

$$R_\tau(\hat{G}_1(\tau), \hat{G}_2(\tau)) = \hat{F}(\tau)$$

is defined whenever the above problem is solvable in L^2. The uniform energy bounds are related to resolvent as follows:

PROPOSITION 5.29. *Uniform energy bounds are equivalent to the resolvent bound*

$$\|R_\tau\|_{L^2 \to L^2} \lesssim |\Im\tau|^{-1}, \qquad \Im\tau < 0.$$

The case when $\tau = 0$ is of particular interest. There we simply use the notation d^0 instead of d^0_τ.

5.3. Local energy decay. The set-up for the Maxwell system is simpler then the one for the wave equation, at least as long as we consider only the homogeneous case with zero charges $Q = Q^* = 0$. Then in the NT case we have

DEFINITION 5.30. *We say that local energy decay holds for the homogeneous charge free Maxwell system* (5.18) *if*

$$\|F\|_{LE} + \|F\|_{L^\infty L^2} \lesssim \|F(0)\|_{L^2}.$$

A similar definition as above applies for the BH case, with similar modifications as in the scalar case in order to account for the trapped set $\mathcal{T}$.

The inhomogeneous case is more delicate. Adding source terms introduces charges into the system, and that obstructs standard decay estimates. In order to understand how to handle this difficulty, we first consider the case of a spherically symmetric space-time.

Restricting our attention to the charge associated to spheres, we can decompose both the Maxwell field and the sources G_1, G_2 into a radial and nonradial part. Due to the radiality of the metric, the Maxwell equations decouple into a radial and and a nonradial part.

To define this decomposition, for a function ψ in $\mathbb{R}^{3+1}$ we will denote by $\overline{\psi}$ its zero spherical harmonic. Then set

$$\bar{F} = \overline{F_{tr}} dt \wedge dr + \overline{F_{\phi\theta}} d\omega^2,$$

respectively

$$\bar{G} = \overline{G_{t\phi\theta}} dt \wedge d\omega^2 + \overline{G_{r\phi\theta}} dr \wedge d\omega^2.$$

The Maxwell equations for the radial parts become quite simple,

$$d\overline{F_{\phi\theta}} = \overline{G_{1,t\phi\theta}} dt + \overline{G_{1,r\phi\theta}} dr, \qquad d\overline{F^*_{\phi\theta}} = \overline{G_{2,t\phi\theta}} dt + \overline{G_{2,r\phi\theta}} dr.$$

Thus $\bar{F}$ is simply obtained by integrating its differential. In order to have decay at infinity, the integration needs to be also from infinity. However, as we have

$$|\bar{F}| \approx r^2(|F_{\phi\theta}| + |F^*_{\phi\theta}|), \qquad |\bar{G}_i| \approx r^2(|G_{i,t\phi\theta}| + |G_{i,r\phi\theta}|),$$

the condition $G_i \in LE^*$ is insufficient. Instead we need to assume at least that $rG_1 \in LE^*$, in which case we obtain the bound

$$\|\langle r\rangle \bar{F}\|_{LE} \lesssim \|\langle r\rangle \bar{G}_1\|_{LE^*} + \|\langle r\rangle \bar{G}_2\|_{LE^*}$$

In the case of metrics without spherical symmetry, the radial and nonradial modes can no longer be cleanly decoupled. In particular, the spheres are no longer invariantly defined objects. However, to the r^{-2} order they still decouple as $r \to \infty$. This motivates the following definition:

Definition 5.31. *We say that local energy decay holds for the inhomogeneous Maxwell system if we have the estimate*

$$\begin{aligned} &\|F\|_{LE_{\mathcal{T}}} + \|\langle r\rangle \bar{F}\|_{LE} + \|F\|_{L^\infty L^2} \\ &\lesssim \|F(0)\|_{L^2} + \|(G_1, G_2)\|_{LE^*_{\mathcal{T}}} + \|\langle r\rangle (\bar{G}_1, \bar{G}_2)\|_{LE^*}. \end{aligned} \tag{5.22}$$

The transition to higher local energy decay bounds follows the same rules as for the uniform energy estimates:

Theorem 5.32. *Let $(\mathcal{M}, g)$ be either a NT or a stationary BH space-time as above. If local energy decay estimates hold for the Maxwell system* (5.20) *then higher local energy decay estimates hold as well.*

Again, the result extends to slowly varying BH space times once some care is taken with the trapped set $\mathcal{T}$.

We next consider the interpretation of local energy decay in terms of resolvent bounds in the stationary case. For that we use again the norms $\mathcal{LE}$, and $\mathcal{LE}^*$ obtained by freezing time in the definition of LE, respectively LE^*.

Proposition 5.33. *Consider either a stationary NT or BH space-time. Then local energy decay is equivalent to the uniform resolvent bound*

$$\|R_\tau(G_1, G_2)\|_{\mathcal{LE}} + \|\langle r\rangle \overline{R_\tau(G_1, G_2)}\|_{\mathcal{LE}} \lesssim \|(G_1, G_2)\|_{\mathcal{LE}^*} + \|\langle r\rangle (\bar{G}_1, \bar{G}_2)\|_{\mathcal{LE}^*}$$

for $\Im \tau \leq 0$.

As in the scalar case, one needs to carefully define the resolvent on the real line as the limit when the real line is approached from the lower half-space. This implies that the field $F = R_\tau(G_1, G_2)$ must satisfy the outgoing radiation (Sommerfeld) condition for Maxwell, otherwise known as the Silver-Muller conditions. Thus, in addition to complex eigenvalues, one is led to introduce the embedded resonances as an obstruction to local energy decay:

Definition 5.34. *$F \in \mathcal{LE}$ is an embedded resonance associated to the real time frequency τ if it satisfies the outgoing radiation condition and $d^0_\tau F = 0$, $d^0_\tau * F = 0$.*

A special case is that of zero resonances. These have simpler asymptotics

$$F(r, \omega) = F^0(\omega) r^{-2} + O(r^{-3})$$

and a symbol type behavior at infinity. There the outgoing radiation condition is meaningless. According to the above decay rates, the charged solutions are examples of zero resonances. However, this is a stable zero mode, which we exclude; this is achieved using the different weight for the radial mode in our function spaces. Thus we define

DEFINITION 5.35. *A zero resonance for the Maxwell system is a charge free stationary solution F for the Maxwell equations, i.e. which solves $d^0 F = 0$, $d^0 * F = 0$, with decay at infinity $F, \langle r\rangle \bar{F} \in \mathcal{LE}_0$.*

Their absence is equivalent to a bound of the form

$$\|F\|_{\mathcal{LE}} + \|r\bar{F}\|_{\mathcal{LE}} \lesssim \|d^0 F\|_{\mathcal{LE}^*} + \|d^0 F^*\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{d^0 F}\|_{\mathcal{LE}^*} + \|\langle r\rangle \overline{d^0 F^*}\|_{\mathcal{LE}^*} \tag{5.23}$$

which we will refer to in the sequel as the *zero energy resolvent bound*. One can also state a higher regularity version of this bound, but that follows from the one above.

5.4. Local energy decay results. The first local energy decay result in this setting, proved in [**26**], is as follows:

THEOREM 5.36. *Consider a spherically symmetric stationary BH space-time. Then:*

a) Uniform energy estimates hold for the inhomogeneous Maxwell system.

b) Local energy decay holds for the inhomogeneous Maxwell system.

Some related results were obtained in [**3**], [**2**].

We conjecture that a similar bound holds also in the nonradial case, under suitable spectral assumptions:

CONJECTURE 5.37. *Consider a stationary BH space-time, so that the homogeneous Maxwell system has no negative eigenfunctions and no zero resonances. Then*

a) Uniform energy estimates hold for the inhomogeneous Maxwell system.

b) Local energy decay holds for the inhomogeneous Maxwell system.

5.5. Pointwise decay bounds. For the purpose of this section, we assume that the metric g is asymptotically flat in the sense of Section 3. We begin with the statement of the Price law, which carries somewhat different weights here. Further, different components of F in a null frame decay at better rates; this is what is usually called *peeling estimates*. See [**24, 8**]. To state this we work in normalized (Regge-Wheeler type) coordinates near infinity, and introduce a null frame $(\partial_u, \partial_v, e_A, e_B)$, where as usual we set

$$u = t - r, \qquad v = t + r$$

and (e_A, e_B) is an orthonormal frame of $\mathbb{S}^2$.

Definition 5.38. *We say that pointwise decay bounds hold for the Maxwell system if for all localized data $F(0)$ we have*

$$
\begin{aligned}
|F_{uA}| &\lesssim \kappa \frac{1}{\langle t\rangle \langle t-r\rangle^3} \\
|F_{uv}| &\lesssim \kappa \frac{1}{\langle t\rangle^2 \langle t-r\rangle^2} \\
|F_{AB}| &\lesssim \kappa \frac{1}{\langle t\rangle^2 \langle t-r\rangle^2} \\
|F_{vA}| &\lesssim \kappa \frac{1}{\langle t\rangle^3 \langle t-r\rangle}
\end{aligned}
\tag{5.24}
$$

where

$$\kappa = \|F(0)\|_{H^m}.$$

Our last result is the Maxwell counterpart of our earlier result for the scalar wave:

Theorem 5.39 ([**23**]). *Let $(\mathcal{M}, g)$ be a BH asymptotically flat space-time. Assume that uniform energy bounds and local energy decay estimates hold for the Maxwell system. Then pointwise decay estimates (Price's law) hold as well.*

This last result does not require the metric to be radial or stationary. Its proof is not dissimilar to the proof of the similar result for the scalar wave equation, but with changes to account for the different zero resolvent bound, and also for the charges.

References

[1] L. Andersson, P. Blue, *Hidden symmetries and decay for the wave equation on the Kerr spacetime*, arXiv:0908.2265.

[2] L. Andersson and P. Blue, *A uniform energy bound for Maxwell fields in the exterior of a slowly rotating Kerr black hole*, arXiv:1310.2664.

[3] P. Blue, *Decay of the Maxwell field on the Schwarzschild manifold.* J. Hyperbolic Diff. Equ. **5** (2008), no.4, 807–856.

[4] P. Blue, A. Soffer, *Semilinear wave equations on the Schwarzschild manifold I: local decay estimates.* Advances in Differential Equations **8** (2003), 595–614.

[5] P. Blue, A. Soffer, *The wave equation on the Schwarzschild metric II: Local decay for the spin-2 Regge-Wheeler equation*, J. Math. Phys., **46** (2005), 9pp.

[6] P. Blue, J. Sterbenz, *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space*, Comm. Math. Phys. **268** (2006), no. 2, 481–504.

[7] Nicolas Burq, Colin Guillarmou and Andrew Hassell, *Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics.* Geom. Funct. Anal. 20 (2010), no. 3, 627–656.

[8] D. Christodoulou, S. Klainerman, *Asymptotic properties of linear field equations in Minkowski space.* Comm. Pure Appl. Math. **43** (1990), 137–199.

[9] M. Dafermos, I. Rodnianski, *The red-shift effect and radiation decay on black hole spacetimes.* Comm. Pure Appl. Math. **62** (2009), no. 7, 859–919. MR2527808

[10] M. Dafermos, I. Rodnianski, *Lectures on black holes and linear waves*, Evolution equations, 97–205, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013.

[11] M. Dafermos, I. Rodnianski, *Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases* $|a| \ll M$ *or axisymmetry*, arXiv:1010.5132

[12] M. Dafermos, I. Rodnianski, *The black hole stability problem for linear scalar perturbations*, Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, T. Damour et al (ed.), World Scientific, Singapore (2011), 132–189,

[13] M. Dafermos, I. Rodnianski, and Y. Shlapentokh-Rothman *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case* $|a| < M$, arXiv:1402.7034v1

[14] Roland Donninger, Wilhelm Schlag, and Avy Soffer. A proof of Price's law on Schwarzschild black hole manifolds for all angular momenta. *Adv. Math.*, 226(1):484–540, 2011.

[15] W. Inglese, F. Nicolò, *Asymptotic properties of the electromagnetic field in the external Schwarzschild.* Ann. Henry Poncaré 1 (5) (2000), 895–944.

[16] Herbert Koch and Daniel Tataru, *Carleman estimates and absence of embedded eigenvalues.* Comm. Math. Phys. 267 (2006), no. 2, 419–449.

[17] L. Mason, J.P. Nicolas, *Peeling of Dirac and Maxwell fields on a Schwarzschild background.* Journal of Geometry and Physics **62** (2012), 867–889.

[18] Jeremy Marzuola, Jason Metcalfe and Daniel Tataru, *S*trichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. J. Funct. Anal. 255 (2008), no. 6, 1497–1553.

[19] J. Marzuola, J. Metcalfe, D. Tataru, M. Tohaneanu, *Strichartz estimates on Schwarzschild black hole backgrounds*, Comm. Math. Phys. **293** (2010), no. 1, 37–83.

[20] Jason Metcalfe and Daniel Tataru. *Global parametrices and dispersive estimates for variable coefficient wave equations.* Math. Ann. 353 (2012), no. 4, 1183–1237.

[21] J. Metcalfe, J. Sterbenz, D. Tataru, *Local Energy Decay for Scalar Fields on Time Dependent Non-trapping Backgrounds*, in preparation

[22] J. Metcalfe, D. Tataru, M. Tohaneanu, *Price's law on nonstationary spacetimes*, Adv. Math. **230** (2012), no. 3, 995–1028.

[23] J. Metcalfe, D. Tataru, M. Tohaneanu, *Pointwise decay for the Maxwell field on black hole space-times*, arXiv:1411.3693

[24] R. Penrose, *Zero rest-mass fields including gravitation: asymptotic behavior*, Proc. Roy. Soc. A **284** (1965) 159–203

[25] Richard H. Price. Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. *Phys. Rev. D (3)*, 5:2419–2438, 1972.

[26] J. Sterbenz, D. Tataru, *Local energy decay for Maxwell fields part I: Spherically symmetric black-hole backgrounds*, arXiv:1305.5261

[27] Daniel Tataru. *Parametrices and dispersive estimates for Schrödinger operators with variable coefficients. Amer. J. Math.*, 130(3):571–634, 2008.

[28] Daniel Tataru, *Local decay of waves on asymptotically flat stationary space-times*, Amer. J. Math, 135 (2013), no. 2, 361–401.

[29] D. Tataru, M. Tohaneanu, *A local energy estimate on Kerr black hole backgrounds.* Int. Math. Res. Not. IMRN 2011, no. 2, 248–292. MR2764864

[30] M. Tohaneanu, *Strichartz estimates on Kerr black hole backgrounds.* Transactions of the AMS, Volume 364, Number 2 (2012), 689–702.

Department of Mathematics, University of California, Berkeley, CA 94720

E-mail address: `tataru@math.berkeley.edu`