Rigidity results in general relativity: A review

Alexandru D. Ionescu and Sergiu Klainerman

ABSTRACT. Despite a common perception in the physics community, the Black Hole Rigidity problem remains wide open when one removes the highly restrictive real analyticity assumption underlying the classical results. In this survey we review the progress made in the last ten years in understanding the conjecture in the more realistic setting of smooth spacetimes. We review both local and global results and discuss the new mathematical ideas behind them. We present three types of global results which assert, under somewhat different assumptions, that any stationary solution close to a non-extremal Kerr must be isometric to a non-extremal Kerr, whose parameters a, M are determined by the ADM mass and angular momentum of the given stationary solution. The results illustrate an important geometric obstruction in understanding the full rigidity problem, the possible presence of trapped null geodesics perpendicular to the stationary Killing vectorfield. The key insight in all these results is that such null geodesics are non-existent in any non-extremal Kerr and thus, roughly, in any small perturbation of it.

CONTENTS

1. Introduction

A fundamental conjecture in General Relativity[1] asserts that the domains of outer communication of regular[2], stationary, four dimensional, vacuum black hole solutions are isometrically diffeomorphic to those of Kerr black holes. One expects, due to gravitational radiation, that general, asymptotically flat, dynamic, solutions of the Einstein-vacuum equations settle down, asymptotically, into a stationary regime. A similar scenario is expected to hold true in the presence of matter. Thus the conjecture, if true, would characterize all possible asymptotic states of the general evolution.

So far the conjecture has been resolved, by combining results of Hawking [**18**], Carter [**8**], and Robinson [**33**], under the additional hypothesis of non-degenerate horizons and *real analyticity* of the space-time. The assumption of real analyticity, however, is both hard to justify and difficult to dispense of. One can show, using standard elliptic theory, that stationary solutions are real analytic in regions where the corresponding Killing vector-field **T** is time-like, but there is no reason to expect the same result to hold true in the ergo-region (in a Kerr spacetime $\mathcal{K}(a,m), 0 < a < m$ the Killing vector-field **T**, which is time-like in the asymptotic region, becomes space-like in the ergo-region). In view of the relevance of the conjectured result to the general problem of evolution, i.e. the final state conjecture, there is also no reason to expect that, by losing gravitational radiation, general solutions become, somehow, analytic. Thus the assumption of analyticity is a fundamental limitation of the present uniqueness results[3] Here is a more precise version of the Carter-Robinson-Hawking result.

THEOREM 1.1 (Carter-Robinson-Hawking). *The domain of outer communication of a real analytic regular, stationary (i.e. there exists a killing vectorfield* **T** *which is timelike in the asymptotic region), asymptotically flat, four dimensional, vacuum black hole solution is isometrically diffeomorphic to the domain of outer communication of a Kerr black hole.*

The theorem relies on the following steps.
  (1) Based on the observation that, *though a general stationary space-time may seem quite complicated, its behavior along the event horizon is remarkably simple,* Hawking has shown that in addition to the original, stationary, Killing field, which has to be tangent to the event horizon, there must exist, infinitesimally along the horizon and tangent to its generators, an additional Killing vector-field.

---

[1]See reviews by B. Carter [**7**] and P. Chrusciel [**11**], [**13**], for a history and review of the current status of the conjecture.

[2]The notion of regularity needed here requires a careful discussions concerning the geometric hypothesis on the space-time.

[3]The results based on analyticity can be reformulated as a proof of the fact that there can be no other explicit stationary solutions. Note also that the case of static solutions has been treated in full generality, without assuming analyticity, by Israel [**24**] and Bunting-Masood ul Alam [**6**].

(2) In the case of a non-degenerate horizon the Hawking vectorfield can be *extended*, by standard hyperbolic PDE techniques, to the full domain of dependence of the horizon, see [**17**].

(3) The extension problem is however *ill posed* in the complement of the domain of dependence, i.e. in the domain of outer communication of the black hole. To overcome this difficulty Hawking assumes analyticity and extends the vectorfield by a Cauchy-Kowalewski type argument. In this step the field equations are no longer used; the assumption of analyticity, which in effect replaces the Einstein equations by the Cauchy-Riemann equations, completely trivializes the problem.

(4) As a consequence of the previous step, the space-time under consideration is not just stationary but also axi-symmetric, a situation for which Carter-Robinson's uniqueness theorem [**8**], [**33**] applies. It is interesting to remark that this final step does not require analyticity.

A similar result holds for the Einstein-Maxwell equations. Namely the only real analytic, stationary, regular asymptotically flat solutions of the Einstein-Maxwell equations belong to the Kerr-Newman family. The reduction to the axially symmetric case, due to Hawking, follows precisely the same argument as in the vacuum case. The rigidity of stationary, axially symmetric solutions is due to Mazur [**30**], see also [**13**].

The goal of this article is to review recent results which aim to prove the conjecture without appealing to analyticity. We focus our discussion to the case of the vacuum, but we will also mention some of the more interesting extensions to the case of the Einstein-Maxwell equations.

We start with a discussion, in section 2, of local extension results for Killing vectorfields. The setting is very general; we consider a Killing vectorfield $\mathbf{Z}$ defined in a domain O of a Ricci flat, smooth, semi-riemannian manifold $(\mathbf{M}, \mathbf{g})$ and consider the question of whether $\mathbf{Z}$ admits a smooth Killing extension in a full neighborhood of a point p in the boundary ∂O. It turns out that the answer is affirmative if the boundary verifies what we call the *strict null convexity condition*. This condition, concerning the behavior of null geodesics tangent to ∂O at p, is automatically satisfied on a Riemannian manifold, but imposes a serious restriction if $\mathbf{g}$ is Lorentzian. If in addition the manifold admits a nowhere vanishing[4] Killing vectorfield $\mathbf{T}$, which commutes with $\mathbf{Z}$ in O, we show that $\mathbf{Z}$ can be extended past p under a weaker assumption which we call $\mathbf{T}$-strict null convexity. This is a condition which affects only the null vectors at $T_p(\partial O)$ orthogonal to $\mathbf{T}$.

It is important to stress here that in the particular case of Kerr space-time $\mathcal{K}(a, m)$, domains of the form $r_* < r < R$ with r the usual Boyer-Lindquist coordinate and r_* its value on the horizon are not, in general,

---

[4]It suffices to consider a Killing vectorfield defined in a neighborhood of the point $p \in \partial O$.

strictly null-convex at $r = R$ but are all strictly **T**-null convex, where **T** is the stationary Killing field of the Kerr solution. This fact, first discovered in [**20**],[**21**], plays a fundamental role in the global results discussed in section 3 of this paper.

The null convexity condition is a particular instance of the more general pseudoconvexity condition[5] of Calderon-Hörmander. It is a necessary condition to derive uniqueness results for ill posed problems[6] based on Carleman type estimates. It is not a priori clear that the same necessary condition is relevant to our extension problem. The main goal of section 2 is to describe the geometric ideas by which the extension problem can in fact be turned into an unique continuation problem. The results are stated in theorems 2.3, 2.4. Though they are both very general (they hold for arbitrary semi-Riemannian manifolds!) they rely in an essential way on the Ricci flat condition. We also review related local extension results, see theorems 2.19, 2.20, for the Hawking vectorfield in a neighborhood of a bifurcate horizon.

In section 3 we discuss three global results, see [**20**], [**2**], [**3**], concerning the black hole uniqueness problem, which assert, under somewhat different assumptions, that any stationary solution close to a non-extremal Kerr must be isometric to a non-extremal Kerr, whose parameters a, M are determined by their ADM mass and angular momentum. They are all based on specific regularity, non-degeneracy and asymptotic flatness assumptions discussed in subsection 3.2.

The first two results are based on the local characterization of the Kerr solution, due to Mars [**27**], by the vanishing of the so called Mars-Simon tensor $\mathcal{S}$. In theorem 3.14 we make an assumption on the bifurcation sphere of the horizon which implies that $\mathcal{S}$ vanishes along the horizon. We then derive a wave equation for $\mathcal{S}$ and show, by unique continuation results, that $\mathcal{S}$ must vanish everywhere. In theorem 3.14 we assume instead that $\mathcal{S}$ is sufficiently small and rely on the extension results discussed in section 2 to show that the spacetime is axially symmetric. The rigidity result then follows by applying the Carter-Robinson theorem. Both results assume the presence of a unique non-degenerate horizon. This condition was later removed by Wong and Yu in [**40**] by an ingenious argument based on the mountain pass lemma.

The third rigidity result differs substantially from the other two in that we only make a smallness assumption on the bifurcate sphere. More precisely we assume that the stationary vectorfield is small on the bifurcate sphere and deduce that the entire domain of outer communication is isometric to that of a Kerr solution with small angular momentum. This is the first uniqueness result, in the framework of smooth, asymptotically flat, stationary solutions, which combines local considerations near the horizon, via Carleman estimates, with information obtained by global elliptic estimates.

---

[5]Which applies to general, scalar linear partial differential operators

[6]Problems where existence is by no means guaranteed.

These results illustrate an important geometric obstruction in understanding the full rigidity problem, the possible presence of trapped null geodesics perpendicular to the stationary Killing vectorfield. The key insight[7] in all these results is that such null geodesics are non-existent in any non-extremal Kerr and thus, roughly, in any small perturbation of it.

In the last section we formulate, together with S. Alexakis, a general conjecture which illustrates the importance of trapped null geodesics perpendicular to **T** and thus the importance of developing strategies based on global considerations, not just on unique continuation methods starting from the horizon.

2. Local rigidity results

In this section we revisit the extension problem for Killing vector-fields in smooth Ricci flat Lorentzian manifolds and its relevance to the black hole rigidity problem. In the most general situation the problem can be stated as follows:

Assume $(\mathbf{M}, \mathbf{g})$ *is a given smooth semi-riemannian manifold,* $O \subseteq \mathbf{M}$ *is an open subset, and* Z *is a smooth Killing vector-field in* O*. Under what assumptions does* Z *extend (uniquely) as a Killing vector-field in* $\mathbf{M}$*?*

A classical result[8] of Nomizu establishes such a unique extension provided that the metric is real analytic, $\mathbf{M}$ and O are connected and $\mathbf{M}$ is simply connected. The result has been used, see [**18**] and [**13**], to reduce the black hole rigidity problem, for real analytic stationary solutions of the Einstein field equations, to the simpler case of axial symmetry treated by the Carter-Robinson theorem. This reduction has been often regarded as decisive, especially in the physics literature, without a clear understanding of the sweeping simplification power of the analyticity assumption. Indeed the remarkable thing about Nomizu's theorem, to start with, is the fact the metric is not assumed to satisfy any specific equation. Moreover no assumptions are needed about the boundary of O in $\mathbf{M}$ and the result is global with only minimal assumptions on the topology of $\mathbf{M}$ and O. The result is clearly wrong in the case of smooth manifolds $(\mathbf{M}, \mathbf{g})$ which are not real analytic. To be able to say anything meaningful we need to both restrict the metric $\mathbf{g}$ by realistic equations and make specific assumptions about the boundary of O. Local and global assumptions are also need to be carefully separated.

In this section we limit our attention to a purely local description of the extension problem in the smooth case. We assume that $(\mathbf{M}, \mathbf{g})$ is a non-degenerate Ricci flat, semi-riemannian metric i.e.

$$(2.1) \qquad\qquad\qquad \mathbf{Ric}(\mathbf{g}) = 0,$$

---

[7]A related fact plays a fundamental role in recent linear stability results concerning solutions of the scalar wave equation in a non-extremal Kerr, see [**15**] and the references therein.

[8]See [**31**]. We rely here on the version of the theorem given in [**13**] .

and denote by $\mathbf{D}$ the Levi-Civita connection induced by $\mathbf{g}$.

We define the following crucial concept[9].

DEFINITION 2.1. *A domain $O \subset \mathbf{M}$ is said to be strictly null-convex at a boundary point $p \in \partial O$ if there exists a smooth function h, defined in a small neighborhood U of p, such that $dh(p) \neq 0$ and $O \cap U = \{x \in U : h(x) < 0\}$ (i.e. h is a nondgenerate defining function for O at p) which verifies the following null convexity property at p:*

$$(2.2) \qquad\qquad \mathbf{D}^2 h(X, X)(p) < 0$$

for all null vectors X (i.e. $\mathbf{g}(X, X) = 0$) tangent to ∂O at p (i.e. $X(h) = 0$ at p).

It is easy to see that (2.2), does not depend on the choice of the defining function h. The strict null-convexity condition is automatically satisfied if the metric $\mathbf{g}$ is Riemannian. It is also satisfied for Lorentzian metrics $\mathbf{g}$ if ∂O is space-like at p, but it imposes serious restrictions for time-like hypersurfaces. It clearly fails if ∂O is null in a neighborhood of p. Indeed in that case we can choose the defining function h to be optical, i.e.,

$$(2.3) \qquad\qquad \mathbf{D}^\alpha h \mathbf{D}_\alpha h = 0$$

at all points of ∂O in a neighborhood of p, and thus, choosing $X^\alpha = \mathbf{D}^\alpha h$, we have,

$$X^\alpha X^\beta \mathbf{D}_\alpha \mathbf{D}_\beta h = \frac{1}{2} X(\mathbf{D}^\alpha h \mathbf{D}_\alpha h) = 0.$$

One can also show that unique continuation fails in this case.

Under the assumption that $\mathbf{M}$ contains a Killing vectorfield $\mathbf{T}$ we also define the following variant of the null convexity condition.

DEFINITION 2.2. *The domain $O \subset \mathbf{M}$ is said to be strict $\mathbf{T}$- null-convex at a boundary point $p \in \partial O$ if the defining function h at p is $\mathbf{T}$ invariant and verifies the convexity condition (2.2) for all null vectors $X \in T_p(\partial O)$ which are orthogonal to $\mathbf{T}$.*

The following general extension principle was proved in [22]. A previous, related, version appeared in [1]. We note however that the version used in [1] applies to a bifurcate null hypersurface which fails to be smooth at all points on the bifurcate sphere. The results mentioned in theorems 2.3, 2.4, based on unique continuation, can however be extended to cover that case, see [20].

THEOREM 2.3. *Assume that $(\mathbf{M}, \mathbf{g})$ is a smooth d-dimensional Ricci flat, semi-riemannian manifold and $O \subset \mathbf{M}$ is a strictly null-convex domain at a point $p \in \partial O$. We assume that Z is a Killing vectorfield in O. Then Z extends as a Killing to a neighborhood of the point p in $\mathbf{M}$.*

Using similar techniques one can also prove the following.

---

[9]In our previous papers we have used the broader terminology of *pseudo-convexity* condition, which applies to a given scalar linear PDE.

THEOREM 2.4. *If* $(\mathbf{M}, \mathbf{g})$ *admits a (nowhere vanishing) Killing vector-field* $\mathbf{T}$ *and* Z *is a Killing vectorfield in* $\mathbf{O}$ *which commutes with* $\mathbf{T}$, *then the same extension result holds true if we replace strict null convexity by the weaker strict* $\mathbf{T}$-*null convexity condition. Moreover the extended* Z *continues to commute with* $\mathbf{T}$.

The proof of both theorems is based on the following ideas[10].

(1) Extend the vectorfield Z in a full neighborhood of the point p by solving a Jacobi type equation along a family of congruent geodesics transversal to ∂O. In the case of theorem (2.4) one needs to make sure that extended Z still commutes with $\mathbf{T}$. This can easily be done by choosing a congruence left invariant by $\mathbf{T}$, i.e. such that the generator L of the congruence commutes with $\mathbf{T}$. We note that (unlike other contexts in the GR literature where L denotes a null vectorfield) L here does not have to be null.

(2) Derive a closed system of covariant wave equations for a modified version of the Lie derivative of the curvature tensor R, denoted W, coupled with transport equations for the deformation tensor ${}^{(Z)}\pi$ of the extended Z.

(3) Use a unique continuation argument to show that both W and ${}^{(Z)}\pi$ have to vanish in a full neighborhood of p. To implement the continuation criterion one needs the strict null convexity conditions in the definitions 2.1 and 2.2.

We start with a few general results:

LEMMA 2.5. *For arbitrary* k-*covariant tensor-field* V *and vector-field* X *we have,*

$$(2.4) \qquad \mathbf{D}_\beta(\mathcal{L}_X V_{\alpha_1\ldots\alpha_k}) - \mathcal{L}_X(\mathbf{D}_\beta V_{\alpha_1\ldots\alpha_k}) = \sum_{j=1}^{k} {}^{(X)}\Gamma_{\alpha_j\beta\rho} V_{\alpha_1\ldots}{}^{\rho}{}_{\ldots\alpha_k}$$

where ${}^{(X)}\pi = \mathcal{L}_X \mathbf{g}$ *is the deformation tensor of* X *and,*

$${}^{(X)}\Gamma_{\alpha\beta\mu} := \frac{1}{2}(\mathbf{D}_\alpha {}^{(X)}\pi_{\beta\mu} + \mathbf{D}_\beta {}^{(X)}\pi_{\alpha\mu} - \mathbf{D}_\mu {}^{(X)}\pi_{\alpha\beta}).$$

LEMMA 2.6. *Let* X *be a vectorfield with deformation tensor* ${}^{(X)}\pi$ *and define,*

$${}^{(X)}P_{\alpha\beta\mu} := (1/2)(\mathbf{D}_\alpha {}^{(X)}\pi_{\beta\mu} - \mathbf{D}_\beta {}^{(X)}\pi_{\alpha\mu}).$$

Then,

$$\mathbf{D}_\nu {}^{(X)}P_{\alpha\beta\mu} - \mathbf{D}_\mu {}^{(X)}P_{\alpha\beta\nu}$$
$$(2.5) \qquad = (\mathcal{L}_X \mathbf{R})_{\alpha\beta\mu\nu} - (1/2){}^{(X)}\pi_\alpha{}^\rho \mathbf{R}_{\rho\beta\mu\nu} - (1/2){}^{(X)}\pi_\beta{}^\rho \mathbf{R}_{\alpha\rho\mu\nu}$$

where $\mathbf{R}$ *is the Riemann curvature tensor of the metric* $\mathbf{g}$.

---

[10]In [**1**] similar results were proved using a frame dependent approach.

Note that,

$$(2.6) \qquad {}^{(X)}\Gamma_{\alpha\mu\beta} \;\; = \;\; {}^{(X)}P_{\alpha\beta\mu} + \frac{1}{2}\mathbf{D}_\mu\, {}^{(X)}\pi_{\alpha\beta}$$

Recall that a Weyl field on $\mathbf{M}$ is a four covariant trace-less tensor, verifying all the symmetries of the Riemann curvature tensor. Note that the Lie derivative of a Weyl field may fail to have vanishing trace. The leads us to the following modified definition.

DEFINITION 2.7. *Given a Weyl field W, X an arbitrary vectorfield and $\omega_{\alpha\beta}$ an arbitrary 2-form on $\mathbf{M}$, we define,*

$$\widehat{\mathcal{L}}_{X,\omega}W := \mathcal{L}_X W - \frac{1}{2}(\,{}^{(X)}\pi + \omega) \odot W$$

where, for any 2-tensor B, $B \odot W$ denotes the tensor,

$$(B \odot W)_{\alpha\beta\gamma\delta} := B_\alpha{}^\lambda W_{\lambda\beta\gamma\delta} + B_\beta{}^\lambda W_{\alpha\lambda\gamma\delta} + B_\gamma{}^\lambda W_{\alpha\beta\lambda\delta} + B_\delta{}^\lambda W_{\alpha\beta\gamma\lambda}.$$

LEMMA 2.8. *The tensor $\widehat{\mathcal{L}}_{X,\omega}W$ defined above is a Weyl field.*

2.1. Proof of Theorem (2.3). To prove the theorems we first extend Z past p according to the following equation

$$(2.7) \qquad\qquad \mathbf{D}_L\mathbf{D}_L Z = \mathbf{R}(L, Z)L,$$

Let ${}^{(Z)}\pi = \mathcal{L}_Z\mathbf{g}$ be the deformation tensor of Z. To show that ${}^{(Z)}\pi \equiv 0$ in a neighborhood U of p we need to prove that ${}^{(Z)}\pi$, ${}^{(Z)}P$, $\mathcal{L}_Z\mathbf{R}$ all vanish, simultaneously in U. The idea is to try to derive transport equations for ${}^{(Z)}\pi$ and ${}^{(Z)}P$, along the geodesics generated by L, coupled to a covariant wave equation for $\mathcal{L}_Z\mathbf{R}$. To do this we will need however to redefine slightly the main quantities. The crucial ingredient which makes possible to derive useful transport equations is the following.

LEMMA 2.9. *If Z is extended according to (2.7) then the deformation tensor $\pi := {}^{(Z)}\pi$ of Z verifies*

$$(2.8) \qquad\qquad \pi_{\alpha\beta}L^\beta = 0.$$

2.1.1. *The main coupled system.* To derive the desired transport equations we would also need that $P = {}^{(Z)}P$ verifies $P_{\alpha\beta\mu}L^\mu = 0$. This is not true however and we are forced to introduce the modification,

$$(2.9)\;\; \mathbf{P}_{\alpha\beta\mu} \;\; = \;\; P_{\alpha\beta\mu} - \frac{1}{2}\mathbf{D}_\mu\omega_{\alpha\beta} = (1/2)(\mathbf{D}_\alpha\pi_{\beta\mu} - \mathbf{D}_\beta\pi_{\alpha\mu} - \mathbf{D}_\mu\omega_{\alpha\beta})$$

with ω a 2-form chosen precisely such that $\mathbf{P}_{\alpha\beta\mu}L^\mu = 0$. This leads to the following.

LEMMA 2.10. *If we define ω in $\mathbf{M}$ as the solution of the transport equation*

$$(2.10) \qquad\qquad \mathbf{D}_L\omega_{\alpha\beta} = \pi_{\alpha\rho}\mathbf{D}_\beta L^\rho - \pi_{\beta\rho}\mathbf{D}_\alpha L^\rho,$$

with $\omega = 0$ in O, then

$$(2.11) \quad L^\mu \mathbf{P}_{\alpha\beta\mu} = 0, \qquad L^\beta \omega_{\alpha\beta} = 0, \qquad L^\mu \mathbf{B}_{\alpha\mu} = 0 = L^\mu \mathbf{B}_{\mu\alpha} \ in \quad \mathbf{M}.$$

where $\mathbf{B} = \frac{1}{2}(\pi + \omega)$,

With these preliminaries one can easily derive transport equations for the tensors $\mathbf{B}$ and $\mathbf{P}$ along the geodesics generated by L.

PROPOSITION 2.11. *Let $\mathbf{B}, \mathbf{P}$ as above and $\mathbf{W} := \widehat{\mathcal{L}}_{Z,\omega}\mathbf{R}$. We have,*

$$(2.12) \qquad \mathbf{D}_L \mathbf{B}_{\alpha\beta} + \mathbf{D}_\alpha L^\rho \mathbf{B}_{\rho\beta} \;=\; L^\rho \mathbf{P}_{\rho\beta\alpha}$$

$$(2.13) \qquad \mathbf{D}_L \mathbf{P}_{\alpha\beta\mu} + \mathbf{D}_\mu L^\rho \mathbf{P}_{\alpha\beta\rho} \;=\; L^\nu \mathbf{W}_{\alpha\beta\mu\nu} + L^\nu \mathbf{B}_\mu{}^\rho \mathbf{R}_{\alpha\beta\rho\nu}$$

PROOF OF PROPOSITION 2.11.   Using the above properties of π, ω and $\mathbf{P}$ we calculate

$$
\begin{aligned}
2L^\rho \mathbf{D}_\rho B_{\alpha\beta} &= L^\rho \mathbf{D}_\rho \pi_{\beta\alpha} + L^\rho \mathbf{D}_\rho \omega_{\alpha\beta} \\
&= L^\rho (2\mathbf{P}_{\rho\beta\alpha} + \mathbf{D}_\alpha \omega_{\rho\beta} + \mathbf{D}_\beta \pi_{\rho\alpha}) + \pi_{\alpha\rho}\mathbf{D}_\beta L^\rho - \pi_{\beta\rho}\mathbf{D}_\alpha L^\rho \\
&= 2L^\rho \mathbf{P}_{\rho\beta\alpha} - 2B_{\rho\beta}\mathbf{D}_\alpha L^\rho,
\end{aligned}
$$

and the desired identity (2.12) follows.

To prove (2.13), we rely on the following identity:

(2.14)
$$
\mathbf{D}_\nu \widetilde{P}_{\alpha\beta\mu} - \mathbf{D}_\mu \widetilde{P}_{\alpha\beta\nu} \;=\; (\mathcal{L}_Z \mathbf{R})_{\alpha\beta\mu\nu} - (1/2)\pi_\alpha{}^\rho \mathbf{R}_{\rho\beta\mu\nu} - (1/2)\pi_\beta{}^\rho \mathbf{R}_{\alpha\rho\mu\nu}.
$$
$$
\widetilde{P}_{\alpha\beta\mu} \;:=\; (1/2)(\mathbf{D}_\alpha \pi_{\beta\mu} - \mathbf{D}_\beta \pi_{\alpha\mu}) = \mathbf{P}_{\alpha\beta\mu} + (1/2)\mathbf{D}_\mu \omega_{\alpha\beta}.
$$

Assuming (2.14) we easily prove now (2.13). Indeed, relying on the definitions above and the cacellation properties of π and $\mathbf{P}$ we calculate

$$
\begin{aligned}
&L^\nu (\mathbf{D}_\nu \mathbf{P}_{\alpha\beta\mu} - \mathbf{D}_\mu \mathbf{P}_{\alpha\beta\nu}) \\
&= L^\nu \Big[\frac{1}{2}\mathbf{D}_\mu \mathbf{D}_\nu \omega_{\alpha\beta} - \frac{1}{2}\mathbf{D}_\nu \mathbf{D}_\mu \omega_{\alpha\beta} + W_{\alpha\beta\mu\nu} + (B \cdot \mathbf{R})_{\alpha\beta\mu\nu} - \frac{1}{2}\pi_\alpha{}^\rho \mathbf{R}_{\rho\beta\mu\nu} \\
&\qquad - \frac{1}{2}\pi_\beta{}^\rho \mathbf{R}_{\alpha\rho\mu\nu} \Big] \\
&= L^\nu W_{\alpha\beta\mu\nu} + L^\nu B_\mu{}^\rho \mathbf{R}_{\alpha\beta\rho\nu}.
\end{aligned}
$$

The identity (2.13) is now an immediate consequence of (2.11). The proof of the identity (2.14) is a simple verification. $\qquad\square$

DEFINITION 2.12. *By convention, we let $\mathcal{M}(^{(1)}B, \ldots, {}^{(k)}B)$ denote any smooth "multiple" of the tensors $^{(1)}B, \ldots, {}^{(k)}B$, i.e. any tensor of the form*

$$(2.15) \quad \mathcal{M}(^{(1)}B, \ldots, {}^{(k)}B)_{\alpha_1 \ldots \alpha_r} = {}^{(1)}B_{\beta_1 \ldots \beta_{m_1}} \, {}^{(1)}C_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_{m_1}}$$
$$+ \ldots + {}^{(k)}B_{\beta_1 \ldots \beta_{m_k}} \, {}^{(k)}C_{\alpha_1 \ldots \alpha_r}{}^{\beta_1 \ldots \beta_{m_k}},$$

for some smooth tensors $^{(1)}C, \ldots, {}^{(k)}C$ in $\mathbf{M}$.

With this definition proposition 2.11 takes the form,

$$\mathbf{D}_L \mathbf{B} \;=\; \mathcal{M}(\mathbf{W}, \mathbf{B}, \mathbf{P}), \qquad \mathbf{D}_L \mathbf{P} = \mathcal{M}(\mathbf{W}, \mathbf{B}, \mathbf{P}),$$

To get a closed system it remains to establish an equation for $\mathbf{W} = \widehat{\mathcal{L}}_{Z,\omega} \mathbf{R}$. This is achieved by the following.

LEMMA 2.13. *Let* $\mathbf{W} := \mathcal{L}_{Z,\omega}\mathbf{R}$, *with* ω *an arbitrary 2 form. Then, with the definitions made above,*

$$(2.16) \qquad\qquad\qquad \mathbf{D}^a \mathbf{W}_{\alpha\beta\gamma\delta} \;=\; \mathbf{J}_{\beta\gamma\delta}$$

where,

$$\mathbf{J}_{\beta\gamma\delta} = \mathbf{B}^{\mu\nu}\mathbf{D}_\nu\mathbf{R}_{\mu\beta\gamma\delta} + \mathbf{g}^{\mu\nu}\mathbf{P}_{\mu\rho\nu}\mathbf{R}^\rho{}_{\beta\gamma\delta} + \mathbf{P}_{\beta\nu\mu}\mathbf{R}^{\mu\nu}{}_{\gamma\delta}$$
$$+ \mathbf{P}_{\gamma\nu\mu}\mathbf{R}^\mu{}_\beta{}^\nu{}_\delta + \mathbf{P}_{\delta\nu\mu}\mathbf{R}^\mu{}_{\beta\gamma}{}^\nu.$$

PROOF. Follows easily from the definition of W and lemma 2.5 applied to the curvature tensor $\mathbf{R}$ and vectorfield Z.

$\square$

Differentiating (2.16) and using the symmetries of $\mathbf{W}$ we easily deduce (see equation 7.1. in [20]),

$$\mathbf{D}^\rho \mathbf{D}_\rho \mathbf{W}_{\alpha\beta\mu\nu} = \mathcal{M}(\mathbf{B}, \mathbf{DB}, \mathbf{P}, \mathbf{DP}, \mathbf{W})_{\alpha\beta\mu\nu}.$$

We have thus derive the closed system,

$$\begin{aligned}
\mathbf{D}_L \mathbf{B} &= \mathcal{M}(\mathbf{W}, \mathbf{B}, \mathbf{P}) \\
\mathbf{D}_L \mathbf{P} &= \mathcal{M}(\mathbf{W}, \mathbf{B}, \mathbf{P}), \\
\square \mathbf{W} &= \mathcal{M}(\mathbf{W}, \mathbf{B}, \mathbf{DB}, \mathbf{P}, \mathbf{DP})
\end{aligned}$$

with the notation $\mathcal{M}(W, B, P)$ explained above.

2.1.2. *Unique continuation argument.* Once we have our coupled system it remains to prove the simultaneous vanishing of $\mathbf{B}, \mathbf{P}, \mathbf{W}$ by a unique continuation argument. More generally we consider solutions of systems of equations of the form,

$$(2.17) \qquad \begin{cases} \square_{\mathbf{g}} S = \mathcal{M}(^{(1)}B, \ldots, {}^{(k)}B, S, \mathbf{D}S) \\ \mathbf{D}_L{}^{(i)}B = \mathcal{M}(^{(1)}B, \ldots, {}^{(k)}B, S, \mathbf{D}S), \qquad i = 1, \ldots, k. \end{cases}$$

Theorem 2.3 is now an immediate consequence of the following.

PROPOSITION 2.14. *Let* $(\mathbf{M}, \mathbf{g})$ *be a general semi-riemannian manifold,* O *a domain in* $\mathbf{M}$ *verifying the strict null-convexity condition at* $p \in \partial O$. *Assume given a collection of tensorfields* S, B *on* $\mathbf{M}$, *and a vectorfield* L *verifying* (2.17) *in a neighborhood of* p. *Then, if* (S, B) *vanish in a neighborhood of* p *intersected with* O, *they also vanish in a full neighborhood of* p.

Expressing the equations (2.17) in local coordinates[11], we can easily reduce the statement of proposition 2.14 to the following statement.

PROPOSITION 2.15. *Assume that h is a strictly null convex defining function for O in a neighborhood U of $p \in \partial O$. Assume given smooth function G_i, H_j $i = 1, \ldots, I$, $j = 1, \ldots, J$, which satisfy the following differential inequalities in U of $p \in \partial O$,*

$$(2.18) \quad \begin{cases} |\Box_{\mathbf{g}} G_i| \leq M \sum_{l=1}^{I}(|G_l| + |\partial G_l|) + M \sum_{m=1}^{J} |H_m|; \\ |L(H_j)| \leq M \sum_{l=1}^{I}(|G_l| + |\partial G_l|) + M \sum_{m=1}^{J} |H_m|, \end{cases}$$

for any $i = 1, \ldots, I$, $j = 1, \ldots, J$. Then, if the function G, H vanish in $U \cap O$ then they also vanish in a full, small, neighborhood of p.

The proof of the proposition is based on Carleman estimates. The first step is to obtain a quantitative version of our null-convexity condition.

LEMMA 2.16. *Assume the defining function h is strictly null-convex at p. There exists a constant $M > 0$, depending only on bounds for the metric g and its derivatives (with respect to our fixed coordinate system in a neighborhood V of p) and $\mu \in [-M, M]$ as well as a small neighborhood $U \subset V$ of p such that, for any vectorfield $Y = Y^\alpha \partial_\alpha$*

$$(2.19) \quad \begin{cases} |dh| \geq M^{-1} \\ Y^\alpha Y^\beta (\mu g_{\alpha\beta} - D_\alpha D_\beta h) + M|Y(h)|^2 \geq M^{-1}|Y|^2, \end{cases}$$

uniformly, at all points of U, with $|Y|^2 = (Y^0)^2 + (Y^1)^2 + \ldots + (Y^d)^2$..

Here is also a quantitative version of the $\mathbf{T}$-null-convexity condition.

LEMMA 2.17. *Assume the defining function h is strictly $\mathbf{T}$-null-convex at p There exists a constant $M > 0$, depending only on bounds for the metric g and its derivatives in a fixed coordinate neighborhood V of p, a constant $\mu \in [-M, M]$ and a sufficiently small neighborhood $U \subset V$ of p such that, for any vectorfield $Y = Y^\alpha \partial_\alpha$*

(2.20)

$$\begin{cases} |dh| \geq M^{-1} \\ Y^\alpha Y^\beta (\mu g_{\alpha\beta} - D_\alpha D_\beta h) + M \left(|Y(h)|^2 + |g(\mathbf{T}, Y)|^2\right) \geq M^{-1}|Y|^2, \end{cases}$$

uniformly, at all points of U, with $|Y|^2 = (Y^0)^2 + (Y^1)^2 + \ldots + (Y^d)^2$ and $|dh|^2 = \sum_{\alpha=0}^{n}(\partial_\alpha h)^2$.

The proof of proposition (2.15) can be reduced to two Carleman estimates. The first, and by far the more important one, concerns the scalar wave operator $\Box_{\mathbf{g}}$. To state it we assume that the defining function h of the domain O, near $p \in \partial O$, verifies (2.19) in a full neighborhood U_1 of a point

---

[11]Using definition 2.12 and also enlarging M and restricting the neighborhood U of p as necessary.

p with $h(p) = 0$. Let U_ϵ be small neighborhoods of p such that $|h| \leq 2^{-1}\epsilon$ in U_ϵ and define the weight functions, $f_\epsilon : U_\epsilon \longrightarrow \mathbb{R}$

$$f_\epsilon := \log(\epsilon + h + e_p)$$

with e_p is a small perturbation such that the weights e^{f_ϵ} verify (2.19) in U_ϵ, uniformly in $\epsilon > 0$

We are now ready to state our main Carleman estimate, see also the more precise statement and its proof in proposition A.1 in the Appendix. For a full account of the more general Carleman estimates, which are also adapted to **T** null-convexity, see [**20**, section 3.2].

PROPOSITION 2.18. *If f_ϵ are as above, there exists a sufficiently small $\epsilon > 0$ and a large constant $C_\epsilon > 0$ such that, for all $\phi \in C_0^2(U_\epsilon)$ and all sufficiently large $\lambda > 0$,*

$$(2.21) \quad \lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot |D\phi|\|_{L^2} \leq C_\epsilon \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \Box_{\mathbf{g}} \phi\|_{L^2},$$

where, $|D\phi|^2 = \sum_{\alpha=0}^n |\partial_\alpha \phi|^2$ relative to the given coordinate system.

We also need a Carleman estimate to deal with the ODE part of our system. This is considerably easier, no additional restrictions are needed, see [**1**, Lemma 3.4].

2.2. Existence results in a neighborhood of the horizon. The methods discussed in the previous subsections can be applied to construct Killing vectorfields in a neighborhood of a bifurcate horizon of stationary vacuum solutions. One can in fact present the result without reference to stationarity as follows.

Let $(\mathbf{M}, \mathbf{g})$ to be a smooth[12] vacuum Einstein space-time. Let S be an embedded spacelike 2-sphere in $\mathbf{M}$ and let $\mathcal{N}, \underline{\mathcal{N}}$ be the null boundaries of the causal set of S, i.e. the union of the causal future and past of S. We fix U to be a small neighborhood of S such that both $\mathcal{N}, \underline{\mathcal{N}}$ are regular, achronal, null hypersurfaces in U spanned by null geodesic generators orthogonal to S. We say that the triplet $(S, \mathcal{N}, \underline{\mathcal{N}})$ forms a local, regular, bifurcate, non-expanding horizon in U if both $\mathcal{N}, \underline{\mathcal{N}}$ are non-expanding null hypersurfaces in U. This simply means that the traces of the null second fundamental forms of $\mathcal{N}$ and $\underline{\mathcal{N}}$, called expansions, are both vanishing respectively on $\mathcal{N}$ and $\underline{\mathcal{N}}$. Our main results are the following:

THEOREM 2.19. *Given a local, regular, bifurcate $(S, \mathcal{N}, \underline{\mathcal{N}})$ in a smooth, vacuum Einstein space-time $(\mathbf{M}, \mathbf{g})$, there exists an open neighborhood $V \subset U$ of S and a non-trivial Killing vector-field $\mathbf{K}$ in $\mathbf{M}$, which is tangent to the null generators of $\mathcal{N}$ and $\underline{\mathcal{N}}$. In other words, every local, regular, bifurcate, non-expanding horizon is a Killing bifurcate horizon.*

---

[12]$\mathbf{M}$ is assumed to be a connected, oriented, C^∞ 4-dimensional manifold without boundary.

THEOREM 2.20. *Under the same assumptions as above, if in addition there exists a Killing vectorfield* $\mathbf{T}$ *in* U *tangent to* $\mathcal{N} \cup \underline{\mathcal{N}}$ *and not identically vanishing on* S, *then there exists an open neighborhood* $V \subset U$ *and a nontrivial rotational Killing vector-field* $\mathbf{Z}$ *in* U *which commutes with* $\mathbf{T}$.

It was already known, see [17], that such a Killing vector-field exists in a small neighborhood of S intersected with the domain of dependence of $\mathcal{N} \cup \underline{\mathcal{N}}$, which we could call O in reference to theorem 2.3. The extension of $\mathbf{K}$ to a full neighborhood of S has been known to hold only under the restrictive additional assumption of analyticity of the space-time (see [18], [23], [17]). The novelty of both theorems is the ability to construct these local Killing fields in a full neighborhood of the 2-sphere S, without making any analyticity assumption. Both theorems can be viewed as applications of theorems 2.3, 2.4 to domains O which are obtained by intersecting neighborhoods of S in $\mathbf{M}$ with the domain of dependence of the bifurcate sphere S. The strict null convexity condition is an easy consequence of the bifurcation (non-degeneracy) property of the boundary $(\mathcal{N} \cup \underline{\mathcal{N}}) \cap U$ of O. Note however that O is not smooth at points of S. This requires a slight modification of the Carleman estimates needed in the proof of theorems 2.3, 2.4. A full account of such Carleman estimates is given in [20, section 3.2].

2.3. Counterexamples. We review a counterexample to Haking's rigidity theorem in the non-analytic case. Let $(\mathcal{K}(m,a), \mathbf{g})$ denote the (maximally extended) Kerr space-time of mass M and angular momentum Ma, $0 \leq a < M$. Let $\mathbf{M}^{(end)}$ denote an asymptotic region, $\mathbf{E} = \mathcal{I}^-(\mathbf{M}^{(end)}) \cap \mathcal{I}^+(\mathbf{M}^{(end)})$ the corresponding domain of outer communication, and $\mathcal{H}^- = \delta(\mathcal{I}^+(\mathbf{M}^{(end)})$ the boundary (event horizon) of the corresponding white hole[13]. Let $\mathbf{T} = d/dt$ denote the stationary (timelike in $\mathbf{M}^{(end)}$) Killing vector-field of $(\mathcal{K}(m,a), \mathbf{g})$, and let $Z = d/d\phi$ denote its rotational (with closed orbits) Killing vector-field. The following theorem was proved in [22].

THEOREM 2.21. *Assume that* $0 < a < M$ *and* $U_0 \subseteq \mathcal{K}(m,a)$ *is an open set such that* $U_0 \cap \mathcal{H}^- \cap \overline{\mathbf{E}} \neq \emptyset$. *Then,*

   (1) *There is an open set* $U \subseteq U_0$, $U \cap \mathcal{H}^- \neq \emptyset$, *and a smooth Lorentz metric* $\widetilde{\mathbf{g}}$ *in* U *such that,*

$$(2.22) \qquad {}^{\widetilde{\mathbf{g}}}\mathbf{Ric} = 0 \;\; in \; U, \qquad \mathcal{L}_{\mathbf{T}} \widetilde{\mathbf{g}} = 0 \;\; in \; U, \qquad \widetilde{\mathbf{g}} = \mathbf{g} \;\; in \; U \setminus \mathbf{E};$$

   (2) *The vector-field* $Z = d/d\phi$ *does not extend to a Killing vector-field for* $\widetilde{\mathbf{g}}$, *commuting with* $\mathbf{T}$, *in* U.

In other words, one can modify the Kerr space-time smoothly, on one side of the horizon $\mathcal{H}^-$, in such a way that the resulting metric still satisfies the Einstein vacuum equations, has $\mathbf{T} = d/dt$ as a Killing vector-field, but does not admit an extension of the Killing vector-field Z. The crucial point

---

[13]A similar statement can be made on the future event horizon $\mathcal{H}^+$.

here is that the neighborhood under consideration is away from the bifurcate sphere, where theorems 2.19 - 2.20 apply. The result illustrates one of the major difficulties one faces in trying to extend Hawking's rigidity result to the more realistic setting of smooth stationary solutions of the Einstein vacuum equations: unlike in the analytic situation, one cannot hope to construct an additional symmetry of stationary solutions of the Einstein-vacuum equations (as in Hawking's Rigidity Theorem) by relying only on the local information provided by the equations.

The proof relies on a symmetry reduction induced by the Killing vectorfield $\mathbf{T}$. We denote the fixed Kerr metric by $\mathbf{g}$ and define the *reduced metric*

$$h_{\alpha\beta} = X\mathbf{g}_{\alpha\beta} - \mathbf{T}_\alpha\mathbf{T}_\beta, \qquad \text{where } X = \mathbf{g}(\mathbf{T}, \mathbf{T}),$$

on a hypersurface Π passing through the point p and transversal to $\mathbf{T}$. The metric h is nondegenerate (Lorentzian) as long as $X > 0$ in Π, which explains our assumption $0 < a < m$. It is well-known, see for example [**37**, Section 3], that the Einstein vacuum equations together with stationarity $\mathcal{L}_\mathbf{T}\mathbf{g} = 0$ are equivalent to the system of equations

(2.23)
$$^h\mathrm{Ric}_{ab} = \frac{1}{2X^2}(\nabla_a X \nabla_b X + \nabla_a Y \nabla_b Y),$$
$$^h\square(X + iY) = \frac{1}{X}h^{ab}\partial_a(X + iY)\partial_b(X + iY),$$

in Π, where $X + iY$ is the complex Ernst potential associated[14] to $\mathbf{T}$.

We then modify the metric h and the functions X and Y in a neighborhood of the point p in such a way that the identities (2.23) are still satisfied. The existence of a large family of smooth triplets $(\widetilde{h}, \widetilde{X}, \widetilde{Y})$ satisfying (2.23) and agreeing with the Kerr triplet in $\Pi\backslash\mathbf{E}$ follows by a classic local existence result, solving a characteristic initial-value problem, using, for example, the main existence result in [**32**].

One can then we construct the new space-time metric $\widetilde{\mathbf{g}}$,

$$\widetilde{\mathbf{g}}_{ab} = \widetilde{X}^{-1}\widetilde{h}_{ab} + \widetilde{X}\widetilde{A}_a\widetilde{A}_b, \qquad \widetilde{\mathbf{g}}_{a4} = \widetilde{X}\widetilde{A}_a, \qquad \widetilde{\mathbf{g}}_{44} = \widetilde{X}, \qquad a, b = 1, 2, 3,$$

associated to the triplet $(\widetilde{h}, \widetilde{X}, \widetilde{Y})$, the vector-field $\mathbf{T} = \partial_4$, and a suitable 1-form $\widetilde{A}$ which is defined in Π. By construction (see [**37**, Theorem 1]) this metric verifies the identities $^{\widetilde{\mathbf{g}}}\mathbf{Ric} = 0$ and $\mathcal{L}_\mathbf{T}\widetilde{\mathbf{g}} = 0$, in a suitable open set U. Finally one can show that we have enough flexibility to choose initial conditions for $\widetilde{X}, \widetilde{Y}$ such that the vector-field Z cannot be extended as a Killing vector-field for $\widetilde{\mathbf{g}}$ commuting with $\mathbf{T}$, in the open set U.

REMARK 2.22. *Note that the construction of the extended metric in theorem 2.21 relies in an essential way on the fact that $a \neq 0$ to allow for a non-trivial ergo-region near the horizon where $\mathbf{T}$ is space-like. No such result is known for $a = 0$.*

---

[14]See section 3.1

3. Mars-Simon tensor and global results

3.1. Killing vector-fields. In what follows we consider $1 + 3$ dimensional Lorentzian manifolds endowed with a Killing vectorfield $\mathbf{K}$, i.e.

$$(3.1) \qquad \mathbf{D}_\alpha \mathbf{K}_\beta + \mathbf{D}_\beta \mathbf{K}_\alpha = 0$$

We define the 2-form,

$$\mathbf{F}_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{K}_\beta = \frac{1}{2}\left(\mathbf{D}_\alpha \mathbf{K}_\beta - \mathbf{D}_\beta \mathbf{K}_\alpha\right)$$

as well as its Hodge dual,

$$^*\mathbf{F}_{\alpha\beta} \;=\; \frac{1}{2}\,\in_{\alpha\beta}{}^{\mu\nu}\mathbf{F}_{\mu\nu}$$

Note that $^*(^*\mathbf{F}) = -\mathbf{F}$. We also define the left and right Hodge duals of the curvature tensor,

$$^*\mathbf{R}_{\alpha\beta\gamma\delta} \;=\; \frac{1}{2}\,\in_{\alpha\beta}{}^{\mu\nu}\mathbf{R}_{\mu\nu\gamma\delta}, \qquad \mathbf{R}^*_{\alpha\beta\gamma\delta} = \frac{1}{2}\mathbf{R}_{\alpha\beta\mu\nu}\in^{\mu\nu}{}_{\gamma\delta}$$

and note that for a vacuum manifold, i.e. $\mathbf{Ric}(\mathbf{g}) = 0$, we have $^*\mathbf{R} = \mathbf{R}^*$, $^*(^*\mathbf{R}) = -\mathbf{R}$. We also define the complex tensors,

$$\mathcal{F}_{\alpha\beta} = \mathbf{F}_{\alpha\beta} + i\,^*\mathbf{F}_{ab}, \qquad \mathcal{R}_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha\beta\gamma\delta} + \,^*\mathbf{R}_{\alpha\beta\gamma\delta}$$

Note that $\mathcal{R}$ verifies all the symmetries of the curvature tensor as well as[15] $\mathbf{g}^{\alpha\gamma}\mathbf{R}_{\alpha\beta\gamma\delta} = 0$. Note also that both $\mathcal{F}$ and $\mathcal{R}$ are self dual i.e $^*\mathcal{F} = -i\mathcal{F}$, $^*\mathcal{R} = -i\mathcal{R}$.

We recall the following well known,

LEMMA 3.1. *For all tensor-fields* $\mathbf{U}$ *in* $\mathbf{M}$, *if* $\mathbf{K}$ *is Killing we have,*

$$(3.2) \qquad \mathbf{D}_\mu \mathbf{D}_\alpha \mathbf{K}_\beta \;=\; \mathbf{R}_{\lambda\mu\alpha\beta}\mathbf{K}^\lambda$$

$$(3.3) \qquad [\mathcal{L}_\mathbf{K}, \mathbf{D}]\mathbf{U} \;=\; 0$$

In particular, if $(\mathbf{M}, \mathbf{g})$ *has vanishing Ricci curvature then,*

$$\mathbf{D}_\mu \mathcal{F}_{\alpha\beta} \;=\; \mathbf{R}_{\lambda\mu\alpha\beta}\mathbf{K}^\lambda$$

COROLLARY 3.2. *If* $(\mathbf{M}, \mathbf{g})$ *is a vacuum Lorentzian manifold endowed with a Killing vectorfield* $\mathbf{K}$ *we have,*

$$\mathbf{D}_{[\mu}\mathcal{F}_{\alpha\beta]} = \mathbf{D}_\mu\mathcal{F}_{\alpha\beta} + \mathbf{D}_\alpha\mathcal{F}_{\beta\mu} + \mathbf{D}_\beta\mathcal{F}_{\mu\alpha} = 0, \qquad \mathbf{D}^\beta\mathcal{F}_{\alpha\beta} = 0.$$

We define also the Ernst 1-form associated to the Killing vector-field $\mathbf{K}$,

$$(3.4) \quad \sigma_\mu \;=\; 2(i_\mathbf{K}\mathbf{F}_\alpha) = 2\mathbf{K}^\alpha\mathcal{F}_{\alpha\mu} = \mathbf{D}_\mu(-\mathbf{K}^\alpha\mathbf{K}_\alpha) - i\in_{\mu\beta\gamma\delta}\mathbf{K}^\beta\mathbf{D}^\gamma\mathbf{K}^\delta.$$

PROPOSITION 3.3. *The following are true,*

$$(3.5) \qquad \begin{cases} \mathbf{D}_\mu\sigma_\nu - \mathbf{D}_\nu\sigma_\mu = 0; \\ \mathbf{D}^\mu\sigma_\mu = -\mathcal{F}^2; \\ \sigma_\mu\sigma^\mu = g(\mathbf{K}, \mathbf{K})\mathcal{F}^2. \end{cases}$$

---

[15]i.e. $\mathcal{R}$ is a complex valued Weyl tensor.

PROOF. We have,

$$
\begin{aligned}
2^{-1}\big(\mathbf{D}_\mu\sigma_\nu - \mathbf{D}_\nu\sigma_\mu\big) &= \mathbf{K}^\alpha\big(\mathbf{D}_\nu\mathcal{F}_{\alpha\mu} - \mathbf{D}_\mu\mathcal{F}_{\alpha\nu}\big) + \mathbf{D}_\nu\mathbf{K}^\alpha\mathcal{F}_{\alpha\mu} - \mathbf{D}_\mu\mathbf{K}^\alpha\mathcal{F}_{\alpha\nu} \\
&= -\mathbf{K}^\alpha\mathbf{D}_\alpha\mathcal{F}_{\mu\nu} - \mathbf{D}_\mu\mathbf{K}^\alpha\mathcal{F}_{\alpha\nu} - \mathbf{D}_\nu\mathbf{K}^\alpha\mathcal{F}_{\mu\alpha} = -\mathcal{L}_\mathbf{K}\mathcal{F}_{\mu\nu} \\
&= 0.
\end{aligned}
$$

Also,

$$
2^{-1}\mathbf{D}^\mu\sigma_\mu = \mathbf{K}^\alpha\mathbf{D}^\mu\mathcal{F}_{\alpha\mu} + \mathbf{D}^\mu\mathbf{K}^\alpha\mathcal{F}_{\alpha\mu} = -F^{\alpha\mu}\mathcal{F}_{\alpha\mu} = -2^{-1}\mathcal{F}^2
$$

The last formula in (3.5) follows easily from the lemma below. $\qquad\square$

LEMMA 3.4. *Introduce the decomposition*

$$
i_\mathbf{K}(\mathbf{F})_\alpha = \mathbf{K}^\mu\mathbf{F}_{\mu\alpha}, \quad i_\mathbf{K}(^*\mathbf{F})_\alpha = \mathbf{K}^{\mu\,*}\mathbf{F}_{\mu\alpha}, \quad i_K(\mathcal{F})_\alpha = \mathbf{K}^\mu\mathcal{F}_{\mu\alpha}.
$$

Clearly, $i_\mathbf{K}(\mathbf{F}), i_\mathbf{K}(^*\mathbf{F}), i_K(\mathcal{F})$ are orthogonal to K and,

(3.6) $\quad g(\mathbf{K},\mathbf{K})\mathbf{F}_{\alpha\beta} = \mathbf{K}_\alpha\, i_\mathbf{K}(\mathbf{F})_\beta - \mathbf{K}_\beta\, i_\mathbf{K}(\mathbf{F})_\alpha + \in_{\alpha\beta\mu\nu}\mathbf{K}^\mu\, i_\mathbf{K}(^*\mathbf{F})^\nu$

Also,

(3.7) $\quad g(\mathbf{K},\mathbf{K})\mathcal{F}_{\alpha\beta} = \mathbf{K}_\alpha\, i_\mathbf{K}(\mathcal{F})_\beta - \mathbf{K}_\beta\, i_\mathbf{K}(\mathcal{F})_\alpha - i\in_{\alpha\beta\mu\nu}\mathbf{K}^\mu\, i_\mathbf{K}(\mathcal{F})^\nu$

In particular,

(3.8) $\qquad\qquad g(\mathbf{K},\mathbf{K})\mathcal{F}^2 = 4i_\mathbf{K}(\mathcal{F})^\mu i_\mathbf{K}(\mathcal{F})_\mu = \sigma^\mu\sigma_\mu$

REMARK 3.5. *Since $d(\sigma_\mu dx^\mu) = 0$, if $\mathbf{M}$ is simply connected, we infer that there exists a function $\sigma : \mathbf{M} \to \mathbb{C}$, called the Ernst potential, such that $\sigma_\mu = \mathbf{D}_\mu\sigma$. Note also that $\mathbf{D}_\mu g(\mathbf{K},\mathbf{K}) = 2\mathbf{F}_{\mu\lambda}\mathbf{K}^\lambda = -\Re\sigma_\mu$. Hence we can choose the potential σ such that,*

(3.9) $\qquad\qquad\qquad \Re\sigma = -\mathbf{g}(\mathbf{K},\mathbf{K}).$

Moreover, if $(\mathbf{M},\mathbf{g})$ is asymptotically flat, we can choose $\sigma = 1$ at space like infinity.

As a corollary of the lemma we also deduce,

(3.10) $\qquad\qquad\qquad \Box\sigma = -\mathbf{g}(\mathbf{K},\mathbf{K})^{-1}\mathbf{D}_\mu\sigma\mathbf{D}^\mu\sigma$

or, writing $\sigma = -f - if^*$ we deduce,

(3.11) $\qquad\qquad \Box f = f^{-1}\big(\mathbf{D}^\mu f\mathbf{D}_\mu f - \mathbf{D}^\mu f^*\mathbf{D}_\mu f^*\big)$

(3.12) $\qquad\qquad \Box f^* = f^{-1}\mathbf{D}^\mu f\,\mathbf{D}_\mu f^*$

In other words the pair $(x = f, y = f^*)$ defines, whenever $f = g(K,K) \neq 0$, a wave map to the Poincaré plane $\mathbf{H} := \{(x,y)/x > 0\}$ with metric,

$$
ds^2 = \frac{dx^2 + dy^2}{x^2}
$$

3.2. Stationary Vacuum Spacetimes. We consider vacuum, asymptotically flat, $1 + 3$ dimensional spacetimes which are stationary, i.e. they possess a smooth, non degenerate, Killing vectorfield $\mathbf{T}$ which is timelike in the asymptotic region (i.e. a neighborhood of null infinity). More precisely we make the following assumptions:

(1) (Asymptotic flatness.) We assume that there is an open subset $\mathbf{M}^{(end)}$ of $\mathbf{M}$ which is diffeomorphic to $\mathbb{R} \times (\{x \in \mathbb{R}^3 : |x| > R\})$ for some R sufficiently large. In local coordinates $\{t, x^i\}$ defined by this diffeomorphism, we assume that, with $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$,

$$(3.13)$$

$$\mathbf{g}_{00} = -1 + \frac{2M}{r} + O(r^{-2}), \quad \mathbf{g}_{ij} = \delta_{ij} + O(r^{-1}), \quad \mathbf{g}_{0i} = -\epsilon_{ijk}\frac{2S^j x^k}{r^3} + O(r^{-3}),$$

for some constants $M > 0, S_1, S_2, S_3$ such that $J = [(S^1)^2 + (S^2)^2 + (S^3)^2]^{1/2} \in [0, M^2)$. We also assume $\mathbf{T} = \partial_t$ with $t = x^0$. We define the domain of outer communication (exterior region)

$$\mathbf{E} = \mathcal{I}^-(\mathbf{M}^{(end)}) \cap \mathcal{I}^+(\mathbf{M}^{(end)}).$$

(2) (Completeness.) We also assume that $\mathbf{E}$ is globally hyperbolic and every orbit of $\mathbf{T}$ in $\mathbf{E}$ is complete and intersects transversally a given spacelike Cauchy hypersurface Σ_0. We also assume, for convenience, that Σ_0 is diffeomorphic to $\{x \in \mathbb{R}^3 : |x| > 1/2\}$ and agrees with the hypersurface corresponding to $t = 0$ in $\mathbf{M}^{(end)}$.

(3) (Smooth bifurcate sphere.) Let $S_0 = \partial(\mathcal{I}^-(\mathbf{M}^{(end)})) \cap \partial(\mathcal{I}^+(\mathbf{M}^{(end)}))$. We assume that $S_0 \subseteq \Sigma_0$ and S_0 is an imbedded 2-sphere which agrees with the sphere of radius 1 in $\mathbb{R}^3$ under the identification of Σ_0 with $\{x \in \mathbb{R}^3 : |x| > 1/2\}$. Furthermore, we assume that there is a neighborhood $\mathbf{O}$ of S_0 in $\mathbf{M}$ such that the sets

$$\mathcal{H}^+ = \mathbf{O} \cap \partial(\mathcal{I}^-(\mathbf{M}^{(end)})) \quad \text{and} \quad \mathcal{H}^- = \mathbf{O} \cap \partial(\mathcal{I}^+(\mathbf{M}^{(end)}))$$

are smooth imbedded hypersurfaces diffeomorphic to $S_0 \times (-1, 1)$, We assume that these hypersurfaces are null, non-expanding and intersect transversally in S_0.

(4) (Tangency at Horizon[16]) Finally, we assume that the vector-field $\mathbf{T}$ is tangent to both hypersurfaces $\mathcal{H}^+ = \mathbf{O} \cap \delta(\mathcal{I}^-(\mathbf{M}^{(end)}))$ and $\mathcal{H}^- = \mathbf{O} \cap \delta(\mathcal{I}^+(\mathbf{M}^{(end)}))$.

DEFINITION 3.6. *A space-time verifying the above assumptions will be called a regular, nondegenerate stationary vacuum spacetime.*

REMARK 3.7. *Note that the definition pre-supposes the presence of a unique connected horizon.*

---

[16]It can be shown in fact that this assumption follows from the first two, using in particular the completeness of $\mathbf{T}$.

3.3. Kerr spacetime. In Boyer-Lindquist coordinates the Kerr metric takes the form,

(3.14)
$$ds^2 = -\frac{q^2\Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin\theta)^2}{q^2}\left(d\phi - \frac{2aMr}{\Sigma^2}dt\right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + q^2(d\theta)^2,$$

where,
$$q^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 + a^2 - 2Mr,$$
$$\Sigma^2 = (r^2 + a^2)q^2 + 2Mra^2(\sin\theta)^2.$$

On the horizon we have $r = r_+ := M + \sqrt{M^2 - a^2}$ and $\Delta = 0$. The domain of outer communication $\mathbf{E}$ is given by $r > r_+$. One can show that the complex Ernst potential σ and the complex scalar $\mathcal{F}^2$ associated to the Killing vectorfield $\mathbf{T} = \partial_t$ are given by

(3.15)
$$\sigma = 1 - \frac{2M}{r + ia\cos\theta}, \qquad \mathcal{F}^2 = -\frac{4M^2}{(r + ia\cos\theta)^4}.$$

Thus,

(3.16)
$$-4M^2\mathcal{F}^2 = (1 - \sigma)^4$$

everywhere in the exterior region. Writing $y + iz := (1 - \sigma)^{-1}$ we observe that,
$$y = \frac{r}{2M} \geq \frac{r_+}{2M} > \frac{1}{2}.$$

everywhere in the exterior region.

3.4. Mars-Simon. In [**27**] M. Mars gave a very useful local characterization of the Kerr family in terms of the vanishing of complex 4-tensor $\mathcal{S}$, called the Mars-Simon tensor. In other words $\mathcal{S}$ plays the same role in detecting a Kerr spacetime as the Riemann curvature tensor plays in detecting flat space.

DEFINITION 3.8. *Given a stationary spacetime with Killing field* $\mathbf{T}$ *and associated Ernst potential* σ, *we define the Mars-Simon tensor*[17],
$$\mathcal{S}_{\alpha\beta\mu\nu} := \mathcal{R}_{\alpha\beta\mu\nu} + 6(1 - \sigma)^{-1}\mathcal{Q}_{\alpha\beta\mu\nu}.$$

where,
$$\mathcal{Q}_{\alpha\beta\mu\nu} := \mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}$$

and,
$$\mathcal{I}_{\alpha\beta\mu\nu} := (\mathbf{g}_{\alpha\mu}\mathbf{g}_{\beta\nu} - \mathbf{g}_{\alpha\nu}\mathbf{g}_{\beta\mu} + i\in_{\alpha\beta\mu\nu})/4$$

REMARK 3.9. *Note that* $\mathcal{R}, \mathcal{Q}, \mathcal{S}$ *are all complex, self dual Weyl fields in the sense defined above.*

---

[17]in regions where $\sigma \neq 1$.

Here is an important property of $\mathcal{S}$, derived and made use of in [20].

PROPOSITION 3.10. *The tensor $\mathcal{S}$ verifies the equation,*

$$\mathbf{D}^\alpha \mathcal{S}_{\alpha\beta\mu\nu} = -6(1-\sigma)^{-1}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta}\left[\mathcal{F}_\beta{}^\rho \delta_\mu^\gamma \delta_\nu^\delta + \frac{2}{3}\mathcal{I}^\rho{}_{\beta\mu\nu}\mathcal{F}^{\gamma\delta}\right]$$

We give a complete proof of the proposition in appendix B. As a corollary we derive,

COROLLARY 3.11. *The tensor $\mathcal{S}$ verifies a covariant wave equation of the form,*

$$(3.17) \qquad\qquad \Box \mathcal{S} = \mathcal{M}(\mathbf{D}\mathcal{S}, \mathcal{S})$$

3.5. A Maxwell System. In the appendix we also derive a Maxwell type equation for the following slightly modified version of the Mars-Simon tensor,

$$\mathcal{S}'_{\alpha\beta\mu\nu} := \mathcal{R}_{\alpha\beta\mu\nu} + 6\varpi^{-1}\mathcal{Q}_{\alpha\beta\mu\nu}.$$

where, for some constant[18] C,

$$(3.18) \qquad\qquad \varpi := C(-\mathcal{F}^2)^{1/4}$$

PROPOSITION 3.12. *The self-dual complex 2-form*

$$\mathcal{H}_{\alpha\beta} := \varpi^{-3}\mathcal{S}'_{\alpha\beta\mu\nu}\mathcal{F}^{\mu\nu},$$

verifies the Maxwell equations,

$$(3.19) \qquad \mathbf{D}^\alpha \mathcal{H}_{\alpha\beta} = -\varpi^{-3}\mathbf{T}^\sigma(\mathcal{S}' \cdot \mathcal{S}')_{\sigma\beta} - 3\varpi^{-1}E^\rho \mathcal{H}_{\rho\beta},$$

where,

$$(\mathcal{S}' \cdot \mathcal{S}')_{\sigma\beta} = \mathcal{S}'_\beta{}^{\rho\mu\nu}\mathcal{S}'_{\sigma\rho\mu\nu}$$

and,

$$(3.20) \qquad\qquad E_\rho := \sigma_\rho + \mathbf{D}_\rho \varpi = -\frac{1}{2}C^4 \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho}$$

REMARK 3.13. *Remark that the right hand side of (3.19) is quadratic in $\mathcal{S}$ and thus, if $\mathcal{S} = O(\epsilon)$, sufficiently small, we can ignore it in a first approximation and derive the linearized equation*

$$(3.21) \qquad\qquad \mathbf{D}^\alpha \mathcal{H}^{(lin)}_{\alpha\beta} = 0.$$

---

[18]The precise constant in Kerr is $C = (4M^2)^{1/4}$

3.6. Rigidity results based on $\mathcal{S}$.

THEOREM 3.14 (Ionescu-Klainerman [20]). *Assume that* $(\mathbf{M}, \mathbf{g})$ *is regular, nondegenerate stationary vacuum spacetime and that the stationary Killing vectorfield* $\mathbf{T}$ *does not vanish on the bifurcate sphere. Assume also that the following conditions are verified*

$$(3.22) \qquad\qquad -4M^2\mathcal{F}^2 = (1-\sigma)^4 \quad on \ S_0,$$

and

$$(3.23) \qquad\qquad \Re\big((1-\sigma)^{-1}\big) > 1/2 \quad at \ some \ point \ on \ S_0.$$

Then $(\mathbf{M}, \mathbf{g})$ *is isometric to the domain of outer communication of a Kerr space-time with mass* M *and* $0 < a < M$.

The proof of the theorem is based on the following ingredients:

(1) Assumption (3.22) is used to show that $\mathcal{S}$ vanishes along the horizon[19].

(2) Due to the non-degeneracy of the horizon one can check that the null convexity condition is verified at all points of the bifurcate sphere[20]. Thus our unique continuation results applied to equation (3.17) can be applied to prove that $\mathcal{S}$ vanishes in a neighborhood of the horizon.

(3) This is the key step! We define functions y, z such that $y + iz = (1-\sigma)^{-1}$. In the particular case of a Kerr space $\mathcal{K}(a, M)$ they are $y = (2M)^{-1}r, z = (2M)^{-1}a\cos\theta$. Remarkably the foliation induced by y, in the direction of growing y, verifies the strict $\mathbf{T}$-null convexity condition. Using some of the main steps of Mars's theorem in [27] one can check that this fact remains true as long as $\mathcal{S}$ vanishes. Thus the vanishing of $\mathcal{S}$ in a region $y \le y_0$ leads, based on our results on unique continuation (applied to equation (3.17)) to its vanishing in a larger region $y \le y_0 + \epsilon$. Therefore, by a simple continuity argument, $\mathcal{S}$ vanishes everywhere.

(4) According to Mars theorem [27] we conclude that our space-time is isometric to $\mathcal{K}(a, M)$, with a, M determined from the asymptotic conditions of the metric $\mathbf{g}$.

We now state our second main theorem. Roughly the theorem shows that any stationary spacetime close to a non-extremal Kerr solution $\mathcal{K}(a, M)$, $|a| < M$, must be a non-extremal Kerr solution. The closeness to Kerr is expressed in terms of the smallness of the Mars-Simon tensor $\mathcal{S}$.

---

[19]We also make use of the fact that, in view of our assumption that $\mathbf{T}$ does not vanish identically on the bifurcate sphere S_0 can only vanish at a finite number isolated points on S_0.

[20]Strictly speaking the classical null-cinvexity condition applies only to smooyh boundaries. Yet the defintion of null convexity can be extended to our bifurcate horizon situation.

THEOREM 3.15 (Alexakis-Ionescu-Klainerman [**2**]). *Assume that* $(\mathbf{M}, \mathbf{g})$ *is regular, nondegenerate stationary vacuum spacetime verifying all the assumptions of theorem 3.14 except* (3.22) *which is replaced by the condition.*

$$(3.24) \qquad |(1 - \sigma)\mathcal{S}(\mathbf{T}, E_\alpha, E_\beta, E_\gamma)| \leq \varepsilon \ \ on \ \Sigma^0 \cap \mathbf{E},$$

for some sufficiently small constant $\varepsilon > 0$ *(depending only on our regularity assumption on the metric* $\mathbf{g}$*) where* E_0, E_1, E_2, E_3 *is a fixed orthonormal frame along* Σ_0 *with* E_0 *the future unit normal. Then the entire domain of outer communication* $\mathbf{E}$ *is isometric to the exterior region of a Kerr solution* $\mathcal{K}(a, M)$.

The proof of the theorem is based on the following ideas.

(1) A simple argument, due to Hawking[21], shows that one can construct a second Killing vectorfield $\mathbf{K}$ on the horizon $\mathcal{H}^+ \cup \mathcal{H}^-$, with $\mathbf{K}$ tangent to the generators.

(2) We check that the strict-null convexity condition is verified at all point of the bifurcate sphere S_0 and extend $\mathbf{K}$ in a full neighborhood of S_0. Moreover $\mathbf{K}$ commutes with $\mathbf{T}$, $[\mathbf{K}, \mathbf{T}] = 0$.

(3) Introduce the coordinates y, z such that $y + iz = (1 - \sigma)^{-1}$ and show, using the smallness assumption on $\mathcal{S}$, that y verifies the strict $\mathbf{T}$-null-convexity condition.

(4) Extend $\mathbf{K}$ everywhere in $\mathbf{E}$ as a Killing vectorfield such that $[\mathbf{K}, \mathbf{T}] = 0$ and find a combination $\mathbf{Z}$ of $\mathbf{T}, \mathbf{K}$ which has closed orbits.

(5) Use the Carter-Robinson theorem to conclude that $\mathbf{E}$ is isometric to the exterior domain of a Kerr solution.

REMARK 3.16. *Theorem 3.15 has been significantly strengthened by Wong and Yu in* [**40**] *in which they show, by a clever application of the mountain pass lemma, that the assumption of a connected horizon, implicit in both theorems 3.14, 3.15, is unnecessary.*

REMARK 3.17. *The reliance on the Carter-Robinson theorem in the last step of the proof is somewhat unsatisfactory since we are in a small* $\mathcal{S}$ *regime. In fact the authors believe that an alternative argument can be given relying on proposition 3.12 and the study of stationary solutions to the linearized system* (3.21).

3.7. Third rigidity result. In this section we review a recent black hole rigidity result for slowly rotating stationary solutions of the Einstein vacuum equations. The result states that the domain of outer communication of any stationary vacuum black hole, in the sense of definition 3.6, with the stationary Killing vector-field $\mathbf{T}$ being small on the bifurcation sphere of the horizon, must be isometric to the domain of outer communication of a Kerr solution $K(a, M)$ with small a. More precisely,

---

[21]Hawking's original argument also applies to degenerate horizons. In the case of a non-degenerate horizon, assumed here, the proof is completely trivial.

Theorem 3.18 (Alexakis-Ionescu-Klainerman [**3**]). *Assume that* $(\mathbf{M}, \mathbf{g})$ *is regular, nondegenerate stationary vacuum spacetime, as in subsection 3.2. Assume in addition that there exists a regular maximal hypersurface* Σ_1 *passing through the bifurcation sphere and that*

$$(3.25) \qquad\qquad \|\mathbf{g}(\mathbf{T}, \mathbf{T})\|_{L^\infty(\mathcal{S}_0)} < \epsilon,$$

where ϵ *is a sufficiently small constant*[22]. *Then* $(\mathbf{M}, \mathbf{g})$ *is stationary and axially symmetric, thus, in view of the Carter-Robinson theorem, isometric to a Kerr spacetime* $\mathcal{K}(a, m)$ *with small* a.

This result should be compared with that stated in theorem 3.15 in which rigidity was proved, for the entire range $0 \leq a < M$, under a global smallness assumption on the Mars-Simon tensor associated to the stationary space-time. We recall that the proof of theorem 3.15 rested on the following ingredients:

(1) An unconditional local rigidity result, discussed in section 2, according to which a second, rotational Killing vector-field $\mathbf{Z}$ can be constructed in a small neighborhood of the bifurcate sphere of the horizon.

(2) An extension argument for the Killing vector-field $\mathbf{Z}$ based on a global foliation of the space-time with $\mathbf{T}$- conditional pseudo-convex hypersurfaces. The crucial $\mathbf{T}$- conditional null-convexity condition is ensured by the assumed smallness of the Mars-Simon tensor.

(3) Once $\mathbf{Z}$ is globally extended, and thus the space-time is shown to be both stationary and axisymmetric, one can appeal to the classical Carter-Robinson theorem to conclude the desired rigidity.

Theorem 3.18 is still based on the first and third ingredients above but replaces the second one with a new ingredient inspired from the classical work of Sudarsky and Wald [**36**] (see also [**7**]) on the staticity of stationary, axially symmetric, black hole solutions with zero angular momentum. The Sudarski-Wald result was based on a simple integral formula linking the total extrinsic curvature of a regular maximal hypersurface Σ imbedded in the space-time and passing through the bifurcate sphere, with the angular momentum of the horizon. It can be easily shown[23] that zero ADM angular momentum implies vanishing angular momentum of the horizon and thus, in view of the above mentioned formula, the maximal hyper-surface has to be totally geodesic. This then implies the desired conclusion of [**36**], i.e the space-time is static. The main observation in the proof of theorem 3.18 is that a simple smallness assumption of $\mathbf{T}$ on the bifurcate sphere[24] implies

---

[22]We note that the smallness depends on the entire geometry of $(\mathbf{M}, \mathbf{g})$, in particular on its ADM mass M.

[23]This step is based on the additional assumption of axial symmetry.

[24]This is equivalent with a small angular momentum assumption on the horizon. It remains open whether this condition can be replaced with a smallness assumption of the ADM angular momentum.

the smallness of the total curvature of the maximal hypersurface. This can then be combined with a simple application of the classical Hopf Lemma to conclude that the entire ergo-region of the black hole can be covered by the local neighborhood of the horizon in which the second, rotational, Killing vector-field $\mathbf{Z}$ has been extended, according to step (1) above. Away from the ergo-region $\mathbf{T}$ is time-like and thus $\mathbf{T}$-conditional null-convexity is automatically satisfied. Thus, the second Killing vector-field $\mathbf{Z}$ can be easily extended to the entire space-time by the results discussed in section 2.

3.8. Einstein-Maxwell case. The results of theorems 3.14, 3.15 presented in this section have been extended to the Einstein-Maxwell equations by W. Wong and P. Yu. The analogue of the Mars Simon tensor was discovered by Wong in [**38**]. It consists of a pair of tensors, one related to the curvature tensor and the second related to the Maxwell field. A Kerr-Newman solution is characterized by their simultaneous vanishing. A slight modification of the pair appears in [**40**]. The applications to the rigidity problem appear in [**40**] as well as [**39**] and [**41**].

4. Conclusions

Despite statements to the contrary made often in physics literature, the rigidity conjecture remains wide open. Though a lot of progress was made in the last ten years, the full scope of the conjecture remains out of reach. The global results presented in this survey are mostly limited to perturbative regimes. Under somewhat different assumptions they all assert that that stationary solutions closed to a non-extremal Kerr must be isometric to a non-extremal Kerr whose parameters a, M are determined by their ADM mass and angular momentum. Despite their limitations they offer however a perspective of what one might expect to encounter in the general case. To start with, the results illustrate the important role played by null geodesics perpendicular to the stationary Killing vectorfield $\mathbf{T}$. Based on the experience we have accumulated so far, we conjecture, together with our collaborator Spyros Alexakis, the following general conjecture.

Conjecture[Alexakis-Ionescu-Klainerman]. *Any asymptotically flat, regular, stationary vacuum solution, as as in subsection 3.2, which admits no trapped null geodesics perpendicular to* $\mathbf{T}$ *must be isometric to the exterior part of a non-extremal Kerr solution.*

All the three global results discussed in the survey are based on the fact that small, stationary perturbations of a non-extremal Kerr spacetime[25] verifies the hypothesis of the conjecture. It is conceivable that the conjecture can be proved with current techniques, based on unique continuation methods. The conjecture leaves however open the question whether such null geodesics can be ruled out in general. It thus illustrates an important aspect of the general case, namely the fact that we cannot hope to prove the full

---

[25]Note however that the Kerr family admits plenty of trapped null geodesics.

rigidity conjecture based only on a continuation argument starting form the horizon. Indeed such an argument may not distinguish between the given stationary Killing vectorfield $\mathbf{T}$ and any other possible Killing vectorfield, such as $\mathbf{T} + c\mathbf{Z}$ in Kerr. While, in Kerr, there are no trapped null geodesics perpendicular to $\mathbf{T}$ there are plenty of those perpendicular[26] to $\mathbf{T} + c\mathbf{Z}$. Thus a full proof of the rigidity conjecture must rely on global properties of the space-time.

Appendix A. Proof of proposition 2.18

We first restate the proposition in a general setting of an arbitrary Lorentzian manifold (M, g), a domain $O \subset M$, $p \in \partial O$ and h a defining, nondgenerate, function for ∂O in a full neighborhood U_1 of p, i.e. $h < 0$ in $O \cap U_1$ and $h = 0$ on $\partial O \cap U_1$. Moreover we assume that h verifies the condition (2.19) in U_1.

$$(A.1) \qquad \begin{cases} |dh| \geq M^{-1} \\ Y^\alpha Y^\beta \left(\mu g_{\alpha\beta} - D_\alpha D_\beta h \right) + M|Y(h)|^2 \geq M^{-1}|Y|^2, \end{cases}$$

uniformly, at all points of U, with $|Y|^2 = (Y^0)^2 + (Y^1)^2 + \ldots + (Y^d)^2$..

Let U_ϵ be small neighborhoods of p such that $|h| \leq 2^{-1}\epsilon$ in U_ϵ and define the weight functions, $f_\epsilon : U_\epsilon \longrightarrow \mathbb{R}$

$$f_\epsilon := \log(\epsilon + h + e_p)$$

where e_p is a small perturbation. More precisely, we say that e_ϵ is a negligible perturbation if

$$\sup_{U_\epsilon} |D^j e_\epsilon| \leq \epsilon^2 \qquad \text{for } j = 0, 1, 2.$$

In particular the weights f_ϵ verify (2.19) in U_ϵ, uniformly in $\epsilon > 0$. Also, uniformly in U_ϵ,

$$|Df_\epsilon| \leq C\epsilon^{-1}$$

PROPOSITION A.1. *If f_ϵ are as above, there exists a sufficiently small $\epsilon > 0$ and a large constant $C_\epsilon > 0$ such that, for all $\phi \in C_0^2(U_\epsilon)$ and all sufficiently large $\lambda > 0$,*

$$(A.2) \qquad \lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot D\phi\|_{L^2} \leq C_\epsilon \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \Box_{\mathbf{g}}\phi\|_{L^2},$$

REMARK A.2. *Note that C_ϵ denotes a constant which depends only on the small parameter ϵ but not on λ. Throughout the proof below we shrink $\epsilon > 0$ whenever necessary and enlarge the constant C_ϵ.*

PROOF. We first fix $\epsilon > 0$. Since all derivatives of $f = f_\epsilon$ are bounded on $U = U_\epsilon$ it suffices to prove (with a different C_ϵ !),

$$(A.3) \qquad \lambda \cdot \|e^{-\lambda f} \cdot \phi\|_{L^2} + \|D(e^{-\lambda f} \cdot \phi)\|_{L^2} \leq C_\epsilon \lambda^{-1/2} \cdot \|e^{-\lambda f} \cdot \Box_{\mathbf{g}}\phi\|_{L^2}.$$

---

[26]In fact for any given trapped null geodesic we can find a constant c such that $\mathbf{T} + c\mathbf{Z}$ is perpendicular to it.

Note that in (A.3), as well as in all similar places below, $\|D\psi\|_{L^2} :=$
$\|\,|D\psi|\,\|_{L^2}$ with $|D\psi|^2 = \sum_{\alpha=0}^n |\partial_\alpha \psi|^2$ as in proposition 2.18.
    To prove estimate (A.3) we start by setting,

(A.4) $\phi = e^{\lambda f} \psi$

Observe that,

$$e^{-\lambda f} \Box(e^{\lambda f}\psi) = \Box\psi + \lambda(2D^\beta f D_\beta \psi + \Box f \psi) + \lambda^2(D^\beta f D_\beta f)\psi$$
$$= L\psi + \Box f \psi$$

where,

$$L\psi = \Box\psi + 2\lambda X(\psi) + \lambda^2 G\psi, \qquad X = D^\alpha f \partial_\alpha, \qquad G = D^\beta f D_\beta f.$$

Thus estimate (A.3) follows from,

(A.5) $\lambda\|\psi\|_{L^2} + \|D\psi\|_{L^2} \le C_\epsilon \lambda^{-1/2} \|L\psi\|_{L^2},$

Recall the energy momentum tensor of $\Box = \Box_g$,

$$Q_{\mu\nu} = D_\mu \psi D_\nu \psi - \frac{1}{2} g_{\mu\nu}(D^\sigma \psi D_\sigma \psi).$$

Given a vectorfield X and a scalar function w we define $P_\mu = P_\mu[X, w]$

$$P_\mu : = Q_{\mu\nu} X^\nu - w\phi \partial_\mu \phi + \frac{1}{2}\partial_\mu w \phi^2$$

LEMMA A.3. *The one form $P_\mu = P_\mu[X, w]$ verifies the identity,*

(A.6) $D^\mu P_\mu = (X(\psi) - w\psi)\Box\psi + \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + \frac{1}{2}\Box w \psi^2 - w g(d\psi, d\psi)$

    In our case we have $\Box\psi = L\psi - 2\lambda X(\psi) - \lambda^2 G\psi$. Hence,

$$D^\mu P_\mu = (X(\psi) - w\psi)\left(L\psi - 2\lambda X(\psi) - \lambda^2 G\psi\right)$$
$$+ \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} - wD^\mu\psi D_\mu\psi + \frac{1}{2}\Box_g w |\psi|^2$$

or,

$$(Div\, P) + \lambda|X(\psi)|^2 - \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + wD^\mu\psi D_\mu\psi = E$$

where,

$$E = (X(\psi) - w\psi)\left(L\psi - 2\lambda X(\psi) - \lambda^2 G\psi\right) + \lambda|X(\psi)|^2 + \frac{1}{2}\Box_g w |\psi|^2$$
$$= (X(\psi) - w\psi)L\psi + (X(\psi) - w\psi)\left[-\lambda(X(\psi) - w\psi) - \lambda(X(\psi) + w\psi) - \lambda^2 G\psi\right]$$
$$+ \lambda|X(\psi)|^2 + \frac{1}{2}\Box_\mathbf{g} w |\psi|^2$$
$$= (X(\psi) - w\psi)L\psi - \lambda(X(\psi) - w\psi)^2 - \lambda\left(|X(\psi)|^2 - w^2\psi^2\right) + \lambda|X(\psi)|^2 + \frac{1}{2}\Box_\mathbf{g} w|\psi|^2$$
$$- \lambda^2 G\psi(X(\psi) - w\psi)$$
$$= (X(\psi) - w\psi)L\psi - \lambda(X(\psi) - w\psi)^2 + |\psi|^2\left(\lambda w^2 + \frac{1}{2}\Box_g w\right) - \lambda^2 G\psi(X(\psi) - w\psi)$$

Note that,

$$G\psi(X(\psi) - w\psi) = \frac{1}{2}GX^\mu D_\mu(\psi^2) - wG\psi^2 = D_\mu\left(\frac{1}{2}GX^\mu\psi^2\right) - \frac{1}{2}\psi^2\left[D_\mu(GX^\mu) + 2Gw\right]$$

Thus,

LEMMA A.4. *We have the point wise identity,*

$$D^\mu P'_\mu + \lambda|X(\psi)|^2 - \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + wD^\mu\psi D_\mu\psi - \frac{\lambda^2}{2}\psi^2\left[\mathbf{D}_\mu(GX^\mu) + 2Gw\right]$$
$$= E'$$

where,

$$P'_\mu = P_\mu + \frac{1}{2}\lambda^2 GX_\mu\psi^2$$

and,

$$E' \quad = \quad (X(\psi) - w\psi)L\psi - \lambda(X(\psi) - w\psi)^2 + |\psi|^2\left(\lambda w^2 + \frac{1}{2}\Box_\mathbf{g}w\right)$$

Now,

$$(X(\psi) - w\psi)L\psi \le \lambda^{-1}|L\psi|^2 + \lambda|X(\psi) - w\psi|^2$$

Hence,

$$E' \le \lambda^{-1}|L\psi|^2 + |\psi|^2\left(\lambda w^2 + \frac{1}{2}\Box_\mathbf{g}w\right)$$

Since by integration $D^\alpha P'_\alpha$ disappears, it suffices to check that the desired inequality

$$\lambda^2\|\psi\|_{L^2}^2 + \|D\psi\|_{L^2}^2 \le C_\epsilon\lambda^{-1}\|L\psi\|_{L^2}^2$$

for λ sufficiently large, follows by integrating the following pointwise inequality,

$$\lambda|X(\psi)|^2 - \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + wD^\mu\psi D_\mu\psi - \frac{\lambda^2}{2}\psi^2\left[D_\mu(GX^\mu) + 2Gw\right]$$
$$\le \lambda^{-1}|L\psi|^2 + |\psi|^2\left(\lambda w^2 + \frac{1}{2}\Box_\mathbf{g}w\right)$$

It thus suffices[27] to prove the following two inequalities, for C_ϵ sufficiently large and λ large,

$$\lambda|X(\psi)|^2 - \frac{1}{2}Q_{\mu\nu}{}^{(X)}\pi^{\mu\nu} + wD^\mu\psi D_\mu\psi \ge C_\epsilon^{-1}|D\psi|^2$$

$$-\frac{1}{2}\left[D_\mu(GX^\mu) + 2Gw\right] \ge C_\epsilon^{-1}$$

---

[27]Note indeed that the remaining term $|\psi|^2\left(\lambda w^2 + \frac{1}{2}\Box_\mathbf{g}w\right)$ on the right of the above inequality is lower order and can be easily absorbed.

Recalling the definition of $X = D^\alpha f \partial_\alpha$ and $G = D_a f D^\alpha f$ we write,

$$-\frac{1}{2} Q_{\mu\nu} \,^{(X)}\pi^{\mu\nu} + w D^\mu \psi D_\mu \psi = \left[-D^\mu D^\nu f + \frac{1}{2} g^{\mu\nu}(2w + \Box f) \right] D_\mu \psi D_\nu \psi$$

$$D_\mu(GX^\mu) + 2Gw = X(G) + GD_\mu X^\mu + 2Gw$$
$$= D^\alpha f D_\alpha (D^\beta f D_\beta f) + G(\Box f + 2w)$$
$$= 2D^\alpha f D^\beta f D_\alpha D_\beta f + D^\alpha f D_\alpha f(\Box f + 2w)$$

Hence, it suffices to show the inequalities (with $2w' = \Box f + 2w$),

(A.7) $\quad \lambda |D^\alpha f D_\alpha \psi|^2 + \left[-D_\alpha D_\beta f + w' g_{\alpha\beta} \right] D^\alpha \psi D^\beta \psi \geq C_\epsilon^{-1} |D\psi|^2$

(A.8) $\quad\qquad\qquad - \left(D_\alpha D_\beta f + w' g_{\alpha\beta} \right) D^\alpha f D^\beta f \geq C_\epsilon^{-1}$

Now recall that $f = f_\epsilon = \log(h + \epsilon)$. Therefore,

$$D_\alpha f = (h + \epsilon)^{-1} D_\alpha h$$
$$D_\alpha D_\beta f = (h + \epsilon)^{-1} D_\alpha D_\beta h - (h + \epsilon)^{-2} D_\alpha h D_\beta h$$

The inequality (A.7) becomes,

$$(\lambda + 1)(h + \epsilon)^{-1} |D^\alpha h D_\alpha \psi|^2 + \left[-D_\alpha D_\beta h + w(h + \epsilon) g_{\alpha\beta} \right] D^\alpha \psi D^\beta \psi$$
$$\geq C_\epsilon^{-1}(h + \epsilon) |D\psi|^2$$

which follows if,

$$\frac{1}{2\epsilon} \lambda |D^\alpha h D_\alpha \psi|^2 + [w(h + \epsilon) g_{\alpha\beta} - D_\alpha D_\beta h)] D^\alpha \psi D^\beta \psi \geq 2\epsilon C_\epsilon^{-1} |D\psi|^2$$

for some C_ϵ large. The inequality (A.8) becomes,

$$- \left[(h + \epsilon)^{-1} D_\alpha D_\beta h - (h + \epsilon)^{-2} D_\alpha h D_\beta h \right] D^\alpha h D^\beta h - w' \mathbf{g}_{\alpha\beta} D^\alpha h D^\beta h$$
$$\geq C_\epsilon^{-1}(h + \epsilon)^2$$

or,

$$(h + \epsilon)^{-1}(D_\alpha h D^\alpha h)^2 - D_\alpha D_\beta h D^\alpha h D^\beta h - w'(h + \epsilon)(D_\alpha h D^\alpha h) \geq C_\epsilon^{-1}(h + \epsilon)^3$$

It thus suffices to have, with a slightly different C_ϵ,

$$\frac{1}{2\epsilon}(D_\alpha h D^\alpha h)^2 - D_\alpha D_\beta h D^\alpha h D^\beta h - w'(h + \epsilon)(D_\alpha h D^\alpha h) \geq C_\epsilon^{-1} \epsilon^3$$

It thus remains to show that, with the right choice of ϵ sufficiently small and λ sufficiently large we can verify the inequalities,

(A.9)

$$\frac{1}{2\epsilon} \lambda |D^\alpha h D_\alpha \psi|^2 + \left[w'(h + \epsilon) g_{\alpha\beta} - D_\alpha D_\beta h) \right] D^\alpha \psi D^\beta \psi \geq 2\epsilon C_1 |D\psi|^2$$

(A.10)

$$\frac{1}{2\epsilon}(D_\alpha h D^\alpha h)^2 - D_\alpha D_\beta h D^\alpha h D^\beta h - w'(h + \epsilon)(D_\alpha h D^\alpha h) \geq C_\epsilon^{-1} \epsilon^3$$

In view of the quantitative strict null convexity condition (2.19) we have, for a sufficiently large M, $\mu \in [-M, M]$ and for all vector fields Y in U,

(A.11)
$$Y^\alpha Y^\beta (\mu g_{\alpha\beta} - D_\alpha D_\beta h) + M|Yh|^2 \geq M^{-1}|Y|^2, \qquad |Dh| \geq M^{-1}.$$

Hence with $Y^\alpha = D^\alpha \psi$, $\lambda \geq M$,

$$D^\alpha \psi D^\beta \psi (\mu g_{\alpha\beta} - D_\alpha D_\beta h) + \lambda |D^\alpha \psi D_\alpha(h)|^2 \geq M^{-1}|D\psi|^2,$$

Therefore, choosing $w' = (h + \epsilon)^{-1}\mu$, i.e. $w = (h + \epsilon)^{-1}\mu - \frac{1}{2}\Box f_\epsilon$, we deduce that (A.9) holds true provided that $\epsilon \ll M^{-1}$.

If $D^\alpha h D_\alpha h(p) \neq 0$, we can also find ϵ sufficiently small such (A.10) holds uniformly on $U = U_\epsilon$. Once again we need $\epsilon \ll M^{-1}$. If $D_\alpha h D^\alpha h = 0$ holds at p, we take $Y = D^\alpha h \partial_\alpha$ in (A.11) and derive, at p,

(A.12)
$$-D_\alpha D_\beta h D_\alpha h D_\beta h(p) \geq M^{-1}|dh(p)|^2$$

On the other hand the inequality (A.10) becomes, at p.

$$-D_\alpha D_\beta h D^\alpha h D^\beta h \quad \geq \quad 2C_\epsilon^{-1}\epsilon^3$$

In view of (A.12) this last inequality is satisfied if,

$$M^{-1}|dh(p)|^2 \geq 2C_\epsilon^{-1}\epsilon^3.$$

Since $|dh(p)| \geq M^{-1}$ we need $M^{-3} \gg C_\epsilon \epsilon^3$. Thus, to have both inequalities satisfied on U_ϵ we need,

$$\epsilon \ll M^{-1}$$

for a sufficiently large C_ϵ In other words given M, such that (A.11) is verified, we first choose $\epsilon \ll M^{-1}$ and then choose C_ϵ sufficiently large.

$\square$

Appendix B. Proof of Proposition 3.10

We give a slightly modified definition of the Mars-Simon tensor.

(B.1)
$$\mathcal{S}_{\alpha\beta\mu\nu} : \quad = \quad \mathcal{R}_{\alpha\beta\mu\nu} + 6\varpi^{-1}\mathcal{Q}_{\alpha\beta\mu\nu},$$
$$\mathcal{Q}_{\alpha\beta\mu\nu} : \quad = \quad \mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - \frac{1}{3}\mathcal{F}^2 \mathcal{I}_{\alpha\beta\mu\nu},$$
$$\mathcal{I}_{\alpha\beta\mu\nu} : \quad = \quad (\mathbf{g}_{\alpha\mu}\mathbf{g}_{\beta\nu} - \mathbf{g}_{\alpha\nu}\mathbf{g}_{\beta\mu} + i \in_{\alpha\beta\mu\nu})/4$$

with an arbitrary complex function ϖ.

PROPOSITION B.1. *The tensor $\mathcal{S}$ verifies the equation,*

$$\mathbf{D}^\alpha \mathcal{S}_{\alpha\beta\mu\nu} \quad = \quad -6\varpi^{-1}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta}\left[\mathcal{F}_\beta{}^\rho \delta_\mu^\gamma \delta_\nu^\delta + \frac{2}{3}\mathcal{I}^\rho{}_{\beta\mu\nu}\mathcal{F}^{\gamma\delta}\right] - 6\varpi^{-2}E^\rho \mathcal{Q}_{\rho\beta\mu\nu}$$

where,

$$E_\rho = \mathbf{D}_\rho \varpi + \sigma_\rho$$

REMARK B.2. *Note that proposition 3.10 is an immediate consequence of the above proposition for the special case $h = (1 - \sigma)$.*

PROOF. In view of (3.2) and the definitions of $\mathcal{S}$ and σ,

$$
\begin{aligned}
\mathbf{D}_\alpha \mathcal{F}_{\beta\gamma} &= \mathbf{T}^\lambda \mathcal{R}_{\lambda\alpha\beta\gamma} = \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma} - 6\varpi^{-1}\mathbf{T}^\lambda \mathcal{Q}_{\lambda\alpha\beta\gamma} \\
&= \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma} - 6\varpi^{-1}\mathbf{T}^\lambda \big(\mathcal{F}_{\lambda\alpha}\mathcal{F}_{\beta\gamma} - \frac{1}{3}\mathcal{F}^2 \mathcal{I}_{\lambda\alpha\beta\gamma}\big) \\
&= \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma} - 3\varpi^{-1}\sigma_\alpha \mathcal{F}_{\beta\gamma} + 2\varpi^{-1}\mathcal{F}^2 \mathbf{T}^\lambda \mathcal{I}_{\lambda\alpha\beta\gamma}
\end{aligned}
$$

i.e.,

$$
(B.2) \qquad \mathbf{D}_\alpha \mathcal{F}_{\beta\gamma} = 2\varpi^{-1}\mathcal{F}^2 \mathbf{T}^\lambda \mathcal{I}_{\lambda\alpha\beta\gamma} - 3\varpi^{-1}\sigma_\alpha \mathcal{F}_{\beta\gamma} + \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}
$$

Thus,

$$
\begin{aligned}
\mathbf{D}_\alpha[\varpi^{-3}\mathcal{F}_{\beta\gamma}] &= \varpi^{-3}\Big[2\varpi^{-1}\mathcal{F}^2 \mathbf{T}^\lambda \mathcal{I}_{\lambda\alpha\beta\gamma} - 3\varpi^{-1}\sigma_\alpha \mathcal{F}_{\beta\gamma} + \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}\Big] \\
&\quad -3\varpi^{-4}\mathbf{D}_\alpha h \mathcal{F}_{\beta\gamma} \\
&= 2\varpi^{-4}\mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\alpha\beta\gamma} - 3\varpi^{-4}(\sigma_\alpha + \mathbf{D}_\alpha h)\mathcal{F}_{\beta\gamma} + \varpi^{-3}\mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma} \\
&= 2\varpi^{-4}\mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\alpha\beta\gamma} - 3\varpi^{-4}E_\alpha \mathcal{F}_{\beta\gamma} + \varpi^{-3}\mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}
\end{aligned}
$$

We record this result for future reference,

$$
(B.3)
$$
$$
\mathbf{D}_\alpha[\varpi^{-3}\mathcal{F}_{\beta\gamma}] = \quad 2\varpi^{-4}\mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\alpha\beta\gamma} - 3\varpi^{-4}E_\alpha \mathcal{F}_{\beta\gamma} + \varpi^{-3}\mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}
$$

Since $\mathcal{I}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma} = \mathcal{F}_{\lambda\alpha}$,

$$
\begin{aligned}
\mathcal{F}^{\beta\gamma}D_\alpha \mathcal{F}_{\beta\gamma} &= -3\varpi^{-1}\sigma_\alpha \mathcal{F}^2 + 2\varpi^{-1}\mathcal{F}^2 \mathbf{T}^\lambda \mathcal{F}_{\lambda\alpha} + \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma} \\
&= -2\varpi^{-1}\sigma_\alpha \mathcal{F}^2 + \mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma}
\end{aligned}
$$

Thus,

$$
(B.4) \qquad\qquad \mathbf{D}_\alpha(\mathcal{F}^2) = -4\varpi^{-1}\sigma_\alpha \mathcal{F}^2 + 2\mathbf{T}^\lambda \mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma}
$$

We now calculate, using (B.2), (B.4) and (3.19),

$$
\begin{aligned}
\mathbf{D}^\alpha \mathcal{Q}_{\alpha\beta\mu\nu} &= \mathbf{g}^{\rho\alpha}\mathbf{D}_\rho\big(\mathcal{F}_{\alpha\beta}\mathcal{F}_{\mu\nu} - (1/3)\mathcal{F}^2\mathcal{I}_{\alpha\beta\mu\nu}\big) \\
&= \mathbf{g}^{\rho\alpha}\mathcal{F}_{\alpha\beta}\mathbf{D}_\rho \mathcal{F}_{\mu\nu} + \mathbf{g}^{\rho\alpha}\mathcal{F}_{\mu\nu}\mathbf{D}_\rho \mathcal{F}_{\alpha\beta} - (1/3)\mathbf{g}^{\rho\alpha}\mathcal{I}_{\alpha\beta\mu\nu}\mathbf{D}_\rho \mathcal{F}^2 \\
&= \mathbf{g}^{\rho\alpha}\mathcal{F}_{\alpha\beta}\mathbf{D}_\rho \mathcal{F}_{\mu\nu} - (1/3)\mathbf{g}^{\rho\alpha}\mathcal{I}_{\alpha\beta\mu\nu}\mathbf{D}_\rho \mathcal{F}^2 \\
&= -\mathcal{F}_\beta{}^\rho[2\varpi^{-1}\mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} - 3\varpi^{-1}\sigma_\rho \mathcal{F}_{\mu\nu} + \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu}] \\
&\quad - (1/3)\mathcal{I}_{\alpha\beta\mu\nu}[-4\varpi^{-1}\sigma^\alpha \mathcal{F}^2 + 2\mathbf{g}^{\alpha\rho}(\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta})\mathcal{F}^{\gamma\delta}] \\
&= -2\varpi^{-1}\mathcal{F}^2 \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + 3\varpi^{-1}\mathcal{F}_{\beta\rho}\sigma^\rho \mathcal{F}_{\mu\nu} \\
&\quad +(4/3)\varpi^{-1}\mathcal{F}^2 \mathcal{I}_{\alpha\beta\mu\nu}\sigma^\alpha \\
&\quad - \big[\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} + (2/3)\mathcal{I}_{\alpha\beta\mu\nu}\mathbf{g}^{\alpha\rho}\mathcal{F}^{\gamma\delta}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta}\big].
\end{aligned}
$$

We deduce,

$$
\begin{aligned}
\mathbf{D}^\alpha \mathcal{Q}_{\alpha\beta\mu\nu} &= \varpi^{-1} A_{\beta\mu\nu} + B_{\beta\mu\nu} \\
A_{\beta\mu\nu} &= -2\mathcal{F}^2 \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + 3\mathcal{F}_{\beta\rho}\sigma^\rho \mathcal{F}_{\mu\nu} + (4/3)\mathcal{F}^2 \mathcal{I}_{\rho\beta\mu\nu}\sigma^\rho. \\
B_{\beta\mu\nu} &= -\big[\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} + (2/3)\mathcal{I}_{\alpha\beta\mu\nu}\mathbf{g}^{\alpha\rho}\mathcal{F}^{\gamma\delta}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta} \big]
\end{aligned}
$$

Recalling the definition of $\mathcal{Q}$ we derive,

$$
\begin{aligned}
A_{\beta\mu\nu} &= -2\mathcal{F}^2 \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + 2\mathcal{F}_{\beta\rho}\sigma^\rho \mathcal{F}_{\mu\nu} + \mathcal{F}^2 \mathcal{I}_{\rho\beta\mu\nu}\sigma^\rho - \sigma^\rho \mathcal{Q}_{\rho\beta\mu\nu} \\
&= K_{\beta\mu\nu} - \sigma^\rho \mathcal{Q}_{\rho\beta\mu\nu}
\end{aligned}
$$

where,

$$
K_{\beta\mu\nu} = -2\mathcal{F}^2 \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + 2\mathcal{F}_{\beta\rho}\sigma^\rho \mathcal{F}_{\mu\nu} + \mathcal{F}^2 \mathcal{I}_{\rho\beta\mu\nu}\sigma^\rho
$$

Making use of the identity,

$$
(B.5) \qquad \mathcal{F}_\mu{}^\sigma \mathcal{I}_{\nu\sigma\alpha\beta} + \mathcal{F}_\nu{}^\sigma \mathcal{I}_{\mu\sigma\alpha\beta} = \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{F}^{\gamma\delta}\mathcal{I}_{\gamma\delta\alpha\beta} = \frac{1}{2}\mathbf{g}_{\mu\nu}\mathcal{F}_{\alpha\beta}.
$$

and $\sigma_\rho = 2\mathbf{T}^\sigma \mathcal{F}_{\sigma\rho}$ we derive,

$$
\begin{aligned}
K_{\beta\mu\nu} &= -2\mathcal{F}^2 \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + \mathcal{F}^2 \mathcal{I}_{\rho\beta\mu\nu}\sigma^\rho + 2\mathcal{F}_{\beta\rho}\sigma^\rho \mathcal{F}_{\mu\nu} \\
&= -2\mathcal{F}^2 \big(\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} - \mathcal{I}_{\rho\beta\mu\nu}\mathbf{T}^\sigma \mathcal{F}_{\sigma\rho} \big) + 4\mathbf{T}^\sigma \mathcal{F}_\sigma{}^\rho \mathcal{F}_{\beta\rho}\mathcal{F}_{\mu\nu} \\
&= -2\mathcal{F}^2 \mathbf{T}^\sigma \big(\mathcal{F}_\beta{}^\rho \mathcal{I}_{\sigma\rho\mu\nu} + \mathcal{F}_\sigma{}^\rho \mathcal{I}_{\beta\rho\mu\nu} \big) + 4\mathbf{T}^\sigma \mathcal{F}_\sigma{}^\rho \mathcal{F}_{\beta\rho}\mathcal{F}_{\mu\nu} \\
&= -2\mathcal{F}^2 \mathbf{T}^\sigma \cdot (1/2)\mathbf{g}_{\beta\sigma}\mathcal{F}_{\mu\nu} + 4\mathbf{T}^\sigma \mathcal{F}_{\mu\nu} \cdot (1/4)\mathbf{g}_{\sigma\beta}\mathcal{F}^2 \\
&= 0.
\end{aligned}
$$

Consequently,

$$
\mathbf{D}^\alpha \mathcal{Q}_{\alpha\beta\mu\nu} = -\varpi^{-1}\sigma^\rho \mathcal{Q}_{\rho\beta\mu\nu} + B_{\beta\mu\nu}
$$

from which we deduce, recalling $\sigma_\rho + \mathbf{D}_\rho h = E_\rho$

$$
\begin{aligned}
\mathbf{D}^\alpha\big(\varpi^{-1}\mathcal{Q}_{\alpha\beta\mu\nu}\big) &= -\varpi^{-2}\sigma^\rho \mathcal{Q}_{\rho\beta\mu\nu} + \varpi^{-1}B_{\beta\mu\nu} - \varpi^{-2}\mathbf{D}^\rho h \mathcal{Q}_{\rho\beta\mu\nu} \\
&= \varpi^{-1}B_{\beta\mu\nu} - \varpi^{-2}E^\rho \mathcal{Q}_{\rho\beta\mu\nu}
\end{aligned}
$$

Finally, recalling the definitions of $\mathcal{S}$ and B, we deduce,

$$
\begin{aligned}
\mathbf{D}^\alpha \mathcal{S}_{\alpha\beta\mu\nu} &= \mathbf{D}^\alpha \mathcal{R}_{\alpha\beta\mu\nu} + 6\mathbf{D}^\alpha\big(\varpi^{-1}\mathcal{Q}_{\alpha\beta\mu\nu}\big) = 6\varpi^{-1}B_{\beta\mu\nu} - 6\varpi^{-2}E^\rho \mathcal{Q}_{\rho\beta\mu\nu} \\
&= -6\varpi^{-1}\big[\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} + (2/3)\mathcal{I}^\rho{}_{\beta\mu\nu}\mathcal{F}^{\gamma\delta}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta} \big] \\
&\qquad -6\varpi^{-2}E^\rho \mathcal{Q}_{\rho\beta\mu\nu} \\
&= -6\varpi^{-1}\mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta}\big[\mathcal{F}_\beta{}^\rho \delta_\mu^\gamma \delta_\nu^\delta + \frac{2}{3}\mathcal{I}^\rho{}_{\beta\mu\nu}\mathcal{F}^{\gamma\delta} \big] - 6\varpi^{-2}E^\rho \mathcal{Q}_{\rho\beta\mu\nu}
\end{aligned}
$$

$\square$

B.1. Second Mars Tensor. In [**28**] Mars was able to give an alternative, stronger, characterization of the Kerr solution in terms of the vanishing of the tensor $\mathcal{S}_{\alpha\beta\mu\nu}\mathcal{F}^{\mu\nu}$. In what follows we show that a simple modification of that tensor verifies a Maxwell equation. We choose,

$$(B.6) \qquad \varpi = C(-\mathcal{F}^2)^{1/4}$$

in the generalized definition of $\mathcal{S}$ in (B.1), where C is a constant to be determined. With this choice of h we define the complex self-dual 2-form,

$$(B.7) \qquad \mathcal{H}_{\alpha\beta} := \varpi^{-3}\mathcal{S}_{\alpha\beta\mu\nu}\mathcal{F}^{\mu\nu}.$$

Since $\mathcal{Q}_{\alpha\beta\mu\nu}\mathcal{F}^{\mu\nu} = \frac{2}{3}\mathcal{F}^2\mathcal{F}_{\alpha\beta}$ we also have,

$$(B.8) \qquad \mathcal{H}_{\alpha\beta} = \varpi^{-3}\mathcal{R}_{\alpha\beta\mu\nu}\mathcal{F}^{\mu\nu} - 4C^{-4}\mathcal{F}_{\alpha\beta}.$$

PROPOSITION B.3. *The self-dual complex 2-form $\mathcal{H}_{\alpha\beta}$ defined above verifies the Maxwell equations,*

$$(B.9) \qquad \mathbf{D}^\alpha \mathcal{H}_{\alpha\beta} = -\varpi^{-3}\mathbf{T}^\sigma(\mathcal{S}\cdot\mathcal{S})_{\sigma\beta} - 3\varpi^{-1}E^\rho\mathcal{H}_{\rho\beta},$$

where,

$$(\mathcal{S}\cdot\mathcal{S})_{\sigma\beta} = \mathcal{S}_\beta{}^{\rho\mu\nu}\mathcal{S}_{\sigma\rho\mu\nu}$$

and,

$$(B.10) \qquad E_\rho := \sigma_\rho + \mathbf{D}_\rho\varpi = -\frac{1}{2}C^4\mathbf{T}^\sigma\mathcal{H}_{\sigma\rho}$$

PROOF. Recall (B.4),

$$\mathbf{D}_\alpha(\mathcal{F}^2) = -4\varpi^{-1}\sigma_\alpha\mathcal{F}^2 + 2\mathbf{T}^\lambda\mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma} = 4\varpi^{-1}\sigma_\alpha(hC^{-1})^4 + 2\mathbf{T}^\lambda\mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma}$$
$$= 4C^{-4}h^3\sigma_\alpha + 2\mathbf{T}^\lambda\mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma}$$

Hence,

$$C^{-1}D_\alpha\varpi = \mathbf{D}_\alpha\left[(-\mathcal{F}^2)^{1/4}\right] = -\frac{1}{4}\mathbf{D}_\alpha\mathcal{F}^2(-\mathcal{F}^2)^{-3/4} = -\frac{1}{4}[\varpi^4 C^{-4}]^{-3/4}(\mathbf{D}_\alpha\mathcal{F}^2)$$
$$= -\frac{1}{4}C^3\varpi^{-3}(\mathbf{D}_\alpha\mathcal{F}^2) = -\frac{1}{4}C^3\varpi^{-3}\left(4C^{-4}\varpi^3\sigma_\alpha + 2\mathbf{T}^\lambda\mathcal{S}_{\lambda\alpha\beta\gamma}\mathcal{F}^{\beta\gamma}\right)$$
$$= -C^{-1}\sigma_\alpha - \frac{1}{2}C^3\mathbf{T}^\lambda\mathcal{H}_{\lambda\alpha}$$

or,

$$(B.11) \qquad \mathbf{D}_\beta\varpi = -\sigma_\beta - \frac{1}{2}C^4\mathbf{T}^\sigma\mathcal{H}_{\sigma\beta}.$$

We deduce,

$$(B.12) \qquad E_\beta = -\frac{1}{2}C^4\mathbf{T}^\sigma\mathcal{H}_{\sigma\beta}$$

We now calculate,

$$\mathbf{D}^\alpha\mathcal{H}_{\alpha\beta} = \mathbf{D}^\alpha[\mathcal{S}_{\alpha\beta\mu\nu}\cdot\varpi^{-3}\mathcal{F}^{\mu\nu}] = \mathbf{D}^\alpha\mathcal{S}_{\alpha\beta\mu\nu}\cdot\varpi^{-3}\mathcal{F}^{\mu\nu} + \mathcal{S}_{\alpha\beta\mu\nu}\mathbf{D}^\alpha[\varpi^{-3}\mathcal{F}^{\mu\nu}].$$

Using proposition 3.10 ,

$$\mathbf{D}^\alpha\mathcal{S}_{\alpha\beta\mu\nu} = -6\varpi^{-1}\mathbf{T}^\sigma\mathcal{S}_{\sigma\rho\gamma\delta}\left[\mathcal{F}_\beta{}^\rho\delta_\mu^\gamma\delta_\nu^\delta + \frac{2}{3}\mathcal{I}^\rho{}_{\beta\mu\nu}\mathcal{F}^{\gamma\delta}\right] - 6\varpi^{-2}E^\rho\mathcal{Q}_{\rho\beta\mu\nu}$$

we deduce,

$$
\begin{aligned}
\mathbf{D}^\alpha \mathcal{S}_{a\beta\mu\nu} \cdot \varpi^{-3} \mathcal{F}^{\mu\nu} &= -6\varpi^{-4} \mathcal{F}^{\mu\nu} \big[\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} + (2/3) \mathcal{I}^\rho{}_{\beta\mu\nu} \mathcal{F}^{\gamma\delta} \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta} \big] \\
&\quad - 6\varpi^{-2} E^\rho \mathcal{Q}_{\rho\beta\mu\nu} \varpi^{-3} \mathcal{F}^{\mu\nu} \\
&= -6\varpi^{-4} \mathcal{F}^{\mu\nu} \big[\mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} + (2/3) \mathcal{I}^\rho{}_{\beta\mu\nu} \mathcal{F}^{\gamma\delta} \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\gamma\delta} \big] \\
&\quad - 6\varpi^{-5} E^\rho \left(\mathcal{F}_{\rho\beta} \mathcal{F}_{\mu\nu} - \frac{1}{3} \mathcal{F}^2 \mathcal{I}_{\rho\beta\mu\nu} \right) \mathcal{F}^{\mu\nu} \\
&= -6\varpi^{-1} \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} + 4\varpi^{-1} \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} \\
&\quad - 4\varpi^{-5} (\mathcal{F}^2) E^\rho \mathcal{F}_{\rho\beta} \\
&= -2\varpi^{-1} \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} + 4\varpi^{-2} (\mathcal{F}^2) E^\rho \mathcal{F}_{\beta\rho} \\
&= -2\varpi^{-1} \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} + 4\varpi^{-5} (-h^4/C^4) E^\rho \mathcal{F}_{\beta\rho} \\
&= -2\varpi^{-1} \mathcal{F}_\beta{}^\rho \mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} + 4\varpi^{-1} C^{-4} E^\rho \mathcal{F}_{\beta\rho} \\
&= -2\varpi^{-1} \mathcal{F}_\beta{}^\rho \left(\mathbf{T}^\sigma \mathcal{H}_{\sigma\rho} + 2C^{-4} E_\rho \right)
\end{aligned}
$$

Thus, in view of (B.12),

$$
D^\alpha \mathcal{S}_{a\beta\mu\nu} \cdot \varpi^{-3} \mathcal{F}^{\mu\nu} = 0.
$$

On the other hand, recalling (B.3)

$$
\mathbf{D}_\rho [\varpi^{-3} \mathcal{F}_{\mu\nu}] = 2\varpi^{-4} \mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} - 3\varpi^{-4} E_\rho \mathcal{F}_{\mu\nu} + \varpi^{-3} \mathbf{T}^\lambda \mathcal{S}_{\lambda\rho\mu\nu}
$$

we have,

$$
\mathcal{S}_{\rho\beta\mu\nu} \mathbf{D}^\rho [\varpi^{-3} \mathcal{F}^{\mu\nu}] = \mathcal{S}^\rho{}_\beta{}^{\mu\nu} \big[2\varpi^{-4} \mathcal{F}^2 \mathbf{T}^\sigma \mathcal{I}_{\sigma\rho\mu\nu} + \varpi^{-3} \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} - 3\varpi^{-4} E_\rho \mathcal{F}_{\mu\nu} \big]
$$

Observe that $\mathcal{S}^\rho{}_\beta{}^{\mu\nu} \mathcal{I}_{\sigma\rho\mu\nu} = 0$ Thus,

$$
\mathcal{S}_{\rho\beta\mu\nu} \mathbf{D}^\rho [\varpi^{-3} \mathcal{F}^{\mu\nu}] = \mathcal{S}^\rho{}_\beta{}^{\mu\nu} \varpi^{-3} \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} - 3\varpi^{-1} \mathcal{H}^\rho{}_\beta E_\rho
$$

Finally we deduce,

$$
\begin{aligned}
\mathbf{D}^\alpha \mathcal{H}_{\alpha\beta} &= \mathbf{D}^\alpha \mathcal{S}_{\alpha\beta\mu\nu} \cdot \varpi^{-3} \mathcal{F}^{\mu\nu} + \mathcal{S}_{\alpha\beta\mu\nu} \mathbf{D}^\alpha [\varpi^{-3} \mathcal{F}^{\mu\nu}] \\
&= \mathcal{S}^\rho{}_\beta{}^{\mu\nu} \varpi^{-3} \mathbf{T}^\sigma \mathcal{S}_{\sigma\rho\mu\nu} - 3\varpi^{-1} \mathcal{H}^\rho{}_\beta E_\rho \\
&= -\varpi^{-3} \mathbf{T}^\sigma (\mathcal{S} \cdot \mathcal{S})_{\sigma\beta} - 3\varpi^{-1} E^\rho \mathcal{H}_{\rho\beta}
\end{aligned}
$$

$\square$

References

[1] S. Alexakis, A. D. Ionescu, and S. Klainerman, *Hawking's local rigidity theorem without analyticity*, Geom. Funct. Anal. **20** (2010), 845–869.

[2] S. Alexakis, A. D. Ionescu, and S. Klainerman, *Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces*, Comm. Math. Phys. **299** (2010), 89–127.

[3] S. Alexakis, A. D. Ionescu, and S. Klainerman, *Rigidity of stationary black holes with small angular momentum on the horizon*, Duke Math. Journ. **163**, Number 14 (2014), 2603-2615.

[4] R. Beig and W. Simon, *On the multipole expansion for stationary space-times*, Proceeding of the Royal Society, series A, **376**, No 1765, (1981), 333-341.

[5] R. Beig and W. Simon, *The stationary gravitational field near spatial infinity*, Gen. Rel. and Grav., **12**, No 12, (1980), 1003–1013.

[6] G. Bunting and A.K.M. Massood ul Alam, *Non-existence of multiple black holes in asymptotically euclidean static vacuum space-time*, Gen. Rel. Grav. **19** (1987), 147-154.

[7] B. Carter, *Has the Black Hole Equilibrium Problem Been Solved*, gr-qc/9712038.

[8] B. Carter, *An axi-symmetric black hole has only two degrees of freedom*, Phys. Rev. Letters, **26**, (1971) 331-333.

[9] S. Chandrasekhar, *The mathematical theory of black holes.* International Series of Monographs on Physics, 69, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York (1983).

[10] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Math. Series 41, Princeton University Press (1993)

[11] P.T. Chrusciel, *"No Hair" Theorems-Folclore, Conjecture, Results*, Diff. Geom. and Math. Phys.(J. Beem and K.L. Duggal) Cont. Math. , **170**, AMS, Providence, (1994), 23-49, gr-qc9402032, (1994).

[12] P.T. Chrusciel,*The classification of static vacuum space-times containing an asymptotically flat space-like hypersurface with compact interior* , Class. Quant. Grav. **16** (1999), 661-687, gr-qc/9809088.

[13] P.T. Chrusciel, *Uniqueness of black Holes revisited*, Helv. Phys. Acta **69** (1996), 529–552, Proceedings of Journee relativistes 1996, Ascona May 1996, gr-qc/ 9610010.

[14] P.T. Chrusciel and R.M. Wald *Maximal hypersurfaces in A.F. space-times*, Comm. Math. Phys. **163**, 561-164 (1994).

[15] Mihalis Dafermos, Igor Rodnianski, Yakov Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$*, preprint arXiv:1407.1415.

[16] J.L. Friedman, K. Schleich, D.M. Witt, *Topological censorship*, Phys, Rev Letters, **71**, 1846-1849 (1993).

[17] H. Friedrich, I. Racz, R. Wald, *On the rigidity theorem for spacetimes with a stationary event or Cauchy horizon*, arXiv:gr/qc9811021, 2002

[18] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge Univ. Press (1973).

[19] L. Hörmander, *The analysis of linear partial differential operators IV. Fourier integral operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 275. Springer-Verlag, Berlin (1985).

[20] A. Ionescu and S. Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*, Invent. Math. **175** (2009), 35–102.

[21] A. Ionescu and S. Klainerman, *Uniqueness results for ill posed characteristic problems in curved space-times*, Communications in Mathematical Physics **285**, 873-900 (2009)

[22] A. Ionescu and S. Klainerman, *On the local extension of Killing vectorfields in Ricci flat manifolds*, J. Amer. Math. Soc. **26** (2013), 563-593.

[23] J. Isenberg and V. Moncrief, *Symmetries of Cosmological Cauchy Horizons*, Comm. Math. Phys. **89**, 387-413, (1983).

[24] W. Israel, *Event horizons in static vacuum space-times*, Phys. Rev. Letters **164** (1967), 1776-1779.

[25] S. Klainerman and F. Nicolò, *The evolution problem in general relativity.* Progress in Mathematical Physics, 25. Birkhäuser Boston, Inc., Boston, MA, (2003).

[26] S. Kobayashi *Transformations groups in differential geometry*, Springer, 1972.

[27] M. Mars, A spacetime characterization of the Kerr metric, Classical Quantum Gravity **16** (1999), 2507–2523.

[28] M. Mars, Uniqueness properties of the Kerr metric, Classical Quantum Gravity **17** (2000), 3353–3373.

[29] E.T. Newman and R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. **3** (1962), 566-578.

[30] P. O. Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution*, J. Phys. A **15** (1982), no. 10, 3173?3180.

[31] K. Nomizu, *On local and global existence of Killing vector fields,* Ann. of Math. **72** (1960), 105–120.

[32] A. D. Rendall, *Reduction of the characteristic initial value problem to the Cauchy problem and its applications to the Einstein equations,* Proc. R. Soc. Lond. A **427** (1990), 221–239.

[33] D.C. Robinson, *Uniqueness of the Kerr black hole*, Phys. Rev. Lett. **34** (1975), 905-906.

[34] I. Racz and R. Wald *Extensions of space-times with Killing horizons*, Class. Quant. Gr., **9** (1992), 2463-2656.

[35] W. Simon, *Characterization of the Kerr metric*, Gen. Rel. Grav. **16** (1984), 465-476.

[36] D. Sudarski and R.M. Wald, *Mass formulas for stationary Einstein Yang-Mills black holes and a simple proof of two staticity theorems*, Phys. Rev **D47** (1993), 5209-5213, gr-qc /9305023.

[37] G. Weinstein, *On rotating black holes in equilibrium in General Relativity*, Comm. Pure Appl. Math. **43** (1990), 903–948.

[38] W. Wai-Yeung Wong, *A spacetime characterization of the Kerr-Newmann metric*, Ann. Henri Poincaré **10** (2009) no3, 453–484.

[39] W. Wai-Yeung Wong, *On the uniqueness of Kerr-Newman black holes,* Ph.D. thesis, Princeton University, 2009.

[40] W. Wai-Yeung Wong and Pin Yu , *Non-existence of Multiple Black Hole solutions close to Kerr-Newmann*

[41] Pin Yu, *On the rigidity of Black Holes*, PhD thesis, Princeton University, June 2010.

PRINCETON UNIVERSITY
E-mail address: `aionescu@math.princeton.edu`

PRINCETON UNIVERSITY
E-mail address: `seri@math.princeton.edu`