

Interior a priori estimates for the Monge-Ampère equation

Jiakun Liu and Xu-Jia Wang

ABSTRACT. In this paper we prove the strict convexity, the interior $C^{1,\alpha}$, $C^{2,\alpha}$ and $W^{2,p}$ estimates for convex solutions to the Monge-Ampère type equation. For the strict convexity and $C^{1,\alpha}$ estimate, we assume that the inhomogeneous term f satisfies a doubling condition. For the $C^{2,\alpha}$ and $W^{2,p}$ estimates, we assume that f is Hölder continuous or continuous. These estimates are mainly due to Caffarelli. We also give a brief discussion on the regularity for more general Monge-Ampère type equations arising in optimal transportation.

1. Introduction

In this paper we consider the regularity of solutions to the Monge-Ampère equation

$$(1.1) \quad \det D^2u = f \quad \text{in } \Omega,$$

where Ω is a domain in \mathbb{R}^n , $\det D^2u$ denotes the determinant of the Hessian matrix D^2u , and f is a given function. This is a fully nonlinear, second order partial differential equation. It is elliptic when the Hessian matrix D^2u is positive definite, namely when u is locally uniformly convex. In this paper we always assume that the solution is convex, and the inhomogeneous term f is nonnegative.

When $f = K(1 + |Du|^2)^{(n+2)/2}$, equation (1.1) is the prescribed Gauss curvature equation, with the Gauss curvature K . The Monge-Ampère equation also arises in a variety of applications, such as in affine geometry, isometric embedding, optimal transportation, and reflector design. It has been extensively studied in the last century. Significant contributions were made by Aleksandrov, Calabi, Pogorelov, Heinz, Caffarelli-Nirenberg-Spruck, Cheng-Yau, Krylov, and many others before 1990s. In particular Pogorelov proved

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an interior second derivative estimate for strictly convex solutions and Calabi obtained an estimate for higher derivatives. Therefore a regularity theory in the case when f is positive and smooth has been established. We refer the reader to [7, 8, 9, 15, 16, 17, 29, 32, 33] for more details.

In this paper we address the interior a priori estimates for the Monge-Ampère equation, mainly obtained by Caffarelli in 1990s [3, 4, 5]. But in dimension two, these estimates were obtained earlier [17, 28, 30]. We will discuss the following estimates:

- a) Strict convexity of solutions;
- b) $C^{1,\alpha}$ regularity of solutions;
- c) Continuity of the second derivatives;
- d) Hölder continuity of the second derivatives;
- e) $W^{2,p}$ estimate for large p ;
- f) $W^{2,1+\epsilon}$ estimate for small $\epsilon > 0$.

For part a), we assume that the solution vanishes on the boundary and f satisfies a doubling condition. The strict convexity of solutions is assumed in all the estimates b)-f). For estimate b), we assume that f satisfies a doubling condition. For estimates c) and d), we assume that f is positive and Dini or Hölder continuous. For estimate e), we assume that f is positive and continuous. By an example in [35], the continuity is needed for the $W^{2,p}$ estimate for large p . But for small $p > 1$, it suffices to assume that $c_0 < f < c_1$ for some positive constants c_0 and c_1 , and that is the estimate in f).

Among the above estimates, the main ones are the $C^{2,\alpha}$ and $W^{2,p}$ estimate by Caffarelli [4]. However, the paper [4] is difficult to understand. For example no details for the $C^{2,\alpha}$ were given. Even for the uniformly elliptic equation, the proof of the $C^{2,\alpha}$ is quite complicated [2]. A simplified proof for the $C^{2,\alpha}$ estimate was later given in [19], using a perturbation argument introduced in [37]. In [37] a simple proof for the $C^{2,\alpha}$ estimate for elliptic and parabolic equations, both linear and nonlinear, was presented. For the $W^{2,p}$ estimate, the proof in this paper is also simpler than that in [4], but the basic idea is the same as that in [4]. The purpose of this article is to provide simplified proof for Caffarelli's work on the Monge-Ampère equation. The presentation of this paper is based on the lectures given by the second author at Tsinghua University in 2011. In this note we also provide a simple proof for the existence and uniqueness of minimum ellipsoids of convex bodies.

This paper is organized as follows. In Section 2, we discuss the strict convexity and $C^{1,\alpha}$ regularity of solutions. In Section 3 we outline the proof in [19] for a continuity estimate for the second derivatives of solutions. In Section 4, following the lines of [4] with some simplification in [38], we give a shorter proof of the $W^{2,p}$ estimate, assuming f is continuous. When f is not continuous but pinched by two positive constants, recently De Philippis-Figalli-Savin [12] and Schmidt [30] proved $u \in W_{\text{loc}}^{2,1+\epsilon}$ for some small $\epsilon > 0$. We include the proof of the $W_{\text{loc}}^{2,1+\epsilon}$ estimate in Section 5. Finally in Section

6 we briefly discuss the corresponding estimates for the more general Monge-Ampère equation

$$(1.2) \quad \det [D^2u - A(x, u, Du)] = f(x, u, Du).$$

This more general equation arises in reflector design and optimal transportation. The interior $C^{2,\alpha}$ and $W^{2,p}$ estimates for (1.2) have also been obtained recently in [23, 24], respectively, under some appropriate assumptions on the cost function.

2. Strict convexity and $C^{1,\alpha}$ estimate

2.1 Minimum ellipsoid.

First we introduce a lemma, which is due to F. John, and is often used in the study of convex bodies and the Monge-Ampère equation.

LEMMA 2.1. *Let Ω be a bounded, convex domain in \mathbb{R}^n . Then among all ellipsoids containing Ω , there is a unique ellipsoid E of smallest volume such that*

$$(2.1) \quad \frac{1}{n}E \subset \Omega \subset E.$$

In this paper we denote by $\alpha\Omega$ the α -dilation of Ω with respect to the center of its minimum ellipsoid. We call E the *minimum ellipsoid* of Ω . By a rotation of the coordinates, we may assume that E is given by

$$E = \left\{ \sum_{i=1}^n \left(\frac{x_i - x_{0,i}}{r_i} \right)^2 < 1 \right\},$$

where $x_0 = (x_{0,1}, \dots, x_{0,n})$. By the unimodular linear transform $T : x \rightarrow y$,

$$(2.2) \quad y_i = \frac{r}{r_i}(x_i - x_{0,i}) + x_{0,i}, \quad i = 1, \dots, n,$$

where $r = (r_1 \cdots r_n)^{1/n}$, E becomes the ball $B_r(x_0)$ with

$$B_{r/n}(x_0) \subset T(\Omega) \subset B_r(x_0).$$

We say Ω is *normalized* if its minimum ellipsoid is a ball (namely when T is the identity mapping).

PROOF OF LEMMA 2.1. Let $V_0 = \inf\{|E| : E \in \Phi\}$, where Φ is the set of ellipsoids containing Ω . Let E_k be a sequence of ellipsoids in Φ with $|E_k| \rightarrow V_0$. Since E_k contains Ω , it must be uniformly bounded and converges in Hausdorff distance to an minimum ellipsoid E .

To show that E satisfies (2.1), we assume by a linear transform that E is the unit sphere with center at the origin. If (2.1) is not true, let $x_0 \in \partial\Omega$ such that $|x_0| = \inf_{x \in \partial\Omega} |x|$. By a rotation of axes, we assume $x_0 = (0, \dots, 0, -t)$ with $t \leq \frac{1}{n} - \epsilon$ for some $\epsilon > 0$, such that the plane $\{x_n = -t\}$ is a tangent plane of $\partial\Omega$ at x_0 . Then we have $\Omega \subset G =: B_1(0) \cap \{x_n > -t\}$. It suffices to prove that the unit ball is not the minimum ellipsoid of G .

The proof is very elementary. Let $y_i = x_i/(1 + \delta)$ for $i = 1, \dots, n - 1$, and $y_n = x_n(1 + \delta)^{n-1}$, where $\delta = \epsilon^2$. In the new coordinates, G is *strictly* contained in the unit sphere with center at $(0, \dots, 0, h)$ provided ϵ is sufficiently small, where $h = (1 + \delta)^{n-1} - 1 + \delta^2$. We reach a contradiction as E is a minimum ellipsoid.

Next, we prove the uniqueness. Suppose there exist two minimum ellipsoids E_1 and E_2 . By a rescaling we assume $E_1 = \{\sum x_i^2 < 1\}$ and $E_2 = \{\sum (x_i - a_i)^2/r_i^2 < 1\}$ with $\prod r_i = 1$, where (a_1, \dots, a_n) is the centre of E_2 . Then for any $x \in E_1 \cap E_2$, we have

$$\sum_i \left[x_i^2 + \frac{(x_i - a_i)^2}{r_i^2} \right] < 2.$$

That is

$$\sum_i \frac{1 + r_i^2}{2r_i^2} \left[\left(x_i - \frac{a_i}{1 + r_i^2} \right) + \frac{a_i^2 r_i^2}{(1 + r_i^2)^2} \right] < 1,$$

namely,

$$(2.3) \quad \sum_i \frac{1 + r_i^2}{2r_i^2} \left(x_i - \frac{a_i}{1 + r_i^2} \right)^2 < 1 - \sum_i \frac{a_i^2}{2(1 + r_i^2)} \leq 1.$$

Hence the domain $\Omega \subset E_1 \cap E_2$ is contained in the ellipsoid

$$(2.4) \quad E = \left\{ \sum_i \frac{1 + r_i^2}{2r_i^2} \left(x_i - \frac{a_i}{1 + r_i^2} \right)^2 < 1 - \sum_i \frac{a_i^2}{2(1 + r_i^2)} \right\}.$$

Note that $\prod r_i = 1$. Hence

$$(2.5) \quad \prod_{i=1}^n \frac{1 + r_i^2}{2r_i^2} = \prod_{i=1}^n \frac{1}{2}(1 + r_i^2) \geq 1,$$

which implies that $|E| \leq |E_1|$ and the equality achieves if and only if $a_i = 0$ and $r_i = 1$ for all $i = 1, \dots, n$. Therefore, we obtain the uniqueness. \square

The above proof of the uniqueness seems new, and is much simpler than the known ones. The uniqueness of the minimum ellipsoid is not needed in our treatment below. We remark that there is also a unique ellipsoid contained in Ω with maximal volume among all ellipsoids contained in Ω .

2.2 Uniform and Hölder estimates

The rest part of this section is devoted to the strict convexity and $C^{1,\alpha}$ estimate. The presentation is similar to that in [33]. Consider the Monge-Ampère equation,

$$(2.6) \quad \det D^2 u = \nu \quad \text{in } \Omega,$$

where Ω is a bounded, convex domain in \mathbb{R}^n , ν is a finite measure.

THEOREM 2.1. (*Uniform estimate*) Let u be a bounded, convex solution to (2.6) subject to the boundary condition $u = 0$ on $\partial\Omega$. If

$$\nu(\Omega) \leq b\nu\left(\frac{1}{2}\Omega\right)$$

for some positive constant b , then

$$(2.7) \quad C^{-1}\{|\Omega|\nu(\Omega)\}^{1/n} \leq \sup|u| \leq C\{|\Omega|\nu(\Omega)\}^{1/n},$$

where C is a constant depending only on n and b .

In particular if $\nu = f dx$ and $c_0 \leq f \leq c_1$ for positive constants c_0, c_1 , then

$$(2.8) \quad C^{-1}|\Omega|^{2/n} \leq \sup|u| \leq C|\Omega|^{2/n},$$

where C depends only on n, c_0, c_1 .

PROOF. Let $E = \{\sum(x_i - a_i)^2/r_i^2 < 1\}$ be the minimum ellipsoid of Ω . The Monge-Ampère equation is affine invariant and homogeneous, by making the changes $x_i \mapsto \frac{x_i - a_i}{r_i}$ and $u \mapsto u/h$, where $h = [(r_1 \cdots r_n)\nu(\Omega)]^{1/n}$, we may assume that

$$B_{1/n}(0) \subset \Omega \subset B_1(0),$$

$$\nu(\Omega) = 1, \quad \nu\left(\frac{1}{2}\Omega\right) \geq b^{-1}.$$

To prove (2.7), it suffices to prove that $C^{-1} \leq \sup|u| \leq C$.

To prove $\sup|u| \leq C$, we use the assumption $\nu(\Omega) = 1$. Assume that $\sup|u|$ attains at $\hat{x} \in \Omega$. Let w be a convex function vanishing on $\partial\Omega$ and its graph is a convex cone with vertex at $(\hat{x}, u(\hat{x}))$. By convexity, $N_w(\Omega) \subset N_u(\Omega)$ and $B_r(0) \subset N_w(\Omega)$ for $r = \frac{1}{2}\sup|u|$ as $\Omega \subset B_1(0)$, where N_u is the normal mapping of u [33]. Hence $\sup|u|^n \leq C|N_w(\Omega)| \leq C$.

To prove $\sup|u| \geq C$, we use the assumption $\nu(\frac{1}{2}\Omega) \geq b^{-1}$. For any $x \in \frac{1}{2}\Omega$,

$$|Du(x)| \leq \frac{-\inf u}{\text{dist}(x, \partial\Omega)} \leq 2n \sup|u|.$$

Hence $\nu(\frac{1}{2}\Omega) = |N_u(\frac{1}{2}\Omega)| \leq C \sup|u|^n$. □

THEOREM 2.2. (*Global Hölder estimate*) Let u be a generalized solution to

$$\begin{aligned} \det D^2u &= \nu \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

Suppose Ω is convex and contained in the unit ball B_1 , $\varphi \in C^\alpha(\bar{\Omega})$ is convex, and $\nu(\Omega) \leq c_1$ for a constant c_1 . Then

$$(2.9) \quad |u(x) - u(y)| \leq C|x - y|^{\bar{\alpha}} \quad \forall x, y \in \Omega,$$

where C depends on n, c_1 , and $\|\varphi\|_{C^{\bar{\alpha}}(\bar{\Omega})}$, and $\bar{\alpha} = \min(\frac{1}{n}, \alpha)$.

PROOF. First consider the case $u \equiv 0$ on $\partial\Omega$. We show that

$$|u(x_0) - u(y_0)| \leq C|y_0 - x_0|^{1/n}$$

for $x_0, y_0 \in \bar{\Omega}$. Since u is convex, it suffices to prove the inequality for $x_0 \in \Omega, y_0 \in \partial\Omega$. For any point $x_0 \in \Omega$, by choosing proper coordinates, we assume that $x_0 = de_n$ and $\Omega \subset \{x_n > 0\}$, where $d = \text{dist}(x_0, \partial\Omega)$ and $e_n = (0, \dots, 0, 1)$. Then $\Omega \subset \hat{\Omega} = \{x \in \mathbb{R}^n : |x'| < 2, 0 < x_n < 4\}$. Let v and w be convex functions such that their graphs are convex cones, with vertex at $(x_0, u(x_0))$ and bases $\partial\Omega$ and $\partial\hat{\Omega}$, respectively. Then $N_u(\Omega) \supset N_v(\Omega) = N_v(x_0) \supset N_w(x_0)$. Since w is a convex cone, one easily verifies that $|N_w(x_0)| \geq \frac{C}{d}|u(x_0)|^n$, namely $|u(x_0)| \leq C[d\nu(\Omega)]^{1/n}$.

When $u = \varphi$ on $\partial\Omega$ for a convex function $\varphi \in C^\alpha(\bar{\Omega})$, we let u_0 be a solution of $\det D^2u = \nu$ in Ω which vanishes on $\partial\Omega$. Then $u_0 + \varphi$ is a sub-barrier and we also obtain (2.9). \square

COROLLARY 2.1. *Let u be a generalized solution to (2.6) which vanishes on $\partial\Omega$. Suppose $\nu(\Omega) \leq b\nu(\frac{1}{2}\Omega)$ for some constant $b > 0$. Let ℓ be a line segment in Ω with two endpoints $z', z'' \in \partial\Omega$. Let z be a point on ℓ such that $u(z) \leq \frac{1}{2} \inf_{\Omega} u$. Then $|z' - z| \geq C|z'' - z|$ for some $C > 0$ depending only on n and b .*

Note that the ratio $\frac{|z' - z|}{|z'' - z|}$ is invariant under linear transforms. Hence by making a linear transform we may assume that $|\Omega| = 1$ and Ω is normalized. By Theorem 2.1 and the assumption $\nu(\Omega) \leq b\nu(\frac{1}{2}\Omega)$, we may assume furthermore that $\inf u = -1$ and $\nu(\Omega) \leq C$. Hence when $u(z) \leq -\frac{1}{2}$, by Theorem 2.2 we have $\text{dist}(z, \partial\Omega) \geq C_0 > 0$, and so Corollary 2.1 follows.

2.3 Strict convexity

We say a measure ν satisfies the *doubling condition* if there exists a constant $b > 0$ such that for any convex set $\omega \subset \Omega$,

$$(2.10) \quad \nu(\omega) \leq b\nu(\frac{1}{2}\omega).$$

This condition is *invariant* under affine transforms. We always assume that $\nu(\Omega) > 0$. For $x_0 \in \Omega, h > 0$, denote

$$S_{h,u}^0(x_0) = \{x \in \Omega \mid u(x) < \ell(x) + h\},$$

where ℓ is the supporting function of u at x_0 . Sometimes we drop the subscript u if no confusion arises, and also write $S_{h,u}^0(x_0)$ as S_h^0 if x_0 is the minimum point of u . First we consider the strict convexity of solutions.

THEOREM 2.3. *(Strict convexity) Let u be a generalized solution to (2.6). Assume that ν satisfies the doubling condition (2.10). Then for any point $x_0 \in \Omega$, either the contact set $\mathcal{C} = \{x \in \Omega : u(x) = \ell(x)\}$ is a single point, where ℓ is a supporting function of u at x_0 , or \mathcal{C} has no extremal points in Ω .*

PROOF. By subtracting the supporting function, we assume that $\mathcal{C} = \{u = 0\}$, $u \geq 0$ in Ω . If the claim is not true, by contradiction we assume \mathcal{C} is not a single point and $z \in \Omega$ is an extremal point of \mathcal{C} .

By a transform of coordinates we may assume that $z = (0, \dots, 0, \delta)$ for a small constant $\delta > 0$, and $G_0 = \mathcal{C} \cap \{x_n \geq 0\}$ is compactly contained in Ω .

Let $G_\varepsilon = \{x \in \Omega : u(x) \leq \varepsilon x_n\}$, where $\varepsilon > 0$ is a constant. Denote $w(x) = u(x) - \varepsilon x_n$. When ε is sufficiently small, we have $\partial G_\varepsilon \subset \Omega$. Hence w vanishes on ∂G_ε . Let $z' \in G_\varepsilon$ such that the three points $0, z$ and z' lie on a straight line. Observing that G_ε shrinks to the set G_0 as $\varepsilon \rightarrow 0$, we have $|z' - z| = o(|z'|)$ and moreover, $\inf_{G_\varepsilon} w \geq 2w(z) = -2\varepsilon\delta$. Applying Corollary 2.1 to w we reach a contradiction. \square

Next we rule out the possibility that extremal points of \mathcal{C} lie on the boundary $\partial\Omega$ under some appropriate boundary assumptions.

THEOREM 2.4. *Let u be a generalized solution to (2.6) in a convex domain Ω . Assume that ν satisfies the doubling condition (2.10). Suppose that either*

- (i) $u = 0$ on $\partial\Omega$; or
- (ii) $u = \varphi$ on $\partial\Omega$ for some $\varphi \in C^{1,\alpha}$ and $\partial\Omega \in C^{1,\alpha}$ with $\alpha > 1 - \frac{2}{n}$, and $\nu \geq c_0$ for a positive constant c_0 .

Then u is strictly convex in Ω .

The exponent $\alpha > 1 - \frac{2}{n}$ in (ii) is optimal, as shown by Pogorelov's example

$$(2.11) \quad u(x) = (1 + x_n^2)|x'|^{2(1-\frac{1}{n})}.$$

The function satisfies the equation

$$\det D^2 u = 2\beta^{n-1}(1 + x_n^2)^{n-2}(\beta - 1 - (\beta + 1)x_n^2),$$

where $\beta = 2 - 2/n$. The right hand side is an analytic positive function when $|x_n| < \frac{\beta-1}{\beta+1}$.

PROOF OF THEOREM 2.4. *Case (i):* For any point $x_0 \in \Omega$, let ℓ be a supporting function of u at x_0 . If u is not strictly convex at x_0 , by Theorem 2.3 all extremal points of the contact set $\mathcal{C} = \{u = \ell\}$ lie on the boundary $\partial\Omega$. By convexity $u = 0$ in Ω , which is a contradiction to $\nu(\Omega) > 0$.

Case (ii): For any point $x_0 \in \Omega$, by subtracting a hyperplane we assume that $u \geq 0$ in Ω and $u(x_0) = 0$. Let $h_0 = \sup\{h \mid S_h^0 \Subset \Omega\}$ and let $h = h_0 + \delta$, where the constant $\delta > 0$ will be determined below. Choose a point $x_b \in \partial\Omega \cap \partial S_{h_0}^0$. By a linear transform of the coordinates we assume that $x_0 = 0$ and $x_b = (0, \dots, 0, 1)$. By the conditions $\varphi \in C^{1+\alpha}$ and $\partial\Omega \in C^{1+\alpha}$, we have $u = \varphi \leq h$ on $\partial\Omega \cap B_r(x_b)$, where $r = \delta^{1/(1+\alpha)}$. By the convexity of u , we have $u \leq h$ in the cone $\mathcal{C} = \{x \in \Omega : x_n \geq \frac{1}{r}|x'|\}$.

Let v be the solution to $\det D^2 v = \nu$ in S_h^0 and $v = h$ on ∂S_h^0 . By the first estimate in (2.7), we have

$$\inf_{S_h^0} (v - h) \leq -C|\mathcal{C}|^{2/n} \leq -C\delta^{\frac{2(n-1)}{n(1+\alpha)}}.$$

By the comparison principle, $u \leq v$ in S_h^0 . Hence

$$-h_0 - \delta = -h = \inf_{S_h^0} (u - h) \leq -C\delta^{\frac{2(n-1)}{n(1+\alpha)}}.$$

When $\alpha > 1 - \frac{2}{n}$, $\frac{2(n-1)}{n(1+\alpha)} < 1$. Hence when $\delta > 0$ is small, we obtain

$$h_0 \geq C\delta^{\frac{2(n-1)}{n(1+\alpha)}}.$$

This inequality implies not only the strict convexity of u but also give an estimate for the modulus of convexity for u . \square

REMARK 2.1. Combining the strict convexity with a normalisation argument one can obtain the following estimate, which is dual to the $C^{1,\alpha}$ estimate in §2.5.

COROLLARY 2.2. [33] *Under the hypotheses of Theorem 2.4, there exists $\beta > 0$, depending on n and b , such that for any $x_0 \in \Omega$,*

$$(2.12) \quad u(x) \geq C|x - x_0|^{1+\beta} + \ell_{x_0}(x),$$

where ℓ_{x_0} is a support function of u at x_0 , and $C > 0$ is a constant depending on n, b, Ω, ν , and $\text{dist}(x, \partial\Omega)$.

PROOF. The proof was given in [33]. We include it here for completeness. By subtracting a linear function, we assume $x_0 = 0$, $u(0) = 0$, $u \geq 0$ in Ω .

By the strict convexity, S_h^0 is compactly contained in Ω if $h > 0$ is small. Make a linear transform $y = Tx$ and $v(y) = u(x)/h$ such that $B_{1/n} \subset T(S_h^0) \subset B_1$. When $h \leq C_0$, $v = 1$ on $T(\partial S_h^0)$. The doubling condition (2.10) and the uniform estimate (Theorem 2.1) implies that $\nu[T(S_h^0)] \leq C_2$. Hence by the Hölder continuity of v (Theorem 2.2), we have $\text{dist}(T(S_{h/2}^0), T(\partial S_h^0)) \geq C_3$, where C_2, C_3 depends only on n, b . Changing back we obtain

$$(2.13) \quad u(\theta x) \geq \frac{1}{2}u(x)$$

for any $x \in \partial S_h^0$, where $\theta = 1 - \frac{1}{2}C_3$. As h is any small constant, it follows that for any x near the origin,

$$u(x) \geq 2^{-k}u(\theta^{-k}x)$$

provided $\theta^{-k}x \in \Omega$. Hence we obtain (2.12) with β given by $\theta^{1+\beta} = 1/2$. \square

2.4. Strict convexity in dimension two

In dimension two, we have the following stronger result.

THEOREM 2.5. *Let u be a convex function in $B_1(0) \subset \mathbb{R}^2$ satisfying $\det D^2 u \geq c_0 > 0$, where c_0 is a positive constant. Then u is strictly convex in Ω .*

PROOF. By subtracting a linear function we may suppose that $u(1,0) = u(-1,0) = 0$, and $\inf_{x \in \partial B_1} [u(x) + u(-x)]$ is attained at $x = (1,0)$. We need only to show that $u(0) \leq -C$.

By convexity we have $0 \geq u(x_1, 0) \geq -2|u(0)|$ for $x_1 \in (-1, 1)$ and for any $x \in B_{3/4}$,

$$\begin{aligned} u(x_1, x_2) &\geq u(x_1, 0) - C|x_2| \geq -2|u(0)| - C|x_2|, \\ u(x_1, x_2) &\leq u(x_1, 0) + C|x_2| \leq C|x_2|. \end{aligned}$$

It follows that for any $x \in (-\frac{1}{4}, \frac{1}{4}) \times (-\frac{1}{4}, \frac{1}{4})$,

$$\partial_{x_1} u(x) \leq 8(u(\frac{1}{2}, x_2) - u(x_1, x_2)) \leq 16(|u(0)| + C|x_2|).$$

Similarly we have $\partial_{x_1} u(x) \geq -16(|u(0)| + C|x_2|)$.

Approximating by smooth functions, we may assume that u is smooth. Hence we have $u_{11}u_{22} \geq c_0$. Hence

$$\int_{-1/4}^{1/4} \frac{1}{u_{22}(x)} dx_1 \leq c_0^{-1} \int_{-1/4}^{1/4} u_{11} dx_1 \leq \frac{32}{c_0} (|u(0)| + C|x_2|).$$

We obtain

$$\int_{-1/4}^{1/4} u_{22}(x) dx_1 \geq \frac{1}{4} \left(\int_{-1/4}^{1/4} \frac{1}{u_{22}(x)} dx_1 \right)^{-1} \geq \frac{c_0}{128} (|u(0)| + C|x_2|)^{-1}.$$

It follows that

$$\begin{aligned} C &\geq \int_{-1/4}^{1/4} \left[\int_0^{1/4} u_{22}(x) dx_2 \right] dx_1 \\ &= \int_0^{1/4} \left[\int_{-1/4}^{1/4} u_{22}(x) dx_1 \right] dx_2 \\ &\geq \frac{c_0}{128} \int_0^{1/4} (|u(0)| + C|x_2|)^{-1} dx_2. \end{aligned}$$

Hence $u(0) \leq -\epsilon_0$ for some $\epsilon_0 > 0$ depending only on c_0 and the gradient of u in B_1 . \square

The above strict convexity was due to Aleksandrov, but the above proof was first given in [33].

2.5 $C^{1,\alpha}$ regularity

First we note that by the Legendre transform, if u is strictly convex at 0, then u^* is C^1 smooth at any point in $N_u(0)$. Moreover, if u satisfies (2.12), then u^* is $C^{1,\alpha}$ with $\alpha = 1/\beta$ (see Lemma 3.1 in [23]).

THEOREM 2.6. *Let u , Ω and ν be the same as in Theorem 2.4. Then there exists $\alpha \in (0, 1]$, depending on n, b such that for any $x_0 \in \Omega$,*

$$(2.14) \quad u(x) \leq C|x - x_0|^{1+\alpha} + \ell_{x_0}(x),$$

where ℓ_{x_0} is a support function of u at x_0 , C is a constant depending on n, b, Ω, ν , and $\text{dist}(x_0, \partial\Omega)$.

PROOF. By subtracting a linear function, we assume $x_0 = 0$, $u(0) = 0$, $u \geq 0$ in Ω . By the strict convexity of u , the set $S_h^0 = \{u < h\} \Subset \Omega$ if $h > 0$ is small. Suppose there exists $\sigma > 0$ depending only on n and b such that for any small $h > 0$ and any $x \in \partial S_h^0$,

$$(2.15) \quad u\left(\frac{1}{2}x\right) < \frac{1-\sigma}{2}u(x).$$

Define α by $1 - \sigma = 2^{-\alpha}$. Then for any $x \in \partial S_h^0$ and any $t \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$,

$$u(tx) \leq 2^{-k}(1 - \sigma)^k u(x) = [2^{-k}]^{1+\alpha} u(x) \leq 2t^{1+\alpha} u(x).$$

Hence $u \in C^{1,\alpha}$.

Inequality (2.15) follows from (2.13) with $\sigma = \frac{1-\theta}{5}$. Indeed, consider the convex function $g(t) = u(tx)$, $t \in [-1, 1]$. Replacing g by $g/g(1)$, we may assume that $g(1) = 1$. Let $\psi(t) = g(t + \frac{1}{2}) - g'(\frac{1}{2})t - g(\frac{1}{2})$. Then by the convexity of g , $\psi(0) = 0$, $\psi \geq 0$. If $g(\frac{1}{2}) > \frac{1-\epsilon}{2}$, by convexity we have $1 + \epsilon \geq g'(\frac{1}{2}) \geq 1 - \epsilon$ and $\psi(-\frac{1}{2}) \leq \epsilon$. Applying (2.13) to ψ , we have $\psi(-\frac{1}{2}\theta^{-1}) \leq 2\psi(-\frac{1}{2}) \leq 2\epsilon$. Hence $g(-\frac{1}{2}\theta^{-1} + \frac{1}{2}) < 0$ when $\epsilon < \frac{1-\theta}{5}$, we reach a contradiction as $u \geq 0$. \square

The $C^{1,\alpha}$ regularity is not true if the measure ν does not satisfies the doubling condition. See examples in [35]

3. $C^{2,\alpha}$ estimate

3.1. Statement of the result. In order to state the main results, let us first introduce some notations and terminologies. Let u be a convex function in Ω . For any point $y \in \Omega$ and constant $h > 0$, denote

$$(3.1) \quad S_{h,u}^0(y) = \{x \in \Omega : u(x) < \ell_y(x) + h\},$$

where ℓ_y is a support function of u at y . The *modulus of convexity* of u is a nonnegative function given by

$$\rho_u(r) = \inf_{y \in \Omega} \rho_{u,y}(r), \quad r > 0,$$

where

$$\rho_{u,y}(r) = \sup\{h \geq 0 : S_{h,u}^0(y) \subset B_r(y)\}$$

if there exists $h \geq 0$ such that $S_{h,u}^0(y) \subset B_r(y)$; or $\rho_{u,y}(r) = 0$ otherwise. A convex function u is *strictly convex* if $\rho_u(r) > 0$ for all $r > 0$.

Denote

$$\omega_f(r) = \sup\{|f(x) - f(y)| : |x - y| < r\}.$$

For any $r_1, r_2 > 0$, apparently

$$\omega_f(r_1) + \omega_f(r_2) \geq \omega_f(r_1 + r_2).$$

In particular, we have

$$\omega_f(Cr) \leq C\omega_f(r).$$

We say f is Dini continuous if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < \infty.$$

THEOREM 3.1. *Let $u \in C^2$ be a strictly convex solution of (1.1). Assume that f satisfies*

$$C_1 \leq f \leq C_2$$

for some constants $C_1, C_2 > 0$. Then for all $x, y \in \Omega' \Subset \Omega$, we have the estimate

$$(3.2) \quad |D^2u(x) - D^2u(y)| \leq C \left[d + \int_0^d \frac{\omega_f(r)}{r} + d \int_d^1 \frac{\omega_f(r)}{r^2} \right],$$

where $d = |x - y|$, $C > 0$ depends only on n, ρ_u, Ω' and C_1, C_2 . It follows that

(i) If f is Dini continuous, then (3.2) gives an estimate for the modulus of continuity of D^2u .

(ii) If $f \in C^\alpha(\Omega)$ and $\alpha \in (0, 1)$, then

$$(3.3) \quad \|u\|_{C^{2,\alpha}(\Omega')} \leq C \left[1 + \frac{\|f\|_{C^\alpha(\Omega)}}{\alpha(1-\alpha)} \right].$$

(iii) If $f \in C^{0,1}(\Omega)$, then

$$(3.4) \quad |D^2u(x) - D^2u(y)| \leq Cd [1 + \|f\|_{C^{0,1}} |\log d|].$$

Note that the constant C depends in the modulus of convexity of u , which implies that C also depends on $\text{osc}_\Omega u$. The C^2 estimate in (i) and the $C^{2,\alpha}$ estimate in (ii) were proved in [19] and [4], respectively. Here, the estimate of the form (3.2) is from the paper [19]. A similar type estimate for the Laplacian and heat equations was previously obtained in [37]. The proof is based on a perturbation argument, which was inspired by the original idea of Caffarelli [4].

3.2. Proof of Theorem 3.1. Let us collect some relevant facts and results [19] which will be needed in the subsequent proof.

Let Ω be a bounded convex domain and let T be a unimodular linear transformation such that $T(\Omega)$ is normalized. Choose an appropriate coordinate system such that the minimum ellipsoid of Ω is given by $E = \{\sum_{i=1}^n \frac{x_i^2}{a_i^2} < 1\}$ with $a_1 \geq \dots \geq a_n$. Then

$$T(x) = (\lambda_1 x_1, \dots, \lambda_n x_n)$$

with

$$\lambda_i = \frac{1}{a_i} (a_1 \cdots a_n)^{1/n}, \quad i = 1, 2, \dots, n.$$

Note that λ_1 and λ_n are the least and largest eigenvalues of T . For convenience we say Ω has a *good shape* if

$$(3.5) \quad \lambda_n \leq c^* \lambda_1$$

for some constant c^* under control. If Ω has a good shape, then there exist two concentric balls B_r and B_R with $R \leq nc^*r$ such that $B_r \subset \Omega \subset B_R$.

LEMMA 3.1. *Let u be a convex solution of $\det D^2u = 1$ in Ω and $u = 0$ on $\partial\Omega$. Suppose that u attains its minimum at the origin and $D^2u(0)$ is the unit matrix (or uniformly bounded), then the domain Ω is of good shape.*

In the following, we call the set $S_{h,u}^0(y)$ in (3.1) the sub-level set of u at y with height h and denote $S_{h,u}(y) = \partial S_{h,u}^0(y)$ its boundary. When no confusion arises we will sometimes drop the subscript u , and when y is the minimum point of u , we will simply write the sub-level set as S_h^0 .

LEMMA 3.2. *Let u_i , $i = 1, 2$, be two convex solutions of $\det D^2u = 1$ in S_{1,u_1}^0 . Suppose that $\|u_i\|_{C^4} \leq C_0$, $|u_1 - u_2| \leq \delta$ for some small constant $\delta > 0$, and $B_{1/n} \subset S_{1,u_1}^0 \subset B_1$ is normalized. Then we have, for $1 \leq k \leq 3$,*

$$(3.6) \quad |D^k(u_1 - u_2)| \leq C\delta \quad \text{in } S_{3/4,u_1}^0,$$

where C depends only on n and C_0 .

We are now ready to present the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. By subtracting a linear function we suppose

$$u(0) = 0, \quad Du(0) = 0,$$

so that the origin is the minimum point of u . Choose $h > 0$ small such that the sub-level set S_h^0 is compactly contained in Ω . Let T_h be a unimodular linear transform such that $T_h(S_h^0)$ is normalized. Hence by making the change $x \rightarrow T_h x / \sqrt{h}$ and $u \rightarrow u/h$, we may suppose $h = 1$, S_1^0 is normalized, and

$$(3.7) \quad \int_0^1 \frac{\omega(r)}{r} \leq \epsilon,$$

where $\omega(r) = \omega_f(r)$, ϵ can be as small as we want, provided h is sufficiently small.

Let u_k , $k = 0, 1, \dots$, be the solution of

$$\begin{aligned} \det D^2 u_k &= f(0) \quad \text{in } S_{4^{-k},u}^0, \\ u_k &= u (= 4^{-k}) \quad \text{on } \partial S_{4^{-k},u}^0. \end{aligned}$$

Denote

$$\begin{aligned} \nu(t) &= \sup_{z \in \Omega} \{ |f(x) - f(y)| : x, y \in S_{t^2,u}^0(z) \}, \\ \nu_k &= \nu(2^{-k}), \end{aligned}$$

which is invariant under unimodular linear transformation of x . If $S_{t^2, u}^0$ has good shape, then we have $\nu(t) \leq \omega(Ct) \leq C\omega(t)$.

Since $S_{1, u}^0$ has a good shape, by the classical solvability of Dirichlet problems [15], $\|u_0\|_{C^4(S_{3/4, u}^0)} \leq C$. Note that

$$\det D^2(1 - C\nu_0)u \leq \det D^2 u_0 \leq \det D^2(1 + C\nu_0)u \quad \text{in } S_{1, u}^0,$$

for some constant C , and $u = u_0 = 1$ on the boundary. By the comparison principle, we have

$$(1 + C\nu_0)(u - 1) \leq u_0 - 1 \leq (1 - C\nu_0)(u - 1).$$

It follows that $|u - u_0| \leq C\nu_0$. Similarly we have $|u - u_1| \leq C\nu_1$. Hence we obtain $|u_1 - u_0| \leq C\nu_0$. Since $S_{1, u}^0$ has a good shape, so does $S_{4^{-1}, u}^0$. It follows $\|u_1\|_{C^4(S_{3/16, u_1}^0)} \leq C$. By Lemma 3.2 we obtain

$$|D^k u_0(x) - D^k u_1(x)| \leq C\nu_0$$

for $x \in S_{4^{-2}, u_1}^0$, where $1 \leq k \leq 3$. Lemma 3.1 then implies that S_{4^{-2}, u_1}^0 has a good shape.

By induction we assume that $S_{4^{-k-1}, u}^0$ has a good shape with the constant c^* in (3.5) independent of k . Applying the same argument to $\hat{u}_0 := 4^k u_k(\frac{x}{2^k})$ and $\hat{u}_1 := 4^k u_{k+1}(\frac{x}{2^k})$, we obtain, for $x \in S_{4^{-k-2}, u_{k+1}}^0$,

$$(3.8) \quad \begin{aligned} |Du_k(x) - Du_{k+1}(x)| &\leq C2^{-k}\nu_k, \\ |D^2 u_k(x) - D^2 u_{k+1}(x)| &\leq C\nu_k, \\ |D^3 u_k(x) - D^3 u_{k+1}(x)| &\leq C2^k\nu_k, \end{aligned}$$

where 2^k in (3.9) is the scaling constant. Hence

$$(3.9) \quad \begin{aligned} |D^2 u_0(x) - D^2 u_{k+1}(x)| &\leq C \sum_{i=0}^k \nu_i \\ &\leq C \int_{2^{-k}}^1 \frac{\omega(r)}{r} dr \leq C\epsilon \end{aligned}$$

for $x \in S_{4^{-k-2}, u_{k+1}}^0$, where $C > 0$ is independent of k . Therefore, $D^2 u(0)$ is uniformly bounded. This implies that equation (1.1) is uniformly elliptic and Theorem 3.1 follows from [37]. \square

REMARK 3.1. Instead of applying the estimate for uniformly elliptic equation in [37], one can directly prove the estimate (3.2) as in [19]. For any given point z near the origin,

$$(3.10) \quad \begin{aligned} |D^2 u(z) - D^2 u(0)| &\leq I_1 + I_2 + I_3 =: \\ &|D^2 u_k(z) - D^2 u_k(0)| + |D^2 u_k(0) - D^2 u(0)| + |D^2 u(z) - D^2 u_k(z)|. \end{aligned}$$

Choose $k \geq 1$ such that $4^{-k-4} \leq u(z) \leq 4^{-k-3}$. Then by (3.8),

$$(3.11) \quad I_2 \leq C \sum_{j=k}^{\infty} \nu_j \leq C \int_0^{|z|} \frac{\omega(r)}{r}.$$

To estimate I_3 , we consider the sub-level sets $S_{4^{-j},u}^0(z)$ of u at z . Similarly to (3.11) we have

$$(3.12) \quad I_3 = |D^2u(z) - D^2u_k(z)| \leq C \int_0^{|z|} \frac{\omega(r)}{r}.$$

To estimate I_1 , denote $h_j = u_j - u_{j-1}$. By (3.8),

$$(3.13) \quad |D^2h_j(z) - D^2h_j(0)| \leq C2^j\nu_j|z|.$$

Hence

$$(3.14) \quad \begin{aligned} I_1 &\leq |D^2u_{k-1}(z) - D^2u_{k-1}(0)| + |D^2h_k(z) - D^2h_k(0)| \\ &\leq |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^k |D^2h_j(z) - D^2h_j(0)| \\ &\leq C|z|(1 + \sum_{j=1}^k 2^j\nu_j) \\ &\leq C|z|(1 + \int_{|z|}^1 \frac{\omega(r)}{r^2}). \end{aligned}$$

Hence we obtain (3.2). Note that (3.3) and (3.4) follows readily from (3.2).

4. $W^{2,p}$ estimate

4.1. Statement of the result. In this section we prove the $W^{2,p}$ estimate by following Caffarelli's approach [4] with some simplification. The basic observation is that when f is continuous, the sub-level set S_h^0 is a small perturbation of a ball (after normalization), and the solution is a small perturbation of a quadratic function. It implies that in a sufficiently dense set, the second derivative is close to that of the quadratic function.

The main result in this section is the following

THEOREM 4.1. *Let u be a strictly convex solution of*

$$(4.1) \quad \det D^2u = f \quad \text{in } \Omega.$$

Then for any $p \geq 1$, there exists $\epsilon = \epsilon(p) > 0$ such that when

$$(4.2) \quad 1 \leq f \leq 1 + \epsilon,$$

we have $D^2u \in L^p(\Omega')$ and the estimate

$$(4.3) \quad \|u\|_{W^{2,p}(\Omega')} \leq C,$$

where $\Omega' \Subset \Omega$, C depends on $n, p, \epsilon, \Omega, \Omega'$, and the modulus of convexity of u .

COROLLARY 4.1. *Let u be a strictly convex solution of (4.1). If $0 < C_1 \leq f \leq C_2$ and $f \in C^0(\Omega)$, then $D^2u \in L^p(\Omega')$ for any $1 \leq p < \infty$, and we have the estimate (4.3), where C depends on n, p, Ω, Ω' , and the modulus of convexity of u .*

Note that the condition (4.2) cannot be removed. There exists a function f satisfying $C^{-1} \leq f \leq C$ such that $u \notin W^{2,p}$ for large p [35].

4.2. Proof of Theorem 4.1. We recall a Calderón-Zygmund type decomposition from [4, 16] which is needed in our proof.

Covering Lemma: Let $A, B \subseteq \Omega$ be two subsets of Ω . Let $\mathcal{F} = \{S_{h_x, u}^0(x) : x \in B\}$ be a family of sub-level sets such that $A \subset \cup_{x \in B} S_{h_x, u}^0(x)$. Then there exists a sequence $\{S_i, i = 1, 2, \dots\}$ satisfying that $A \subset \cup S_i$, where $S_i = S_{h_{x_i}, u}^0(x_i)$, and there exists a constant K such that any two sub-level sets in $\{S_{h_i/K}^0(x_i)\}$ are disjoint.

Moreover, we assume the following additional assumption, whose proof is postponed to the next subsection.

Assumption H: For any $N > 10$ and $\delta > 0$, there exists $\epsilon > 0$ such that for any convex solution u of (4.1) satisfying $u(0) = 0$, $u \geq 0$ and S_1^0 is normalized, we have $|D| \geq (1 - \delta)|S_1^0|$ where

$$(4.4) \quad \begin{aligned} D &= \{y \in S_1^0 : u(x) \geq \ell_y(x) + \frac{1}{N}|x - y|^2 \quad \forall x \in S_1^0(y)\} \\ &= \{y \in S_1^0 : S_h^0(y) \subset B_{\sqrt{N^k h}}(y) \quad \forall h \in (0, 1)\}, \end{aligned}$$

where ℓ_y is the tangent plane of u at y .

Now, let u be a convex solution of (4.1) defined in $B_M(0)$ for a large $M > 1$, with the properties that $u(0) = 0$, $u \geq 0$, and the set S_1^0 is normalized. Denote

$$\begin{aligned} D_k &= \{x \in S_{1, u}^0 : D^2u(x) > 2N^{-k}I\} \\ &= \{x \in S_{1, u}^0 : \exists h_x > 0 \text{ such that } S_h^0(x) \subset B_{\sqrt{N^k h}}(x) \quad \forall h < h_x\}, \\ A_k &= S_{1, u}^0 - D_k, \quad k = 1, 2, \dots, \end{aligned}$$

where h_x depends on x . In the following lemma we show fast polynomial decay of the measure of A_k , which implies an L^p estimate of D^2u .

LEMMA 4.1. *For any $q < \infty$, there exists $\epsilon = \epsilon(q) > 0$ such that if $|f - 1| < \epsilon$, then*

$$(4.5) \quad |A_k \cap B_{r_k}(0)| \leq N^{-qk},$$

where $r_1 = \frac{1}{n}$, r_{k+1} is given by

$$r_{k+1} = r_k - N^{-\frac{k}{2(\beta-2)}}.$$

where β is the constant in (4.11) below.

PROOF. We divide the proof into three steps.

Step 1: For $y \in A_k$, let

$$(4.6) \quad \hat{h}_y = \inf\{h > 0 : S_h^0(y) \subset B_{\sqrt{N^k h}}(y)\}.$$

By our definition of A_k , we have $\hat{h}_y > 0$ for any $y \in A_k$.

Let

$$\bar{u}(x) = \frac{1}{\hat{h}_y}[u - \ell_y](T_y(x)),$$

where T_y is a linear transform such that \bar{u} satisfies the conditions in Assumption H.

By Assumption H, there exists a set $D = D_{\bar{u}} \subset S_{1, \bar{u}}^0$ satisfying $|D| \geq (1 - \delta)|S_{1, \bar{u}}^0|$ and for any $z \in D$,

$$\bar{u}(x) \geq \bar{\ell}_z(x) + \frac{1}{N}|x - z|^2 \quad \forall x \in S_{1, \bar{u}}^0(z),$$

where $\bar{\ell}_z$ is the tangent plane of \bar{u} at z .

Note that if $x_0 \in D_{k-1}$, then after normalization, the sub-level set $S_{h,u}^0(x_0)$ cannot have a good shape, in the sense that

$$(4.7) \quad S_{h,u}^0(x_0) \not\subset B_{\sqrt{N}h}(x_0)$$

when h is sufficiently small.

Scaling back to u in the set $S_{\hat{h}_y, u}^0(y)$, we see that there is a set $E_y \subset S_{\hat{h}_y, u}^0(y)$ with

$$(4.8) \quad \frac{|S_{\hat{h}_y, u}^0(y) - E_y|}{|S_{\hat{h}_y, u}^0(y)|} \leq \delta$$

and for any $x \in E_y$, there exists $h_x > 0$ such that

$$S_{h_x, u}^0(x) \subset B_{\sqrt{N^{k+1}h_x}}(x) \quad \forall h_x \leq h_x.$$

Hence

$$(4.9) \quad E_y \subset D_{k+1}, \quad \text{i.e. } E_y \cap A_{k+1} = \emptyset.$$

Moreover by (4.7), if $x \in E_y$, then $x \notin D_{k-1}$. Hence we also have

$$(4.10) \quad E_y \subset A_{k-1}.$$

Step 2: Let y and \hat{h}_y be as in (4.6). Subtract a linear function such that $u(y) = 0$, $Du(y) = 0$. By the strict convexity, there exists a constant $\beta > 2$ such that $u(x) \geq C|x|^\beta$ near $x = 0$. Hence

$$\hat{h}_y \geq C[N^k \hat{h}_y]^{\beta/2},$$

that is,

$$\hat{h}_y \leq C^{\frac{-2}{\beta-2}} N^{-\frac{k\beta}{\beta-2}}.$$

We obtain

$$(4.11) \quad N^k \hat{h}_y \leq C^{\frac{-2}{\beta-2}} N^{-\frac{2k}{\beta-2}} \leq N^{-\frac{k}{\beta-2}}$$

when N is chosen large. That is

$$\text{diam}(S_{\hat{h}_y, u}^0(y)) \leq N^{-\frac{k}{2(\beta-2)}}.$$

By our choice of r_k , we see that if $y \in A_{k+1} \cap B_{r_{k+1}}(0)$, then

$$(4.12) \quad S_{\hat{h}_y, u}^0(y) \subset B_{r_k}(0).$$

We point out that (4.12) can also be derived from the $C^{1,\alpha}$ regularity of generalized solutions [24].

Step 3: The set of all sub-level sets $\{S_{\hat{h}_y/K, u}^0(y), y \in A_k\}$ is obviously a covering of $A_k \cap B_{r_k}(0)$, where \hat{h}_y is given by (4.6) and K is the constant in the covering lemma. By the covering lemma, there exists a countable set $\{y_i\} \subset A_k \cap B_{r_k}(0)$, $i = 1, 2, \dots$, such that

(P1), $A_k \cap B_{r_k}(0) \subset \bigcup S_{\hat{h}_{y_i}, u}^0(y_i)$; and

(P2), any two sub-level sets in $\{S_{\hat{h}_{y_i}/K}^0(y_i)\}$ are disjoint.

Denote $S_i = S_{\hat{h}_{y_i}, u}^0(y_i)$ and $S'_i = S_{\hat{h}_{y_i}/K}^0(y_i)$. From (4.7), there exists a set $E_{y_i} \subset S_i$ such that

$$(4.13) \quad \frac{|S_i - E_{y_i}|}{|S_i|} \leq \delta,$$

and E_{y_i} satisfies (4.9), (4.10).

Since $A_{k+1} \subset A_k$, by the covering property (P1) we have

$$|A_{k+1} \cap B_{r_{k+1}}(0)| \leq |A_{k+1} \cap (\cup S_i)| \leq \sum_i |A_{k+1} \cap S_i|.$$

By (4.13) we have

$$(4.14) \quad |A_{k+1} \cap S_i| \leq \delta |S_i|.$$

By the convexity of u we have

$$|S_i| \leq K^n |S'_i|.$$

By (4.8)

$$|S'_i - E_{y_i}| \leq \delta |S_i| \leq K^n \delta |S'_i|.$$

Writing $S'_i = (S'_i - E_{y_i}) \cup (E_{y_i} \cap S'_i)$ we then have

$$|S'_i| \leq \frac{1}{1 - K^n \delta} |E_{y_i} \cap S'_i| \leq 2|\hat{S}_i|$$

when $K^n \delta < \frac{1}{2}$, where $\hat{S}_i = S'_i \cap E_{y_i}$. We therefore obtain

$$|A_{k+1} \cap B_{r_{k+1}}(0)| \leq 2K^n \delta \sum |\hat{S}_i|.$$

By (4.10), $E_{y_i} \subset A_{k-1}$, and by (4.12), $S'_i \subset B_{r_{k-1}}$. Hence $\hat{S}_i \subset (A_{k-1} \cap B_{r_{k-1}})$ and so

$$|A_{k+1} \cap B_{r_{k+1}}(0)| \leq 2K^n \delta \sum |\hat{S}_i \cap (A_{k-1} \cap B_{r_{k-1}})|.$$

By property (P2), \hat{S}_i are mutually disjoint. Hence

$$|A_{k+1} \cap B_{r_{k+1}}(0)| \leq 2K^n \delta |(\cup \hat{S}_i) \cap (A_{k-1} \cap B_{r_{k-1}})|.$$

But $\cup \hat{S}_i \subset A_{k-1} \cap B_{r_{k-1}}$. We finally obtain

$$|A_{k+1} \cap B_{r_{k+1}}(0)| \leq 2K^n \delta |A_{k-1} \cap B_{r_{k-1}}|.$$

Therefore for any given $q < \infty$, we can choose δ small enough such that $2\delta K^n \leq N^{-2q}$. We obtain the desired estimate (4.5). \square

Theorem 4.1 now follows from Lemma 4.1 easily.

PROOF OF THEOREM 4.1. It suffices to prove Theorem 4.1 on a sub-level set which after proper rescaling satisfies the conditions in Assumption H.

Denote $A'_k = A_k - A_{k+1}$. Then in A'_k , we have $|D^2u| \leq CN^{(n-1)k}$. Hence

$$\begin{aligned} \int_{B_{1/2n}(0)} |D^2u|^p dx &\leq C \sum N^{(n-1)kp} |A'_k| \\ &\leq C \sum N^{(n-1)kp} N^{-kq}. \end{aligned}$$

Letting $q > (n-1)p$, we obtain $u \in W^{2,p}(\Omega')$. \square

4.3. Proof of Assumption H. Given $u \in C^0(\bar{\Omega})$, denote

$$(4.15) \quad \Gamma[u] = \sup v,$$

where the sup is taken in the set of all affine functions v satisfying $v \leq u$ in Ω . Then $\Gamma[u]$ is convex, and is called the *convex envelope* of u in Ω . Apparently

$$\Gamma[u] \leq u.$$

LEMMA 4.2. *Let $u \in C^2(\bar{\Omega})$ be a convex solution of (4.1) and $v \in C^2(\bar{\Omega})$ be a convex function. Suppose $f < 1 + \epsilon$. Then for any Borel set $E \Subset \Omega$,*

$$(4.16) \quad \mu_\Gamma(E) \leq \int_{\mathcal{C} \cap E} \left[(1 + \epsilon)^{1/n} - \det^{1/n} D^2v \right]^n dx,$$

where $\Gamma := \Gamma[u - v]$, μ_Γ is the Monge-Ampère measure associated with Γ [4, 16, 33] and $\mathcal{C} = \{x \in \Omega : \Gamma[u - v](x) = (u - v)(x)\}$ is the contact set.

PROOF. From the smoothness of u, v , we have $\Gamma \in C^{1,1}$ and $\mu_\Gamma = \det \partial^2 \Gamma dx$, see for example Section 2 in [33]. For any point $x \in \mathcal{C}$ where Γ is twice differentiable, we have

$$D^2\Gamma(x) \leq D^2(u - v)(x).$$

By the concavity of $\det^{1/n}$

$$\begin{aligned} \det^{1/n}[D^2u - D^2v] &\leq \det^{1/n}D^2u - \det^{1/n}D^2v \\ &\leq (1 + \epsilon)^{1/n} - \det^{1/n}D^2v. \end{aligned}$$

Hence at x ,

$$(4.17) \quad \det D^2\Gamma \leq \left[(1 + \epsilon)^{1/n} - \det^{1/n}D^2v \right]^n.$$

Recall that $\mu_\Gamma = 0$ in $\Omega - \mathcal{C}$ [4, 16, 33]. We obtain (4.16). \square

LEMMA 4.3. *Let u be a C^2 solution of (4.1) such that $S := S_{2,u}^0 \Subset \Omega$. Let w be a solution to $\det D^2w = 1$ in S such that $w = u$ on ∂S . Suppose $1 < f < 1 + \epsilon$. Then*

$$(4.18) \quad |\{\Gamma[u - \frac{1}{2}w] = u - \frac{1}{2}w\} \cap S| \geq (1 - C\epsilon)|S|,$$

where the constant $C > 0$ depends only on n .

PROOF. By subtracting a constant we assume that $u = w = 0$ on ∂S . By the comparison principle we have $(1 + C\epsilon)w \leq u \leq w$. Hence

$$\left(\frac{1}{2} + C\epsilon\right)w \leq u - \frac{1}{2}w \leq \frac{1}{2}w.$$

Since w is convex, we have

$$\left(\frac{1}{2} + C\epsilon\right)w \leq \Gamma \leq \frac{1}{2}w,$$

where $\Gamma = \Gamma[u - \frac{1}{2}w]$. As S is convex and $u = w = 0$, we also have $\Gamma = 0$ on ∂S . Hence

$$\left(\frac{1}{2} + C\epsilon\right)Dw(S) \supset D\Gamma(S) \supset \frac{1}{2}Dw(S).$$

From the second inclusion we have

$$\mu_\Gamma(S) \geq \mu_{\frac{1}{2}w}(S).$$

On the other hand, by Lemma 4.2,

$$\begin{aligned} \mu_\Gamma(S) &\leq \int_{S \cap \mathcal{C}} \left[(1 + \epsilon)^{1/n} - \det^{1/n}D^2\left(\frac{1}{2}w\right) \right]^n dx \\ &= \int_{S \cap \mathcal{C}} \left[(1 + \epsilon)^{1/n} - \frac{1}{2} \right]^n dx \\ &\leq (2^{-n} + C\epsilon)|S \cap \mathcal{C}| \end{aligned}$$

where $\mathcal{C} = \{\Gamma = u - \frac{1}{2}w\}$, and C depends only on n . Since $\det D^2w = 1$, we have

$$\mu_{\frac{1}{2}w}(S) = 2^{-n}|S|.$$

It follows that

$$|S| \leq (1 + C\epsilon)|S \cap \mathcal{C}|.$$

Note that $\mathcal{C} \subset S$. Hence $|\mathcal{C}| \geq (1 - C\epsilon)|S|$. \square

PROOF OF ASSUMPTION H. Let u, w, \mathcal{C} be as in Lemma 4.3. If $S_{1,u}^0$ is normalized and u is a solution of (4.1) in Ω which encloses $B_M(0)$ for some large M , then $S_{2,u}^0$ is compactly contained in $B_M(0)$ and $\|w\|_{C^2(S_{1,u}^0)} \leq C$ for some $C > 0$ depending only on n .

For any point $y \in \mathcal{C}$, let ℓ_y be the support function of Γ at y . Then

$$\begin{cases} (u - \frac{1}{2}w)(x) \geq \ell(x) & \text{in } S_{2,u}^0, \\ (u - \frac{1}{2}w)(y) = \ell(y). \end{cases}$$

That is

$$(4.19) \quad \begin{cases} u(x) \geq \ell(x) + \frac{1}{2}w(x) & \text{in } S_{2,u}^0, \\ u(y) = \ell(y) + \frac{1}{2}w(y). \end{cases}$$

Let $D = \mathcal{C} \cap S_{1,u}^0$. Since w is smooth and uniformly convex, we obtain (4.4) for some $N > 0$ depending only on $\|w\|_{C^2(S_{1,u}^0)}$. \square

Note that in the proof of Theorem 4.1 (§3.2), we used only the affine invariance of the Monge-Ampère equation and the strict convexity of solutions. The condition $1 \leq f \leq 1 + \epsilon$ is used in the proof of Assumption H. This condition can be weakened to the following VMO type condition, that is,

$$\frac{1}{|\omega|} \int_{\omega} |f - 1| dx \leq \epsilon$$

for any convex subset $\omega \subset \Omega$ with nonempty interior (see [18]).

5. $W^{2,1+\epsilon}$ estimate

5.1. Statement of the result. For strictly convex solution u of (1.1), if f solely satisfies

$$(5.1) \quad 0 < C_1 \leq f \leq C_2 < \infty \quad \text{in } \Omega$$

for some constants C_1 and C_2 , Caffarelli proved [5] that $u \in C_{\text{loc}}^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. For the regularity of the second derivatives of u under the assumption (5.1), counterexamples in [35] show that $u \notin W^{2,p}$ for large $p > 1$ depending on the ratio $\frac{C_2}{C_1}$. By taking $\frac{C_2}{C_1}$ large enough, p can be chosen as close to 1 as desired. Recently, De Philippis-Figalli-Savin [12] and Schmidt [30] proved $u \in W_{\text{loc}}^{2,1+\epsilon}$ for some small ϵ depending on n, C_1 and C_2 . The main result can be stated as follows:

THEOREM 5.1. *Let u be a strictly convex solution of (1.1), where f satisfies (5.1). Then we have the estimate*

$$(5.2) \quad \|u\|_{W^{2,1+\epsilon}(\Omega')} \leq C,$$

where $\Omega' \Subset \Omega$, $\epsilon, C > 0$ depends on n, Ω', C_1, C_2 and the modulus of convexity of u .

5.2. Proof of Theorem 5.1. The proof follows from some modifications of the strategy in §3. We divide the proof into four steps.

1. Let u be a strictly convex solution of (1.1) such that $u(0) = 0$, $u \geq 0$, and $S_1^0 \Subset \Omega$ is normalized. Denote by $|D^2u|$ the largest eigenvalue of the Hessian matrix D^2u . The $W^{2,1}$ estimate is straightforward by integration by parts

$$(5.3) \quad \int_{S_1^0} |D^2u(x)| dx \leq \int_{S_1^0} \Delta u(x) dx \leq \int_{\partial S_1^0} \partial_\nu u \leq C_3,$$

where ν is the unit outer normal of ∂S_1^0 , the first inequality holds since u is convex and the last inequality follows from the interior estimate of Du .

Inequality (5.3) implies the following density estimate (comparing to Assumption H in §3): there exists a large constant $N > 0$ depending only on n, C_1 and C_2 such that

$$(5.4) \quad \left| D = \{x \in S_{1/K}^0 : D^2u(x) > 2N^{-1}I\} \right| \geq \frac{1}{N^{n-1}} \int_{S_1^0} |D^2u|,$$

where $|\cdot|$ denotes the Lebesgue measure and K is the constant in the covering lemma.

To prove (5.4), note that at $x \in D$, by equation (1.1) and assumption (5.1)

$$|D^2u(x)| < C_4N^{n-1} \quad \text{for a constant } C_4 = C_3(n, C_1, C_2).$$

Let $A = S_{1/K}^0 - D = \{x \in S_{1/K}^0 : |D^2u(x)| \geq C_4N^{n-1}\}$. By contradiction, if (5.4) is not true, then $|A| > |S_{1/K}^0| - \frac{1}{N^{n-1}} \int_{S_1^0} |D^2u|$. Since

$$C_4N^{n-1}|A| \leq \int_{S_{1/K}^0} |D^2u(x)| dx,$$

from (5.3) we obtain

$$C_4N^{n-1}|S_{1/K}^0| < \int_{S_{1/K}^0} |D^2u(x)| dx + C_4 \int_{S_1^0} |D^2u(x)| dx \leq (1 + C_4)C_3.$$

This gives a contradiction by choosing N large.

Choosing a larger constant $N_0 \geq C_4N^{n-1}$, (5.4) implies that

$$(5.5) \quad \int_{S_1^0} |D^2u(x)| dx \leq N_0 \left| \{x \in S_{1/K}^0 : N_0^{-1}I \leq D^2u(x) \leq N_0I\} \right|.$$

We remark that by subtracting a linear function, one can obtain (5.5) in any normalized sub-level set $S_{1,u}^0(y) \subset \Omega' \Subset \Omega$.

2. Recall the definitions of “good” sets D_k and “bad” sets A_k in §3.2. In this section we write them in a slightly different form. For a large constant $M > 0$ which will be determined later, define

$$(5.6) \quad \begin{aligned} A_k &= \{x \in S_1^0 : |D^2u(x)| \geq M^k\}, \\ D_k &= S_1^0 - A_k = \{x \in S_1^0 : |D^2u(x)| < M^k\}, \end{aligned}$$

where $k = 1, 2, \dots$. Note that when $M = CN^{n-1}$ with N in Assumption H, definition (5.6) is the same as in §3.2.

Denote $A'_k = A_k - A_{k+1}$ for $k = 1, 2, \dots$. Observe that $M^k \leq |D^2u| < M^{k+1}$ in A'_k . From the proof of Theorem 4.1 one can see that the polynomial decay of measures

$$(5.7) \quad M^k |A'_k| < \frac{C}{M^{k(p-1)}}$$

implies $u \in W^{2,p'}$ for all $p' < p$. Similarly, for some $\varepsilon > 0$

$$(5.8) \quad M^k |A'_k| < \frac{C}{M^{k\varepsilon}}$$

implies $u \in W^{2,1+\varepsilon'}$ for all $\varepsilon' < \varepsilon$. In fact,

$$(5.9) \quad \begin{aligned} \int_{S_1^0} |D^2u|^{1+\varepsilon'} &\leq \sum_k M^{(k+1)(1+\varepsilon')} |A'_k| \\ &\leq CM^{1+\varepsilon'} \sum_k M^{k(\varepsilon'-\varepsilon)} \leq C'. \end{aligned}$$

3. We claim that (5.8) can be deduced from the following induction formula

$$(5.10) \quad \int_{A_{k+1}} |D^2u| \leq \tau \int_{A_k} |D^2u| \quad \text{for a constant } \tau \in (0, 1).$$

To see this, by (5.3) $\int_{A_1} |D^2u| \leq \tilde{C}$, thus (5.10) implies that

$$\int_{A_k} |D^2u| \leq \tilde{C}\tau^{-1}\tau^k \leq C\tau^k \quad \text{for } k \geq 1.$$

Let $\varepsilon = \log \tau^{-1} / \log M$. Noting that $M^k |A'_k| < \int_{A_k} |D^2u|$, then we obtain the desired estimate (5.8) from (5.10).

4. Therefore, it suffices to prove (5.10). The proof is a modification of that of Lemma 4.1. For each $y \in A_{k+1}$, let

$$(5.11) \quad \hat{h}_y = \inf\{h > 0 : S_h^0(y) \subset B_{\sqrt{N_0 M^k h}}(y)\},$$

where N_0 is the constant in (5.5). By our definition of A_{k+1} , $\hat{h}_y > 0$ for any $y \in A_{k+1}$. The family $\{S_{\hat{h}_y, u}^0(y)\}$ is obviously a covering of A_{k+1} . By the covering lemma, there exists a sequence $\{y_i \in A_{k+1}\}$ such that $A_{k+1} \subset \bigcup S_i$ and $\{S'_i\}$ are mutually disjoint, where $S_i = S_{\hat{h}_y, u}^0(y_i)$ and $S'_i = S_{\hat{h}_y, u}^0(y_i)$.

For each i , make a normalization

$$\tilde{x} = \frac{1}{\hat{h}_{y_i}^{1/2}} T x, \quad \tilde{u} = \frac{1}{\hat{h}_{y_i}} u,$$

where T is an $n \times n$ matrix such that $S_{1, \tilde{u}}^0 = \hat{h}_{y_i}^{-1/2} T(S_i)$ is normalized. By (5.11), $|T| \leq \sqrt{N_0 M^k}$. Since $D^2u = T^t \circ D^2\tilde{u} \circ T$, we have $|D^2u| \leq |T|^2 |D^2\tilde{u}|$ and

$$(5.12) \quad M^k \leq |D^2u(x)| \leq N_0^2 M^k, \quad \text{if } N_0^{-1} I \leq D^2\tilde{u}(\tilde{x}) \leq N_0 I.$$

Applying (5.5) to \tilde{u} in $S_{1,\tilde{u}}^0$,

$$\int_{S_{1,\tilde{u}}^0} |D^2\tilde{u}(\tilde{x})| d\tilde{x} \leq N_0 \left| \tilde{x} \in S_{1/K,\tilde{u}}^0 : N_0^{-1}I \leq D^2\tilde{u}(\tilde{x}) \leq N_0I \right|.$$

By rescaling back,

$$\begin{aligned} & \frac{1}{N_0 M^k} \frac{|\det T|}{\hat{h}_i^{n/2}} \int_{S_i} |D^2 u(x)| dx \\ & \leq N_0 \frac{|\det T|}{\hat{h}_i^{n/2}} \left| \{x \in S'_i : M^k \leq |D^2 u(x)| \leq N_0^2 M^k\} \right|, \end{aligned}$$

thus we obtain

$$(5.13) \quad \int_{S_i} |D^2 u(x)| dx \leq N_0^2 M^k \left| \{x \in S'_i : M^k \leq |D^2 u(x)| \leq N_0^2 M^k\} \right|,$$

Since $A_{k+1} \subset \bigcup S_i$ and $\{S'_i\}$ are mutually disjoint, setting $M = N_0^2$ and summing over i we have

$$(5.14) \quad \begin{aligned} \int_{A_{k+1}} |D^2 u(x)| dx & \leq N_0^2 M^k \left| \{x \in S_1^0 : M^k \leq |D^2 u(x)| \leq M^{k+1}\} \right| \\ & \leq N_0^2 \int_{A_k \setminus A_{k+1}} |D^2 u(x)| dx. \end{aligned}$$

Adding $N_0^2 \int_{A_{k+1}} |D^2 u|$ to both sides, we finally obtain

$$(5.15) \quad \int_{A_{k+1}} |D^2 u| \leq \frac{N_0^2}{1 + N_0^2} \int_{A_k} |D^2 u|.$$

Therefore, (5.10) is proved for the constant $\tau = \frac{N_0^2}{1 + N_0^2} < 1$. Theorem 5.1 is then proved by a standard covering argument.

6. Monge-Ampère equations of general form

In this section we consider the Monge-Ampère equation

$$(6.1) \quad \det [D^2 u - A(x, u, Du)] = f(x, u, Du)$$

arising in optimal transportation [27] and reflector antenna design [36, 20]. It is known that the optimal mapping $T = T_u(x)$ is determined by a potential function u by

$$Du(x) = D_x c(x, T),$$

where $c(x, y)$ is the cost function. Differentiating the above formula gives

$$D^2 u(x) = D_x^2 c(x, T) + D_{xy}^2 c \cdot DT.$$

Hence

$$\det [D^2 u - D_x^2 c] = |\det D_{xy}^2 c| |\det DT| = |\det D_{xy}^2 c| \frac{\rho}{\rho^*(T)},$$

where ρ, ρ^* are two probability densities over the initial domain Ω and the target domain Ω^* , respectively. Hence the potential function u satisfies equation (6.1) with the matrix A and the right hand side f given by

$$(6.2) \quad A(x, Du) = D_x^2 c(x, T_u(x)),$$

$$(6.3) \quad f = |\det D_{xy}^2 c| \frac{\rho(x)}{\rho^*(T)}.$$

We say a solution u of (6.1) is elliptic if the matrix $\{D^2 u - A\}$ is positive definite. The potential function is an elliptic solution due to its c -convexity, namely at each point $x_0 \in \Omega$ there exists a c -support of u of the form

$$\varphi(x) = c(x, y_0) + a_0,$$

where a_0 is a constant and $y_0 \in \mathbb{R}^n$ such that

$$u(x_0) = \varphi(x_0), \quad u(x) \geq \varphi(x), \quad \forall x \in \Omega.$$

A notion of c -convexity of domains was introduced in [27]: a set $U \subset \mathbb{R}^n$ is c -convex with respect to another set $V \subset \mathbb{R}^n$ if the image $D_y c(U, y)$ is convex for any $y \in V$. Similarly, V is called c^* -convex with respect to U if $D_x c(x, V)$ is convex for any $x \in U$.

In the special case when $c(x, y) = x \cdot y$, the matrix $A \equiv 0$, equation (6.1) reduces to the standard Monge-Ampère equation (1.1), the notion of c -convexity coincides with that of convexity, and a c -support is just a support hyperplane.

When $\rho, \rho^* > 0$, $\rho \in C^{1,1}(\Omega)$, $\rho^* \in C^{1,1}(\Omega^*)$, and Ω^* is c^* -convex with respect to Ω , the C^3 regularity of potentials has been established in [27] under the following assumptions:

- (A1) For any $x, p \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $D_x c(x, y) = p$; and for any $y, q \in \mathbb{R}^n$, there is a unique $x \in \mathbb{R}^n$ such that $D_y c(x, y) = q$.
- (A2) For any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $\det\{D_{xy}^2 c(x, y)\} \neq 0$.
- (A3) For any $x, p \in \mathbb{R}^n$, and any $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$(6.4) \quad A_{ij,kl}(x, p) \xi_i \xi_j \eta_k \eta_l > 0,$$

where $A_{ij,kl} = D_{p_k p_l}^2 A_{ij}$ and A is given by (6.2).

If the $C^{1,1}$ smoothness of ρ, ρ^* is removed, the C^1 regularity and strict c -convexity of u has also been obtained in [34]; and the $C^{1,\alpha}$ regularity obtained in [22, 25]. Hence one can define the sub-level set $S_{h,u}^0$ of u , namely

$$S_{h,u}^0(x_0) = \{x \in \Omega : u(x) < \varphi(x) + h\},$$

where $h > 0$ is a constant, φ is a c -support of u at x_0 . Under assumptions (A1)–(A3), it was proved [22] that $S_h^0 = S_{h,u}^0(x_0)$ is c -convex with respect to y_0 , where $y_0 = T_u(x_0)$, see also [14]. Therefore, by the coordinate transform $x \mapsto D_y c(x, y_0)$ we can make S_h^0 convex. By the techniques in Sections 2 and 3 as well as some new estimates, the following results have been obtained.

THEOREM 6.1 ([23]). *Let $u \in C^2(\Omega)$ be an elliptic solution of (6.1) with A and f given by (6.2) and (6.3), respectively. Assume that (A1)–(A3), f is Dini continuous and satisfies $C_1 \leq f \leq C_2$ for some positive constants C_1, C_2 . Then for all $x, y \in \Omega' \Subset \Omega$, we have the estimate (3.2). Furthermore, if $f \in C^\alpha$ or $C^{0,1}$, we have (3.3) or (3.4), respectively.*

THEOREM 6.2 ([24]). *In the hypotheses of Theorem 6.1, if f is only continuous and $C_1 \leq f \leq C_2$ for some positive constants C_1, C_2 , then $D^2u \in L^p(\Omega')$ for any $1 \leq p < \infty$, $\Omega' \Subset \Omega$, and we have the estimate (4.3).*

The proof of Theorem 6.1 is similar to that of Theorem 3.1. However, one needs to deal with c -convex functions instead of convex functions, and to establish an interior C^2 estimate of Pogorelov type for solutions of (6.1). We refer the readers to [23] for more details.

Theorem 6.2 can be proved using the steps in Section 4, but is more complicated. A crucial part is to understand the local geometry of the potential function and the cost function, in a blow-up process. Namely we show that they converges to quadratic functions in the blowing-up, and that means locally equation (6.1) is very close to the standard Monge-Ampère equation. In other words, we show that equation (6.1) behaves very like the standard Monge-Ampère equation locally. We stress that these properties are established under assumption (A3), or a weak form of it. In general they may not be true. See [24] for a detailed proof.

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INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS, SCHOOL OF MATHEMATICS
AND APPLIED STATISTICS, UNIVERSITY OF WOLLONGONG, WOLLONGONG, NSW 2522,
AUSTRALIA.

E-mail address: `jiakun1@uow.edu.au`

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, THE AUSTRALIAN NATIONAL
UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA.

E-mail address: `Xu-Jia.Wang@anu.edu.au`