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The regularity of solutions in degenerate geometric problems

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ABSTRACT. We discuss the *optimal regularity* of solutions to *degenerate elliptic and parabolic* fully nonlinear partial differential equations, in particular the evolution of a hypersurface M_t^n in $\mathbb{R}^{n+1}$ by powers of its *Gaussian curvature* and other nonlinear functions of its principal curvatures. We will also discuss the regularity question related to the *Weyl problem* with nonnegative curvature, which involves a fully-nonlinear degenerate elliptic equation of Monge-Ampère type.

1. Introduction

We will survey works concerning the *optimal regularity* of solutions to *degenerate elliptic and parabolic* fully nonlinear partial differential equations that are motivated from geometric problems. These include the evolution of a hypersurface in $\mathbb{R}^{n+1}$ by powers of its Gaussian curvature and other nonlinear functions of its principal curvatures. We will also discuss a regularity question related to the Weyl problem with nonnegative curvature, which involves a fully-nonlinear degenerate elliptic equation of Monge-Ampère type. Our point of view is to treat such problems by means of a *geometric approach*: we use the intuition provided by simple geometric models to obtain sharp regularity results in these classical nonlinear partial differential equations and discover new phenomena. Some of the techniques that will be described here were initiated from the author's joint work with Richard Hamilton [**32**] on the C^∞-regularity of the free boundary for the porous medium equation and they were later extended to a number of parabolic fully-nonlinear degenerate geometric flows. *The author would like to express her gratitude to Richard Hamilton for years of fruitful collaboration and many happy hours of stimulating mathematical discussions.*

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One of the simplest models of *nonlinear diffusion* is the *porous medium equation*

$$u_t = \Delta u^m \tag{1.1}$$

for a nonnegative scalar function $u(x,t)$ defined on $x \in \mathbb{R}^n$. In spite of its simplicity, this equation appears in a number of physical and geometric problems and its rich analytical properties provide an intuition for many more complex nonlinear models of diffusion, including fully nonlinear geometric problems. It is well known that (1.1) with $m > 1$ describes the flow of an isotropic gas through a porous medium [**71**]. Another application refers to heat radiation in plasmas [**90**]. Written in divergence form $u_t = \mathrm{div}(mu^{m-1}Du)$ we see that the diffusion coefficient $d(u) := mu^{m-1}$ vanishes when $u = 0$, in other words the equation becomes degenerate at points where the solution vanishes. This results to *finite speed of propagation* a striking difference with the heat equation. As a result, solutions are not necessarily smooth and in fact C^α is the optimal regularity of solutions [**18**]. However, the optimal Hölder exponent α is still an open question. When $0 \leq m < 1$, equation (1.1) becomes *fast-diffusion* [**59**] and arises in various physical applications, in particular in models of gas-kinetics [**21, 29**], in diffusion in plasmas [**11**], and in thin liquid film dynamics driven by Van der Waals forces [**46, 47**]. Also it arises in geometry; the case $m = (n-2)/(n+2)$, in dimensions $n \geq 3$ describes the evolution of a conformally flat metric by the Yamabe flow [**56, 89, 28, 79, 12, 13**], and the case $m = 0$, $n = 2$ describes the Ricci flow on surfaces [**57, 30, 88**]. When $m < 0$, equation (1.1) corresponds to *ultra-fast diffusion* [**82, 31**]. In dimension $n = 1$, it describes a plane curve shrinking along the normal vector with speed depending on the curvature [**45**] and in higher dimensions resembles the equation satisfied by the mean curvature in the Inverse mean curvature flow [**65**]. For more information on works related to equation (1.1), we refer the interested reader to the books [**34, 83, 84**].

Another quasi-linear model of a similar nature is the *p-harmonic heat flow*

$$u_t = \mathrm{div}\big(|Du|^{p-1}Du\big) \tag{1.2}$$

for a nonnegative scalar function $u(x,t)$ defined on $x \in \mathbb{R}^n$ and exponents $p > 0$. The case $p > 1$ corresponds to show diffusion and the case $0 < p < 1$ to fast diffusion. We refer the reader to the book by DiBenedetto [**41**] for an overview on this well studied model. There is a natural generalization of (1.2) to *p-harmonic maps* between Riemannian manifolds, where the *pioneering work* of K. Uhlenbeck [**81**] shed light on understanding the regularity of solutions. We will see in section 2 that the Gauss curvature flow (2.1) with exponent $p > 0$ may be viewed as a fully-nonlinear analogue to equations (1.1) and (1.2).

Going back to the degenerate case when $m > 1$ in (1.1), it is well known that if the initial data $u(\cdot, 0)$ is compactly supported in $\mathbb{R}^n$, then the solution $u(\cdot,t)$ will remain compactly supported for all time t. Hence the boundary of the support of $u(\cdot,t)$, namely the surface $\Gamma_t = \partial\{\, u(\cdot,t) > 0 \,\}$ behaves like a *free-boundary* propagating with finite speed which is given by the gradient of the pressure function $f := m\, u^{m-1}$. The *optimal regularity* of the free-boundary Γ is closely related to the optimal regularity of the pressure function f, and its understanding has been a long standing and widely studied open problem. We refer the reader to the well known works on the subject by S. Angenent, D. Aronson, L. Caffarelli, A. Friedman, H. Koch, J-L. Vazquez and N. Wolanski [**9, 10, 18, 20, 19, 20**] (among others) and their references. These works concern with the optimal regularity of the free-boundary for large times.

In [**32**] the author, jointly with R. Hamilton, established the short time C^∞ regularity of the pressure $f := mu^{m-1}$ up to the free-boundary and the free-boundary Γ under the natural initial (t=0) non-degeneracy condition $|Df_0| \geq c > 0$ that holds at the free-boundary and guarantees that the free-boundary will instantly move. As a result, the diffusion (even degenerate) will result to the smoothing of the free-boundary. Their approach was different that in the previous regularity works mentioned above. It relies on *sharp a priori estimates* on the linearized equation of the pressure f near the free-boundary which are scaled according to a singular metric (distance function) that is appropriate for this degenerate problem (c.f. in the next section). A similar approach was introduced (independently to [**32**]) by H. Koch [**67**] who established the short time $W^{2,p}$ and $C^{1,\alpha}$ regularity of the solution as well as the C^∞ regularity of the solution at large times. Recently, it was further developed by C. Epstein and R. Mazzeo [**43**] in connection to problems in mathematical biology involving degenerate equations on domains with corners.

To shortly describe the main point of view in [**32, 67**], let us recall that the pressure function $f := m\, u^{m-1}$ satisfies the equation

$$f_t = f\,\Delta f + (m-1)^{-1}|Df|^2.$$

Assuming the non-degeneracy condition $|Df| \geq c > 0$ near the free-boundary, it follows that around any free-boundary point $P \in \Gamma_t$ the linearization of this equation (in an appropriate coordinate system that fixes the free-boundary) can be modeled on the linear degenerate operator

$$L[h] := h_t - (x_1\,\Delta h + \theta\,\partial_{x_1} h)\,, \qquad \text{on } x_1 > 0$$

with $\theta > 0$ and *no boundary condition* at $x_1 = 0$. The diffusion is governed by the *cycloidal metric*

$$ds^2 = \frac{\sum_{i=1}^n dx_i^2}{2x_1} \tag{1.3}$$

whose geodesics are *cycloids*. The corresponding *distance function* is given by

$$s(x^1, x^2) = \frac{\sum_{i=1}^n |x_i^1 - x_i^2|}{\sqrt{x_1^1} + \sqrt{x_1^2} + \sum_{i=1}^{n-1} |x_i^1 - x_i^2|}$$

with parabolic distance between two points $P_1 = (x^1, t_1)$ and $P^2 = (x^2, t_2)$ given by

$$\bar{s}(P^1, P^2) = s(x^1, x^2) + \sqrt{|t_1 - t_2|}.$$

In [**32**] the author and R. Hamilton obtained *sharp* a priori $C_s^{2,\alpha}$ estimates for solutions to the equation $L[h] = g$ that are scaled according to the cycloidal metric s. Similarly, sharp $W^{2,p}$ estimates, a Harnack inequality and $C^{1,\alpha}$ estimates were established by H. Koch in [**67**]. These estimates lead to the short-time C^∞-regularity the pressure function f up to the interface Γ and the interface itself [**32, 67**]. They also imply the C^∞-regularity of the free-boundary for large times [**67**], improving the older results by Caffarelli, Vazquez and Wolanski [**19**] which show that the free-boundary is of class $C^{1,\alpha}$. We will see in sections 2 and 3 that a similar approach leads to the optimal regularity of solutions to the Gauss curvature flow and other fully-nonlinear equations. We note that advancing free-boundaries may hit each other creating *singularities*, hence one does not expect that the free-boundary remains regular for all time $t > 0$.

Degeneracies of the type described above often arise in various elliptic and parabolic geometric equations, in particular in the evolution of a *convex hyper-surface* M_t embedded $\mathbb{R}^{n+1}$ by a speed σ which is a function of its *principal curvatures*. Each point $P \in M_t$ moves in the inward normal direction ν by the flow

$$\frac{\partial P}{\partial t} = \sigma\, \nu. \tag{1.4}$$

The most classical flow (1.4) is the Mean curvature flow, where $\sigma = H$, the mean curvature of the surface (c.f. [**61**]). This is a well known geometric model of a quasilinear parabolic PDE that corresponds to the heat equation on the evolving surface M_t. As a result, *weakly convex* surfaces become instantly *strictly convex* and *smooth* [**61, 42**]. It is widely studied, especially due to its exotic singularity models that appear in the non-convex case. We refer the reader to the fundamental works by G. Huisken and C. Sinestrari [**62, 63, 64**] and their references.

A fully-nonlinear case of an equation (1.4) which is of particular interest is the evolution of an n-dimensional compact convex hypersurface M_t embedded $\mathbb{R}^{n+1}$ by a power p of its *Gaussian curvature* $K = \lambda_1 \cdots \lambda_n$, namely $\sigma(\lambda_1, \cdots, \lambda_n) = K^p$, $p > 0$ in (1.4) and ν the inward normal vector at P. Among the different powers of exponents p those of special interest is the classical case $p = 1$ and the *affine flow* $p = 1/(n+2)$ which appears to be the simplest invariant flow in *affine differential geometry*; up to a suitable re-parametrization it is the motion of a hypersurface in the direction of its affine normal vector. This flow also appears of interest in image analysis.

The *Gauss Curvature Flow* serves as a geometric prototype for a parabolic fully-nonlinear equation of Monge-Ampère type. In addition to its fully-nonlinear character, the complexity of this flow is related to the fact that the Gaussian curvature, the speed of the hypersurface, controls only the product of the principal curvatures, leaving much freedom for *degeneracies and singularities* that occur as one or more principal curvatures become arbitrarily small and others arbitrarily large.

The *Monge-Ampère equation* on $\mathbb{R}^n$, in the elliptic setting, has been extensively studied in the pioneering works by A.D. Alexandrov, L. Caffarelli, L.C. Evans, N.V. Krylov, L. Nirenberg, A. V. Pogorelov, J. Spruck, N. Trudinger and S.T. Yau among many others. We refer the reader to a survey by C. E. Gutiérrez [**53**] for references on this important equation.

The *classical Gauss curvature flow* with exponent $p = 1$ was first introduced by W. Firey in [**44**] as a model for the wearing process of stones. It follows from the works of B. Chow [**26**] and K.-S. Chou (K. Tso) [**80**] that uniformly strictly convex hypersurfaces will become instantly C^∞-smooth and will remain smooth up to their extinction time. However, weakly convex surfaces which are not necessarily uniformly strictly convex, may lead to *degeneracies*. This phenomenon was first observed by R. Hamilton [**55**] who showed that if the initial surface is weakly convex with *flat sides*, then each flat side will persist for some time. Other related phenomena were observed by D. Chopp, L.C. Evans, and H. Ishii [**25**]. Since the equation remains degenerate, at least for some time, the *optimal regularity* of solutions poses an interesting analytical problem. It will be further discussed in section 2.

It is apparent from the results in [**26, 4, 7, 39**] that the *homogeneity* of the equation (1.4) plays an important role in the regularity of solutions. In [**3**] B. Andrews studied the evolution of convex hypersurfaces by speeds σ which are homogeneous of degree one functions of the principal curvatures of the surface. In particular, the equations considered in [**3**] include the evolution by the *n-th root of the Gaussian curvature* which was previously studied by B. Chow in [**26**] and also speeds that are quotients of successive *elementary symmetric polynomials of the principal curvatures*. In section 3 we will discuss results concerning the optimal regularity of solutions in such models. In particular, we will discuss a free-boundary problem associated with these flows that appears to be of a different nature than in previously studied cases such as the porous medium equation and the Gauss curvature flow.

One of the most classical problems in differential geometry which is closely related to the theory of the elliptic Monge-Ampère equation is the Weyl problem with nonnegative curvature, posed in 1916 by Weyl himself: *Given a Riemannian metric g on the 2-sphere $\mathbb{S}^2$ whose Gauss curvature is everywhere positive, does there exist a global C^2 isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, ds^2)$, where ds^2 is the standard flat metric on $\mathbb{R}^3$?*

H. Lewy [**69**] solved the problem under the assumption that the metric g is analytic. The solution to the Weyl problem, under the regularity assumption that g has continuous fourth order derivatives, was given in 1953 by L. Nirenberg [**72**].

P. Guan and Y.Y. Li [**49**] considered the question: *If the Gauss curvature of the metric g is nonnegative instead of strictly positive and g is smooth, is it still possible to have a smooth isometric embedding ?*

We will see in section 4 that the answer to this problem is related to the local optimal regularity of solutions to the *degenerate Monge-Ampère* equation

$$\det D^2 u = |x|^2\, g(x), \qquad x \in \mathbb{R}^2 \tag{1.5}$$

with $g > 0$ and we will discuss some recent progress related to this old problem.

2. Gauss curvature flow

We consider in this section the motion of a convex n-dimensional hypersurface M_t embedded in $\mathbb{R}^{n+1}$ under the *Gauss curvature flow* with exponent p, namely the equation

$$\frac{\partial P}{\partial t} = K^p\, \nu \tag{2.1}$$

where each point P moves in the inward direction ν to the surface with velocity equal to the p-power of its Gaussian curvature K. We will address the question of the optimal regularity of a viscosity solution M_t to (2.1) and we also discuss the free-boundary problem associated to surfaces M_t with flat sides.

The classical case $p = 1$ of the Gauss curvature flow (2.1) was first introduced by W. Firey [**44**] in 1974 as a model for the wearing process of a stone tumbling over a uniformly abrasive plane (the beach). Assuming that the pebble occupies an open bounded and convex region in R^{n+1} at time $t = 0$, then the number of collisions with a region U of the stone is proportional to the measure of the normal image $\nu(U) = \{\nu_P : \; P \in U\} \subset S^n$ of U that is equal to $\int_U K\, d\mathcal{H}^n$. The rate at which the stone wears away at a point P is given by $\rho(\nu_P)\, K_P$ for some positive function ρ on S^n. In the case where ρ is a constant, it follows that the surface of the pebble evolves by (2.1).

Assuming the existence, uniqueness and regularity of the solution, Firey showed in [**44**] that compact surfaces which are symmetric about an origin contract to round points. He also conjectured that the result should hold without any symmetry assumption. The existence and uniqueness of a C^∞ solution to the Gauss curvature flow, under the assumption that the initial surface is compact and uniformly convex, was established by K.-S. Chou (K. Tso) [**80**]. In the same paper it was also proved that the Gauss curvature flow contracts the initial convex hypersurface into a point in finite time. However, Firey's conjecture remained open for more than a decade, until B. Andrews [**6**] showed that the normalized flow of a two-dimensional compact surface in R^3 converges to a round sphere, hence proving the conjecture of Firey. Up to date, the conjecture *remains open* in higher dimensions and poses a challenging open question. In a recent work by P. Guan and L. Ni

[**51**], the C^∞ convergence of the n-dimensional normalized Gauss curvature flow to a soliton was shown, shedding new light towards the solution of this question.

Although K.-S. Chou's work shows that compact and uniformly convex surfaces becomes instantly C^∞ smooth, in general convex surfaces that are not necessarily uniformly strictly convex, may not become instantly strictly convex and smooth (c.f. [**55**], [**25**]) and the *optimal regularity* of solutions poses an interesting problem that will be further discussed in this section.

Equations of the form (2.1) for different powers of $p > 0$ were studied by B. Andrews in [**7**] following previous works by B. Chow [**26**] for the homogeneous of degree one case $p = 1/n$ and by B. Andrews [**4**] for the affine case $p = 1/(n+2)$. It was proven in [**7, 26**] that when $p \leq 1/n$, then any compact and convex hyper-surface will become instantly strictly convex and smooth. However, examples are given in [**7**] for flows (2.1) with $p > 1/n$, where the hyper-surfaces do not immediately become smooth or strictly convex.

In dimension $n = 2$, the regularity of solutions to the Gauss curvature flow $p = 1$ is well understood. It follows from the work of B. Andrews in [**6**] that, in this case, all compact surfaces become instantly of class $C^{1,1}$ and remain so up to a time when they become strictly convex and therefore smooth, before they contract to a point. Also, it follows from the works by the author with R. Hamilton [**33**] and K. Lee [**36**], that $C^{1,1}$ is the optimal regularity in this case.

In what follows, the regularity of the Gauss curvature (2.1) flow in any dimension $n \geq 2$ and for different exponents $p > 0$ will be discussed. We will also discuss the optimal regularity of surfaces with flat sides and the regularity of interfaces.

2.1. $C^{1,\alpha}$ **Regularity.** We will discuss in this section an approach to the $C^{1,\alpha}$ *regularity of viscosity solutions* to the *Gauss curvature flow* (2.1), for any power $p > 0$, which was given by the author and O. Savin in [**39**]. In that work, the more general problem of the regularity of viscosity solutions of the *parabolic Monge-Ampère equation*

$$u_t = b(x,t)\,(\det D^2 u)^p, \qquad x \in \Omega \subset \mathbb{R}^n \tag{2.2}$$

with exponent $p > 0$ was discussed. The coefficient $b(x,t)$ is assumed to be only bounded, measurable and to satisfy the *ellipticity condition*

$$\lambda \leq b(x,t) \leq \Lambda \tag{2.3}$$

for some fixed constants $\lambda > 0$ and $\Lambda < \infty$. The function u is assumed to be convex in x and increasing in t.

If we express a surface M_t evolving by (2.1) locally as a graph $x_{n+1} = u(x,t)$, with $x \in \Omega \subset \mathbb{R}^n$, then the function u satisfies the parabolic Monge-Ampère type equation

$$u_t = \frac{(\det D^2 u)^p}{(1+|Du|^2)^{\frac{(n+2)p-1}{2}}}. \tag{2.4}$$

Since any convex solution satisfies locally the bound $|Du| \leq C$, equation (2.4) becomes of the form (2.2).

In addition to the regularity results for the Gauss curvature flow (2.1) which were mentioned above the $C^{1,\alpha}$ and $W^{2,p}$ interior estimates for equations similar to (2.2) were established by Q. Huang and G. Lu for the case of exponent $p = 1/n$ in [**60**] and by C.E. Gutiérrez and Q. Huang for $p = -1$ in [**54**]. Let us remark that the equations (2.2) for negative and positive powers are in some sense dual to each other. Indeed, if u is a solution of (2.2) and $u^*(\xi,t)$ is the Legendre transform of $u(\cdot,t)$ then

$$u_t^* = -\tilde{b}(\xi,t)(\det D^2 u^*)^{-p}, \qquad \lambda \leq \tilde{b}(\xi,t) \leq \Lambda.$$

If w is a solution to the Monge-Ampère equation

$$\det D^2 w = 1, \qquad x \in \Omega \subset \mathbb{R}^n$$

then $u(x,t) = w(x) + t$ solves equation (2.2) with $b \equiv 1$ for any p. The question of *regularity* for the Monge-Ampère equation is closely related to the *strict convexity* of w. Strict convexity does not always hold in the interior as it can be seen from a classical example due to Pogorelov [**76**]. However, Caffarelli [**16**] showed that if the convex set D where w coincides with a tangent plane contains at least a line segment then all extremal points of D must lie on $\partial\Omega$.

In [**39**] *a parabolic version of Caffarelli's result* was shown by the author and O. Savin, for a solution of (2.2). This result says: if at a time t the convex set D where u equals a tangent plane contains at least a line segment then, either the extremal points of D lie on $\partial\Omega$ or $u(\cdot,t)$ coincides with the initial data on D. We refer the reader to Theorem 5.3 in [**39**]. The second behavior occurs for example in those solutions with flat sides. In other words: *a line segment in the graph of u at time t either originates from the boundary data at time t or from the initial data.*

A similar result *for angles instead of line segments* was also established in [**39**]. That result played a crucial role for deriving the C^1 and $C^{1,\alpha}$ estimates that followed. More precisely, it was shown: *if at a time t the solution u admits a tangent angle from below, then either the set where u coincides with the edge of the angle has all extremal points on $\partial\Omega$ or the initial data has the same tangent angle from below.* We refer the reader to Theorem 6.1 in [**39**].

The $C^{1,\alpha}$ regularity of solutions to (2.2) is closely related to understanding whether or not *solutions separate instantly away from the edges of a tangent angle of the initial data*. It turns out that the exponent $p := 1/(n-2)$ is critical in the following sense: (i) When $p > 1/(n-2)$ the set where u coincides with the edge of the angle may persist for some time (Proposition 4.8 in [**39**]), hence C^1 regularity does not hold in this case without further hypotheses. (ii) When $p < 1/(n-2)$, then at any time t after the initial time, solutions are $C^{1,\alpha}$ in the interior of any section of $u(\cdot,t)$ which is included in the considered domain Ω (Theorem 8.1 in [**39**]). (iii) For the critical exponent $p = 1/(n-2)$, solutions are C^1 with a logarithmic modulus of continuity for the gradient (Theorem 8.2 in [**39**]).

In the case of *any power* $p > 0$, $C^{1,\alpha}$ estimates were shown in [**39**] *at all points* (x,t) *where* u *separates from the initial data* (see Theorem 8.4). Also, *assuming that the initial data is* $C^{1,\beta}$ *in some direction* e *it was shown that the solution is* $C^{1,\alpha}$ *in the same direction* e *for all later times* (Theorem 8.3).

In particular, the methods in [**39**] can be applied for solutions with *flat sides*. If the initial data has a flat side $D \subset \mathbb{R}^n$, then solutions are $C^{1,\alpha}$ for all later times in the interior of D. A similar statement holds for solutions that contain edges of tangent angles: they are $C^{1,\alpha}$ along the direction of the edge for all later times.

We summarize some the results mentioned above in the following theorem.

Theorem 2.1 ($C^{1,\alpha}$ Regularity of solutions [**39**]). *Let* u *be a viscosity solution of* (2.2) *in* $\Omega \times [0,T]$ *with* $u(x,0) \geq 0$ *in* Ω, $u(x,0) \geq 1$ *on* $\partial\Omega$. *Then, there exists* $\alpha > 0$ *depending on* n, λ, Λ, p *such that:*

(i) $u(x,t)$ *is* $C^{1,\alpha}$ *in* x *at all points* (x,t) *with* x *an interior point of the set* $\{u(x,0) = 0\}$ *and* $u(x,t) < 1$.

(ii) If $u(x,0) \geq |x_n|$ *then* $u(x,t)$ *is* $C^{1,\alpha}$ *in the* x' *variables at all points* $((x',0),t)$ *with* x' *an interior point of the set* $\{x' \,:\, u((x',0),0) = 0\}$ *and* $u(x,t) < 1$.

For the proofs of the above results and further discussion we refer the reader to [**39**].

2.2. Worn stones with flat sides. In his work titled as "worn stones with flat sides" [**55**], R. Hamilton considered the evolution of a convex two-dimensional surface M_t in R^3 under the Gauss curvature flow (2.1) with $p = 1$, in the case where the initial surface has flat sides and as a consequence the parabolic equation describing the motion of the hypersurface becomes degenerate at points where the curvature becomes zero. He showed, that if the initial surface M_0 has a flat side, then a little later there will be a smaller flat side and it takes some positive time for the surface to become strictly convex. Hence, *the junction* Γ *between each flat side and the strictly convex* *part of the surface*, where the equation becomes degenerate, behaves like a *free-boundary* propagating with finite speed. This phenomenon is a result of the *degeneracy of the equation* and it is of a very similar nature as to the free-boundary occurring in the porous medium equation (1.1) that was discussed in the introduction. We remark that similar phenomena hold in any dimension $n \geq 2$ and for different exponents p in (2.1) (c.f. in [**66**]).

We will next discuss next the *optimal regularity* question in this *free-boundary problem.* To simplify the exposition we will first restrict ourselves to the simplest case where $p = 1$ and $n = 2$ in (2.1) (the case considered in [**55**] and [**33, 36**]) and the surface M_t has only one flat side. At the end of this section we will discuss known results and open problems concerning higher dimensions $n \geq 3$ and other exponents p.

In [**33**] the author, jointly with R. Hamilton, established the short time solvability and the optimal regularity of the Gauss curvature flow with flat sides, by viewing the flow as a *free-boundary* problem. Assume that the initial weakly convex and compact surface M_0 embedded in $\mathbb{R}^3$ has only one flat side, namely

$$M_0 = M_0^1 \cup M_0^2$$

where M_0^1 is the flat side and M_0^2 is the strictly convex part of the surface. Since the equation is invariant under rotation, we may assume that

$$M_0 \subset \mathbb{R}^3_+ := \{(x, y, z) \in \mathbb{R}^3 : \ z \geq 0\}$$

and that the flat side M_0^1 is the closure of an open set and it is contained in the $z = 0$ plane. By the results in [**55**], the flat side will persist for some time, in other words

$$M_t = M_t^1 \cup M_t^2 \subset \mathbb{R}^3_+$$

with M_t^1 contained in the $z = 0$ plane. We denote by Γ_t the free-boundary

$$\Gamma_t := \partial M_t^1 = M_t^1 \cap M_t^2.$$

Let us denote by T_c the *extinction time of the flat side* M_t^1. It is known that T_c strictly less than the time T at which the surface M_t itself shrinks to a point (c.f. in [**6**]). Then, for $0 < t \leq T_c$, the lower part of the surface M_t can be written as the graph of a function

$$z = u(p, t)$$

on the set

$$\Omega_t = \{\, p := (x, y) \in \mathbb{R}^2 : \ |Du|(p, t) < \infty \,\}. \tag{2.5}$$

This is because the results in [**6**] guarantee that the lower part of the surface will not turn vertical before the flat side shrinks to a point. Since M_t solves the Gauss curvature flow, the function u will satisfy the equation

$$u_t = \frac{\det D^2 u}{(1 + |Du|^2)^{3/2}} \tag{2.6}$$

by a standard computation. On the flat side we will always have $z = 0$, while on the strictly convex side we will have that $z > 0$.

Motivated by the theory for the porous medium equation, one introduces the *pressure* function $f := \sqrt{2u}$ which satisfies the equation

$$f_t = \frac{f \det D^2 f + f_\nu^2 f_{\tau\tau}}{(1 + f^2 |Df|^2)^{3/2} } \tag{2.7}$$

where τ denotes the tangential direction to the level sets of $f(\cdot, t)$ and ν denotes the outward (increasing) normal direction. Our basic assumption on the initial surface is that the initial pressure f_0 satisfies the *non-degeneracy condition*

$$\partial_\nu f_0 \geq \lambda \qquad \text{and} \qquad \partial^2_{\tau\tau} f_0(p) \geq \lambda \qquad \forall p \in \Gamma_0 \tag{2.8}$$

for some positive number $\lambda > 0$. Conditions (2.8) guarantee that the interface Γ_0 will start to move at any point at time $t = 0$ making the Gauss curvature flow to behave as a free-boundary problem.

To understand better the degeneracy of equation (2.7) one performs a *local coordinate change*, near an interface point $P_0 = (x_0, y_0, t_0)$ which *fixes the free-boundary*: assume that the time t_0 is sufficiently close to zero, so that the non-degeneracy condition (2.8) holds at t_0, with a possibly smaller constant $\bar\lambda > 0$. Also, we may assume (after rotating the coordinates) that at P_0 the outward normal to the free-boundary Γ_{t_0} has the direction of the e_1 coordinate (the x-axis). Hence, we can solve around the point P_0, the equation $z = f(x, y, t)$ with respect to x, yielding to a map

$$x = \hat f(z, y, t)$$

defined for all (z, y, t) sufficiently close to $Q_0 = (0, y_0, t_0)$. The function $\hat f$ evolves by the *fully nonlinear degenerate equation*

$$\hat f_t = \frac{-z \det D^2 \hat f + \hat f_z \hat f_{yy}}{(z^2 + \hat f_z^2 + z^2 \hat f_y^2)^{3/2}}, \qquad z > 0. \tag{2.9}$$

The *free-boundary* $f = 0$ has now been transformed into the *fixed boundary* $z = 0$. It shown in [**33**] that the linearized operator of equation (2.9) around the initial data is of the form

$$L[h] := h_t - \big(z\, a_{11} h_{zz} + 2\sqrt{z}\, a_{12} h_{zy} + a_{22} h_{yy} + b_1 h_z + b_2 h_y + c\, h\big) \tag{2.10}$$

where, under (2.8) and certain initial regularity conditions, the matrix (a_{ij}) (that depends on the initial data) is strictly positive and $b_1 \geq \theta > 0$, for some $\theta > 0$.

Consider the model equation

$$h_t = z\, h_{zz} + h_{yy} + \theta\, h_z \tag{2.11}$$

with $\theta > 0$, on the half-space $z \geq 0$, and *no extra boundary conditions* on h along $z = 0$. The diffusion is governed by the *Riemannian metric* ds where

$$ds^2 = \frac{dz^2}{z} + dy^2. \tag{2.12}$$

We remark that this is similar to the cycloidal metric (1.3) which appears in the free-boundary problem for the porous medium equation, but simpler in some way, as it becomes singular only in one direction. The *distance function* between two points $P^1 = (z_1, y_1)$ and $P^2 = (z_2, y_2)$ in the metric s is computed to be equivalent to

$$s(P^1, P^2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2| \tag{2.13}$$

with parabolic distance between two points $P^1 = (z_1, y_1, t_1)$ and $P^2 = (z_2, y_2, t_2)$ defined by

$$s(P^1, P^2) = |\sqrt{z_1} - \sqrt{z_2}| + |y_1 - y_2| + \sqrt{|t_1 - t_2|}. \tag{2.14}$$

We will denote by C_s^α, $\alpha \in (0, 1)$, the space of Hölder continuous functions with respect to the metric s, and by $C_s^{2+\alpha}$ the space of all functions h such that

$$h, h_z, h_y, h_t, z\, h_{zz}, \sqrt{z}\, h_{zy}, h_{yy} \in C_s^\alpha.$$

We will also say, in the sequel, that the pressure $f \in C_s^{2+\alpha}$ near the interface Γ, if the transformed function $\hat{f}$ in the new coordinates which fix the free-boundary satisfies $\hat{f} \in C_s^{2+\alpha}$.

The following result concerning the short time existence and optimal regularity of the Gauss curvature flow with flat sides was proven in [**33**].

Theorem 2.2 (Short time existence and regularity [**33**]). *Assume that at $t = 0$, the pressure function $f_0 := \sqrt{2u_0}$ is of class $C^{2+\alpha}$ up to the interface $z = 0$, for some $0 < \alpha < 1$, and satisfies the non-degeneracy condition* (2.8). *Then, there exists a time $\tau > 0$ for which the Gauss curvature flow* (2.7) *admits a solution M_t on $0 \leq t \leq \tau$ for which the pressure function $f = \sqrt{2u}$ is C^∞- smooth up to the interface $z = 0$. In particular the free-boundary Γ_t will be a smooth curve for all t in $0 < t \leq \tau$.*

Discussion on the proof of Theorem 2.2. The proof of Theorem 2.2 is based on *Schauder a-priori estimates* for solutions to *degenerate equations* of the form $L[h] = g$ (with L given by (2.10)) between the spaces $C_s^{2+\alpha}$ and C_s^α, $0 < \alpha < 1$. We refer the reader to Theorem 5.1 in [**33**]. The proof of the Schauder estimate is based on local derivative estimates for the model operator and the method of approximation by polynomials that was inspired by the works M. Safonov [**77, 78**], L. Caffarelli [**15**] and L. Wang [**85, 86**]. It been observed in [**33**] that Schauder estimates between the above defined weighted Hölder spaces, according to the singular metric s, are *optimal* .

Remark 2.1 (Global coordinate change). The local coordinate change that was introduced above is not sufficient to give the existence of a global solution. For this, an appropriate global change of coordinates that transforms the free-boundary Γ_t to the fixed boundary ∂D of the unit disc $D \subset R^2$ is needed. We refer the reader to section 8 in [**33**] for further details.

REMARK 2.2 (Comparison with previous results). Andrews in [**6**] proves the existence of a unique convex viscosity solution M_t of the two-dimensional Gauss curvature flow with initial data any weakly convex compact surface M_0. This solution exists up to the time T when the area of the surface shrinks to zero. He also shows that M_t is of class $C^{1,1}$ for all $0 < t < T$ and becomes strictly convex and smooth at a time T_c with $T_c < T$. The novelty of Theorem 2.2 is that it provides the C^∞ regularity of the pressure function f and the strictly convex part of the surface M_t^2 up to the interface Γ_t, as well as the smoothness of the interface itself. In particular, it shows that in the two-dimensional case $C^{1,1}$ is the optimal regularity for the whole surface M^t for as long as the fat side persists. Also, while Andrews' result in [**6**] is strictly two-dimensional, the result of Theorem 2.2 can be easily generalized to the n-dimensional Gauss curvature flow, for any $n \geq 3$, to provide the C^∞ regularity of the pressure $f := u^{(n-1)/n}$ up to the interface. This in particular implies that the surface M_t is of class $C^{1,1/(n-1)}$ and that this is the optimal.

Since Theorem 2.2 deals only with the short time regularity of the flow the natural question is whether the C^∞ regularity of the pressure f continues to hold up to the focusing time T_c of the flat side, after which the surface becomes strictly convex and smooth. This was answered by the author and K. Lee in [**36**] and is summarized in the next theorem.

THEOREM 2.3 (Long time regularity [**36**]). *Under the assumptions of Theorem 2.2, the pressure function $f = \sqrt{2\,u}$ of the solution is C^∞- smooth up to the interface $z = 0$ on $0 < t < T_c$. In particular the free-boundary $\Gamma_t = M_t^0 \cap M_t^2$ is C^∞- smooth for all t in $0 < t < T_c$.*

Discussion on the proof of Theorem 2.3. The basic step in the proof of Theorem 2.3 is to show that for all $t < T_c$ the transformed pressure function $\hat{f}$ in the coordinates that fix the free-boundary belongs to the space $C_s^{1+\beta}$, for some $0 < \beta < 1$, namely that $\hat{f}_z, \hat{f}_y$ and $\hat{f}_t \in C_s^\beta$. Each derivative $\hat{h}$ of $\hat{f}$ satisfies an equation of the form $L[\hat{h}] = 0$ where L is an operator of the form (2.10), with coefficients involving the first and second derivatives of $\hat{f}$, hence of f. To obtain the Hölder continuity of $\hat{h}$ one needs to establish the analogue of the well known result by Krylov and Safonov [**68**] on the *Hölder regularity of solutions to parabolic operators with bounded measurable coefficients*, for *degenerate operators* of the class (2.10). This is the main result in the work by the author and Lee in [**37**].

However, to apply the results in [**37**], one needs to show that the matrix $[a_{ij}]$ in (2.10) is uniformly elliptic and that all the coefficients of (2.10) are bounded with $b_1 \geq \theta$, for some $\theta > 0$. Since all a_{ij} and b_i involve first and second derivatives of $\hat{f}$, and hence of the solution f of (2.7), one needs to establish *Pogorelov type a-priori bounds* on the *first and second derivatives* of f, holding on $0 \leq t < \tau$, for any $\tau < T_c$. In particular one needs to prove that the *non-degeneracy condition* (2.8) continues to hold near the interface for all $0 \leq t \leq \tau < T_c$.

REMARK 2.3 (Focusing behavior). Theorem 2.3 does not provide any information as to the exact regularity of the surface Σ_{T_c} at the vanishing time T_c of the flat side. Indeed, most of the estimates established in [**36**] hold for time $0 < t < T_c$ and are violated at time T_c. The results in [**6**] show that the surface Σ_{T_c} is of class $C^{1,1}$. In [**35**] the author and K. Lee showed that Σ_{T_c} cannot be more regular than of class $C^{2,\beta}$, for some $\beta < 1$. It has been observed by the author and K. Lee that the flat side *shrinks to a point* at time T_c. However, the *focusing shape* of the free-boundary remains an open equation: *will the free-boundary* Γ_t *become circular as it shrinks to a point ?*

REMARK 2.4 (Higher dimensions and other exponents). The $C^{1,\alpha}$ regularity of solutions to equations (2.1) that was shown in [**39**] and summarized in Theorem 2.1 hold in any dimension $n \geq 2$ and for any power $p > 0$. We will next briefly discuss the generalization of Theorems 2.2 and 2.3 to higher dimensions and other exponents p.

The analogues of the results in Theorems 2.2 and 2.3 for the *p-power Gauss curvature flow* (2.1) and $1/2 < p \leq 1$ have been shown by L. Kim, K. Lee and E. Rhee in [**66**]. We will next discuss*higher dimensions* $n \geq 3$. When $p \leq 1/n$ in (2.1), then it follows from the results in [**7, 26**] that *weakly convex* solutions become *instantly strictly convex* and therefore, by well known regularity results, C^∞ smooth. When $p > 1/n$, then it can be easily seen from *radially symmetric examples* that *flat sides persist* for some time. In fact, the sort time existence and regularity result in Theorem 2.2 can be generalized (using very similar techniques as in [**33**]) to give that that corresponding pressure function $f := u^{\frac{np-1}{p+np-1}}$ will become instantly C^∞ smooth, under the assumption that the initial surface satisfies the appropriate non-degeneracy condition, namely that the initial pressure f_0 satisfies the analogue of (2.8). However, the *long time regularity* result in Theorem 2.3 is still an *open question.* The main difficulty here is to show that the higher dimensional analogue of the non-degeneracy condition (2.8) continues to hold near the interface (boundary of the flat side) at all times, as long as the flat side exists. The proof of such an estimate involves the appropriate uniform control of the principal curvatures of the surface near the interface. Controlling the eigenvalues of the hessian of solutions to Monge-Ampère type of equations, both elliptic and parabolic, often poses a challenging task.

2.3. The fate of rolling stones. We will now briefly discuss the progress that has been made towards establishing Firey's conjecture [**44**] in any dimension n: *a strictly convex compact solution* M_t *to the Gauss curvature flow contracts the initial surface to a spherical point.*

It is known by the work of K.-S. Chou [**80**] that the Gauss curvature flow contracts the initial to a point. In 1999 B. Andrews [**6**] proved Firey's conjecture in dimension $n = 2$, by showing that the normalized Gauss curvature flow converges to the round sphere. His result is based on a pinching curvature estimate, which unfortunately uses that the surface is two-dimensional in a crucial way. Since then, many attempts have been made to generalize Andrew's result in higher dimensions but have failed.

Regarding the flow (2.1) with different exponents $p > 0$, it is known (c.f. [**80, 26, 7**]) that convex and compact surfaces evolving by (2.1) contract to a point. In addition, when $p \in (1/(n+2), 1/n]$ for $n \geq 2$ (and additionally for $p \in [1/2, 1]$ when $n = 2$) the normalized flow (or equivalently the rescaled flow) converges to a round sphere (c.f. [**26, 7, 8**]). This result is no longer true in the case of the *affine flow* $p = 1/(n+2)$ where the rescaled flow converges to an ellipsoid [**7**]. The answer to *Firey's conjecture* remains open in the range of exponents $p > 1/n$, and particular when $p = 1$ and $n \geq 3$.

It follows from the arguments in [**7, 5, 51**] that the limit of the rescaled flow satisfies the elliptic equation

$$\langle P, \nu \rangle = c\, K^p, \qquad c > 0 \tag{2.15}$$

namely it is a compact and convex soliton. Hence, the problem reduces to that of *the classification of compact and convex solutions* to equation (2.15). This appears to be a rather complex question which has been open for the past decade.

We finish this section by noting an interesting recent work by P. Guan and L. Ni [**51**] where the authors establish *uniform regularity estimates* for the normalized Gauss curvature flow in higher dimensions $n \geq 3$. In particular, the convergence to a soliton in the C^∞ topology is obtained. The estimates in [**51**] are established via the study of a new interesting entropy functional for the flow. As a by-product of this new entropy functional, the non-negativity of B. Chow's entropy [**27**] as well as the nonnegativity of W. Firey's entropy [**44**], is deduced. Related to these results is an earlier work by R. Hamilton [**58**] where he obtained upper bounds on the diameter and the the Gauss curvature for the normalized flow.

3. Q_k flow

We will briefly discuss in this section the *optimal regularity of viscosity solutions* to other *full-nonlinear curvature flows* (1.4) where the speed σ is a *homogeneous of degree one* function of the principal curvatures of the surface.

In [**23**] the author, jointly with C. Caputo and N. Sesum, studied the regularity of weakly convex surfaces M_t embedded in R^{n+1} ($n \geq 2$) by the Q_k-flow for $1 \leq k \leq n$, namely the equation

$$\frac{\partial P}{\partial t} = Q_k\, \nu \tag{3.1}$$

where each point $P \in M_t$ of the surface M_t moves in the direction of its inner normal vector ν by a speed which is equal to the quotient

$$Q_k(\lambda) = \frac{S_k^n(\lambda)}{S_{k-1}^n(\lambda)}$$

of successive elementary symmetric polynomials of the principal curvatures $\lambda = (\lambda_1, \dots, \lambda_n)$ of M_t. We recall that

$$S_k^n(\lambda) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

When $k = 1$, equation (3.1) corresponds to the well studied *Mean curvature flow* and will not be considered here. Our attention will focus to the cases $k \geq 2$ where equation (3.1) may become *degenerate* if the initial data fails to be strictly convex. Notice that in dimension $n = 2$ the case $k = 2$ corresponds to the *Harmonic mean curvature flow.*

The evolution of convex hypersurfaces by equations of type (3.1) was previously studied by B. Andrews [**7**], where he showed that for any strictly convex hypersurface M_t in $\mathbb{R}^{n+1}$ the solution to (3.1) exists up to some finite time T at which it shrinks to a point in an asymptotically spherical manner. In [**40**], S. Dieter considered the evolution of smooth weakly convex hypersurfaces in $\mathbb{R}^{n+1}$ with $S_{k-1}^n(\lambda) > 0$ under the Q_k-flow. She showed that because of the condition $S_{k-1}^n(\lambda) > 0$, the speed Q_k of the hypersurface becomes instantly strictly positive at time $t > 0$, and consequently she established the short time existence of the flow in this case.

The work in [**23**] focusses on the evolution of $C^{1,1}$ weakly convex surfaces under the Q_k-flow, with no other assumption on the initial data. Such surfaces may have flat sides or other sets of degenerate points. Nevertheless, it was shown in [**23**] the long time existence and uniqueness of a $C^{1,1}$ solution of (3.1) in a certain viscosity sense. This is stated next.

THEOREM 3.1 (Existence and uniqueness [**23**]). *Let M_0 be a compact weakly convex hyper-surface in $\mathbb{R}^{n+1}$ which is of class $C^{1,1}$. Then, there exists a unique viscosity solution of the* (3.1) *flow which is of class $C^{1,1}$. The solution exists up to the time $T < \infty$ at which the enclosed volume becomes zero.*

REMARK 3.1 (Comparison with the Gauss curvature flow). It follows from the previous theorem that the Q_k-flow enjoys better regularity than the Gauss curvature flow (2.1) with exponent $p > 1/n$. In the latter case, it was shown in section 2.1 that the optimal regularity of solutions is $C^{1,\alpha}$ with α depending on p and the the dimension n.

In the same work [**23**], the special case were the initial surface M_0 has flat sides was considered. Assume for simplicity that $M_0 \subset \mathbb{R}^{n+1}_+$ has only one flat side, namely $M_0 = M_0^1 \cup M_0^2$ with M_0^1 flat (i.e. it lies on a hyperplane in $\mathbb{R}^{n+1}$) and M_0^2 strictly convex. Since the equation is invariant under rotation, we may assume that M_0^1 lies on the $x_{n+1} = 0$ plane and that that M_0^2 lies above this plane. Then, the lower part of the surface M_0 can be written as the graph of a function

$$x_{n+1} = u_0(x_1, \cdots, x_n)$$

over a compact domain $\Omega \subset \mathbb{R}^n$ containing the initial flat side M_0^1. We define the *pressure function*

$$f := \sqrt{u}$$

Let Γ_0 denote the boundary of the flat side M_0^1. Our main assumption on the initial surface M_0 is that it is of class $C^{1,1}$, the function f is C^2 up to the flat side and it satisfies the following non-degeneracy condition, which we call *non-degeneracy condition* (3.2):

$$|Df| \geq \lambda \quad \text{and} \quad [\partial_{ij}^2 f_0] \geq \lambda, \qquad \text{on } \Gamma_0 \tag{3.2}$$

for some number $\lambda > 0$.
For each $1 \leq i, j \leq n-1$, $[\partial_{ij}^2 f_0]$ denote the hessian of f in the directions given by the vectors τ_i and τ_j. For each $1 \leq i \leq n-1$, τ_i is defined so that the set Span $[\tau_1, \cdots, \tau_{n-1}]$ is parallel to the tangent hyperplane to the level sets of f.

Definition 3.2. We define $\mathfrak{S}$ to be the class of convex compact hypersurfaces M in $\mathbb{R}^{n+1}$ so that $M = M^1 \cup M^2$, where M^1 is a surface contained in the hyperplane $x_{n+1} = 0$ with smooth boundary Γ, and M^2 is a strictly convex surface, smooth up to its boundary Γ which lies above the hyperplane $x_{n+1} = 0$.

Remark 3.2. Any initial surface M_0 in the class $\mathfrak{S}$ is in particular a $C^{1,1}$ surface. Hence, by Theorem 3.1, there exists a unique $C^{1,1}$ solution M_t of (3.1) with initial data M_0.

Assume that $x_{n+1} = u(x_1, \cdots, x_n, t)$ defines the hypersurface M_t near the hyperplane $x_{n+1} = 0$, with $0 \leq t \leq \tau$ for some short time $\tau > 0$. The following result was shown in [**23**].

Theorem 3.3 (C^∞ regularity of the pressure). *Assume that at time $t = 0$, M_0 is a compact weakly convex hypersurface in $\mathbb{R}^{n+1}$ which belongs to the class $\mathfrak{S}$ so that the pressure function $f_0 := \sqrt{u_0}$ is smooth up to the interface Γ_0 and it satisfies the condition* (3.2). *Let M_t be the unique viscosity solution of the Q_k-flow* (3.1) *for $2 \leq k \leq n$ with initial data M_0. Then, there exists a time $\tau > 0$ such that the pressure function $f := \sqrt{u}$ is smooth up to the interface $x_{n+1} = 0$ and satisfies condition* (3.2) *for all $t \in [0, \tau)$. In particular, the interface Γ_t which is the intersection of the flat and the strictly convex sides is a smooth hypersurface in $\mathbb{R}^n$, for all t in $0 < t \leq \tau$ and it moves by the Q_{k-1}*-flow.

Remark 3.3 (Comparison with other flows). Theorem 3.3 can be viewed as the analogue of the free-boundary regularity results for the porous medium equation (discussed in the introduction) and the Gauss curvature flow with flat sides (Theorem 2.2 in section 2.2). However, the striking difference between the Q_k-flow and the other cases, is that the boundary of the flat side now doesn't move freely but it obeys *another geometric evolution equation*. This difference will be briefly explained during the discussion on the proof of Theorem 3.3 that follows.

REMARK 3.4 ($C^{m,\gamma}$ regularity and further discussion). In the case of a two-dimensional surface in $\mathbb{R}^3$ the Q_2-flow (3.1) becomes the well studied *Harmonic mean curvature flow*. In this case, Theorem 3.3 was previously established by the author and C. Caputo in [**22**]. Following the result in [**22**], one may consider a *pressure function* $f = u^p$, for any number $p \in (0,1)$, and prove the short time existence of a solution to the Q_2-flow which is of class $C^{m,\gamma}$ (with m, γ depending on p) so that the pressure function f is still smooth up to the interface and the interface moves by the Q_{k-1} flow. The fact that the solution M_t remains in the class $C^{m,\gamma}$, for $t > 0$ and does not become instantly C^∞ smooth is another property that distinguishes this flow from other, previously studied, degenerate free-boundary problems.

Discussion on the proof of Theorem 3.3. Following the ideas in [**33**] (c.f. in the Discussion of the proof of Theorem 2.2 and the Remark 2.1) one expresses the pressure function f in a global system of coordinates that fixes the interface Γ_t. Let us denote by $\hat{f}$ the transformed pressure in these new coordinates. The function $\hat{f}$ satisfies a fully-nonlinear equation on the parabolic cylinder $D \times (0,\tau)$, for some $\tau > 0$, where $D \subset \mathbb{R}^n$ is the unit disc. This equation becomes degenerate at the lateral boundary $\partial D \times (0,\tau)$. Its linearized operator near a boundary point $P_0 := (p_0, t_0) \in \partial D \times (0,\tau)$, and after a further change of coordinates that maps p_0 to $0 \in \mathbb{R}^n$ and straightens ∂D, mapping ∂D near p_0 to part of the hyperplane $x_1 = 0$, is of the form

$$L[h] = h_t - \big(x_1^2\, a_{11} h_{11} + 2\, x_1\, a_{1i} h_{1i} + a_{ij} h_{ij} + x_1 b_1 h_1 + b_i\, h_i + c\, h\big), \qquad x_1 > 0 \; i, j \neq 1$$

defined on $x_1 > 0$. Here the summation convention of indices $i, j = 2, \cdots, n$ is used. It follows from the non-degeneracy condition 3.2 that the matrix $[a_{ij}]$ is positive definite. Consider the *model operator*

$$L_0[h] = h_t - \big(x_1^2\, h_{11} + \Delta_{n-1} h + b_1\, x_1\, h_1\big)$$

which is defined on $x_1 > 0$ and with *no boundary condition* at $x_1 = 0$. This operator represents diffusion by the *singular* Riemannian metric

$$d\bar{s}^2 = ds^2 + |dt|$$

where

$$ds^2 = \frac{dx_1^2}{x_1^2} + dx_2^2 + \cdots + dx_n^2. \tag{3.3}$$

We notice that the distance (with respect to the singular metric ds) of an interior point ($x_1 > 0$) from the boundary ($x_1 = 0$) is *infinite*. The consequence of this property on the nonlinear problem (3.1) is that the interface Γ_t does not move freely but satisfies another evolution equation that is the *singular limit* of the Q_k-flow as you approach this interface. In fact it is shown in [**23, 22**] that the interface Γ_t evolves by the Q_{k-1}-flow. This distinguishes the Q_k-flow from other, previously studied, degenerate free-boundary problems, in particular the Gauss curvature flow with flat sides discussed in section 2.2.

4. Monge-Ampère equations with homogenous right hand side

In 1916 H. Weyl posed the following question: *Given a Riemannian metric g on the 2-sphere $\mathbb{S}^2$ whose Gauss curvature is everywhere positive, does there exist a global C^2 isometric embedding $X : (\mathbb{S}^2, g) \to (\mathbb{R}^3, ds^2)$, where ds^2 is the standard flat metric on $\mathbb{R}^3$?*

H. Lewy [**69**] solved the problem under the assumption that the metric g is analytic. The solution to the Weyl problem, under the regularity assumption that g has continuous fourth order derivatives, was given in 1953 by L. Nirenberg [**72**]. His result depends on a priori estimates for uniformly elliptic equations in dimension two [**73**]. By means of a completely different approach to the problem, A.D. Alexandroff [**1**] obtained a generalized solution of Weyl's problem as a limit of polyhedra. The regularity of this generalized solution was proved by A.V. Pogorelov [**74, 75**]. The regularity of solutions to the related n-dimensional Minkowski problem was considered by S. Y. Cheng and S. T. Yau in [**24**].

P. Guan and Y.Y. Li [**49**] considered the question: *If the Gauss curvature of the metric g is nonnegative instead of strictly positive and g is smooth, is it still possible to have a smooth isometric embedding ?*

It was shown in [**49**] that for any C^4-Riemannian metric g on $\mathbb{S}^2$ with nonnegative Gaussian curvature, there is always a $C^{1,1}$ global isometric embedding into $(\mathbb{R}^3, ds^2)$.

Examples show that for some analytic metrics with positive Gauss curvature on $\mathbb{S}^2$ except at one point, there exists only a $C^{2,1}$ but not a C^3 global isometric embedding into $(\mathbb{R}^3, ds^2)$. The phenomenon is global, since C.S. Lin [**70**] has shown that for any smooth 2-dimensional Riemannian metric with nonnegative Gauss curvature there exists a smooth *local* isometric embedding into $(\mathbb{R}^3, ds^2)$. This leads to the following question, which was posed by P. Guan and Y.Y. Li [**49**]: *Under what conditions on a smooth metric g on $\mathbb{S}^2$ with nonnegative Gauss curvature, there is a $C^{2,\alpha}$ global isometric embedding into $(\mathbb{R}^3, ds^2)$, for some $\alpha > 0$, or even a $C^{2,1}$?* The problem can be reduced to a partial differential equation of Monge-Ampère type that becomes degenerate at the points where the Gauss curvature vanishes. It is well known that in general one may have solutions to degenerate Monge-Ampère equations which are at most $C^{1,1}$.

One may consider a smooth Riemannian metric g on $\mathbb{S}^2$ with nonnegative Gauss curvature, which has only one non-degenerate zero. In this case, if we represent the $C^{1,1}$ embedding as a graph, answering the above question amounts to studying the regularity at the origin of the degenerate Monge-Ampère equation

$$\det D^2 u = f, \qquad \text{on } B_1 \tag{4.1}$$

in the case where the forcing term f vanishes quadratically at $x = 0$. More precisely, it suffices to assume that $f(x) = |x|^2 g(x)$, where g is a positive Lipschitz function. Hence, the problem reduces to that of studying the regularity of solutions to the model Monge-Ampère equation

$$\det D^2 u = |x|^2, \qquad x \in B_1$$

near the origin where the equation becomes degenerate.

In addition to the results mentioned above, degenerate equations of the form (4.1) on $\mathbb{R}^2$ were also considered by P. Guan in [**48**] in the case where $f \in C^\infty(B_1)$ and

$$A^{-1}\,(x_1^{2l} + B\,x_2^{2m}) \le f(x_1, x_2) \le A\,(x_1^{2l} + B\,x_2^{2m}) \tag{4.2}$$

for some constants $A > 0, B \ge 0$ and positive integers $l \le m$. The C^∞ regularity of the solution u of (4.1) was shown in [**48**], under the additional condition that $u_{x_2x_2} \ge C_0 > 0$. It was conjectured in [**48**] that the same result must be true under the weaker condition that $\Delta u \ge C_0 > 0$. This was shown later by P. Guan and I. Sawyer in [**52**].

In [**38**] the author and O. Savin studied the exact behavior at the origin of solutions to the *degenerate Monge-Ampère equation*

$$\det D^2 u = |x|^\alpha, \qquad x \in B_1 \tag{4.3}$$

on the unit disc $B_1 = \{\,|x| \le 1\}$ of $\mathbb{R}^n$ and in the range of exponents $\alpha > -2$.

In addition to its connection with the Weyl problem discussed above, equation (4.3) has also an interpretation in the language of *optimal transportation* with quadratic cost $c(x,y) = |x-y|^2$. In this setting the problem consists in transporting the density $|x|^\alpha\,dx$ from a domain Ω_x into the uniform density dy in the domain Ω_y in such a way that we minimize the total "transport cost", namely

$$\int_{\Omega_x} |y(x) - x|^2\,|x|^\alpha\,dx.$$

Then, by a theorem of Y. Brenier [**14**], the optimal map $x \mapsto y(x)$ is given by the gradient of a solution of the Monge-Ampère equation (4.3). The behavior of these solutions at the origin gives information on the geometry of the optimal map near the singularity of the measure $|x|^\alpha\,dx$.

We will next discuss the main results in [**38**] concerning the optimal regularity of solutions to (4.3) and their exact behavior near the origin. At the end of the section, the higher dimensional case will be discussed. We assume that u is a solution of equation (4.3). Then, u is C^∞-smooth away from the origin. As we will see in the sequel, the behavior of solutions in the two cases $\alpha > 0$ and $\alpha < 0$ is different.

We begin with describing the regularity of solutions and their behavior at the origin the case of positive exponent $\alpha > 0$.

THEOREM 4.1 (Regularity of solutions at the origin [**38**]). *If $\alpha > 0$, then $u \in C^{2,\delta}$ for a small δ depending on α.*

Theorem 4.1 is a consequence of Theorem 4.2 which shows that there are exactly two types of behaviors near the origin.

THEOREM 4.2 (Behavior of solutions at the origin [**38**]). *If $\alpha > 0$, and*

$$u(0) = 0, \quad Du(0) = 0 \tag{4.4}$$

then, either there exist positive constants $c(\alpha)$, $C(\alpha)$ depending on α such that u has the radial behavior

$$c(\alpha)|x|^{2+\frac{\alpha}{2}} \leq u(x) \leq C(\alpha)|x|^{2+\frac{\alpha}{2}} \tag{4.5}$$

or, in an appropriate system of coordinates, u admits the non-radial behavior

$$u(x) = \frac{a}{(\alpha+2)(\alpha+1)}|x_1|^{2+\alpha} + \frac{1}{2a}x_2^2 + O\left((|x_1|^{2+\alpha} + x_2^2)^{1+\delta}\right) \tag{4.6}$$

for some $a > 0$.

REMARK 4.1 (Previous results on the non-radial behavior). The non-radial behavior (4.6) was first shown by P. Guan [**48**], under the extra condition that $u_{x_2x_2} \geq c_0 > 0$ near the origin, and was later generalized P. Guan and E. Sawyer [**52**] to only assume that $\Delta u \geq c_0 > 0$.

One may wonder whether the radial behavior is stable or unstable. The next result shows that it is indeed *unstable.*

THEOREM 4.3. *Suppose $\alpha > 0$, let u_0 be the radial solution to* (4.3),

$$u_0(x) = c_\alpha |x|^{2+\frac{\alpha}{2}}$$

and consider the Dirichlet problem

$$\det D^2 u = |x|^\alpha, \qquad u = u_0 - \varepsilon\cos(2\theta) \ \text{on } \partial B_1.$$

Then $u - u(0)$ has the nonradial behavior (4.6) *for small ε.*

In the case $-2 < \alpha < 0$, solutions to (4.3) have only the radial behavior. Actually, a stronger result was shown in [**38**], which asserts that a solution u converges to the radial solution u_0 in the following sense.

THEOREM 4.4. *If $-2 < \alpha < 0$ and* (4.4) *holds, then*

$$\lim_{x\to 0} \frac{u(x)}{u_0(x)} = 1.$$

In the analysis described above, a crucial role is played by entire *homogenous solutions* to the equation

$$\det D^2 w(x) = |x|^\alpha \qquad \text{in } \mathbb{R}^2, \qquad \alpha > -2$$

namely, solutions of the form

$$w(x) = r^{2+\alpha/2} g(\theta) := r^\beta g(\theta), \qquad \beta = 2 + \alpha/2.$$

Such solutions are related to the limiting behavior near the origin of rescalings of solutions to (4.3). Indeed, subsequences of blow up solutions satisfying (4.5) converge to homogenous solutions, as stated in the next result.

THEOREM 4.5 (Limiting behavior at the origin [**38**]). *Under the assumptions of Theorem* 4.2*, if u satisfies* (4.5)*, then for any sequence of $r_k \to 0$ the blow up solutions*

$$r_k^{-2-\frac{\alpha}{2}} u(r_k x)$$

have a subsequence that converges uniformly on compact sets to a homogenous solution of (4.3)*.*

The following result on the periodicity of homogeneous solutions plays an important role in the proof of Theorem 4.5.

PROPOSITION 4.6. *Homogenous solutions to* (4.3) *are periodic on the unit circle. More precisely, the following holds:*

i. If $-2 < \alpha < 0$, then the only homogenous solution is the radial one.
ii. If $\alpha > 0$, then there exists a homogenous solution of principal period $2\pi/k$ if

$$\frac{\pi}{k} \in \left(\frac{\pi}{\sqrt{2\beta}}, \frac{\pi}{2}\right)$$

with $\beta = 2 + \alpha/2$.

Discussion on the proofs. The results in [**38**] are based on understanding the geometry of the sections of the solution u. The basic argument is as follows: assume that a section of u, say $\{u < 1\}$, is "much longer" in the x_1 direction compared to the x_2 direction. If v is an affine rescaling of u so that $\{v < 1\}$ is comparable to a ball, then v is an approximate solution of

$$\det D^2 v(x) \approx c\,|x_1|^\alpha.$$

Hence, the geometry of small sections of solutions of this new equation provides information on the behavior of the small sections of u. For example, if the sections of v are "much longer" in the x_1 direction (case $\alpha > 0$) then the corresponding sections of u degenerate more and more in this direction, producing the non-radial behavior (4.6). If the sections of v are longer in the x_2 direction (case $\alpha < 0$) then the sections of u tend to become round and we end up with a radial behavior near the origin.

More precisely, in the case of non-radial behavior $\alpha > 0$, it is shown in [**38**] (section 3) that blow up limits of solutions to (4.3) with $\alpha > 0$ near the origin are of the form

$$\det D^2 u = h(x_1) \tag{4.7}$$

where $h(x_1)$ depends only on the variable x_1. These equations remain invariant under affine transformations. Also, by taking derivatives along the x_2 direction one obtains the Pogorelov type estimate

$$u_{22} \leq C$$

in the interior of the sections of u.

Assume that u satisfies equation (4.7) in $B_1 \subset \mathbb{R}^n$, in any dimension $n \geq 2$ and perform the following partial Legendre transformation:

$$y_1 = x_1, \quad y_i = u_i(x) \quad i \geq 2, \qquad u^*(y) = x' \cdot D_{x'} u - u(x) \tag{4.8}$$

with $x' = (x_2, ..., x_n)$. The function u^* is obtained by taking the Legendre transform of u on each slice $x_1 = const$. A simple computation shows that u^* (which is convex in y' and concave in y_1) satisfies

$$u^*_{11} + h(y_1)\, \det D^2_{y'} u^* = 0. \tag{4.9}$$

It is shown in [**38**] (Lemma 2.4) that in the special case of dimension $n = 2$ and $h(x_1) = |x_1|^\alpha$, if w is a solution of the *degenerate* equation

$$w_{11} + |y_1|^\alpha\, w_{22} = 0 \qquad \text{in } B_1 \subset \mathbb{R}^2$$

with $|w| \leq 1$, then

$$\begin{aligned} w(y) &= a_0 + a_1 \cdot y + a_2\, y_1\, y_2 + \\ &\quad + a_3 \left(\frac{1}{2}\, y_2^2 - \frac{1}{(\alpha+2)(\alpha+1)}\, |y_1|^{2+\alpha} \right) + O((y_2^2 + |y_1|^{2+\alpha})^{1+\delta}) \end{aligned} \tag{4.10}$$

in $B_{1/2}$ with $|a_i|$ and $O(\cdot)$ bounded by a universal constant and $\delta = \delta(\alpha) > 0$. This analysis plays a crucial role in the proof of Theorem 4.2.

In the case $-2 < \alpha < 0$ where solutions to (4.3) admit only the radial behavior near the origin, the proof of Theorem 4.5 relies on understanding the *doubling properties* of the measure $d\mu := |x|^\alpha\, dx$. It is inspired by the work of L. Caffarelli [**17**] on the *geometry of sections* of Alexandrov solutions to the Monge-Ampère equation

$$\det D^2 u = \mu, \qquad \text{on } \Omega \subset \mathbb{R}^n$$

with μ a doubling measure.

In the case of equations (4.3), the measure $d\mu := |x|^\alpha\, dx$ is *doubling with respect to ellipsoids* in the case of exponents $-1 < \alpha < 0$. In the case $-2 < \alpha \leq 1$ the measure $d\mu := |x|^\alpha\, dx$ is *not doubling* with respect to ellipsoids but it is still doubling with respect to certain convex sets that have the origin as the center of mass. In both cases one concludes that solutions to (4.3) admit only the radial behavior at the origin.

We finish the section with some further discussion and open questions.

REMARK 4.2 (Generalization to other right hand sides). From the proofs one can see that the theorems above, with the exception of the instability result, are still valid for the equation

$$\det D^2 u = |x|^\alpha g(x)$$

with $g \in C^\delta(B_1)$, $g > 0$.

REMARK 4.3 (Exact regularity of solutions). *i.* It is shown in the proof of Theorem 4.1 that solutions of (4.3), with $\alpha > 0$, which satisfy the radial behavior (4.5) at the origin are of class $C^{2,\frac{\alpha}{2}}$. *ii.* Theorems 4.1, 4.2 and the results of Guan in [**48**] and Guan and Sawyer in [**52**] imply that solutions of (4.3), with α a positive integer, which satisfy the non-radial behavior (4.6) at the origin are C^∞-smooth.

REMARK 4.4 (Applications to other problems). Equations of the form

$$\det D^2 w = |Dw|^\beta, \qquad \beta = -\alpha \tag{4.11}$$

for which the set $\{Dw = 0\}$ is compactly included in the domain of definition, can be reduced to (4.3) by defining u to be the Legendre transform of w. Hence, Theorem 4.4 establishes the sharp regularity of solutions w of equation (4.11) when $0 < \beta < 2$.

REMARK 4.5 (Generalization to higher dimensions). It is natural to believe that the analogues of the results in the Theorems given above hold in higher dimensions $n \geq 3$. In fact, many of the results in [**38**] can be easily generalized to higher dimensions. What still needs to be understood is the precise behavior at the origin $y = 0$ of solutions u^* to degenerate equations of the form (4.9) in the case that $h(y_1) = |y_1|^\alpha$ (the higher dimensional analogue of (4.10)). Degeneracies of this form are of a similar nature to those appearing in the parabolic case of the Gauss curvature flow with flat sides (c.f. in section 2.2).

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