Surveys in Differential Geometry XVII

An introduction to C^∞-schemes and C^∞-algebraic geometry

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1. Introduction

If X is a manifold then the $\mathbb{R}$-algebra $C^\infty(X)$ of smooth functions $c : X \to \mathbb{R}$ is a C^∞-*ring*. That is, for each smooth function $f : \mathbb{R}^n \to \mathbb{R}$ there is an n-fold operation $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ acting by $\Phi_f : c_1, \ldots, c_n \mapsto f(c_1, \ldots, c_n)$, and these operations Φ_f satisfy many natural identities. Thus, $C^\infty(X)$ actually has a far richer structure than the obvious $\mathbb{R}$-algebra structure.

In [**7**] the author set out the foundations of a version of algebraic geometry in which rings or algebras are replaced by C^∞-rings, focussing on C^∞-*schemes*, a category of geometric objects which generalize manifolds, and whose morphisms generalize smooth maps, *quasicoherent* and *coherent sheaves* on C^∞-schemes, and C^∞-*stacks*, in particular *Deligne–Mumford* C^∞-*stacks*, a 2-category of geometric objects which generalize orbifolds. This paper is a survey of [**7**].

C^∞-rings and C^∞-schemes were first introduced in *synthetic differential geometry*, see for instance Dubuc [**3**], Moerdijk and Reyes [**14**] and Kock [**9**]. Following Dubuc's discussion of 'models of synthetic differential geometry' [**2**] and oversimplifying a bit, symplectic differential geometers are interested in C^∞-schemes as they provide a category $\mathbf{C^\infty Sch}$ of geometric objects which includes smooth manifolds and certain 'infinitesimal' objects, and all fibre products exist in $\mathbf{C^\infty Sch}$, and $\mathbf{C^\infty Sch}$ has some other nice properties to do with open covers, and exponentials of infinitesimals.

Synthetic differential geometry concerns proving theorems about manifolds using synthetic reasoning involving 'infinitesimals'. But one needs to check these methods of synthetic reasoning are valid. To do this you need a 'model', some category of geometric spaces including manifolds and infinitesimals, in which you can think of your synthetic arguments as happening. Once you know there exists at least one model with the properties you want, then as far as synthetic differential geometry is concerned the job is

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done. For this reason C^∞-schemes were not developed very far in synthetic differential geometry.

Recently, C^∞-rings and C^∞-ringed spaces appeared in a very different context, as part of Spivak's definition of *derived manifolds* [**17**], which are an extension to differential geometry of Lurie's 'derived algebraic geometry' programme [**11**]. The author [**8**] is developing an alternative theory of derived differential geometry which simplifies, and goes beyond, Spivak's derived manifolds. Our notion of derived manifolds are called *d-manifolds*. We also study *d-manifolds with boundary*, and *d-manifolds with corners*, and orbifold versions of all these, *d-orbifolds*. To define d-manifolds and d-orbifolds we need theories of C^∞-schemes, C^∞-stacks, and quasicoherent sheaves upon them, much of which had not been done, so the author set up the foundations of these in [**7**].

D-manifolds and d-orbifolds will have important applications in symplectic geometry, and elsewhere. Many areas of symplectic geometry involve moduli spaces $\overline{\mathcal{M}}_{g,m}(J,\beta)$ of stable J-holomorphic curves in a symplectic manifold (M,ω). The original motivation for [**8**] was to find a good geometric description for the geometric structure on such moduli spaces $\overline{\mathcal{M}}_{g,m}(J,\beta)$. In the Lagrangian Floer cohomology theory of Fukaya, Oh, Ohta and Ono [**5**], moduli spaces $\overline{\mathcal{M}}_{g,m}(J,\beta)$ are given the structure of *Kuranishi spaces*. The notion of Kuranishi space seemed to the author to be unsatisfactory. In trying improve it, using ideas from Spivak [**17**], the author arrived at the theory of [**8**]. The author believes the 'correct' definition of Kuranishi space in the work of Fukaya et al. [**5**] should be that a Kuranishi space is a d-orbifold with corners.

Section 2 explains C^∞-rings and their modules, §3 introduces C^∞-schemes, and quasicoherent and coherent sheaves upon them, and §4 discusses C^∞-stacks, particularly Deligne–Mumford C^∞-stacks, their relation to orbifolds, and quasicoherent and coherent sheaves on Deligne–Mumford C^∞-stacks.

2. C^∞-rings

We begin by explaining the basic objects out of which our theories are built, C^∞-*rings*, or *smooth rings*, following [**7**, §2, §3 & §5]. With the exception of the material on *good* C^∞-rings in §2.2, almost everything in §2.1–§2.2 was already known in synthetic differential geometry, and can be found in Moerdijk and Reyes [**14**, Ch. I], Dubuc [**2**–**4**] or Kock [**9**, §III].

2.1. Two definitions of C^∞-ring.

Definition 2.1. A C^∞-*ring* is a set $\mathfrak{C}$ together with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all $n \geqslant 0$ and smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, where by convention when $n = 0$ we define $\mathfrak{C}^0$ to be the single point $\{\emptyset\}$. These operations must satisfy the following relations: suppose $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for

$i=1,\dots,m$ and $g:\mathbb{R}^m\to\mathbb{R}$ are smooth functions. Define a smooth function $h:\mathbb{R}^n\to\mathbb{R}$ by

$$h(x_1,\dots,x_n)=g\bigl(f_1(x_1,\dots,x_n),\dots,f_m(x_1\dots,x_n)\bigr),$$

for all $(x_1,\dots,x_n)\in\mathbb{R}^n$. Then for all $(c_1,\dots,c_n)\in\mathfrak{C}^n$ we have

$$\Phi_h(c_1,\dots,c_n)=\Phi_g\bigl(\Phi_{f_1}(c_1,\dots,c_n),\dots,\Phi_{f_m}(c_1,\dots,c_n)\bigr).$$

We also require that for all $1\leqslant j\leqslant n$, defining $\pi_j:\mathbb{R}^n\to\mathbb{R}$ by $\pi_j:(x_1,\dots,x_n)\mapsto x_j$, we have $\Phi_{\pi_j}(c_1,\dots,c_n)=c_j$ for all $(c_1,\dots,c_n)\in\mathfrak{C}^n$.

Usually we refer to $\mathfrak{C}$ as the C^∞-ring, leaving the operations Φ_f implicit.

A *morphism* between C^∞-rings $\bigl(\mathfrak{C},(\Phi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\bigr)$, $\bigl(\mathfrak{D},(\Psi_f)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\bigr)$ is a map $\phi:\mathfrak{C}\to\mathfrak{D}$ such that $\Psi_f\bigl(\phi(c_1),\dots,\phi(c_n)\bigr)=\phi\circ\Phi_f(c_1,\dots,c_n)$ for all smooth $f:\mathbb{R}^n\to\mathbb{R}$ and $c_1,\dots,c_n\in\mathfrak{C}$. We will write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Here is the motivating example:

Example 2.2. Let X be a manifold, and write $C^\infty(X)$ for the set of smooth functions $c:X\to\mathbb{R}$. For $n\geqslant 0$ and $f:\mathbb{R}^n\to\mathbb{R}$ smooth, define $\Phi_f: C^\infty(X)^n\to C^\infty(X)$ by

$$\bigl(\Phi_f(c_1,\dots,c_n)\bigr)(x)=f\bigl(c_1(x),\dots,c_n(x)\bigr), \tag{1}$$

for all $c_1,\dots,c_n\in C^\infty(X)$ and $x\in X$. It is easy to see that $C^\infty(X)$ and the operations Φ_f form a C^∞-ring.

Now let $f:X\to Y$ be a smooth map of manifolds. Then pullback $f^*: C^\infty(Y)\to C^\infty(X)$ mapping $f^*:c\mapsto c\circ f$ is a morphism of C^∞-rings. Furthermore, every C^∞-ring morphism $\phi:C^\infty(Y)\to C^\infty(X)$ is of the form $\phi=f^*$ for a unique smooth map $f:X\to Y$.

Write $\mathbf{C^\infty Rings}^{\rm op}$ for the opposite category of $\mathbf{C^\infty Rings}$, with directions of morphisms reversed, and $\mathbf{Man}$ for the category of manifolds without boundary. Then we have a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}:\mathbf{Man}\to\mathbf{C^\infty Rings}^{\rm op}$ acting by $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(X)=C^\infty(X)$ on objects and $F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}(f)=f^*$ on morphisms. This embeds $\mathbf{Man}$ as a full subcategory of $\mathbf{C^\infty Rings}^{\rm op}$.

Note that C^∞-rings are far more general than those coming from manifolds. For example, if X is any topological space we could define a C^∞-ring $C^0(X)$ to be the set of *continuous* $c:X\to\mathbb{R}$, with operations Φ_f defined as in (1). For X a manifold with $\dim X>0$, the C^∞-rings $C^\infty(X)$ and $C^0(X)$ are different.

There is a more succinct definition of C^∞-rings using category theory:

Definition 2.3. Write $\mathbf{Euc}$ for the full subcategory of $\mathbf{Man}$ spanned by the Euclidean spaces $\mathbb{R}^n$. That is, the objects of $\mathbf{Euc}$ are the manifolds $\mathbb{R}^n$ for $n=0,1,2,\dots$, and the morphisms in $\mathbf{Euc}$ are smooth maps $f:\mathbb{R}^n\to\mathbb{R}^m$. Write $\mathbf{Sets}$ for the category of sets. In both $\mathbf{Euc}$ and $\mathbf{Sets}$ we have notions of (finite) products of objects (that is, $\mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m$, and products $S \times T$ of sets S, T), and products of morphisms. Define a (*category-theoretic*) C^∞*-ring* to be a product-preserving functor $F : \mathbf{Euc} \to \mathbf{Sets}$.

Here is how this relates to Definition 2.1. Suppose $F : \mathbf{Euc} \to \mathbf{Sets}$ is a product-preserving functor. Define $\mathfrak{C} = F(\mathbb{R})$. Then $\mathfrak{C}$ is an object in $\mathbf{Sets}$, that is, a set. Suppose $n \geqslant 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ is smooth. Then f is a morphism in $\mathbf{Euc}$, so $F(f) : F(\mathbb{R}^n) \to F(\mathbb{R}) = \mathfrak{C}$ is a morphism in $\mathbf{Sets}$. Since F preserves products $F(\mathbb{R}^n) = F(\mathbb{R}) \times \cdots \times F(\mathbb{R}) = \mathfrak{C}^n$, so $F(f)$ maps $\mathfrak{C}^n \to \mathfrak{C}$. We define $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ by $\Phi_f = F(f)$. Then $\bigl(\mathfrak{C}, (\Phi_f)_{f:\mathbb{R}^n \to \mathbb{R}\ C^\infty}\bigr)$ is a C^∞ ring.

Since all small colimits exist in $\mathbf{Sets}$, regarding C^∞-rings as functors $F : \mathbf{Euc} \to \mathbf{Sets}$ as in Definition 2.3, to take small colimits in the category of C^∞-rings we can take colimits in $\mathbf{Sets}$ object-wise in $\mathbf{Euc}$, so as in [**7**, Prop. 2.5], [**14**, p. 21-22] we have:

PROPOSITION 2.4. *In the category* $\mathbf{C^\infty Rings}$ *of* C^∞*-rings, all small colimits exist, and so in particular pushouts and all finite colimits exist.*

DEFINITION 2.5. Let $\mathfrak{C}$ be a C^∞-ring. Then we may give $\mathfrak{C}$ the structure of a *commutative* $\mathbb{R}$*-algebra*. Define addition '+' on $\mathfrak{C}$ by $c + c' = \Phi_f(c, c')$ for $c, c' \in \mathfrak{C}$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = x + y$. Define multiplication '$\cdot$' on $\mathfrak{C}$ by $c \cdot c' = \Phi_g(c, c')$, where $g : \mathbb{R}^2 \to \mathbb{R}$ is $f(x, y) = xy$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c = \Phi_{\lambda'}(c)$, where $\lambda' : \mathbb{R} \to \mathbb{R}$ is $\lambda'(x) = \lambda x$. Define elements 0 and 1 in $\mathfrak{C}$ by $0 = \Phi_{0'}(\emptyset)$ and $1 = \Phi_{1'}(\emptyset)$, where $0' : \mathbb{R}^0 \to \mathbb{R}$ and $1' : \mathbb{R}^0 \to \mathbb{R}$ are the maps $0' : \emptyset \mapsto 0$ and $1' : \emptyset \mapsto 1$. One can then show using the relations on the Φ_f that all the axioms of a commutative $\mathbb{R}$-algebra are satisfied. In Example 2.2, this yields the obvious $\mathbb{R}$-algebra structure on the smooth functions $c : X \to \mathbb{R}$.

An *ideal* I in $\mathfrak{C}$ is an ideal $I \subset \mathfrak{C}$ in $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra. Then we make the quotient $\mathfrak{C}/I$ into a C^∞-ring as follows. If $f : \mathbb{R}^n \to \mathbb{R}$ is smooth, define $\Phi_f^I : (\mathfrak{C}/I)^n \to \mathfrak{C}/I$ by

$$\bigl(\Phi_f^I(c_1 + I, \ldots, c_n + I)\bigr)(x) = f\bigl(c_1(x), \ldots, c_n(x)\bigr) + I.$$

To show this is well-defined, we must show it is independent of the choice of representatives $c_1, \ldots, c_n$ in $\mathfrak{C}$ for $c_1 + I, \ldots, c_n + I$ in $\mathfrak{C}/I$. By Hadamard's Lemma there exist smooth functions $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ for $i = 1, \ldots, n$ with

$$f(y_1, \ldots, y_n) - f(x_1, \ldots, x_n) = \textstyle\sum_{i=1}^n (y_i - x_i) g_i(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$. If $c_1', \ldots, c_n'$ are alternative choices for $c_1, \ldots, c_n$, so that $c_i' + I = c_i + I$ for $i = 1, \ldots, n$ and $c_i' - c_i \in I$, we have

$$\begin{aligned} &f\bigl(c_1'(x), \ldots, c_n'(x)\bigr) - f\bigl(c_1(x), \ldots, c_n(x)\bigr) \\ &\qquad = \textstyle\sum_{i=1}^n (c_i' - c_i) g_i\bigl(c_1'(x), \ldots, c_n'(x), c_1(x), \ldots, c_n(x)\bigr). \end{aligned}$$

The second line lies in I as $c_i' - c_i \in I$ and I is an ideal, so Φ_f^I is well-defined, and clearly $\bigl(\mathfrak{C}/I, (\Phi_f^I)_{f:\mathbb{R}^n\to\mathbb{R}\ C^\infty}\bigr)$ is a C^∞-ring.

We will use the notation $(f_a : a \in A)$ to denote the ideal in a C^∞-ring $\mathfrak{C}$ generated by a collection of elements $f_a \in \mathfrak{C}$, $a \in A$. That is,

$$(f_a : a \in A) = \bigl\{\textstyle\sum_{i=1}^n f_{a_i} \cdot c_i : n \geqslant 0,\ a_1,\ldots,a_n \in A,\ c_1,\ldots,c_n \in \mathfrak{C}\bigr\}.$$

2.2. Special classes of C^∞-ring. We define *finitely generated*, *finitely presented*, *local*, *fair*, and *good* C^∞-rings.

Definition 2.6. A C^∞-ring $\mathfrak{C}$ is called *finitely generated* if there exist $c_1,\ldots,c_n$ in $\mathfrak{C}$ which generate $\mathfrak{C}$ over all C^∞-operations. That is, for each $c \in \mathfrak{C}$ there exists smooth $f : \mathbb{R}^n \to \mathbb{R}$ with $c = \Phi_f(c_1,\ldots,c_n)$. Given such $\mathfrak{C}, c_1,\ldots,c_n$, define $\phi : C^\infty(\mathbb{R}^n) \to \mathfrak{C}$ by $\phi(f) = \Phi_f(c_1,\ldots,c_n)$ for smooth $f : \mathbb{R}^n \to \mathbb{R}$, where $C^\infty(\mathbb{R}^n)$ is as in Example 2.2 with $X = \mathbb{R}^n$. Then ϕ is a surjective morphism of C^∞-rings, so $I = \operatorname{Ker}\phi$ is an ideal in $C^\infty(\mathbb{R}^n)$, and $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ as a C^∞-ring. Thus, $\mathfrak{C}$ is finitely generated if and only if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$ and ideal I in $C^\infty(\mathbb{R}^n)$.

An ideal I in $C^\infty(\mathbb{R}^n)$ is called *finitely generated* if $I = (f_1,\ldots,f_k)$ for some $f_1,\ldots,f_k \in C^\infty(\mathbb{R}^n)$. A C^∞-ring $\mathfrak{C}$ is called *finitely presented* if $\mathfrak{C} \cong C^\infty(\mathbb{R}^n)/I$ for some $n \geqslant 0$, where I is a finitely generated ideal in $C^\infty(\mathbb{R}^n)$.

A difference with conventional algebraic geometry is that $C^\infty(\mathbb{R}^n)$ is not noetherian, so ideals in $C^\infty(\mathbb{R}^n)$ may not be finitely generated, and $\mathfrak{C}$ finitely generated does not imply $\mathfrak{C}$ finitely presented.

Definition 2.7. A C^∞-ring $\mathfrak{C}$ is called a *C^∞-local ring* if regarded as an $\mathbb{R}$-algebra, as in Definition 2.5, $\mathfrak{C}$ is a local $\mathbb{R}$-algebra with residue field $\mathbb{R}$. That is, $\mathfrak{C}$ has a unique maximal ideal $\mathfrak{m}_{\mathfrak{C}}$ with $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R}$.

If $\mathfrak{C}, \mathfrak{D}$ are C^∞-local rings with maximal ideals $\mathfrak{m}_{\mathfrak{C}}, \mathfrak{m}_{\mathfrak{D}}$, and $\phi : \mathfrak{C} \to \mathfrak{D}$ is a morphism of C^∞ rings, then using the fact that $\mathfrak{C}/\mathfrak{m}_{\mathfrak{C}} \cong \mathbb{R} \cong \mathfrak{D}/\mathfrak{m}_{\mathfrak{D}}$ we see that $\phi^{-1}(\mathfrak{m}_{\mathfrak{D}}) = \mathfrak{m}_{\mathfrak{C}}$, that is, ϕ is a *local* morphism of C^∞-local rings. Thus, there is no difference between morphisms and local morphisms.

Example 2.8. For $n \geqslant 0$ and $p \in \mathbb{R}^n$, define $C_p^\infty(\mathbb{R}^n)$ to be the set of germs of smooth functions $c : \mathbb{R}^n \to \mathbb{R}$ at $p \in \mathbb{R}^n$. That is, $C_p^\infty(\mathbb{R}^n)$ is the quotient of the set of pairs (U, c) with $p \in U \subset \mathbb{R}^n$ open and $c : U \to \mathbb{R}$ smooth by the equivalence relation $(U, c) \sim (U', c')$ if there exists $p \in V \subseteq U \cap U'$ open with $c|_V \equiv c'|_V$. Define operations $\Phi_f : (C_p^\infty(\mathbb{R}^n))^m \to C_p^\infty(\mathbb{R}^n)$ for $f : \mathbb{R}^m \to \mathbb{R}$ smooth by (1). Then $C_p^\infty(\mathbb{R}^n)$ is a C^∞-local ring, with maximal ideal $\mathfrak{m} = \bigl\{[(U, c)] : c(p) = 0\bigr\}$.

Definition 2.9. An ideal I in $C^\infty(\mathbb{R}^n)$ is called *fair* if for each $f \in C^\infty(\mathbb{R}^n)$, f lies in I if and only if $\pi_p(f)$ lies in $\pi_p(I) \subseteq C_p^\infty(\mathbb{R}^n)$ for all $p \in \mathbb{R}^n$, where $C_p^\infty(\mathbb{R}^n)$ is as in Example 2.8 and $\pi_p : C^\infty(\mathbb{R}^n) \to C_p^\infty(\mathbb{R}^n)$ is the natural projection $\pi_p : c \mapsto [(\mathbb{R}^n, c)]$. A C^∞-ring $\mathfrak{C}$ is called *fair* if it is isomorphic to $C^\infty(\mathbb{R}^n)/I$, where I is a fair ideal.

Let X be a closed subset of $\mathbb{R}^n$. Define $\mathfrak{m}_X^\infty$ to be the ideal of all $g\in C^\infty(\mathbb{R}^n)$ such that $\partial^k g|_X\equiv 0$ for all $k\geqslant 0$, that is, g and all its derivatives vanish at each $x\in X$. An ideal I in $C^\infty(\mathbb{R}^n)$ is called *good* if it is of the form $I=(f_1,\ldots,f_k,\mathfrak{m}_X^\infty)$ for some $f_1,\ldots,f_k\in C^\infty(\mathbb{R}^n)$ and closed $X\subseteq\mathbb{R}^n$. A C^∞-ring $\mathfrak{C}$ is called *good* if $\mathfrak{C}\cong C^\infty(\mathbb{R}^n)/I$ for $n\geqslant 0$, where I is a good ideal.

Our term 'fair' was introduced in [**7**] for brevity, but the idea was already well-known. They were introduced by Dubuc [**3**, Def. 11] under the name 'C^∞-rings of finite type presented by an ideal of local character', and in more recent work would be called 'finitely generated and germ-determined C^∞-rings'.

As in [**7**, §2], if $C^\infty(\mathbb{R}^m)/I\cong C^\infty(\mathbb{R}^n)/J$ then I is finitely generated, or fair, or good, if and only if J is. Thus, to decide whether a C^∞-ring $\mathfrak{C}$ is finitely presented, or fair, or good, it is enough to test one presentation $\mathfrak{C}\cong C^\infty(\mathbb{R}^n)/I$. Also, $\mathfrak{C}$ finitely presented implies $\mathfrak{C}$ good implies $\mathfrak{C}$ fair implies $\mathfrak{C}$ finitely generated. Write $\mathbf{C^\infty Rings^{fp}},\mathbf{C^\infty Rings^{go}},\mathbf{C^\infty Rings^{fa}}$ and $\mathbf{C^\infty Rings^{fg}}$ for the full subcategories of finitely presented, good, fair, and finitely generated C^∞-rings in $\mathbf{C^\infty Rings}$, respectively. Then

$$\mathbf{C^\infty Rings^{fp}}\subset\mathbf{C^\infty Rings^{go}}\subset\mathbf{C^\infty Rings^{fa}}\subset\mathbf{C^\infty Rings^{fg}}\subset\mathbf{C^\infty Rings}.$$

From [**7**, Prop.s 2.28 & 2.30] we have:

PROPOSITION 2.10. *The subcategories* $\mathbf{C^\infty Rings^{fg}},\mathbf{C^\infty Rings^{fp}}$, $\mathbf{C^\infty Rings^{go}}$ *are closed under pushouts and all finite colimits in* $\mathbf{C^\infty Rings}$, *but* $\mathbf{C^\infty Rings^{fa}}$ *is not. Nonetheless, pushouts and finite colimits exist in* $\mathbf{C^\infty Rings^{fa}}$, *though they may not coincide with pushouts and finite colimits in* $\mathbf{C^\infty Rings}$.

Given morphisms $\phi:\mathfrak{C}\to\mathfrak{D}$ and $\psi:\mathfrak{C}\to\mathfrak{E}$ in $\mathbf{C^\infty Rings}$, the pushout $\mathfrak{D}\amalg_{\phi,\mathfrak{C},\psi}\mathfrak{E}$ in $\mathbf{C^\infty Rings}$ should be thought of as a *completed tensor product* $\mathfrak{D}\hat{\otimes}_{\mathfrak{C}}\mathfrak{E}$. The tensor product $\mathfrak{D}\otimes_{\mathfrak{C}}\mathfrak{E}$ is an $\mathbb{R}$-algebra, but in general not a C^∞-ring, and to get a C^∞-ring we must take a completion $\mathfrak{D}\hat{\otimes}_{\mathfrak{C}}\mathfrak{E}$. When $\mathfrak{C}=\mathbb{R}$, the trivial C^∞-ring, the pushout $\mathfrak{D}\amalg_{\mathbb{R}}\mathfrak{E}$ is the coproduct $\mathfrak{D}\amalg\mathfrak{E}=\mathfrak{D}\hat{\otimes}_{\mathbb{R}}\mathfrak{E}$. For example, one can show that $C^\infty(\mathbb{R}^m)\hat{\otimes}_{\mathbb{R}}C^\infty(\mathbb{R}^n)\cong C^\infty(\mathbb{R}^{m+n})$.

Here is [**7**, Prop. 3.2]. Part (b) is one reason for introducing *good* C^∞-rings.

PROPOSITION 2.11. **(a)** *If X is a manifold without boundary then the C^∞-ring $C^\infty(X)$ of Example* 2.2 *is finitely presented.*

(b) *If X is a manifold with boundary, or with corners, and $\partial X\neq\emptyset$, then the C^∞-ring $C^\infty(X)$ of Example* 2.2 *is good, but is not finitely presented.*

To save space we will say no more about manifolds with boundary or corners and C^∞-geometry in this paper. More information can be found in [**7**,**8**].

EXAMPLE 2.12. A *Weil algebra* [**2**, Def. 1.4] is a finite-dimensional commutative $\mathbb{R}$-algebra W which has a maximal ideal $\mathfrak{m}$ with $W/\mathfrak{m} \cong \mathbb{R}$ and $\mathfrak{m}^n = 0$ for some $n > 0$. Then by Dubuc [**2**, Prop. 1.5] or Kock [**9**, Th. III.5.3], there is a unique way to make W into a C^∞-ring compatible with the given underlying commutative $\mathbb{R}$-algebra. This C^∞-ring is finitely presented [**9**, Prop. III.5.11]. C^∞-rings from Weil algebras are important in synthetic differential geometry, in arguments involving infinitesimals.

2.3. Modules over C^∞-rings, and cotangent modules. In [**7**, §5] we discuss modules over C^∞-rings.

DEFINITION 2.13. Let $\mathfrak{C}$ be a C^∞-ring. A $\mathfrak{C}$-*module* M is a module over $\mathfrak{C}$ regarded as a commutative $\mathbb{R}$-algebra as in Definition 2.5. $\mathfrak{C}$-modules form an abelian category, which we write as $\mathfrak{C}$-mod. For example, $\mathfrak{C}$ is a $\mathfrak{C}$-module, and more generally $\mathfrak{C} \otimes_{\mathbb{R}} V$ is a $\mathfrak{C}$-module for any real vector space V.

A $\mathfrak{C}$-module M is called *finitely presented* if there exists an exact sequence $\mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^m \to \mathfrak{C} \otimes_{\mathbb{R}} \mathbb{R}^n \to M \to 0$ in $\mathfrak{C}$-mod for some $m, n \geqslant 0$. We write $\mathfrak{C}\text{-mod}^{\text{fp}}$ for the full subcategory of finitely presented $\mathfrak{C}$-modules in $\mathfrak{C}$-mod. Then $\mathfrak{C}\text{-mod}^{\text{fp}}$ is closed under cokernels and extensions in $\mathfrak{C}$-mod. But it may not be closed under kernels, so $\mathfrak{C}\text{-mod}^{\text{fp}}$ may not be an abelian category.

Let $\phi : \mathfrak{C} \to \mathfrak{D}$ be a morphism of C^∞-rings. If M is a $\mathfrak{C}$-module then $\phi_*(M) = M \otimes_{\mathfrak{C}} \mathfrak{D}$ is a $\mathfrak{D}$-module. This induces a functor $\phi_* : \mathfrak{C}\text{-mod} \to \mathfrak{D}\text{-mod}$, which maps $\mathfrak{C}\text{-mod}^{\text{fp}} \to \mathfrak{D}\text{-mod}^{\text{fp}}$.

EXAMPLE 2.14. Let X be a manifold, and $E \to X$ be a vector bundle. Write $C^\infty(E)$ for the vector space of smooth sections e of E. Then $C^\infty(X)$ acts on $C^\infty(E)$ by $(c, e) \mapsto c \cdot e$ for $c \in C^\infty(X)$ and $e \in C^\infty(E)$, so $C^\infty(E)$ is a $C^\infty(X)$-module, which is finitely presented.

Now let X, Y be manifolds and $f : X \to Y$ a smooth map. Then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings. If E is a vector bundle over Y, then $f^*(E)$ is a vector bundle over X. Under the functor $(f^*)_* : C^\infty(Y)\text{-mod} \to C^\infty(X)\text{-mod}$ of Definition 2.13, we see that $(f^*)_*\big(C^\infty(E)\big) = C^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X)$ is isomorphic as a $C^\infty(X)$-module to $C^\infty\big(f^*(E)\big)$.

Every commutative algebra A has a natural module Ω_A called the *module of Kähler differentials*, which is a kind of analogue for A of the cotangent bundle T^*X of a manifold X. In [**7**, §5.3] we define the *cotangent module* $\Omega_{\mathfrak{C}}$ of a C^∞-ring $\mathfrak{C}$, which is the C^∞-version of the module of Kähler differentials.

DEFINITION 2.15. Let $\mathfrak{C}$ be a C^∞-ring, and M a $\mathfrak{C}$-module. A C^∞-*derivation* is an $\mathbb{R}$-linear map $\mathrm{d} : \mathfrak{C} \to M$ such that whenever $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $c_1, \ldots, c_n \in \mathfrak{C}$, we have

$$\mathrm{d}\Phi_f(c_1, \ldots, c_n) = \textstyle\sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n) \cdot \mathrm{d}c_i.$$

Note that d is *not* a morphism of $\mathfrak{C}$-modules. We call such a pair M,d a *cotangent module* for $\mathfrak{C}$ if it has the universal property that for any $\mathfrak{C}$-module M' and C^∞-derivation $\mathrm{d}':\mathfrak{C}\to M'$, there exists a unique morphism of $\mathfrak{C}$-modules $\phi:M\to M'$ with $\mathrm{d}'=\phi\circ\mathrm{d}$.

Define $\Omega_{\mathfrak{C}}$ to be the quotient of the free $\mathfrak{C}$-module with basis of symbols $\mathrm{d}c$ for $c\in\mathfrak{C}$ by the $\mathfrak{C}$-submodule spanned by all expressions of the form $\mathrm{d}\Phi_f(c_1,\ldots,c_n)-\sum_{i=1}^n\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n)\cdot\mathrm{d}c_i$ for $f:\mathbb{R}^n\to\mathbb{R}$ smooth and $c_1,\ldots,c_n\in\mathfrak{C}$, and define $\mathrm{d}_{\mathfrak{C}}:\mathfrak{C}\to\Omega_{\mathfrak{C}}$ by $\mathrm{d}_{\mathfrak{C}}:c\mapsto\mathrm{d}c$. Then $\Omega_{\mathfrak{C}},\mathrm{d}_{\mathfrak{C}}$ is a cotangent module for $\mathfrak{C}$. Thus cotangent modules always exist, and are unique up to unique isomorphism.

Let $\mathfrak{C},\mathfrak{D}$ be C^∞-rings with cotangent modules $\Omega_{\mathfrak{C}},\mathrm{d}_{\mathfrak{C}}$, $\Omega_{\mathfrak{D}},\mathrm{d}_{\mathfrak{D}}$, and $\phi:\mathfrak{C}\to\mathfrak{D}$ be a morphism of C^∞-rings. Then ϕ makes $\Omega_{\mathfrak{D}}$ into a $\mathfrak{C}$-module, and there is a unique morphism $\Omega_\phi:\Omega_{\mathfrak{C}}\to\Omega_{\mathfrak{D}}$ in $\mathfrak{C}$-mod with $\mathrm{d}_{\mathfrak{D}}\circ\phi=\Omega_\phi\circ\mathrm{d}_{\mathfrak{C}}$. This induces a morphism $(\Omega_\phi)_*:\Omega_{\mathfrak{C}}\otimes_{\mathfrak{C}}\mathfrak{D}\to\Omega_{\mathfrak{D}}$ in $\mathfrak{D}$-mod with $(\Omega_\phi)_*\circ(\mathrm{d}_{\mathfrak{C}}\otimes\mathrm{id}_{\mathfrak{D}})=\mathrm{d}_{\mathfrak{D}}$. If $\phi:\mathfrak{C}\to\mathfrak{D}$, $\psi:\mathfrak{D}\to\mathfrak{E}$ are morphisms of C^∞-rings then $\Omega_{\psi\circ\phi}=\Omega_\psi\circ\Omega_\phi$.

Example 2.16. Let X be a manifold. Then the cotangent bundle T^*X is a vector bundle over X, so as in Example 2.14 it yields a $C^\infty(X)$-module $C^\infty(T^*X)$. The exterior derivative $\mathrm{d}:C^\infty(X)\to C^\infty(T^*X)$ is a C^∞-derivation. These $C^\infty(T^*X),\mathrm{d}$ have the universal property in Definition 2.15, and so form a *cotangent module* for $C^\infty(X)$.

Now let X,Y be manifolds, and $f:X\to Y$ be smooth. Then $f^*(TY)$, TX are vector bundles over X, and the derivative of f is a vector bundle morphism $\mathrm{d}f:TX\to f^*(TY)$. The dual of this morphism is $(\mathrm{d}f)^*:f^*(T^*Y)\to T^*X$. This induces a morphism of $C^\infty(X)$-modules $((\mathrm{d}f)^*)_*:C^\infty\bigl(f^*(T^*Y)\bigr)\to C^\infty(T^*X)$. This $((\mathrm{d}f)^*)_*$ is identified with $(\Omega_{f^*})_*$ in Definition 2.15 under the natural isomorphism $C^\infty\bigl(f^*(T^*Y)\bigr)\cong C^\infty(T^*Y)\otimes_{C^\infty(Y)}C^\infty(X)$.

Definition 2.15 abstracts the notion of cotangent bundle of a manifold in a way that makes sense for any C^∞-ring. From [**7**, Th.s 5.13 & 5.16] we have:

Theorem 2.17. **(a)** *Suppose $\mathfrak{C}$ is a finitely presented or good C^∞-ring. Then $\Omega_{\mathfrak{C}}$ is a finitely presented $\mathfrak{C}$-module.*

(b) *Suppose we are given a pushout diagram of finitely generated C^∞-rings:*

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\beta"] \arrow[d, "\alpha"] & \mathfrak{E} \arrow[d, "\delta"] \\ \mathfrak{D} \arrow[r, "\gamma"] & \mathfrak{F}, \end{tikzcd}$$

so that $\mathfrak{F}=\mathfrak{D}\amalg_{\mathfrak{C}}\mathfrak{E}$. Then the following sequence of $\mathfrak{F}$-modules is exact:

$$\Omega_{\mathfrak{C}}\otimes_{\mu_{\mathfrak{C}},\mathfrak{C},\gamma\circ\alpha}\mathfrak{F}\xrightarrow{\substack{(\Omega_\alpha)_*\oplus\\ -(\Omega_\beta)_*}}\begin{array}{c}\Omega_{\mathfrak{D}}\otimes_{\mu_{\mathfrak{D}},\mathfrak{D},\gamma}\mathfrak{F}\oplus\\ \Omega_{\mathfrak{E}}\otimes_{\mu_{\mathfrak{E}},\mathfrak{E},\delta}\mathfrak{F}\end{array}\xrightarrow{(\Omega_\gamma)_*\oplus(\Omega_\delta)_*}\Omega_{\mathfrak{F}}\longrightarrow 0.$$

Here $(\Omega_\alpha)_*:\Omega_{\mathfrak C}\otimes_{\mu_{\mathfrak C},\mathfrak C,\gamma\circ\alpha}\mathfrak F\rightarrow\Omega_{\mathfrak D}\otimes_{\mu_{\mathfrak D},\mathfrak D,\gamma}\mathfrak F$ *is induced by* $\Omega_\alpha:\Omega_{\mathfrak C}\rightarrow\Omega_{\mathfrak D}$, *and so on.*

3. C^∞-schemes

We now summarize material in [**7**, §4] on C^∞-schemes, and in [**7**, §6] on coherent and quasicoherent sheaves on C^∞-schemes. Much of §3.1 goes back to Dubuc [**3**].

3.1. The definition of C^∞-schemes. The basic definitions are modelled on the definitions of schemes in Hartshorne [**6**, §II.2], but replacing rings by C^∞-rings throughout.

Definition 3.1. A C^∞*-ringed space* $\underline{X}=(X,\mathcal O_X)$ is a topological space X with a sheaf $\mathcal O_X$ of C^∞-rings on X. That is, for each open set $U\subseteq X$ we are given a C^∞ ring $\mathcal O_X(U)$, and for each inclusion of open sets $V\subseteq U\subseteq X$ we are given a morphism of C^∞-rings $\rho_{UV}:\mathcal O_X(U)\rightarrow\mathcal O_X(V)$, called the *restriction maps*, and all this data satisfies the usual sheaf axioms [**6**, §II.1].

A *morphism* $\underline{f}=(f,f^\sharp):(X,\mathcal O_X)\rightarrow(Y,\mathcal O_Y)$ of C^∞ ringed spaces is a continuous map $f:X\rightarrow Y$ and a morphism $f^\sharp:\mathcal O_Y\rightarrow f_*(\mathcal O_X)$ of sheaves of C^∞-rings on Y. That is, for each open $U\subset Y$ we are given a morphism of C^∞-rings $f^\sharp(U):\mathcal O_Y(U)\rightarrow\mathcal O_X(f^{-1}(U))$ satisfying the obvious compatibilities with the restriction maps ρ_{UV} in $\mathcal O_X$ and $\mathcal O_Y$.

A *local* C^∞*-ringed space* $\underline{X}=(X,\mathcal O_X)$ is a C^∞-ringed space for which the stalks $\mathcal O_{X,x}$ of $\mathcal O_X$ at x are C^∞-local rings for all $x\in X$. Since morphisms of C^∞-local rings are automatically local morphisms, morphisms of local C^∞-ringed spaces $(X,\mathcal O_X),(Y,\mathcal O_Y)$ are just morphisms of C^∞-ringed spaces, without any additional locality condition. Write $\mathbf{C^\infty RS}$ for the category of C^∞-ringed spaces, and $\mathbf{LC^\infty RS}$ for the full subcategory of local C^∞-ringed spaces.

For brevity, we will use the notation that underlined upper case letters $\underline{X},\underline{Y},\underline{Z},\ldots$ represent C^∞-ringed spaces $(X,\mathcal O_X),(Y,\mathcal O_Y),(Z,\mathcal O_Z),\ldots$, and underlined lower case letters $\underline{f},\underline{g},\ldots$ represent morphisms of C^∞-ringed spaces $(f,f^\sharp),(g,g^\sharp),\ldots$. When we write '$x\in\underline{X}$' we mean that $\underline{X}=(X,\mathcal O_X)$ and $x\in X$. When we write '$\underline{U}$ *is open in* $\underline{X}$' we mean that $\underline{U}=(U,\mathcal O_U)$ and $\underline{X}=(X,\mathcal O_X)$ with $U\subseteq X$ an open set and $\mathcal O_U=\mathcal O_X|_U$.

Definition 3.2. Write $\mathbf{C^\infty Rings}^{\rm op}$ for the opposite category of $\mathbf{C^\infty Rings}$. The *global sections functor* $\Gamma:\mathbf{LC^\infty RS}\rightarrow\mathbf{C^\infty Rings}^{\rm op}$ acts on objects $(X,\mathcal O_X)$ in $\mathbf{LC^\infty RS}$ by $\Gamma:(X,\mathcal O_X)\mapsto\mathcal O_X(X)$ and on morphisms $(f,f^\sharp):(X,\mathcal O_X)\rightarrow(Y,\mathcal O_X)$ by $\Gamma:(f,f^\sharp)\mapsto f^\sharp(X)$. As in [**3**, Th. 8] there is a *spectrum functor* $\mathrm{Spec}:\mathbf{C^\infty Rings}^{\rm op}\rightarrow\mathbf{LC^\infty RS}$, defined explicitly in [**7**, Def. 4.5], which is a right adjoint to Γ, that is, for all $\mathfrak C\in\mathbf{C^\infty Rings}$ and $\underline{X}\in\mathbf{LC^\infty RS}$ there are functorial isomorphisms

$$\mathrm{Hom}_{\mathbf{C^\infty Rings}}(\mathfrak C,\Gamma(\underline{X}))\cong\mathrm{Hom}_{\mathbf{LC^\infty RS}}(\underline{X},\mathrm{Spec}\,\mathfrak C).\tag{2}$$

For any C^∞-ring $\mathfrak{C}$ there is a natural morphism of C^∞-rings $\Phi_{\mathfrak{C}}:\mathfrak{C}\to\Gamma(\operatorname{Spec}\mathfrak{C})$ corresponding to $\underline{\mathrm{id}}_{\underline{X}}$ in (2) with $\underline{X}=\operatorname{Spec}\mathfrak{C}$. By [**3**, Th. 13], the restriction of Spec to $(\mathbf{C^\infty Rings^{fa}})^{\mathrm{op}}$ is full and faithful.

A local C^∞-ringed space $\underline{X}$ is called an *affine C^∞-scheme* if it is isomorphic in $\mathbf{LC^\infty RS}$ to $\operatorname{Spec}\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$. We call $\underline{X}$ a *finitely presented*, or *good*, or *fair*, affine C^∞-scheme if $X\cong\operatorname{Spec}\mathfrak{C}$ for $\mathfrak{C}$ that kind of C^∞-ring.

Let $\underline{X}=(X,\mathcal{O}_X)$ be a local C^∞-ringed space. We call $\underline{X}$ a *C^∞-scheme* if X can be covered by open sets $U\subseteq X$ such that $(U,\mathcal{O}_X|_U)$ is an affine C^∞-scheme. We call a C^∞-scheme $\underline{X}$ *locally fair*, or *locally good*, or *locally finitely presented*, if X can be covered by open $U\subseteq X$ with $(U,\mathcal{O}_X|_U)$ a fair, or good, or finitely presented, affine C^∞-scheme, respectively.

Write $\mathbf{C^\infty Sch^{lf}},\mathbf{C^\infty Sch^{lg}},\mathbf{C^\infty Sch^{lfp}},\mathbf{C^\infty Sch}$ for the full subcategories of locally fair, and locally good, and locally finitely presented, and all, C^∞-schemes in $\mathbf{LC^\infty RS}$, respectively. We call a C^∞-scheme $\underline{X}$ *separated*, or *paracompact*, if the underlying topological space X is Hausdorff, or paracompact.

Example 3.3. Let X be a manifold. Define a C^∞-ringed space $\underline{X}=(X,\mathcal{O}_X)$ to have topological space X and $\mathcal{O}_X(U)=C^\infty(U)$ for each open $U\subseteq X$, where $C^\infty(U)$ is the C^∞-ring of smooth maps $c:U\to\mathbb{R}$, and if $V\subseteq U\subseteq X$ are open define $\rho_{UV}:C^\infty(U)\to C^\infty(V)$ by $\rho_{UV}:c\mapsto c|_V$. Then $\underline{X}=(X,\mathcal{O}_X)$ is a local C^∞-ringed space. It is canonically isomorphic to $\operatorname{Spec}C^\infty(X)$, and so is an affine C^∞-scheme. It is locally finitely presented.

Define a functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}:\mathbf{Man}\to\mathbf{C^\infty Sch^{lfp}}\subset\mathbf{C^\infty Sch}$ by $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}=\operatorname{Spec}\circ F_{\mathbf{Man}}^{\mathbf{C^\infty Rings}}$. Then $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ is full and faithful, and embeds $\mathbf{Man}$ as a full subcategory of $\mathbf{C^\infty Sch}$.

By [**7**, Prop.s 4.3, 4.4, 4.18, 4.25, Cor.s 4.11, 4.14 & Th. 4.26] we have:

Theorem 3.4. **(a)** *All finite limits exist in the category* $\mathbf{C^\infty RS}$.

(b) *The full subcategories* $\mathbf{C^\infty Sch^{lfp}},\mathbf{C^\infty Sch^{lg}},\mathbf{C^\infty Sch^{lf}},\mathbf{C^\infty Sch}$, $\mathbf{LC^\infty RS}$ *in* $\mathbf{C^\infty RS}$ *are closed under all finite limits in* $\mathbf{C^\infty RS}$. *Hence, fibre products and all finite limits exist in each of these subcategories.*

(c) *If $\mathfrak{C}$ is a finitely generated C^∞-ring then* $\operatorname{Spec}\mathfrak{C}$ *is a fair affine C^∞-scheme.*

(d) *Let $(X,\mathcal{O}_X)$ be a finitely presented, or good, or fair, affine C^∞-scheme, and $U\subseteq X$ be an open subset. Then $(U,\mathcal{O}_X|_U)$ is also a finitely presented, or good, or fair, affine C^∞-scheme, respectively. However, this does not hold for general affine C^∞-schemes.*

(e) *Let $(X,\mathcal{O}_X)$ be a locally finitely presented, locally good, locally fair, or general, C^∞-scheme, and $U\subseteq X$ be open. Then $(U,\mathcal{O}_X|_U)$ is also a locally finitely presented, or locally good, or locally fair, or general, C^∞-scheme, respectively.*

(f) *The functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ takes transverse fibre products in* $\mathbf{Man}$ *to fibre products in* $\mathbf{C^\infty Sch}$.

In [**7**, Def. 4.27 & Prop. 4.28] we discuss *partitions of unity* on C^∞-schemes, building on ideas of Dubuc [**4**].

Definition 3.5. Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-scheme. Consider a formal sum $\sum_{a \in A} c_a$, where A is an indexing set and $c_a \in \mathcal{O}_X(X)$ for $a \in A$. We say $\sum_{a \in A} c_a$ is a *locally finite sum on* $\underline{X}$ if X can be covered by open $U \subseteq X$ such that for all but finitely many $a \in A$ we have $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$.

By the sheaf axioms for $\mathcal{O}_X$, if $\sum_{a \in A} c_a$ is a locally finite sum there exists a unique $c \in \mathcal{O}_X(X)$ such that for all open $U \subseteq X$ such that $\rho_{XU}(c_a) = 0$ in $\mathcal{O}_X(U)$ for all but finitely many $a \in A$, we have $\rho_{XU}(c) = \sum_{a \in A} \rho_{XU}(c_a)$ in $\mathcal{O}_X(U)$, where the sum makes sense as there are only finitely many nonzero terms. We call c the *limit* of $\sum_{a \in A} c_a$, written $\sum_{a \in A} c_a = c$.

Let $c \in \mathcal{O}_X(X)$. Suppose $V_i \subseteq X$ is open and $\rho_{XV_i}(c) = 0 \in \mathcal{O}_X(V_i)$ for $i \in I$, and let $V = \bigcup_{i \in I} V_i$. Then $V \subseteq X$ is open, and $\rho_{XV}(c) = 0 \in \mathcal{O}_X(V)$ as $\mathcal{O}_X$ is a sheaf. Thus taking the union of all open $V \subseteq X$ with $\rho_{XV}(c) = 0$ gives a unique maximal open set $V_c \subseteq X$ such that $\rho_{XV_c}(c) = 0 \in \mathcal{O}_X(V_c)$. Define the *support* $\operatorname{supp} c$ of c to be $X \setminus V_c$, so that $\operatorname{supp} c$ is closed in X. If $U \subseteq X$ is open, we say that *c is supported in U* if $\operatorname{supp} c \subseteq U$.

Let $\{U_a : a \in A\}$ be an open cover of X. A *partition of unity on $\underline{X}$ subordinate to $\{U_a : a \in A\}$* is $\{\eta_a : a \in A\}$ with $\eta_a \in \mathcal{O}_X(X)$ supported on U_a for $a \in A$, such that $\sum_{a \in A} \eta_a$ is a locally finite sum on $\underline{X}$ with $\sum_{a \in A} \eta_a = 1$.

Proposition 3.6. *Suppose $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme, and $\{\underline{U}_a : a \in A\}$ an open cover of $\underline{X}$. Then there exists a partition of unity $\{\eta_a : a \in A\}$ on $\underline{X}$ subordinate to $\{\underline{U}_a : a \in A\}$.*

Here are some differences between ordinary schemes and C^∞-schemes:

Remark 3.7. **(i)** If A is a ring or algebra, then points of the corresponding scheme $\operatorname{Spec} A$ are prime ideals in A. However, if $\mathfrak{C}$ is a C^∞-ring then (by definition) points of $\operatorname{Spec} \mathfrak{C}$ are maximal ideals in $\mathfrak{C}$ with residue field $\mathbb{R}$, or equivalently, $\mathbb{R}$-algebra morphisms $x : \mathfrak{C} \to \mathbb{R}$. This has the effect that if X is a manifold then points of $\operatorname{Spec} C^\infty(X)$ are just points of X.

(ii) In conventional algebraic geometry, affine schemes are a restrictive class. Central examples such as $\mathbb{CP}^n$ are not affine, and affine schemes are not closed under open subsets, so that $\mathbb{C}^2$ is affine but $\mathbb{C}^2 \setminus \{0\}$ is not. In contrast, affine C^∞-schemes are already general enough for many purposes. For example:

- All manifolds are affine C^∞-schemes.
- Open C^∞-subschemes of fair affine C^∞-schemes are fair and affine.
- If $\underline{X}$ is a separated, paracompact, locally fair C^∞-scheme then $\underline{X}$ is affine.

Affine C^∞-schemes are always separated (Hausdorff), so we need general C^∞-schemes to include non-Hausdorff behaviour.

(iii) In conventional algebraic geometry the Zariski topology is too coarse for many purposes, so one has to introduce the étale topology. In C^∞-algebraic geometry there is no need for this, as affine C^∞-schemes are Hausdorff.

(iv) Even very basic C^∞-rings such as $C^\infty(\mathbb{R}^n)$ for $n>0$ are not noetherian as $\mathbb{R}$-algebras. So C^∞-schemes should be compared to non-noetherian schemes in conventional algebraic geometry.

3.2. Quasicoherent and coherent sheaves on C^∞-schemes. In [**7**, §6] we discuss sheaves of modules on C^∞-schemes.

Definition 3.8. Let $\underline{X}=(X,\mathcal{O}_X)$ be a C^∞-scheme. An *$\mathcal{O}_X$-module $\mathcal{E}$* on $\underline{X}$ assigns a module $\mathcal{E}(U)$ over $\mathcal{O}_X(U)$ for each open set $U\subseteq X$, with $\mathcal{O}_X(U)$-action $\mu_U:\mathcal{O}_X(U)\times\mathcal{E}(U)\to\mathcal{E}(U)$, and a linear map $\mathcal{E}_{UV}:\mathcal{E}(U)\to\mathcal{E}(V)$ for each inclusion of open sets $V\subseteq U\subseteq X$, such that the following commutes:

$$\begin{tikzcd}
\mathcal{O}_X(U)\times\mathcal{E}(U) \arrow[rr, "\mu_U"] \arrow[d, "\rho_{UV}\times\mathcal{E}_{UV}"] && \mathcal{E}(U) \arrow[d, "\mathcal{E}_{UV}"'] \\
\mathcal{O}_X(V)\times\mathcal{E}(V) \arrow[rr, "\mu_V"] && \mathcal{E}(V),
\end{tikzcd}$$

and all this data $\mathcal{E}(U),\mathcal{E}_{UV}$ satisfies the usual sheaf axioms [**6**, §II.1].

A *morphism of $\mathcal{O}_X$-modules* $\phi:\mathcal{E}\to\mathcal{F}$ assigns a morphism of $\mathcal{O}_X(U)$-modules $\phi(U):\mathcal{E}(U)\to\mathcal{F}(U)$ for each open set $U\subseteq X$, such that $\phi(V)\circ\mathcal{E}_{UV}=\mathcal{F}_{UV}\circ\phi(U)$ for each inclusion of open sets $V\subseteq U\subseteq X$. Then $\mathcal{O}_X$-modules form an *abelian category*, which we write as $\mathcal{O}_X$-mod.

As in [**7**, §6.2], the spectrum functor $\mathrm{Spec}:\mathbf{C^\infty Rings}^{\mathrm{op}}\to\mathbf{C^\infty Sch}$ has a counterpart for modules: if $\mathfrak{C}$ is a C^∞-ring and $(X,\mathcal{O}_X)=\mathrm{Spec}\,\mathfrak{C}$ we can define a functor $\mathrm{MSpec}:\mathfrak{C}\text{-mod}\to\mathcal{O}_X\text{-mod}$. Let $\underline{X}=(X,\mathcal{O}_X)$ be a C^∞-scheme, and $\mathcal{E}$ an $\mathcal{O}_X$-module. We call $\mathcal{E}$ *quasicoherent* if $\underline{X}$ can be covered by open $\underline{U}$ with $\underline{U}\cong\mathrm{Spec}\,\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$, and under this identification $\mathcal{E}|_{\underline{U}}\cong\mathrm{MSpec}\,M$ for some $\mathfrak{C}$-module M. We call $\mathcal{E}$ *coherent* if furthermore we can take these $\mathfrak{C}$-modules M to be finitely presented. We call $\mathcal{E}$ a *vector bundle of rank* $n\geqslant 0$ if $\underline{X}$ may be covered by open $\underline{U}$ such that $\mathcal{E}|_{\underline{U}}\cong\mathcal{O}_U\otimes_{\mathbb{R}}\mathbb{R}^n$. Write $\mathrm{qcoh}(\underline{X}),\mathrm{coh}(\underline{X})$, and $\mathrm{vect}(\underline{X})$ for the full subcategories of quasicoherent sheaves, coherent sheaves, and vector bundles in $\mathcal{O}_X$-mod, respectively.

Definition 3.9. Let $\underline{f}=(f,f^\sharp):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism of C^∞-schemes, and $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Following Hartshorne [**6**, p. 65, p. 110], define the *pullback* $\underline{f}^*(\mathcal{E})$ to be the sheaf of $\mathcal{O}_X$-modules on $(X,\mathcal{O}_X)$ associated to the presheaf $U\mapsto\lim_{V\supseteq f(U)}\mathcal{E}(V)\otimes_{\mathcal{O}_Y(V)}\mathcal{O}_X(U)$, where U is an open set in X, and the limit is over open sets V in Y containing $f(U)$.

If $\phi:\mathcal{E}\to\mathcal{F}$ is a morphism in $\mathcal{O}_Y$-mod we have an induced morphism $\underline{f}^*(\phi)=f^{-1}(\phi)\otimes\mathrm{id}_{\mathcal{O}_X}:\underline{f}^*(\mathcal{E})\to\underline{f}^*(\mathcal{F})$ in $\mathcal{O}_X$-mod. Then $\underline{f}^*:\mathcal{O}_Y\text{-mod}\to$

$\mathcal{O}_X$-mod is a right exact functor between abelian categories, which restricts to a right exact functor $\underline{f}^*: \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$.

Pullbacks $\underline{f}^*(\mathcal{E})$ are a kind of fibre product, and may be characterized by a universal property. So they should be regarded as being *unique up to canonical isomorphism*, rather than unique. We use the Axiom of Choice to choose $\underline{f}^*(\mathcal{E})$ for all $\underline{f}, \mathcal{E}$, and so speak of 'the' pullback $\underline{f}^*(\mathcal{E})$. However, it may not be possible to make these choices functorial in $\underline{f}$. That is, if $\underline{f}: \underline{X} \to \underline{Y}$, $\underline{g}: \underline{Y} \to \underline{Z}$ are morphisms and $\mathcal{E} \in \mathcal{O}_Z$-mod then $(\underline{g} \circ \underline{f})^*(\mathcal{E})$ and $\underline{f}^*(\underline{g}^*(\mathcal{E}))$ are canonically isomorphic in $\mathcal{O}_X$-mod, but may not be equal. We will write $I_{\underline{f},\underline{g}}(\mathcal{E}): (\underline{g} \circ \underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$ for these canonical isomorphisms. Then $I_{\underline{f},\underline{g}}: (\underline{g} \circ \underline{f})^* \Rightarrow \underline{f}^* \circ \underline{g}^*$ is a natural isomorphism of functors.

Similarly, when $\underline{f}$ is the identity $\underline{\mathrm{id}}_{\underline{X}}: \underline{X} \to \underline{X}$ and $\mathcal{E} \in \mathcal{O}_X$-mod we may not have $\underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) = \mathcal{E}$, but there is a canonical isomorphism $\delta_{\underline{X}}(\mathcal{E}): \underline{\mathrm{id}}_{\underline{X}}^*(\mathcal{E}) \to \mathcal{E}$, and $\delta_{\underline{X}}: \underline{\mathrm{id}}_{\underline{X}}^* \Rightarrow \mathrm{id}_{\mathcal{O}_X\text{-mod}}$ is a natural isomorphism of functors.

Example 3.10. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme from Example 3.3, so that $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. Let $E \to X$ be a vector bundle. Define an $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ by $\mathcal{E}(U) = C^\infty(E|_U)$, the smooth sections of the vector bundle $E|_U \to U$, and for open $V \subseteq U \subseteq X$ define $\mathcal{E}_{UV}: \mathcal{E}(U) \to \mathcal{E}(V)$ by $\mathcal{E}_{UV}: e_U \mapsto e_U|_V$. Then $\mathcal{E} \in \mathrm{vect}(\underline{X})$ is a vector bundle on $\underline{X}$, which we think of as a lift of E from manifolds to C^∞-schemes.

Suppose $f: X \to Y$ is a smooth map of manifolds, and $\underline{f}: \underline{X} \to \underline{Y}$ is the corresponding morphism of C^∞-schemes. Let $F \to Y$ be a vector bundle over Y, so that $f^*(F) \to X$ is a vector bundle over X. Let $\mathcal{F} \in \mathrm{vect}(\underline{Y})$ be the vector bundle over $\underline{Y}$ lifting F. Then $\underline{f}^*(\mathcal{F})$ is canonically isomorphic to the vector bundle over $\underline{X}$ lifting $f^*(F)$.

The next theorem comes from [**7**, Cor. 6.11 & Prop. 6.12]. In part (a), the reason $\mathrm{coh}(\underline{X})$ is not closed under kernels is that the C^∞-rings we are interested in are generally *not noetherian* as commutative $\mathbb{R}$-algebras, and this causes problems with coherence; in conventional algebraic geometry, one usually only considers coherent sheaves over noetherian schemes.

Theorem 3.11. **(a)** *Let $\underline{X}$ be a C^∞-scheme. Then* $\mathrm{qcoh}(\underline{X})$ *is closed under kernels, cokernels and extensions in* $\mathcal{O}_X$-mod, *so it is an abelian category. Also* $\mathrm{coh}(\underline{X})$ *is closed under cokernels and extensions in* $\mathcal{O}_X$-mod, *but may not be closed under kernels in* $\mathcal{O}_X$-mod, *so* $\mathrm{coh}(\underline{X})$ *may not be an abelian category.*

(b) *Suppose $\underline{f}: \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes. Then pullback $\underline{f}^*$:* $\mathcal{O}_Y$-mod $\to \mathcal{O}_X$-mod *maps* $\mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$ *and* $\mathrm{coh}(\underline{Y}) \to \mathrm{coh}(\underline{X})$ *and* $\mathrm{vect}(\underline{Y}) \to \mathrm{vect}(\underline{X})$*. Also* $\underline{f}^*: \mathrm{qcoh}(\underline{Y}) \to \mathrm{qcoh}(\underline{X})$ *is a right exact functor.*

(c) *Let $\underline{X}$ be a locally fair C^∞-scheme. Then every $\mathcal{O}_X$-module $\mathcal{E}$ on $\underline{X}$ is quasicoherent, that is,* $\mathrm{qcoh}(\underline{X}) = \mathcal{O}_X$-mod.

Let $\underline{X}$ be a separated, paracompact, locally fair C^∞-scheme. Then partitions of unity exist on $\underline{X}$ subordinate to any open cover by Proposition 3.6. As in [**7**, §6.3], this shows that quasicoherent sheaves $\mathcal{E}$ on $\underline{X}$ are *fine*, which implies that their cohomology groups $H^i(\mathcal{E})$ are zero for all $i>0$. In [**7**, Prop. 6.13] we deduce an exactness property for sections of quasicoherent sheaves on $\underline{X}$:

Proposition 3.12. *Suppose $\underline{X}=(X,\mathcal{O}_X)$ is a separated, paracompact, locally fair C^∞-scheme, and $\cdots\rightarrow\mathcal{E}^i\xrightarrow{\phi^i}\mathcal{E}^{i+1}\xrightarrow{\phi^{i+1}}\mathcal{E}^{i+2}\rightarrow\cdots$ an exact sequence in* $\mathrm{qcoh}(\underline{X})$. *Then $\cdots\rightarrow\mathcal{E}^i(U)\xrightarrow{\phi^i(U)}\mathcal{E}^{i+1}(U)\xrightarrow{\phi^{i+1}(U)}\mathcal{E}^{i+2}(U)\rightarrow\cdots$ is an exact sequence of $\mathcal{O}_X(U)$-modules for each open $U\subseteq X$.*

We define *cotangent sheaves*, the sheaf version of cotangent modules in §2.3.

Definition 3.13. Let $\underline{X}$ be a C^∞-scheme. Define $\mathcal{P}T^*\underline{X}$ to associate to each open $U\subseteq X$ the cotangent module $\Omega_{\mathcal{O}_X(U)}$, and to each inclusion of open sets $V\subseteq U\subseteq X$ the morphism of $\mathcal{O}_X(U)$-modules $\Omega_{\rho_{UV}}:\Omega_{\mathcal{O}_X(U)}\rightarrow\Omega_{\mathcal{O}_X(V)}$ associated to the morphism of C^∞-rings $\rho_{UV}:\mathcal{O}_X(U)\rightarrow\mathcal{O}_X(V)$. Then $\mathcal{P}T^*\underline{X}$ is a *presheaf of $\mathcal{O}_X$-modules on $\underline{X}$*. Define the *cotangent sheaf $T^*\underline{X}$ of $\underline{X}$* to be the sheafification of $\mathcal{P}T^*\underline{X}$, as an $\mathcal{O}_X$-module.

If $\underline{f}:\underline{X}\rightarrow\underline{Y}$ is a morphism of C^∞-schemes, then $\underline{f}^*\big(T^*\underline{Y}\big)$ is the sheafification of the presheaf $\underline{f}^*(\mathcal{P}T^*\underline{Y})$ acting by

$$\begin{aligned}U\longmapsto\underline{f}^*(\mathcal{P}T^*\underline{Y})(U)&=\lim_{V\supseteq f(U)}\mathcal{P}T^*\underline{Y}(V)\otimes_{\mathcal{O}_Y(V)}\mathcal{O}_X(U)\\&=\lim_{V\supseteq f(U)}\Omega_{\mathcal{O}_Y(V)}\otimes_{\mathcal{O}_Y(V)}\mathcal{O}_X(U).\end{aligned}$$

Define a morphism of presheaves $\mathcal{P}\Omega_{\underline{f}}:\underline{f}^*(\mathcal{P}T^*\underline{Y})\rightarrow\mathcal{P}T^*\underline{X}$ on X by

$$(\mathcal{P}\Omega_{\underline{f}})(U)=\lim_{V\supseteq f(U)}(\Omega_{\rho_{f^{-1}(V)\,U}\circ f^\sharp(V)})_*,$$

where $(\Omega_{\rho_{f^{-1}(V)\,U}\circ f^\sharp(V)})_*:\Omega_{\mathcal{O}_Y(V)}\otimes_{\mathcal{O}_Y(V)}\mathcal{O}_X(U)\rightarrow\Omega_{\mathcal{O}_X(U)}=(\mathcal{P}T^*\underline{X})(U)$ is constructed as in Definition 2.15 from the C^∞-ring morphisms $f^\sharp(V):\mathcal{O}_Y(V)\rightarrow\mathcal{O}_X(f^{-1}(V))$ in $\underline{f}$ and $\rho_{f^{-1}(V)\,U}:\mathcal{O}_X(f^{-1}(V))\rightarrow\mathcal{O}_X(U)$ in $\mathcal{O}_X$. Define $\Omega_{\underline{f}}:\underline{f}^*\big(T^*\underline{Y}\big)\rightarrow T^*\underline{X}$ to be the induced morphism of the associated sheaves.

Example 3.14. Let X be a manifold, and $\underline{X}$ the associated C^∞-scheme. Then $T^*\underline{X}$ is a vector bundle on $\underline{X}$, and is canonically isomorphic to the lift to C^∞-schemes from Example 3.10 of the cotangent vector bundle T^*X of X.

Here [**7**, Th.s 6.16 & 6.17] are some properties of cotangent sheaves.

Theorem 3.15. **(a)** *Let $\underline{X}$ be a locally good C^∞-scheme. Then $T^*\underline{X}$ is a coherent sheaf.*

(b) *Let $\underline{f}:\underline{X}\to\underline{Y}$ and $\underline{g}:\underline{Y}\to\underline{Z}$ be morphisms of C^∞-schemes. Then*

$$\Omega_{\underline{g}\circ\underline{f}}=\Omega_{\underline{f}}\circ\underline{f}^*(\Omega_{\underline{g}})\circ I_{\underline{f},\underline{g}}(T^*\underline{Z})$$

*as morphisms $(\underline{g}\circ\underline{f})^*(T^*\underline{Z})\to T^*\underline{X}$ in $\mathcal{O}_X$-*mod. *Here $\Omega_{\underline{g}}:\underline{g}^*(T^*\underline{Z})\to T^*\underline{Y}$ in $\mathcal{O}_Y$-*mod, *so applying $\underline{f}^*$ gives $\underline{f}^*(\Omega_{\underline{g}}):\underline{f}^*(\underline{g}^*(T^*\underline{Z}))\to\underline{f}^*(T^*\underline{Y})$ in $\mathcal{O}_X$-*mod, *and $I_{\underline{f},\underline{g}}(T^*\underline{Z}):(\underline{g}\circ\underline{f})^*(T^*\underline{Z})\to\underline{f}^*(\underline{g}^*(T^*\underline{Z}))$ is as in Definition* 3.9.

(c) *Suppose $\underline{W},\underline{X},\underline{Y},\underline{Z}$ are locally fair C^∞-schemes with a Cartesian square*

$$\begin{tikzcd} \underline{W} \arrow[r, "\underline{f}"] \arrow[d, "\underline{e}"'] & \underline{Y} \arrow[d, "\underline{h}"] \\ \underline{X} \arrow[r, "\underline{g}"] & \underline{Z} \end{tikzcd}$$

in $\mathbf{C^\infty Sch^{lf}}$, *so that $\underline{W}=\underline{X}\times_{\underline{Z}}\underline{Y}$. Then the following is exact in* $\mathrm{qcoh}(\underline{W})$:

$$(\underline{g}\circ\underline{e})^*(T^*\underline{Z})\xrightarrow{\substack{\underline{e}^*(\Omega_{\underline{g}})\circ I_{\underline{e},\underline{g}}(T^*\underline{Z})\oplus\\ -\underline{f}^*(\Omega_{\underline{h}})\circ I_{\underline{f},\underline{h}}(T^*\underline{Z})}}\underline{e}^*(T^*\underline{X})\oplus\underline{f}^*(T^*\underline{Y})\xrightarrow{\Omega_{\underline{e}}\oplus\Omega_{\underline{f}}}T^*\underline{W}\longrightarrow 0.$$

4. C^∞-stacks

In [**7**, §7–§8] we discuss C^∞-*stacks*, which are related to C^∞-schemes in the same way that Artin stacks and Deligne–Mumford stacks in algebraic geometry are related to schemes. Stacks are a rather technical subject which take a lot of work and many pages to set up properly, so to keep this section short we will give less detail than in §2 and §3.

We are most interested in a subclass of C^∞-stacks called *Deligne–Mumford C^∞-stacks*. Here are some of their important properties:

- Deligne–Mumford C^∞-stacks are geometric objects locally modelled on quotients $\underline{U}/G$, for $\underline{U}$ an affine C^∞-scheme and G a finite group.
- Deligne–Mumford C^∞-stacks are related to C^∞-schemes in exactly the same way that orbifolds are related to manifolds.
- Any C^∞-scheme yields an example of a Deligne–Mumford C^∞-stack.
- Deligne–Mumford C^∞-stacks form a 2-*category* $\mathbf{DMC^\infty Sta}$. That is, we have *objects* $\mathcal{X},\mathcal{Y}$, 1-*morphisms* $f,g:\mathcal{X}\to\mathcal{Y}$, and 2-*morphisms* $\eta:f\Rightarrow g$. All 2-morphisms are invertible, that is, they are 2-*isomorphisms*.

  The geometric meaning of 1- and 2-morphisms is not obvious; to get a feel for it, it helps to consider the case when $\mathcal{X},\mathcal{Y}$ are quotients $[\underline{X}/G],[\underline{Y}/H]$ for C^∞-schemes $\underline{X},\underline{Y}$ and finite groups G,H acting on $\underline{X},\underline{Y}$. Oversimplifying somewhat, a 1-morphism $f:[\underline{X}/G]\to[\underline{Y}/H]$ is roughly a pair $(\underline{f},\rho)$ where $\rho:G\to H$ is a group morphism and $\underline{f}:\underline{X}\to\underline{Y}$ is a morphism of C^∞-schemes with

$\underline{f} \circ \gamma = \rho(\gamma) \circ \underline{f}$ for all $\gamma \in G$. If $f = (\underline{f}, \rho)$ and $g = (\underline{g}, \sigma)$ are two such 1-morphisms, then a 2-morphism $\eta : f \Rightarrow g$ is roughly an element $\delta \in H$ such that $\sigma(\gamma) = \delta\, \rho(\gamma)\delta^{-1}$ for all $\gamma \in G$, and $\underline{g} = \delta \circ \underline{f}$.

- There is a good notion of *fibre product* in a 2-category. All fibre products of Deligne–Mumford C^∞-stacks exist, as Deligne–Mumford C^∞-stacks.

4.1. The definition of C^∞-stacks. The next few definitions assume a lot of standard material from stack theory, which is summarized in [**7**, §7].

Definition 4.1. Define a *Grothendieck topology* $\mathcal{J}$ on the category $\mathbf{C^\infty Sch}$ of C^∞-schemes to have coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ where $V_a = i_a(U_a)$ is open in U with $\underline{i}_a : \underline{U}_a \to (V_a, \mathcal{O}_U|_{V_a})$ an isomorphism for all $a \in A$, and $U = \bigcup_{a\in A} V_a$. Up to isomorphisms of the $\underline{U}_a$, the coverings $\{\underline{i}_a : \underline{U}_a \to \underline{U}\}_{a\in A}$ of $\underline{U}$ correspond exactly to open covers $\{V_a : a \in A\}$ of U. Then $(\mathbf{C^\infty Sch}, \mathcal{J})$ is a *site*.

The *stacks on* $(\mathbf{C^\infty Sch}, \mathcal{J})$ form a 2-category $\mathbf{Sta}_{(\mathbf{C^\infty Sch},\mathcal{J})}$. The site $(\mathbf{C^\infty Sch}, \mathcal{J})$ is *subcanonical*. Thus, if $\underline{X}$ is any C^∞-scheme we have an associated stack on $(\mathbf{C^\infty Sch}, \mathcal{J})$ which we write as $\bar{\underline{X}}$. A C^∞-*stack* is a stack $\mathcal{X}$ on $(\mathbf{C^\infty Sch}, \mathcal{J})$ such that the diagonal 1-morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, and there exists a surjective 1-morphism $\Pi : \bar{\underline{U}} \to \mathcal{X}$ called an *atlas* for some C^∞-scheme $\underline{U}$. C^∞-stacks form a 2-category $\mathbf{C^\infty Sta}$. All 2-morphisms in $\mathbf{C^\infty Sta}$ are invertible, that is, they are 2-isomorphisms.

Remark 4.2. So far as the author knows, [**7**] is the first paper to consider stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$. Note that Behrend and Xu [**1**, Def. 2.15] use the term 'C^∞-stack' to mean something different, a geometric stack over the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ of manifolds without boundary with Grothendieck topology $\mathcal{J}_{\mathbf{Man}}$ given by open covers. These are called 'smooth stacks' by Metzler [**12**].

Write $\mathbf{ManSta}$ for the 2-category of geometric stacks on $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$, as in [**1**, **10**, **12**]. The functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ of Example 3.3 embeds the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ into $(\mathbf{C^\infty Sch}, \mathcal{J})$. Thus, restricting from $(\mathbf{C^\infty Sch}, \mathcal{J})$ to $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ defines a natural truncation 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{ManSta}} : \mathbf{C^\infty Sta} \to \mathbf{ManSta}$.

A C^∞-stack $\mathcal{X}$ encodes all morphisms $F : \underline{U} \to \mathcal{X}$ for C^∞-schemes $\underline{U}$, whereas its image $F_{\mathbf{C^\infty Sta}}^{\mathbf{ManSta}}(\mathcal{X})$ remembers only morphisms $F : U \to \mathcal{X}$ for manifolds U. Thus the truncation functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{ManSta}}$ *loses information*, as it forgets morphisms from C^∞-schemes which are not manifolds. This includes any information about *nonreduced* C^∞-schemes. For our applications in [**8**] this nonreduced information will be essential, so we must consider stacks on $(\mathbf{C^\infty Sch}, \mathcal{J})$ rather than on $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$.

Definition 4.3. A *groupoid object* $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ in $\mathbf{C^\infty Sch}$, or simply *groupoid* in $\mathbf{C^\infty Sch}$, consists of objects $\underline{U}, \underline{V}$ in $\mathbf{C^\infty Sch}$ and morphisms

$\underline{s},\underline{t}:\underline{V}\to\underline{U}$, $\underline{u}:\underline{U}\to\underline{V}$, $\underline{i}:\underline{V}\to\underline{V}$ and $\underline{m}:\underline{V}\times_{\underline{s},\underline{U},\underline{t}}\underline{V}\to\underline{V}$ satisfying the identities

$$\begin{gathered}\underline{s}\circ\underline{u}=\underline{t}\circ\underline{u}=\mathrm{id}_{\underline{U}},\ \ \underline{s}\circ\underline{i}=\underline{t},\ \ \underline{t}\circ\underline{i}=\underline{s},\ \ \underline{s}\circ\underline{m}=\underline{s}\circ\underline{\pi}_2,\ \ \underline{t}\circ\underline{m}=\underline{t}\circ\underline{\pi}_1,\\ \underline{m}\circ(\underline{i}\times\underline{\mathrm{id}}_V)=\underline{u}\circ\underline{s},\ \ \underline{m}\circ(\underline{\mathrm{id}}_V\times\underline{i})=\underline{u}\circ\underline{t},\\ \underline{m}\circ(\underline{m}\times\underline{\mathrm{id}}_V)=\underline{m}\circ(\underline{\mathrm{id}}_V\times\underline{m}):\underline{V}\times_{\underline{U}}\underline{V}\times_{\underline{U}}\underline{V}\longrightarrow\underline{V},\\ \underline{m}\circ(\underline{\mathrm{id}}_V\times\underline{u})=\underline{m}\circ(\underline{u}\times\underline{\mathrm{id}}_V):\underline{V}=\underline{V}\times_{\underline{U}}\underline{U}\longrightarrow\underline{V}.\end{gathered}$$

We write groupoids in $\mathbf{C^\infty Sch}$ as $\underline{V}\rightrightarrows\underline{U}$ for short, to emphasize the morphisms $\underline{s},\underline{t}:\underline{V}\to\underline{U}$. To any such groupoid we can associate a *groupoid stack* $[\underline{V}\rightrightarrows\underline{U}]$, which is a C^∞-stack. Conversely, if $\mathcal{X}$ is a C^∞-stack and $\Pi:\bar{\underline{U}}\to\mathcal{X}$ is an atlas one can construct a groupoid $\underline{V}\rightrightarrows\underline{U}$ in $\mathbf{C^\infty Sch}$, and $\mathcal{X}$ is equivalent (in the 2-category sense) to $[\underline{V}\rightrightarrows\underline{U}]$. Thus, every C^∞-stack is equivalent to a groupoid stack.

Suppose $\underline{U}$ is a C^∞-scheme and G is a finite group which acts on the left on $\underline{U}$ by automorphisms, with action $\underline{\mu}:G\times\underline{U}\to\underline{U}$. Then

$$\big(\underline{U},G\times\underline{U},\underline{\pi}_{U},\underline{\mu},1\times\underline{\mathrm{id}}_{U},(i\circ\underline{\pi}_G)\times\underline{\mu},(m\circ((\underline{\pi}_G\circ\underline{\pi}_1)\times(\underline{\pi}_G\circ\underline{\pi}_2)))\times(\underline{\pi}_U\circ\underline{\pi}_2)\big)\tag{3}$$

is a groupoid object in $\mathbf{C^\infty Sch}$, where $1\in G$ is the identity, $i:G\to G$ is the inverse map, $m:G\times G\to G$ is group multiplication, and in the final morphism $\underline{\pi}_1,\underline{\pi}_2$ are the projections from $(G\times\underline{U})\times_{\underline{\pi}_U,\underline{U},\underline{\mu}}(G\times\underline{U})$ to the first and second factors $G\times\underline{U}$. Write $[\underline{U}/G]$ for the groupoid stack associated to (3). It is a C^∞-stack, which we call a *quotient stack*.

We define some classes of morphisms of C^∞-schemes.

Definition 4.4. Let $\underline{f}=(f,f^\sharp):\underline{X}=(X,\mathcal{O}_X)\to\underline{Y}=(Y,\mathcal{O}_Y)$ be a morphism in $\mathbf{C^\infty Sch}$. Then:

- We call $\underline{f}$ an *open embedding* if $V=f(X)$ is an open subset in Y and $(f,f^\sharp):(X,\mathcal{O}_X)\to(U,\mathcal{O}_Y|_V)$ is an isomorphism.
- We call $\underline{f}$ *étale* if each $x\in X$ has an open neighbourhood U in X such that $V=f(U)$ is open in Y and $(f|_U,f^\sharp|_U):(U,\mathcal{O}_X|_U)\to(V,\mathcal{O}_Y|_V)$ is an isomorphism. That is, $\underline{f}$ is a local isomorphism.
- We call $\underline{f}$ *proper* if $f:X\to Y$ is a proper map of topological spaces, that is, if $S\subseteq Y$ is compact then $f^{-1}(S)\subseteq X$ is compact.
- We call $\underline{f}$ *separated* if $f:X\to Y$ is a separated map of topological spaces, that is, $\Delta_X=\big\{(x,x):x\in X\big\}$ is a closed subset of the topological fibre product $X\times_{f,Y,f}X=\big\{(x,x')\in X\times X:f(x)=f(x')\big\}$.
- We call $\underline{f}$ *universally closed* if whenever $\underline{g}:\underline{W}\to\underline{Y}$ is a morphism then $\pi_W:X\times_{f,Y,g}W\to W$ is a closed map of topological spaces, that is, it takes closed sets to closed sets.

Each one is invariant under base change and local in the target in $(\mathbf{C^\infty Sch},\mathcal{J})$. Thus, they are also defined for representable 1-morphisms of C^∞-stacks.

Definition 4.5. Let $\mathcal{X}$ be a C^∞-stack. We say that $\mathcal{X}$ is *separated* if the diagonal 1-morphism $\Delta_{\mathcal{X}}:\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ is universally closed. If $\mathcal{X}=\bar{\underline{X}}$ for some C^∞-scheme $\underline{X}=(X,\mathcal{O}_X)$ then $\mathcal{X}$ is separated if and only if $\Delta_X:X\to X\times X$ is closed, that is, if and only if X is Hausdorff, so $\underline{X}$ is separated.

Definition 4.6. Let $\mathcal{X}$ be a C^∞-stack. A C^∞*-substack* $\mathcal{Y}$ in $\mathcal{X}$ is a substack of $\mathcal{X}$ which is also a C^∞-stack. It has a natural inclusion 1-morphism $i_{\mathcal{Y}}:\mathcal{Y}\hookrightarrow\mathcal{X}$. We call $\mathcal{Y}$ an *open* C^∞*-substack* of $\mathcal{X}$ if $i_{\mathcal{Y}}$ is a representable open embedding. An *open cover* $\{\mathcal{Y}_a:a\in A\}$ of $\mathcal{X}$ is a family of open C^∞-substacks $\mathcal{Y}_a$ in $\mathcal{X}$ with $\coprod_{a\in A} i_{\mathcal{Y}_a}:\coprod_{a\in A}\mathcal{Y}_a\to\mathcal{X}$ surjective.

Deligne–Mumford stacks in algebraic geometry are locally modelled on quotient stacks $[X/G]$ for X an affine scheme and G a finite group. This motivates:

Definition 4.7. A *Deligne–Mumford* C^∞*-stack* is a C^∞-stack $\mathcal{X}$ which admits an open cover $\{\mathcal{Y}_a:a\in A\}$ with each $\mathcal{Y}_a$ equivalent to a quotient stack $[\underline{U}_a/G_a]$ for $\underline{U}_a$ an affine C^∞-scheme and G_a a finite group. We call $\mathcal{X}$ a *locally fair*, or *locally good*, or *locally finitely presented*, Deligne–Mumford C^∞-stack if it has such an open cover with each $\underline{U}_a$ a fair, good, or finitely presented, affine C^∞-scheme, respectively. Write $\mathbf{DMC^\infty Sta^{lf}}$, $\mathbf{DMC^\infty Sta^{lg}}$, $\mathbf{DMC^\infty Sta^{lfp}}$ and $\mathbf{DMC^\infty Sta}$ for the full 2-subcategories of locally fair, locally good, locally finitely presented, and all, Deligne–Mumford C^∞-stacks in $\mathbf{C^\infty Sta}$.

From [**7**, Th.s 8.5, 8.21, 8.26 & Prop. 8.17] we have:

Theorem 4.8. **(a)** *All fibre products exist in the* 2*-category* $\mathbf{C^\infty Sta}$*.*

(b) $\mathbf{DMC^\infty Sta},\mathbf{DMC^\infty Sta^{lf}},\mathbf{DMC^\infty Sta^{lg}}$ *and* $\mathbf{DMC^\infty Sta^{lfp}}$ *are closed under fibre products in* $\mathbf{C^\infty Sta}$*.*

(c) $\mathbf{DMC^\infty Sta},\mathbf{DMC^\infty Sta^{lf}},\mathbf{DMC^\infty Sta^{lg}}$ *and* $\mathbf{DMC^\infty Sta^{lfp}}$ *are closed under taking open* C^∞*-substacks in* $\mathbf{C^\infty Sta}$*.*

(d) *A* C^∞*-stack* $\mathcal{X}$ *is separated and Deligne–Mumford if and only if it is equivalent to a groupoid stack* $[\underline{V}\rightrightarrows\underline{U}]$ *where* $\underline{U},\underline{V}$ *are separated* C^∞*-schemes,* $\underline{s}:\underline{V}\to\underline{U}$ *is étale, and* $\underline{s}\times\underline{t}:\underline{V}\to\underline{U}\times\underline{U}$ *is universally closed.*

(e) *A* C^∞*-stack* $\mathcal{X}$ *is separated, Deligne–Mumford and locally fair (or locally good, or locally finitely presented) if and only if it is equivalent to some* $[\underline{V}\rightrightarrows\underline{U}]$ *with* $\underline{U},\underline{V}$ *separated, locally fair (or locally good, or locally finitely presented)* C^∞*-schemes,* $\underline{s}:\underline{V}\to\underline{U}$ *étale, and* $\underline{s}\times\underline{t}:\underline{V}\to\underline{U}\times\underline{U}$ *proper.*

A C^∞-stack $\mathcal{X}$ has an *underlying topological space* $\mathcal{X}_{\rm top}$.

Definition 4.9. Let $\mathcal{X}$ be a C^∞-stack. Write $\underline{*}$ for the point $\operatorname{Spec}\mathbb{R}$ in $\mathbf{C^\infty Sch}$, and $\bar{\underline{*}}$ for the associated point in $\mathbf{C^\infty Sta}$. Define $\mathcal{X}_{\rm top}$ to be the set of 2-isomorphism classes $[x]$ of 1-morphisms $x:\bar{\underline{*}}\rightarrow\mathcal{X}$. When $i_{\mathcal{U}}:\mathcal{U}\rightarrow\mathcal{X}$ is an open C^∞-substack in $\mathcal{X}$, write

$$\mathcal{U}_{\mathcal{X},\mathrm{top}}=\bigl\{[u\circ i_{\mathcal{U}}]\in\mathcal{X}_{\mathrm{top}}:u:\bar{\underline{*}}\rightarrow\mathcal{U}\text{ is a 1-morphism}\bigr\}\subseteq\mathcal{X}_{\mathrm{top}}.$$

Define $\mathcal{T}_{\mathcal{X}_{\mathrm{top}}}=\bigl\{\mathcal{U}_{\mathcal{X},\mathrm{top}}:i_{\mathcal{U}}:\mathcal{U}\rightarrow\mathcal{X}$ is an open C^∞-substack in $\mathcal{X}\bigr\}$. Then $(\mathcal{X}_{\mathrm{top}},\mathcal{T}_{\mathcal{X}_{\mathrm{top}}})$ is a topological space, which we call the *underlying topological space* of $\mathcal{X}$, and usually write as $\mathcal{X}_{\mathrm{top}}$.

If $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a 1-morphism of C^∞-stacks then there is a natural continuous map $f_{\mathrm{top}}:\mathcal{X}_{\mathrm{top}}\rightarrow\mathcal{Y}_{\mathrm{top}}$ defined by $f_{\mathrm{top}}([x])=[f\circ x]$. If $f,g:\mathcal{X}\rightarrow\mathcal{Y}$ are 1-morphisms and $\eta:f\Rightarrow g$ is a 2-isomorphism then $f_{\mathrm{top}}=g_{\mathrm{top}}$. Mapping $\mathcal{X}\mapsto\mathcal{X}_{\mathrm{top}}$, $f\mapsto f_{\mathrm{top}}$ and 2-morphisms to identities defines a 2-functor $F_{\mathbf{C^\infty Sta}}^{\mathbf{Top}}:\mathbf{C^\infty Sta}\rightarrow\mathbf{Top}$, where the category of topological spaces $\mathbf{Top}$ is regarded as a 2-category with only identity 2-morphisms.

If $\underline{X}=(X,\mathcal{O}_X)$ is a C^∞-scheme, so that $\bar{\underline{X}}$ is a C^∞-stack, then $\bar{\underline{X}}_{\mathrm{top}}$ is naturally homeomorphic to X, and we will identify $\bar{\underline{X}}_{\mathrm{top}}$ with X. If $\underline{f}=(f,f^\sharp):\underline{X}=(X,\mathcal{O}_X)\rightarrow\underline{Y}=(Y,\mathcal{O}_Y)$ is a morphism of C^∞-schemes, so that $\bar{\underline{f}}:\bar{\underline{X}}\rightarrow\bar{\underline{Y}}$ is a 1-morphism of C^∞-stacks, then $\bar{\underline{f}}_{\mathrm{top}}:\bar{\underline{X}}_{\mathrm{top}}\rightarrow\bar{\underline{Y}}_{\mathrm{top}}$ is $f:X\rightarrow Y$.

We call a Deligne–Mumford C^∞-stack $\mathcal{X}$ *paracompact* if the underlying topological space $\mathcal{X}_{\mathrm{top}}$ is paracompact.

Definition 4.10. Let $\mathcal{X}$ be a C^∞-stack, and $[x]\in\mathcal{X}_{\mathrm{top}}$. Pick a representative x for $[x]$, so that $x:\bar{\underline{*}}\rightarrow\mathcal{X}$ is a 1-morphism. Let G be the group of 2-morphisms $\eta:x\Rightarrow x$. There is a natural C^∞-scheme $\underline{G}=(G,\mathcal{O}_G)$ with $\bar{\underline{G}}\cong\bar{\underline{*}}\times_{x,\mathcal{X},x}\bar{\underline{*}}$, which makes $\underline{G}$ into a C^∞*-group* (a group object in $\mathbf{C^\infty Sch}$, just as a Lie group is a group object in **Man**). With $[x]$ fixed, this C^∞-group $\underline{G}$ is independent of choices up to noncanonical isomorphism; roughly, $\underline{G}$ is canonical up to conjugation in $\underline{G}$. We define the *stabilizer group* (or *isotropy group*, or *orbifold group*) $\mathrm{Iso}([x])$ of $[x]$ to be this C^∞-group $\underline{G}$, regarded as a C^∞-group up to noncanonical isomorphism.

If $f:\mathcal{X}\rightarrow\mathcal{Y}$ is a 1-morphism of C^∞-stacks and $[x]\in\mathcal{X}_{\mathrm{top}}$ with $f_{\mathrm{top}}([x])=[y]\in\mathcal{Y}_{\mathrm{top}}$, for $y=f\circ x$, then we define $f_*:\mathrm{Iso}([x])\rightarrow\mathrm{Iso}([y])$ by $f_*(\eta)=\mathrm{id}_f*\eta$. Then f_* is a group morphism, and extends to a C^∞-group morphism. It is independent of choices of $x\in[x]$, $y\in[y]$ up to conjugation in $\mathrm{Iso}([x]),\mathrm{Iso}([y])$.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack then $\mathrm{Iso}([x])$ is a finite group for all $[x]$ in $\mathcal{X}_{\mathrm{top}}$, which is discrete as a C^∞-group. Here are [**7**, Prop. 8.31 & Th. 8.32].

Proposition 4.11. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack and $[x]\in\mathcal{X}_{\mathrm{top}}$, so that $\mathrm{Iso}([x])\cong H$ for some finite group H. Then there exists an open C^∞-substack $\mathcal{U}$ in $\mathcal{X}$ with $[x]\in\mathcal{U}_{\mathcal{X},\mathrm{top}}\subseteq\mathcal{X}_{\mathrm{top}}$ and an equivalence $\mathcal{U}\simeq[\underline{Y}/H]$, where $\underline{Y}=(Y,\mathcal{O}_Y)$ is an affine C^∞-scheme with an action of H, and $[x]\in\mathcal{U}_{\mathcal{X},\mathrm{top}}\cong Y/H$ corresponds to a fixed point y of H in Y.*

THEOREM 4.12. *Suppose $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with* $\mathrm{Iso}([x]) \cong \{1\}$ *for all* $[x] \in \mathcal{X}_{\mathrm{top}}$*. Then $\mathcal{X}$ is equivalent to $\bar{\underline{X}}$ for some C^∞-scheme $\underline{X}$.*

In conventional algebraic geometry, a stack with all stabilizer groups trivial is (equivalent to) an *algebraic space*, but may not be a scheme, so the category of algebraic spaces is larger than the category of schemes. Here algebraic spaces are spaces which are locally isomorphic to schemes in the étale topology, but not necessarily locally isomorphic to schemes in the Zariski topology.

In contrast, as Theorem 4.12 shows, in C^∞-algebraic geometry there is no difference between C^∞-schemes and C^∞-algebraic spaces. This is because in C^∞-geometry the Zariski topology is already fine enough, as in Remark 3.7(iii), so we gain no extra generality by passing to the étale topology.

4.2. Orbifolds as Deligne–Mumford C^∞-stacks. Example 3.3 defined a functor $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}} : \mathbf{Man} \to \mathbf{C^\infty Sch}$ embedding manifolds as a full subcategory of C^∞-schemes. Similarly, one might expect to define a (2)-functor $F_{\mathbf{Orb}}^{\mathbf{DMC^\infty Sta}} : \mathbf{Orb} \to \mathbf{DMC^\infty Sta}$ embedding the (2-)category of orbifolds as a full (2-)subcategory of Deligne–Mumford C^∞-stacks. In fact, in [**7**, §8.8] we took a slightly different approach: we defined a full 2-subcategory **Orb** in $\mathbf{DMC^\infty Sta}$, and then showed this is equivalent to other definitions of the (2-)category of orbifolds. The reason for this is that there is not one definition of orbifolds, but several, and our new definition of orbifolds as examples of C^∞-stacks may be as useful as some of the other definitions.

Orbifolds (without boundary) are spaces locally modelled on $\mathbb{R}^n/G$ for G a finite group acting linearly on $\mathbb{R}^n$, just as manifolds are spaces locally modelled on $\mathbb{R}^n$. They were introduced by Satake [**16**], who called them V-manifolds. Moerdijk [**13**] defines orbifolds as *proper étale Lie groupoids* in **Man**. Both [**13**, **16**] regard orbifolds as a 1-category (an ordinary category), if the issue arises at all. However, for issues such as fibre products or pullbacks of vector bundles this 1-category structure is badly behaved, and it becomes clear that orbifolds should really be a 2-category, as for stacks in algebraic geometry.

There are two main routes in the literature to defining a 2-category of orbifolds **Orb**. The first, as in Pronk [**15**] and Lerman [**10**, §3.3], is to define a 2-category **Gpoid** of proper étale Lie groupoids, and then to define **Orb** as a (weak) 2-category localization of **Gpoid** at a suitable class of 1-morphisms. The second, as in Behrend and Xu [**1**, §2], Lerman [**10**, §4] and Metzler [**12**, §3.5], is to define orbifolds as a class of Deligne–Mumford stacks on the site $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$ of manifolds with Grothendieck topology $\mathcal{J}_{\mathbf{Man}}$ coming from open covers. Our approach is similar to the second route, but defines orbifolds as a class of C^∞-stacks, that is, as stacks on the site $(\mathbf{C^\infty Sch}, \mathcal{J})$ rather than on $(\mathbf{Man}, \mathcal{J}_{\mathbf{Man}})$.

Definition 4.13. A C^∞-stack $\mathcal{X}$ is called an *orbifold* if it is equivalent to a groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$ for some groupoid $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ in $\mathbf{C^\infty Sch}$ which is the image under $F_{\mathbf{Man}}^{\mathbf{C^\infty Sch}}$ of a groupoid (U, V, s, t, u, i, m) in $\mathbf{Man}$, where $s : V \to U$ is an étale smooth map, and $s \times t : V \to U \times U$ is a proper smooth map. That is, $\mathcal{X}$ is the C^∞-stack associated to a *proper étale Lie groupoid* in $\mathbf{Man}$. As a C^∞-stack, every orbifold $\mathcal{X}$ is a separable, paracompact, locally finitely presented Deligne–Mumford C^∞-stack. Write $\mathbf{Orb}$ for the full 2-subcategory of orbifolds in $\mathbf{C^\infty Sta}$.

Here is [**7**, Th. 8.39 & Cor. 8.40]. Since equivalent (2-)categories are considered to be 'the same', the moral of Theorem 4.14 is that our orbifolds are essentially the same objects as those considered by other recent authors.

Theorem 4.14. *The* 2*-category* $\mathbf{Orb}$ *of orbifolds defined above is equivalent to the* 2*-categories of orbifolds considered as stacks on* $\mathbf{Man}$ *defined in Metzler* [**12**, §3.4] *and Lerman* [**10**, §4], *and also equivalent as a weak* 2*-category to the weak* 2*-categories of orbifolds regarded as proper étale Lie groupoids defined in Pronk* [**15**] *and Lerman* [**10**, §3.3].

Furthermore, the homotopy 1*-category* $\mathbf{Orb^{ho}}$ *of* $\mathbf{Orb}$ *(that is, the category whose objects are objects in* $\mathbf{Orb}$, *and whose morphisms are* 2*-isomorphism classes of* 1*-morphisms in* $\mathbf{Orb}$*) is equivalent to the* 1*-category of orbifolds regarded as proper étale Lie groupoids defined in Moerdijk* [**13**]. *Transverse fibre products in* $\mathbf{Orb}$ *agree with the corresponding fibre products in* $\mathbf{C^\infty Sta}$.

4.3. Quasicoherent and coherent sheaves on C^∞-stacks. In [**7**, §9] the author studied sheaves on Deligne–Mumford C^∞-stacks.

Definition 4.15. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define a category $\mathcal{C}_\mathcal{X}$ to have objects pairs $(\underline{U}, u)$ where $\underline{U}$ is a C^∞-scheme and $u : \bar{\underline{U}} \to \mathcal{X}$ is an étale 1-morphism, and morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ where $\underline{f} : \underline{U} \to \underline{V}$ is an étale morphism of C^∞-schemes, and $\eta : u \Rightarrow v \circ \bar{\underline{f}}$ is a 2-isomorphism. If $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ and $(\underline{g}, \zeta) : (\underline{V}, v) \to (\underline{W}, w)$ are morphisms in $\mathcal{C}_\mathcal{X}$ then we define the composition $(\underline{g}, \zeta) \circ (\underline{f}, \eta)$ to be $(\underline{g} \circ \underline{f}, \theta) : (\underline{U}, u) \to (\underline{W}, w)$, where θ is the composition of 2-morphisms across the diagram:

$$
\begin{tikzcd}
\bar{\underline{U}} \arrow[dr, "\bar{\underline{f}}"] \arrow[drr, bend left, "u"] \arrow[dd, "\overline{\underline{g}\circ\underline{f}}"'] & & \\
\arrow[r, phantom, "\Leftarrow \mathrm{id}"] & \bar{\underline{V}} \arrow[r, "v"] \arrow[r, phantom, shift left=3, "\Swarrow\eta"] \arrow[r, phantom, shift right=3, "\Swarrow\zeta"'] \arrow[dl, "\bar{\underline{g}}"] & \mathcal{X}. \\
\bar{\underline{W}} \arrow[urr, bend right, "w"'] & &
\end{tikzcd}
$$

Define an $\mathcal{O}_\mathcal{X}$*-module* $\mathcal{E}$ to assign an $\mathcal{O}_U$-module $\mathcal{E}(\underline{U}, u)$ on $\underline{U} = (U, \mathcal{O}_U)$ for all objects $(\underline{U}, u)$ in $\mathcal{C}_\mathcal{X}$, and an isomorphism $\mathcal{E}_{(\underline{f},\eta)} : \underline{f}^*(\mathcal{E}(\underline{V}, v)) \to \mathcal{E}(\underline{U}, u)$ for all morphisms $(\underline{f}, \eta) : (\underline{U}, u) \to (\underline{V}, v)$ in $\mathcal{C}_\mathcal{X}$, such that for all

$(\underline{f},\eta),(\underline{g},\zeta),(\underline{g}\circ\underline{f},\theta)$ as above the following diagram of isomorphisms of $\mathcal{O}_U$-modules commutes:

$$
\begin{tikzcd}
(\underline{g}\circ\underline{f})^*\big(\mathcal{E}(\underline{W},w)\big) \arrow[rr, "\mathcal{E}_{(\underline{g}\circ\underline{f},\theta)}"] \arrow[dr, "I_{\underline{f},\underline{g}}(\mathcal{E}(\underline{W},w))"'] & & \mathcal{E}(\underline{U},u), \\
& \underline{f}^*\big(\underline{g}^*(\mathcal{E}(\underline{W},w))\big) \arrow[r, "\underline{f}^*(\mathcal{E}_{(\underline{g},\zeta)})"] & \underline{f}^*\big(\mathcal{E}(\underline{V},v)\big) \arrow[u, "\mathcal{E}_{(\underline{f},\eta)}"']
\end{tikzcd} \tag{4}
$$

for $I_{\underline{f},\underline{g}}(\mathcal{E})$ as in Definition 3.9.

A *morphism of $\mathcal{O}_{\mathcal{X}}$-modules* $\phi:\mathcal{E}\to\mathcal{F}$ assigns a morphism of $\mathcal{O}_U$-modules $\phi(\underline{U},u):\mathcal{E}(\underline{U},u)\to\mathcal{F}(\underline{U},u)$ for each object $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$, such that for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$ the following commutes:

$$
\begin{tikzcd}
\underline{f}^*\big(\mathcal{E}(\underline{V},v)\big) \arrow[r, "\mathcal{E}_{(\underline{f},\eta)}"'] \arrow[d, "\underline{f}^*(\phi(\underline{V},v))"'] & \mathcal{E}(\underline{U},u) \arrow[d, "\phi(\underline{U},u)"] \\
\underline{f}^*\big(\mathcal{F}(\underline{V},v)\big) \arrow[r, "\mathcal{F}_{(\underline{f},\eta)}"] & \mathcal{F}(\underline{U},u).
\end{tikzcd}
$$

We call $\mathcal{E}$ *quasicoherent*, or *coherent*, or a *vector bundle of rank n*, if $\mathcal{E}(\underline{U},u)$ is quasicoherent, or coherent, or a vector bundle of rank n, respectively, for all $(\underline{U},u)\in\mathcal{C}_{\mathcal{X}}$. Write $\mathcal{O}_{\mathcal{X}}$-mod for the category of $\mathcal{O}_{\mathcal{X}}$-modules, and $\mathrm{qcoh}(\mathcal{X})$, $\mathrm{coh}(\mathcal{X})$, $\mathrm{vect}(\mathcal{X})$ for the full subcategories of quasicoherent sheaves, coherent sheaves, and vector bundles, respectively.

Here are [**7**, Prop. 9.3 & Ex. 9.4].

Proposition 4.16. *Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then* $\mathcal{O}_{\mathcal{X}}$-mod *is an abelian category, and* $\mathrm{qcoh}(\mathcal{X})$ *is closed under kernels, cokernels and extensions in* $\mathcal{O}_{\mathcal{X}}$-mod, *so it is also an abelian category. Also* $\mathrm{coh}(\mathcal{X})$ *is closed under cokernels and extensions in* $\mathcal{O}_{\mathcal{X}}$-mod, *but it may not be closed under kernels in* $\mathcal{O}_{\mathcal{X}}$-mod, *so may not be abelian. If $\mathcal{X}$ is locally fair then* $\mathrm{qcoh}(\mathcal{X})=\mathcal{O}_{\mathcal{X}}$-mod.

Example 4.17. Let $\underline{X}$ be a C^∞-scheme. Then $\mathcal{X}=\bar{\underline{X}}$ is a Deligne–Mumford C^∞-stack. We will define an *inclusion functor* $\mathcal{I}_{\underline{X}}:\mathcal{O}_X\text{-mod}\to\mathcal{O}_{\mathcal{X}}\text{-mod}$. Let $\mathcal{E}$ be an object in $\mathcal{O}_X$-mod. If $(\underline{U},u)$ is an object in $\mathcal{C}_{\mathcal{X}}$ then $u:\bar{\underline{U}}\to\mathcal{X}=\bar{\underline{X}}$ is 1-isomorphic to $\bar{\underline{u}}:\bar{\underline{U}}\to\bar{\underline{X}}$ for some unique morphism $\underline{u}:\underline{U}\to\underline{X}$. Define $\mathcal{E}'(\underline{U},u)=\underline{u}^*(\mathcal{E})$. If $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ is a morphism in $\mathcal{C}_{\mathcal{X}}$ and $\underline{u},\underline{v}$ are associated to u,v as above, so that $\underline{u}=\underline{v}\circ\underline{f}$, then define

$$\mathcal{E}'_{(\underline{f},\eta)}=I_{\underline{f},\underline{v}}(\mathcal{E})^{-1}:\underline{f}^*(\mathcal{E}'(\underline{V},v))=\underline{f}^*\big(\underline{v}^*(\mathcal{E})\big)\longrightarrow(\underline{v}\circ\underline{f})^*(\mathcal{E})=\mathcal{E}'(\underline{U},u).$$

Then (4) commutes for all $(\underline{f},\eta),(\underline{g},\zeta)$, so $\mathcal{E}'$ is an $\mathcal{O}_{\mathcal{X}}$-module.

If $\phi:\mathcal{E}\to\mathcal{F}$ is a morphism of $\mathcal{O}_X$-modules then we define a morphism $\phi':\mathcal{E}'\to\mathcal{F}'$ in $\mathcal{O}_{\mathcal{X}}$-mod by $\phi'(\underline{U},u)=\underline{u}^*(\phi)$ for $\underline{u}$ associated to u as above. Then defining $\mathcal{I}_{\underline{X}}:\mathcal{E}\mapsto\mathcal{E}'$, $\mathcal{I}_{\underline{X}}:\phi\mapsto\phi'$ gives a functor $\mathcal{O}_X\text{-mod}\to\mathcal{O}_{\mathcal{X}}\text{-mod}$, which induces equivalences between the categories $\mathcal{O}_X\text{-mod},\mathrm{qcoh}(\underline{X}),\mathrm{coh}(\underline{X})$ defined in §3.2 and $\mathcal{O}_{\mathcal{X}}\text{-mod},\mathrm{qcoh}(\mathcal{X}),\mathrm{coh}(\mathcal{X})$ above.

In [**7**, §9.2] we explain how to describe sheaves on a Deligne–Mumford C^∞-stack $\mathcal{X}$ in terms of sheaves on $\underline{U}$ for an étale atlas $\Pi : \bar{\underline{U}} \to \mathcal{X}$ for $\mathcal{X}$. Here are [**7**, Def. 9.5 & Th. 9.6].

Definition 4.18. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Then $\mathcal{X}$ admits an étale atlas $\Pi : \bar{\underline{U}} \to \mathcal{X}$, and as in Definition 4.3 from Π we can construct a groupoid $(\underline{U}, \underline{V}, \underline{s}, \underline{t}, \underline{u}, \underline{i}, \underline{m})$ in $\mathbf{C^\infty Sch}$, with $\underline{s}, \underline{t} : \underline{V} \to \underline{U}$ étale, such that $\mathcal{X}$ is equivalent to the groupoid stack $[\underline{V} \rightrightarrows \underline{U}]$. Define a $(\underline{V} \rightrightarrows \underline{U})$-*module* to be a pair (E, Φ) where E is an $\mathcal{O}_U$-module and $\Phi : \underline{s}^*(E) \to \underline{t}^*(E)$ is an isomorphism of $\mathcal{O}_V$-modules, such that

$$I_{\underline{m},\underline{t}}(E)^{-1} \circ \underline{m}^*(\Phi) \circ I_{\underline{m},\underline{s}}(E) = \bigl(I_{\underline{\pi}_1,\underline{t}}(E)^{-1} \circ \underline{\pi}_1^*(\Phi) \circ I_{\underline{\pi}_1,\underline{s}}(E)\bigr) \circ \bigl(I_{\underline{\pi}_2,\underline{t}}(E)^{-1} \circ \underline{\pi}_2^*(\Phi) \circ I_{\underline{\pi}_2,\underline{s}}(E)\bigr) \tag{5}$$

in morphisms of $\mathcal{O}_W$-modules $(\underline{s} \circ \underline{m})^*(E) \to (\underline{t} \circ \underline{m})^*(E)$, where $\underline{W} = \underline{V} \times_{\underline{s},\underline{U},\underline{t}} \underline{V}$ and $\underline{\pi}_1, \underline{\pi}_2 : \underline{W} \to \underline{V}$ are the projections. Define a *morphism of* $(\underline{V} \rightrightarrows \underline{U})$-*modules* $\phi : (E, \Phi) \to (F, \Psi)$ to be a morphism of $\mathcal{O}_U$-modules $\phi : E \to F$ such that $\Psi \circ \underline{s}^*(\phi) = \underline{t}^*(\phi) \circ \Phi : \underline{s}^*(E) \to \underline{t}^*(F)$. Then $(\underline{V} \rightrightarrows \underline{U})$-modules form an *abelian category* $(\underline{V} \rightrightarrows \underline{U})$-mod. Write $\mathrm{qcoh}(\underline{V} \rightrightarrows \underline{U})$ and $\mathrm{coh}(\underline{V} \rightrightarrows \underline{U})$ for the full subcategories of (E, Φ) in $(\underline{V} \rightrightarrows \underline{U})$-mod with E quasicoherent, or coherent, respectively. Then $\mathrm{qcoh}(\underline{V} \rightrightarrows \underline{U})$ is abelian. Define a functor $F_\Pi : \mathcal{O}_{\mathcal{X}}\text{-mod} \to (\underline{V} \rightrightarrows \underline{U})\text{-mod}$ by $F_\Pi : \mathcal{E} \mapsto \bigl(\mathcal{E}(\underline{U}, \Pi), \mathcal{E}_{(\underline{t},\eta)}^{-1} \circ \mathcal{E}_{(\underline{s},\mathrm{id}_{\Pi \circ \underline{s}})}\bigr)$ and $F_\Pi : \phi \mapsto \phi(\underline{U}, \Pi)$. As in [**7**, §9.2], $F_\Pi(\mathcal{E})$ does satisfy (5) and so lies in $(\underline{V} \rightrightarrows \underline{U})$-mod, and it also maps $\mathrm{qcoh}, \mathrm{coh}(\mathcal{X})$ to $\mathrm{qcoh}, \mathrm{coh}(\underline{V} \rightrightarrows \underline{U})$.

Theorem 4.19. *The functor F_Π above induces equivalences between $\mathcal{O}_{\mathcal{X}}\text{-mod}, \mathrm{qcoh}(\mathcal{X}), \mathrm{coh}(\mathcal{X})$ and $(\underline{V} \rightrightarrows \underline{U})\text{-mod}, \mathrm{qcoh}(\underline{V} \rightrightarrows \underline{U}), \mathrm{coh}(\underline{V} \rightrightarrows \underline{U})$, respectively.*

For example, if $\mathcal{X} = [\underline{Y}/G]$ for $\underline{Y}$ a C^∞-scheme acted on by a finite group G, then Theorem 4.19 shows that $\mathrm{qcoh}(\mathcal{X})$ is equivalent to the abelian category $\mathrm{qcoh}^G(\underline{Y})$ of G-equivariant quasicoherent sheaves on $\underline{Y}$.

In §3.2, for a morphism of C^∞-schemes $\underline{f} : \underline{X} \to \underline{Y}$ we defined a right exact *pullback functor* $\underline{f}^* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$. Pullbacks may not be strictly functorial in $\underline{f}$, that is, we do not have $\underline{f}^*(\underline{g}^*(\mathcal{E})) = (\underline{g} \circ \underline{f})^*(\mathcal{E})$ for all $\underline{f} : \underline{X} \to \underline{Y}$, $\underline{g} : \underline{Y} \to \underline{Z}$ and $\mathcal{E} \in \mathcal{O}_Z$-mod, but instead we have canonical isomorphisms $I_{\underline{f},\underline{g}}(\mathcal{E}) : (\underline{g} \circ \underline{f})^*(\mathcal{E}) \to \underline{f}^*(\underline{g}^*(\mathcal{E}))$. We now generalize this to sheaves on Deligne–Mumford C^∞-stacks. We must interpret pullback for 2-morphisms as well as 1-morphisms.

Definition 4.20. Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks, and $\mathcal{F}$ be an $\mathcal{O}_{\mathcal{Y}}$-module. A *pullback* of $\mathcal{F}$ to $\mathcal{X}$ is an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, together with the following data: if $\underline{U}, \underline{V}$ are C^∞-schemes and $u : \bar{\underline{U}} \to \mathcal{X}$ and $v : \bar{\underline{V}} \to \mathcal{Y}$ are étale 1-morphisms, then there is a C^∞-scheme $\underline{W}$ and morphisms $\pi_{\underline{U}} : \underline{W} \to \underline{U}$, $\pi_{\underline{V}} : \underline{W} \to \underline{V}$ giving a 2-Cartesian diagram:

$$\begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Uparrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "f\circ u"] & \mathcal{Y}. \end{tikzcd} \tag{6}$$

Then an isomorphism $i(\mathcal{F},f,u,v,\zeta) : \pi_{\underline{U}}^*\bigl(\mathcal{E}(\underline{U},u)\bigr) \to \pi_{\underline{V}}^*\bigl(\mathcal{F}(\underline{V},v)\bigr)$ of $\mathcal{O}_W$-modules should be given, which is functorial in $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(\underline{V},v)$ in $\mathcal{C}_{\mathcal{Y}}$ and the 2-isomorphism ζ in (6). We usually write pullbacks $\mathcal{E}$ as $f^*(\mathcal{F})$. By [**7**, Prop. 9.9], pullbacks $f^*(\mathcal{F})$ exist, and are unique up to unique isomorphism. Using the Axiom of Choice, we choose a pullback $f^*(\mathcal{F})$ for all such $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{F}$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism, and $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism in $\mathcal{O}_{\mathcal{Y}}$-mod. Then $f^*(\mathcal{E}), f^*(\mathcal{F}) \in \mathcal{O}_{\mathcal{X}}$-mod. Define the *pullback morphism* $f^*(\phi) : f^*(\mathcal{E}) \to f^*(\mathcal{F})$ to be the unique morphism in $\mathcal{O}_{\mathcal{X}}$-mod such that whenever $u : \bar{\underline{U}} \to \mathcal{X}$, $v : \bar{\underline{V}} \to \mathcal{Y}$, $\underline{W}, \pi_{\underline{U}}, \pi_{\underline{V}}$ are as above, the following diagram of morphisms of $\mathcal{O}_W$-modules commutes:

$$\begin{tikzcd} \pi_{\underline{U}}^*\bigl(f^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[r, "i(\mathcal{E},f,u,v,\zeta)"'] \arrow[d, "\pi_{\underline{U}}^*(f^*(\phi)(\underline{U},u))"'] & \pi_{\underline{V}}^*\bigl(\mathcal{E}(\underline{V},v)\bigr) \arrow[d, "\pi_{\underline{V}}{}^*(\phi(\underline{V},v))"] \\ \pi_{\underline{U}}^*\bigl(f^*(\mathcal{F})(\underline{U},u)\bigr) \arrow[r, "i(\mathcal{F},f,u,v,\zeta)"] & \pi_{\underline{V}}^*\bigl(\mathcal{F}(\underline{V},v)\bigr). \end{tikzcd}$$

This defines a functor $f^* : \mathcal{O}_{\mathcal{Y}}\text{-mod} \to \mathcal{O}_{\mathcal{X}}\text{-mod}$, which also maps $\mathrm{qcoh}(\mathcal{Y}) \to \mathrm{qcoh}(\mathcal{X})$ and $\mathrm{coh}(\mathcal{Y}) \to \mathrm{coh}(\mathcal{X})$. It is right exact by [**7**, Prop. 9.12].

Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, and $\mathcal{E} \in \mathcal{O}_{\mathcal{Z}}$-mod. Then $(g\circ f)^*(\mathcal{E})$ and $f^*(g^*(\mathcal{E}))$ both lie in $\mathcal{O}_{\mathcal{X}}$-mod. One can show that $f^*(g^*(\mathcal{E}))$ is a possible pullback of $\mathcal{E}$ by $g \circ f$. Thus as in Definition 3.9, we have a canonical isomorphism $I_{f,g}(\mathcal{E}) : (g \circ f)^*(\mathcal{E}) \to f^*(g^*(\mathcal{E}))$. This defines a natural isomorphism of functors $I_{f,g} : (g \circ f)^* \Rightarrow f^* \circ g^*$.

Let $f, g : \mathcal{X} \to \mathcal{Y}$ be 1-morphisms of Deligne–Mumford C^∞-stacks, $\eta : f \Rightarrow g$ a 2-morphism, and $\mathcal{E} \in \mathcal{O}_{\mathcal{Y}}$-mod. Then we have $\mathcal{O}_{\mathcal{X}}$-modules $f^*(\mathcal{E})$, $g^*(\mathcal{E})$. Define $\eta^*(\mathcal{E}) : f^*(\mathcal{E}) \to g^*(\mathcal{E})$ to be the unique isomorphism such that whenever $\underline{U}, \underline{V}, \underline{W}, u, v, \pi_{\underline{U}}, \pi_{\underline{V}}$ are as above, so that we have 2-Cartesian diagrams

$$\begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\odot(\eta*\mathrm{id}_{u\circ\bar{\pi}_{\underline{U}}})\Uparrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "f\circ u"] & \mathcal{Y}, \end{tikzcd} \qquad \begin{tikzcd} \bar{\underline{W}} \arrow[r, "\bar{\pi}_{\underline{V}}"'] \arrow[d, "\bar{\pi}_{\underline{U}}"'] \arrow[dr, phantom, "\zeta\Uparrow"] & \bar{\underline{V}} \arrow[d, "v"] \\ \bar{\underline{U}} \arrow[r, "g\circ u"] & \mathcal{Y}, \end{tikzcd}$$

as in (6), where in $\zeta\odot(\eta*\mathrm{id}_{u\circ\underline{\pi}_U})$ '$*$' is horizontal and '$\odot$' vertical composition of 2-morphisms, then we have commuting isomorphisms of $\mathcal{O}_W$-modules:

$$\begin{tikzcd}
\underline{\pi}_{\underline{U}}^*\bigl(f^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[d, "\underline{\pi}_{\underline{U}}^*((\eta^*(\mathcal{E}))(\underline{U},u))"'] \arrow[dr, "i(\mathcal{E},f,u,v,\zeta\odot(\eta*\mathrm{id}_{u\circ\underline{\pi}_U}))"] & \\
 & \underline{\pi}_{\underline{V}}^*\bigl(\mathcal{E}(\underline{V},v)\bigr). \\
\underline{\pi}_{\underline{U}}^*\bigl(g^*(\mathcal{E})(\underline{U},u)\bigr) \arrow[ur, "i(\mathcal{E},g,u,v,\zeta)"'] &
\end{tikzcd}$$

This defines a natural isomorphism $\eta^*:f^*\Rightarrow g^*$.

If $\mathcal{X}$ is a Deligne–Mumford C^∞-stack with identity 1-morphism $\mathrm{id}_\mathcal{X}:\mathcal{X}\to\mathcal{X}$ then for each $\mathcal{E}\in\mathcal{O}_\mathcal{X}$-mod, $\mathcal{E}$ is a possible pullback $\mathrm{id}_\mathcal{X}^*(\mathcal{E})$, so we have a canonical isomorphism $\delta_\mathcal{X}(\mathcal{E}):\mathrm{id}_\mathcal{X}^*(\mathcal{E})\to\mathcal{E}$. These define a natural isomorphism $\delta_\mathcal{X}:\mathrm{id}_\mathcal{X}^*\Rightarrow\mathrm{id}_{\mathcal{O}_\mathcal{X}\text{-mod}}$.

Here is [**7**, Th. 9.11]:

THEOREM 4.21. *Mapping $\mathcal{X}$ to $\mathcal{O}_\mathcal{X}$-mod for objects $\mathcal{X}$ in* $\mathbf{DMC^\infty Sta}$, *and mapping 1-morphisms $f:\mathcal{X}\to\mathcal{Y}$ to $f^*:\mathcal{O}_\mathcal{Y}$-mod $\to\mathcal{O}_\mathcal{X}$-mod, and mapping 2-morphisms $\eta:f\Rightarrow g$ to $\eta^*:f^*\Rightarrow g^*$ for 1-morphisms $f,g:\mathcal{X}\to\mathcal{Y}$, and the natural isomorphisms $I_{f,g}:(g\circ f)^*\Rightarrow f^*\circ g^*$ for all 1-morphisms $f:\mathcal{X}\to\mathcal{Y}$ and $g:\mathcal{Y}\to\mathcal{Z}$ in* $\mathbf{DMC^\infty Sta}$, *and $\delta_\mathcal{X}$ for all $\mathcal{X}\in$* $\mathbf{DMC^\infty Sta}$, *together make up a* ***pseudofunctor*** $(\mathbf{DMC^\infty Sta})^{\mathrm{op}}\to\mathbf{AbCat}$, *where* **AbCat** *is the 2-category of abelian categories. That is, they satisfy the conditions:*

(a) *If $f:\mathcal{W}\to\mathcal{X}$, $g:\mathcal{X}\to\mathcal{Y}$, $h:\mathcal{Y}\to\mathcal{Z}$ are 1-morphisms in* $\mathbf{DMC^\infty Sta}$ *and $\mathcal{E}\in\mathcal{O}_\mathcal{Z}$-mod then the following diagram commutes in $\mathcal{O}_\mathcal{X}$-mod:*

$$\begin{tikzcd}
(h\circ g\circ f)^*(\mathcal{E}) \arrow[r, "I_{f,h\circ g}(\mathcal{E})"'] \arrow[d, "I_{g\circ f,h}(\mathcal{E})"'] & f^*\bigl((h\circ g)^*(\mathcal{E})\bigr) \arrow[d, "f^*(I_{g,h}(\mathcal{E}))"] \\
(g\circ f)^*\bigl(h^*(\mathcal{E})\bigr) \arrow[r, "I_{f,g}(h^*(\mathcal{E}))"] & f^*\bigl(g^*(h^*(\mathcal{E}))\bigr).
\end{tikzcd}$$

(b) *If $f:\mathcal{X}\to\mathcal{Y}$ is a 1-morphism in* $\mathbf{DMC^\infty Sta}$ *and $\mathcal{E}\in\mathcal{O}_\mathcal{Y}$-mod then the following pairs of morphisms in $\mathcal{O}_\mathcal{X}$-mod are inverse:*

$$\begin{tikzcd}
{\begin{array}{c}f^*(\mathcal{E})=\\(f\circ\mathrm{id}_\mathcal{X})^*(\mathcal{E})\end{array}} \arrow[r, shift left, "I_{\mathrm{id}_\mathcal{X},f}(\mathcal{E})"] & \mathrm{id}_\mathcal{X}^*(f^*(\mathcal{E})), \arrow[l, shift left, "\delta_\mathcal{X}(f^*(\mathcal{E}))"] & {\begin{array}{c}f^*(\mathcal{E})=\\(\mathrm{id}_\mathcal{Y}\circ f)^*(\mathcal{E})\end{array}} \arrow[r, shift left, "I_{f,\mathrm{id}_\mathcal{Y}}(\mathcal{E})"] & f^*(\mathrm{id}_\mathcal{Y}^*(\mathcal{E})). \arrow[l, shift left, "f^*(\delta_\mathcal{Y}(\mathcal{E}))"]
\end{tikzcd}$$

Also $(\mathrm{id}_f)^*(\mathrm{id}_\mathcal{E})=\mathrm{id}_{f^*(\mathcal{E})}:f^*(\mathcal{E})\to f^*(\mathcal{E})$.

(c) *If $f,g,h:\mathcal{X}\to\mathcal{Y}$ are 1-morphisms and $\eta:f\Rightarrow g$, $\zeta:g\Rightarrow h$ are 2-morphisms in* $\mathbf{DMC^\infty Sta}$, *so that $\zeta\odot\eta:f\Rightarrow h$ is the vertical composition, and $\mathcal{E}\in\mathcal{O}_\mathcal{Y}$-mod, then*

$$\zeta^*(\mathcal{F})\circ\eta^*(\mathcal{E})=(\zeta\odot\eta)^*(\mathcal{E}):f^*(\mathcal{E})\to h^*(\mathcal{E})\quad \textit{in } \mathcal{O}_\mathcal{X}\text{-mod}.$$

(d) *If $f,\tilde f:\mathcal{X}\to\mathcal{Y}$, $g,\tilde g:\mathcal{Y}\to\mathcal{Z}$ are* 1*-morphisms and $\eta:f\Rightarrow f'$, $\zeta:g\Rightarrow g'$* 2*-morphisms in* $\mathbf{DMC^\infty Sta}$, *so that $\zeta*\eta:g\circ f\Rightarrow\tilde g\circ\tilde f$ is the horizontal composition, and $\mathcal{E}\in\mathcal{O}_{\mathcal{Z}}$*-mod, *then the following commutes in $\mathcal{O}_{\mathcal{X}}$*-mod*:*

$$\begin{tikzcd}
(g\circ f)^*(\mathcal{E}) \arrow[rr, "(\zeta*\eta)^*(\mathcal{E})"'] \arrow[d, "I_{f,g}(\mathcal{E})"'] & & (\tilde g\circ\tilde f)^*(\mathcal{E}) \arrow[d, "I_{\tilde f,\tilde g}(\mathcal{E})"] \\
f^*(g^*(\mathcal{E})) \arrow[r, "f^*(\zeta^*(\mathcal{E}))"] & f^*(\tilde g^*(\mathcal{E})) \arrow[r, "\eta^*(\tilde g^*(\mathcal{E}))"] & \tilde f^*(\tilde g^*(\mathcal{E})).
\end{tikzcd}$$

Definition 4.22. Let $\mathcal{X}$ be a Deligne–Mumford C^∞-stack. Define an $\mathcal{O}_{\mathcal{X}}$-module $T^*\mathcal{X}$ called the *cotangent sheaf* of $\mathcal{X}$ by $(T^*\mathcal{X})(\underline{U},u)=T^*\underline{U}$ for all objects $(\underline{U},u)$ in $\mathcal{C}_{\mathcal{X}}$ and $(T^*\mathcal{X})_{(\underline{f},\eta)}=\Omega_{\underline{f}}:\underline{f}^*(T^*\underline{V})\to T^*\underline{U}$ for all morphisms $(\underline{f},\eta):(\underline{U},u)\to(\underline{V},v)$ in $\mathcal{C}_{\mathcal{X}}$, where $T^*\underline{U}$ and $\Omega_{\underline{f}}$ are as in §3.2.

Let $f:\mathcal{X}\to\mathcal{Y}$ be a 1-morphism of Deligne–Mumford C^∞-stacks. Then $f^*(T^*\mathcal{Y}),T^*\mathcal{X}$ are $\mathcal{O}_{\mathcal{X}}$-modules. Define $\Omega_f:f^*(T^*\mathcal{Y})\to T^*\mathcal{X}$ to be the unique morphism characterized as follows. Let $u:\bar{\underline{U}}\to\mathcal{X}$, $v:\bar{\underline{V}}\to\mathcal{Y}$, $\underline{W}$, $\underline{\pi}_{\underline{U}}$, $\underline{\pi}_{\underline{V}}$ be as in Definition 4.20, with (6) Cartesian. Then the following diagram of morphisms of $\mathcal{O}_W$-modules commutes:

$$\begin{tikzcd}
\underline{\pi}_{\underline{U}}^*\big(f^*(T^*\mathcal{Y})(\underline{U},u)\big) \arrow[r, "i(T^*\mathcal{Y},f,u,v,\zeta)"'] \arrow[d, "\underline{\pi}_{\underline{U}}^*(\Omega_f(\underline{U},u))"'] & \underline{\pi}_{\underline{V}}^*\big((T^*\mathcal{Y})(\underline{V},v)\big) \arrow[r, equal] & \underline{\pi}_{\underline{V}}^*(T^*\underline{V}) \arrow[d, "\Omega_{\underline{\pi}_{\underline{V}}}"] \\
\underline{\pi}_{\underline{U}}^*\big((T^*\mathcal{X})(\underline{U},u)\big) \arrow[r, "(T^*\mathcal{X})_{(\underline{\pi}_{\underline{U}},\mathrm{id}_{u\circ\underline{\pi}_{\underline{U}}})}"] & (T^*\mathcal{X})(\underline{W},u\circ\underline{\pi}_{\underline{U}}) \arrow[r, equal] & T^*\underline{W}.
\end{tikzcd}$$

If $\Pi:\bar{\underline{U}}\to\mathcal{X}$, $(\underline{U},\underline{V},\underline{s},\underline{t},\underline{u},\underline{i},\underline{m})$ and the functor $F_\Pi:\mathcal{O}_{\mathcal{X}}\text{-mod}\to(\underline{V}\rightrightarrows\underline{U})\text{-mod}$ are as in Definition 4.18 then by definition $F_\Pi(T^*\mathcal{X})=(T^*\underline{U},\Omega_{\underline{t}}{}^{-1}\circ\Omega_{\underline{s}})$, and so we write $T^*(\underline{V}\rightrightarrows\underline{U})=(T^*\underline{U},\Omega_{\underline{t}}{}^{-1}\circ\Omega_{\underline{s}})$ in $(\underline{V}\rightrightarrows\underline{U})$-mod.

Here [**7**, Prop. 9.14 & Th. 9.15] is the analogue of Theorem 3.15.

Theorem 4.23. **(a)** *Suppose $\mathcal{X}$ is an n-orbifold. Then $T^*\mathcal{X}$ is a rank n vector bundle on $\mathcal{X}$.*

(b) *Let $\mathcal{X}$ be a locally good Deligne–Mumford C^∞-stack. Then $T^*\mathcal{X}$ is coherent.*

(c) *Let $f:\mathcal{X}\to\mathcal{Y}$ and $g:\mathcal{Y}\to\mathcal{Z}$ be* 1*-morphisms of Deligne–Mumford C^∞-stacks. Then*
$$\Omega_{g\circ f}=\Omega_f\circ f^*(\Omega_g)\circ I_{f,g}(T^*\mathcal{Z})$$
as morphisms $(g\circ f)^*(T^*\mathcal{Z})\to T^*\mathcal{X}$ in $\mathcal{O}_{\mathcal{X}}$-mod.

(d) *Let $f,g:\mathcal{X}\to\mathcal{Y}$ be* 1*-morphisms of Deligne–Mumford C^∞-stacks and $\eta:f\Rightarrow g$ a* 2*-morphism. Then $\Omega_f=\Omega_g\circ\eta^*(T^*\mathcal{Y}):f^*(T^*\mathcal{Y})\to T^*\mathcal{X}$.*

(e) *Suppose $\mathcal{W},\mathcal{X},\mathcal{Y},\mathcal{Z}$ are locally fair Deligne–Mumford C^∞-stacks with a* 2*-Cartesian square*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "f"] \arrow[d, "e"'] \arrow[dr, phantom, "\eta\Uparrow"] & \mathcal{Y} \arrow[d, "h"] \\
\mathcal{X} \arrow[r, "g"] & \mathcal{Z}
\end{tikzcd}$$

in $\mathbf{DMC^\infty Sta^{lf}}$, *so that* $\mathcal{W} = \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. *Then the following is exact in* $\mathrm{qcoh}(\mathcal{W})$:

$$(g \circ e)^*(T^*\mathcal{Z}) \xrightarrow{\substack{e^*(\Omega_g)\circ I_{e,g}(T^*\mathcal{Z})\oplus \\ -f^*(\Omega_h)\circ I_{f,h}(T^*\mathcal{Z})\circ\eta^*(T^*\mathcal{Z})}} \begin{array}{c} e^*(T^*\mathcal{X})\oplus \\ f^*(T^*\mathcal{Y}) \end{array} \xrightarrow{\Omega_e\oplus\Omega_f} T^*\mathcal{W} \longrightarrow 0.$$

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