A geometric construction for invariant jet differentials

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1. Introduction

The action of the reparametrization group $\mathbb{G}_k$, consisting of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$, on the bundle $J_k = J_kT^*X$ of k-jets at 0 of germs of holomorphic curves $f : \mathbb{C} \to X$ in a complex manifold X has been a focus of investigation since the work of Demailly [**5**] which built on that of Green and Griffiths [**13**]. Here $\mathbb{G}_k$ is a non-reductive complex algebraic group which is the semi-direct product $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ of its unipotent radical $\mathbb{U}_k$ with $\mathbb{C}^*$; it has the form

$$\mathbb{G}_k \cong \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & \cdots & & \\ 0 & 0 & \alpha_1^3 & \cdots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} : \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C} \right\}$$

where the entries above the leading diagonal are polynomials in $\alpha_1, \ldots, \alpha_k$, and $\mathbb{U}_k$ is the subgroup consisting of matrices of this form with $\alpha_1 = 1$. The bundle of Demailly-Semple jet differentials of order k over X has fibre at $x \in X$ given by the algebra $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ of $\mathbb{U}_k$-invariant polynomial functions on the fibre $(J_k)_x = (J_kT^*X)_x$ of J_kT^*X. More generally following [**25**] we can replace $\mathbb{C}$ with $\mathbb{C}^p$ for $p \geq 1$ and consider the bundle $J_{k,p}T^*X$ of k-jets at 0 of holomorphic maps $f : \mathbb{C}^p \to X$ and the reparametrization group $\mathbb{G}_{k,p}$ consisting of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$; then $\mathbb{G}_{k,p}$ is the semi-direct product of its unipotent radical $\mathbb{U}_{k,p}$ and the complex reductive group $\mathrm{GL}(p)$, while its subgroup $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes \mathrm{SL}(p)$ (which equals $\mathbb{U}_{k,p}$ when $p = 1$) fits into an exact sequence $1 \to \mathbb{G}'_{k,p} \to \mathbb{G}_{k,p} \to \mathbb{C}^* \to 1$. The generalized Demailly-Semple algebra is then $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

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The Demailly-Semple algebras $\mathcal{O}(J_k)^{\mathbb{U}_k}$ and their generalizations have been studied for a long time. The invariant jet differentials play a crucial role in the strategy devised by Green, Griffiths [**13**], Bloch [**4**], Demailly [**5, 6**], Siu [**28, 29, 30**] and others to prove Kobayashi's 1970 hyperbolicity conjecture [**19**] and the related conjecture of Green and Griffiths in the special case of hypersurfaces in projective space. This strategy has been recently used successfully by Diverio, Merker and Rousseau in [**7**] and then by the first author in [**1**] to give effective lower bounds for the degrees of generic hypersurfaces in $\mathbb{P}_n$ for which the Green-Griffiths conjecture holds.

In particular it has been a long-standing problem to determine whether the algebras of invariants $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ and bi-invariants $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}\times U_{n,x}}$ (where $U_{n,x}$ is a maximal unipotent subgroup of $\mathrm{GL}(T_xX) \cong \mathrm{GL}(n)$) are finitely generated as graded complex algebras, and if so to provide explicit finite generating sets. In [**20**] Merker showed that when $p=1$ and both k and $n=\dim X$ are small then these algebras are finitely generated, and for $p=1$ and all k and n he provided an algorithm which produces finite sets of generators when they exist. In this paper we will describe methods inspired by [**2**] and the approach of [**9**] to non-reductive geometric invariant theory (GIT) to prove the finite generation of $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ for all n and $k \geq 2$ (from which the finite generation of the corresponding bi-invariants follows). In fact we will show that $\mathbb{U}_k$ is a Grosshans subgroup of $\mathrm{SL}(k)$, so that the algebra $\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k}$ is finitely generated and hence every linear action of $\mathbb{U}_k$ which extends to a linear action of $\mathrm{SL}(k)$ has finitely generated invariants. We will also give a geometric description of a finite set of generators for $\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k}$, and a geometric description of the associated affine variety

$$\mathrm{SL}(k)/\!/\mathbb{U}_k = \mathrm{Spec}(\mathcal{O}(SL(k))^{\mathbb{U}_k})$$

which leads to a geometric description of the affine variety

$$(J_k)_x/\!/\mathbb{U}_k = \mathrm{Spec}(\mathcal{O}((J_k)_x)^{\mathbb{U}_k})$$

as a GIT quotient

$$((J_k)_x \times (\mathrm{SL}(k)/\!/\mathbb{U}_k))/\!/\mathrm{SL}(k)$$

by the reductive group $\mathrm{SL}(k)$, in the sense of classical geometric invariant theory [**23**]. Similarly we expect that if $p>1$ and k is sufficiently large (depending on p) then $\mathbb{G}'_{k,p}$ is a subgroup of $\mathrm{SL}(\mathrm{sym}^{\leq k}(p))$, where

$$\mathrm{sym}^{\leq k}(p) = \sum_{i=1}^{k} \dim \mathrm{Sym}^i \mathbb{C}^p,$$

such that the algebra $\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k}(p)))^{\mathbb{G}'_{k,p}}$ is finitely generated, and thus that the algebra and $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ is also finitely generated, and we have a geometric description of the associated affine variety

$$(J_{k,p})_x/\!/\mathbb{G}'_{k,p}.$$

The layout of this paper is as follows. §2 reviews the reparametrization groups $\mathbb{G}_k$ and $\mathbb{G}_{k,p}$ and their actions on jet bundles and jet differentials over a complex manifold X. Next §3 reviews some of the results of [**9**] on non-reductive geometric invariant theory. In §4 we recall from [**2**] a geometric description of the quotients by $\mathbb{U}_k$ and $\mathbb{G}_k$ of open subsets of $(J_k)_x$, and in §5 this is used to find explicit affine and projective embeddings of these quotients and explicit embeddings of $\mathrm{SL}(k)/\mathbb{U}_k$. In §6 we see that the complement of $\mathrm{SL}(k)/\mathbb{U}_k$ in its closure for a suitable embedding in an affine space has codimension at least two. In §7 we conclude that $\mathbb{U}_k$ is a Grosshans subgroup of $\mathrm{SL}(k)$ when $k \geq 2$, so that $\mathcal{O}(SL(k))^{\mathbb{U}_k}$ and $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ are finitely generated, and provide a geometric description of a finite set of generators of $\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k}$. Finally §8 and §9 discuss how to extend the results of §6 and §7 to the action of $\mathbb{G}_{k,p}$ on the jet bundle $J_{k,p} \to X$ of k-jets of germs of holomorphic maps from $\mathbb{C}^p$ to X for $p > 1$.

2. Jets of curves and jet differentials

Let X be a complex n-dimensional manifold and let k be a positive integer. Green and Griffiths in [**13**] introduced the bundle $J_k \to X$ of k-jets of germs of parametrized curves in X; its fibre over $x \in X$ is the set of equivalence classes of germs of holomorphic maps $f : (\mathbb{C}, 0) \to (X, x)$, with the equivalence relation $f \sim g$ if and only if the derivatives $f^{(j)}(0) = g^{(j)}(0)$ are equal for $0 \leq j \leq k$. If we choose local holomorphic coordinates $(z_1, \ldots, z_n)$ on an open neighbourhood $\Omega \subset X$ around x, the elements of the fibre $J_{k,x}$ are represented by the Taylor expansions

$$f(t) = x + tf'(0) + \frac{t^2}{2!} f''(0) + \cdots + \frac{t^k}{k!} f^{(k)}(0) + O(t^{k+1})$$

up to order k at $t = 0$ of $\mathbb{C}^n$-valued maps

$$f = (f_1, f_2, \ldots, f_n)$$

on open neighbourhoods of 0 in $\mathbb{C}$. Thus in these coordinates the fibre is

$$J_{k,x} = \Big\{ (f'(0), \ldots, f^{(k)}(0)/k!) \Big\} = (\mathbb{C}^n)^k,$$

which we identify with $\mathbb{C}^{nk}$. Note, however, that J_k is not a vector bundle over X, since the transition functions are polynomial, but not linear.

Let $\mathbb{G}_k$ be the group of k-jets at the origin of local reparametrizations of $(\mathbb{C}, 0)$

$$t \mapsto \varphi(t) = \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_k t^k, \quad \alpha_1 \in \mathbb{C}^*, \alpha_2, \ldots, \alpha_k \in \mathbb{C},$$

in which the composition law is taken modulo terms t^j for $j > k$. This group acts fibrewise on J_k by substitution. A short computation shows that this is a linear action on the fibre:

$$\begin{aligned} f\circ\varphi(t) = {} & f'(0)\cdot(\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_k t^k) \\ & +\frac{f''(0)}{2!}\cdot(\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_k t^k)^2 \\ & +\cdots+\frac{f^{(k)}(0)}{k!}\cdot(\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_k t^k)^k \text{ (modulo } t^{k+1}) \end{aligned}$$

so the linear action of φ on the k-jet $(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!)$ is given by the following matrix multiplication:

$$(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) \cdot \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \cdots & \alpha_1\alpha_{k-1}+\cdots+\alpha_{k-1}\alpha_1 \\ 0 & 0 & \alpha_1^3 & \cdots & 3\alpha_1^2\alpha_{k-2}+\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \alpha_1^k \end{pmatrix} \tag{1}$$

where the matrix has general entry

$$(G_k)_{i,j} = \sum_{s_1\geq 1,\dots,s_i\geq 1,\ s_1+\cdots+s_i=j} \alpha_{s_1}\dots\alpha_{s_i}$$

for $i, j \leq k$.

There is an exact sequence of groups:

$$1 \to \mathbb{U}_k \to \mathbb{G}_k \to \mathbb{C}^* \to 1, \tag{2}$$

where $\mathbb{G}_k \to \mathbb{C}^*$ is the morphism $\varphi \to \varphi'(0) = \alpha_1$ in the notation used above, and

$$\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$$

is a semi-direct product. With the above identification, $\mathbb{C}^*$ is the subgroup of $\mathbb{G}_k$ consisting of diagonal matrices satisfying $\alpha_2 = \cdots = \alpha_k = 0$ and $\mathbb{U}_k$ is the unipotent radical of $\mathbb{G}_k$, consisting of matrices of the form above with $\alpha_1 = 1$. The action of $\lambda \in \mathbb{C}^*$ on k-jets is thus described by

$$\lambda\cdot(f'(0), f''(0)/2!, \dots, f^{(k)}(0)/k!) = (\lambda f'(0), \lambda^2 f''(0)/2!, \dots, \lambda^k f^{(k)}(0)/k!)$$

Let $\mathcal{E}_{k,m}^n$ denote the vector space of complex valued polynomial functions

$$Q(u_1, u_2, \dots, u_k)$$

of $u_1 = (u_{1,1}, \dots, u_{1,n}), \dots, u_k = (u_{k,1}, \dots, u_{k,n})$ of weighted degree m with respect to this $\mathbb{C}^*$ action, where $u_i = f^{(i)}(0)/i!$; that is, such that

$$Q(\lambda u_1, \lambda^2 u_2, \dots, \lambda^k u_k) = \lambda^m Q(u_1, u_2, \dots, u_k).$$

Thus elements of $\mathcal{E}^n_{k,m}$ have the form

$$Q(u_1,u_2,\ldots,u_k)=\sum_{|i_1|+2|i_2|+\cdots+k|i_k|=m} u_1^{i_1}u_2^{i_2}\ldots u_k^{i_k},$$

where $i_1=(i_{1,1},\ldots,i_{1,n}),\ldots,i_k=(i_{k,1},\ldots,i_{k,n})$ are multi-indices of length n. There is an induced action of $\mathbb{G}_k$ on the algebra $\bigoplus_{m\geq 0}\mathcal{E}^n_{k,m}$. Following Demailly (see [**5**]), we denote by $E^n_{k,m}$ (or $E_{k,m}$) the Demailly-Semple bundle whose fibre at x consists of the $\mathbb{U}_k$-invariant polynomials on the fibre of J_k at x of weighted degree m, i.e those which satisfy

$$\begin{aligned}&Q((f\circ\varphi)'(0),(f\circ\varphi)''(0)/2!,\ldots,(f\circ\varphi)^{(k)}(0)/k!)\\&\quad=\varphi'(0)^m\cdot Q(f'(0),f''(0)/2!,\ldots,f^{(k)}(0)/k!),\end{aligned}$$

and we let $E^n_k=\oplus_m E^n_{k,m}$ denote the Demailly-Semple bundle of graded algebras of invariants.

We can also consider higher dimensional holomorphic surfaces in X, and therefore we fix a parameter $1\leq p\leq n$, and study germs of maps $\mathbb{C}^p\to X$.

Again we fix the degree k of our map, and introduce the bundle $J_{k,p}\to X$ of k-jets of maps $\mathbb{C}^p\to X$. The fibre over $x\in X$ is the set of equivalence classes of germs of holomorphic maps $f:(\mathbb{C}^p,0)\to(X,x)$, with the equivalence relation $f\sim g$ if and only if all derivatives $f^{(j)}(0)=g^{(j)}(0)$ are equal for $0\leq j\leq k$.

We need a description of the fibre $J_{k,p,x}$ in terms of local coordinates as in the case when $p=1$. Let $(z_1,\ldots,z_n)$ be local holomorphic coordinates on an open neighbourhood $\Omega\subset X$ around x, and let $(u_1,\ldots,u_p)$ be local coordinates on $\mathbb{C}^p$. The elements of the fibre $J_{k,p,x}$ are $\mathbb{C}^n$-valued maps

$$f=(f_1,f_2,\ldots,f_n)$$

on $\mathbb{C}^p$, and two maps represent the same jet if their Taylor expansions around $\mathbf{z}=0$

$$f(\mathbf{z})=x+\mathbf{z}f'(0)+\frac{\mathbf{z}^2}{2!}f''(0)+\cdots+\frac{\mathbf{z}^k}{k!}f^{(k)}(0)+O(\mathbf{z}^{k+1})$$

coincide up to order k. Note that here

$$f^{(i)}(0)\in\operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p,\mathbb{C}^n)$$

and in these coordinates the fibre is a finite-dimensional vector space

$$J_{k,p,x}=\left\{(f'(0),\ldots,f^{(k)}(0)/k!)\right\}\cong\mathbb{C}^{n\binom{k+p-1}{k-1}}.$$

Let $\mathbb{G}_{k,p}$ be the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$. Elements of $\mathbb{G}_{k,p}$ are represented by holomorphic maps

$$\begin{aligned}\mathbf{u} \to \varphi(\mathbf{u}) &= \Phi_1\mathbf{u} + \Phi_2\mathbf{u}^2 + \cdots + \Phi_k\mathbf{u}^k \\ &= \sum_{\mathbf{i}\in\mathbb{Z}^p\setminus 0} a_{i_1\ldots i_p} u_1^{i_1}\ldots u_p^{i_p}, \quad \Phi_1 \text{ is non-degenerate}\end{aligned} \tag{3}$$

where $\Phi_i \in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p, \mathbb{C}^p)$. The group $\mathbb{G}_{k,p}$ admits a natural fibrewise right action on $J_{k,p}$, by reparametrizing the k-jets of holomorphic p-discs. A computation similar to that in [**2**] shows that

$$\begin{aligned}f\circ\varphi(\mathbf{u}) = f'(0)\Phi_1\mathbf{u} + \left(f'(0)\Phi_2 + \frac{f''(0)}{2!}\Phi_1^2\right)\mathbf{u}^2 + \cdots \\ + \sum_{i_1+\cdots+i_l=d}\frac{f^{(l)}(0)}{l!}\Phi_{i_1}\ldots\Phi_{i_l}\mathbf{u}^l.\end{aligned}$$

This defines a linear action of $\mathbb{G}_{k,p}$ on the fibres $J_{k,p,x}$ of $J_{k,p}$ with the matrix representation given by

$$\begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \ldots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \ldots & \\ 0 & 0 & \Phi_1^3 & \ldots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \Phi_1^k \end{pmatrix}, \tag{4}$$

where

- $\Phi_i \in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p, \mathbb{C}^p)$ is a $p\times\dim(\operatorname{Sym}^i\mathbb{C}^p)$-matrix, the ith degree component of the map Φ, which is represented by a map $(\mathbb{C}^p)^{\otimes i}\to\mathbb{C}^p$;
- $\Phi_{i_1}\ldots\Phi_{i_l}$ is the matrix of the map $\operatorname{Sym}^{i_1+\cdots+i_l}(\mathbb{C}^p)\to\operatorname{Sym}^l\mathbb{C}^p$, which is represented by
$$\sum_{\sigma\in\mathcal{S}_l}\Phi_{i_1}\otimes\cdots\otimes\Phi_{i_l} : (\mathbb{C}^p)^{\otimes i_1}\otimes\cdots\otimes(\mathbb{C}^p)^{\otimes i_l}\to(\mathbb{C}^p)^{\otimes l};$$
- the (l,m) block of $\mathbb{G}_{k,p}$ is $\sum_{i_1+\cdots+i_l=m}\phi_{i_1}\ldots\Phi_{i_l}$. The entries in these boxes are indexed by pairs (τ,μ) where $\tau\in\binom{p+l-1}{l-1}, \mu\in\binom{p+m-1}{m-1}$ correspond to bases of $\operatorname{Sym}^l(\mathbb{C}^p)$ and $\operatorname{Sym}^m(\mathbb{C}^p)$.

EXAMPLE 2.1. *For $p=2, k=3$, using the standard basis*

$$\{e_i, e_ie_j, e_ie_je_k : 1\le i\le j\le k\le 2\}$$

of $(J_{3,2})_x$*, we get the following* 9×9 *matrix for a general element of* $\mathbb{G}_{3,2}$*:*

$$
\left(\begin{array}{cccccc}
\alpha_{10} & \alpha_{01} & \alpha_{20} & \alpha_{11} & \alpha_{02} & \alpha_{30} \\
\beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} & \beta_{30} \\
0 & 0 & \alpha_{10}^2 & \alpha_{10}\alpha_{01} & \alpha_{01}^2 & \alpha_{10}\alpha_{20} \\
0 & 0 & \alpha_{10}\beta_{10} & \alpha_{10}\beta_{01}+\alpha_{01}\beta_{10} & \alpha_{01}\beta_{01} & \alpha_{10}\beta_{20}+\alpha_{20}\beta_{10} \\
0 & 0 & \beta_{10}^2 & \beta_{10}\beta_{01} & \beta_{01}^2 & \beta_{10}\beta_{20} \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}^3 \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}^2\beta_{10} \\
0 & 0 & 0 & 0 & 0 & \alpha_{10}\beta_{10}^2 \\
0 & 0 & 0 & 0 & 0 & \beta_{10}^3
\end{array}\right.
$$

$$
(5) \qquad
\left.\begin{array}{ccc}
a_{21} & \alpha_{12} & \alpha_{03} \\
\beta_{21} & \beta_{12} & \beta_{03} \\
\alpha_{10}\alpha_{11}+\alpha_{01}\alpha_{20} & \alpha_{10}\alpha_{02}+\alpha_{11}\alpha_{01} & \alpha_{01}\alpha_{02} \\
P & Q & \alpha_{01}\beta_{02}+\alpha_{02}\beta_{01} \\
\beta_{10}\beta_{11}+\beta_{20}\beta_{01} & \beta_{01}\beta_{11}+\beta_{02}\beta_{10} & \beta_{01}\beta_{02} \\
\alpha_{10}^2\alpha_{01} & \alpha_{10}\alpha_{01}^2 & \alpha_{01}^3 \\
\alpha_{10}\alpha_{10}\beta_{01} & \alpha_{10}\alpha_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\
\alpha_{10}\beta_{10}\beta_{01} & \alpha_{10}\beta_{01}\beta_{01} & \alpha_{01}\beta_{01}^2 \\
\beta_{10}^2\beta_{01} & \beta_{10}\beta_{01}^2 & \beta_{01}^3
\end{array}\right)
$$

where

$$
\begin{aligned}
P &= \alpha_{10}\beta_{11}+\alpha_{11}\beta_{10}+\alpha_{20}\beta_{01}+\alpha_{01}\beta_{20} \quad \textit{and} \\
Q &= \alpha_{01}\beta_{11}+\alpha_{11}\beta_{01}+\alpha_{02}\beta_{10}+\alpha_{10}\beta_{02}.
\end{aligned}
$$

This is a subgroup of the standard parabolic $P_{2,3,4} \subset GL(9)$*. The diagonal blocks are the representations* $\mathrm{Sym}^i\mathbb{C}^2$ *for* $i = 1,2,3$ *of* $GL(2)$*, where* $\mathbb{C}^2$ *is the standard representation of* $GL(2)$*.*

In general the linear group $\mathbb{G}_{k,p}$ is generated along its first p rows; that is, the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The assumption on the parameters is that the determinant of the smallest diagonal $p \times p$ block is nonzero; for the $p = 2, k = 3$ example above this means that

$$
\det\begin{pmatrix} \alpha_{10} & \alpha_{01} \\ \beta_{10} & \beta_{01} \end{pmatrix} \neq 0.
$$

The parameters in the $(1,m)$ block are indexed by a basis of $\mathrm{Sym}^m(\mathbb{C}^p) \times \mathbb{C}^p$, so they are of the form α^l_ν where $\nu \in \binom{p+m-1}{m-1}$ is an m-tuple and $1 \leq l \leq p$. An easy computation shows that:

Proposition 2.2. *The polynomial in the* (l,m) *block and entry indexed by*

$$
\tau = (\tau[1], \ldots, \tau[l]) \in \binom{p+l-1}{l-1}
$$

and $\nu \in \binom{p+m-1}{m-1}$ *is*

$$(\mathbb{G}_{k,p})_{\tau,\nu} = \sum_{\nu_1+\cdots+\nu_l=\nu} \alpha_{\nu_1}^{\tau[1]} \alpha_{\nu_2}^{\tau[2]} \ldots \alpha_{\nu_l}^{\tau[l]} \tag{6}$$

Note that $\mathbb{G}_{k,p}$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ by $GL(p)$; that is, we have an exact sequence

$$1 \to \mathbb{U}_{k,p} \to \mathbb{G}_{k,p} \to GL(p) \to 1,$$

and $\mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{U}_{k,p} \rtimes GL(p)$. Here $\mathbb{G}_{k,p}$ has dimension $p \times \mathrm{sym}^{\leq k}(p)$ where $\mathrm{sym}^{\leq k}(p) = \dim(\oplus_{i=1}^k \mathrm{Sym}^i \mathbb{C}^p)$, and is a subgroup of the standard parabolic subgroup $P_{p,\mathrm{sym}^2(p),\ldots,\mathrm{sym}^k(p)}$ of $GL(\mathrm{sym}^{\leq k}(p))$ where $\mathrm{sym}^i(p) = \dim(\mathrm{Sym}^i\mathbb{C}^p)$. We define $\mathbb{G}'_{k,p}$ to be the subgroup of $\mathbb{G}_{k,p}$ which is the semi-direct product

$$\mathbb{G}'_{k,p} = \mathbb{U}_{k,p} \rtimes SL(p)$$

(so that $\mathbb{G}'_{k,p} = \mathbb{U}_{k,p}$ when $p = 1$) fitting into the exact sequence

$$1 \to \mathbb{U}_{k,p} \to \mathbb{G}'_{k,p} \to SL(p) \to 1.$$

The action of the maximal torus $(\mathbb{C}^*)^p \subset GL(p)$ of the Levi subgroup of $\mathbb{G}_{k,p}$ is

$$(\lambda_1, \ldots, \lambda_p) \cdot f^{(i)} = \left(\lambda_1^i \frac{\partial^i f}{\partial u_1^i}, \ldots, \lambda_1^{i_1} \cdots \lambda_p^{i_p} \frac{\partial^i f}{\partial u_1^{i_1} \cdots \partial u_p^{i_p}} \ldots \lambda_p^i \frac{\partial^i f}{\partial u_p^i} \right) \tag{7}$$

We introduce the *Green-Griffiths* vector bundle $E_{k,p,m}^{GG} \to X$, whose fibres are complex-valued polynomials

$$Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$$

on the fibres of $J_{k,p}$, having weighted degree $(m, \ldots, m)$ with respect to the action (7) of $(\mathbb{C}^*)^p$. That is, for $Q \in E_{k,p,m}^{GG}$

$$\begin{aligned} Q(\lambda f'(0), \lambda f''(0)/2!, \ldots, \lambda f^{(k)}(0)/k!) &= \lambda_1^m \cdots \lambda_p^m \\ Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) & \end{aligned}$$

for all $\lambda \in \mathbb{C}^p$ and $(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!) \in J_{k,p,m}$.

Definition 2.3. *The generalized Demailly-Semple bundle* $E_{k,p,m} \to X$ *over* X *has fibre consisting of the* $\mathbb{G}'_{k,p}$*-invariant jet differentials of order* k *and weighted degree* $(m, \ldots, m)$*; that is, the complex-valued polynomials* $Q(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$ *on the fibres of* $J_{k,p}$ *which transform under any reparametrization* $\phi \in \mathbb{G}_{k,p}$ *of* $(\mathbb{C}^p, 0)$ *as*

$$Q(f \circ \phi) = (J_\phi)^m Q(f) \circ \phi,$$

where $J_\phi = \det \Phi_1$ denotes the Jacobian of ϕ at 0. The generalized Demailly-Semple bundle of algebras $E_{k,p} = \oplus_{m\geq 0} E_{k,p,m}$ is the associated graded algebra of $\mathbb{G}'_{k,p}$-invariants, whose fibre at $x \in X$ is the generalized Demailly-Semple algebra $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$.

The determination of a suitable generating set for the invariant jet differentials when $p=1$ is important in the longstanding strategy to prove the Green-Griffiths conjecture. It has been suggested in a series of papers [**13, 5, 27, 20, 7, 21**] that the Schur decomposition of the Demailly-Semple algebra, together with good estimates of the higher Betti numbers of the Schur bundles and an asymptotic estimation of the Euler charactristic, should result in a positive lower bound for the global sections of the Demailly-Semple jet differential bundle.

3. Geometric invariant theory

Suppose now that Y is a complex quasi-projective variety on which a linear algebraic group G acts. For geometric invariant theory (GIT) we need a linearization of the action; that is, a line bundle L on Y and a lift $\mathcal{L}$ of the action of G to L. Usually L is ample, and hence (as it makes no difference for GIT if we replace L with $L^{\otimes k}$ for any integer $k>0$) we can assume that for some projective embedding $Y \subseteq \mathbb{P}^n$ the action of G on Y extends to an action on $\mathbb{P}^n$ given by a representation $\rho: G \to GL(n+1)$, and take for L the hyperplane line bundle on $\mathbb{P}^n$.

For classical GIT developed by Mumford [**23**] (cf. also [**8, 22, 24, 26**]) we require the complex algebraic group G to be reductive. Let Y be a projective complex variety with an action of a complex reductive group G and linearization $\mathcal{L}$ with respect to an ample line bundle L on Y. Then $y \in Y$ is *semistable* for this linear action if there exists some $m>0$ and $f \in H^0(Y, L^{\otimes m})^G$ not vanishing at y, and y is *stable* if also the action of G on the open subset

$$Y_f := \{x \in Y \mid f(x) \neq 0\}$$

is closed with all stabilizers finite. Y^{ss} has a projective categorical quotient $Y^{ss} \to Y/\!/G$, which restricts on the set of stable points to a geometric quotient $Y^s \to Y^s/G$ (see [**23**] Theorem 1.10). The morphism $Y^{ss} \to Y/\!/G$ is surjective, and identifies $x, y \in Y^{ss}$ if and only if the closures of the G-orbits of x and y meet in Y^{ss}; moreover each point in $Y/\!/G$ is represented by a unique closed G-orbit in Y^{ss}. There is an induced action of G on the homogeneous coordinate ring

$$\hat{\mathcal{O}}_L(Y) = \bigoplus_{k\geq 0} H^0(Y, L^{\otimes k})$$

of Y. The subring $\hat{\mathcal{O}}_L(Y)^G$ consisting of the elements of $\hat{\mathcal{O}}_L(Y)$ left invariant by G is a finitely generated graded complex algebra because G is reductive, and the GIT quotient $Y/\!/G$ is the projective variety $\mathrm{Proj}(\hat{\mathcal{O}}_L(Y)^G)$ [**23**].

The subsets Y^{ss} and Y^s of Y are characterized by the following properties (see [**23**, Chapter 2] or [**24**]).

PROPOSITION 3.1. *(Hilbert-Mumford criteria) (i) A point $x \in Y$ is semistable (respectively stable) for the action of G on Y if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus of G.*
(ii) A point $x \in Y$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull

$$\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 (respectively contains 0 in its interior).

Similarly if a complex reductive group G acts linearly on an affine variety Y then we have a GIT quotient

$$Y/\!/G = \mathrm{Spec}(\mathcal{O}(Y)^G)$$

which is the affine variety associated to the finitely generated algebra $\mathcal{O}(Y)^G$ of G-invariant regular functions on Y. In this case $Y^{ss} = Y$ and the inclusion $\mathcal{O}(Y)^G \hookrightarrow \mathcal{O}(Y)$ induces a morphism of affine varieties $Y \to Y/\!/G$.

Now suppose that H is any complex linear algebraic group, with unipotent radical $U \trianglelefteq H$ (so that $R = H/U$ is reductive and H is isomorphic to the semi-direct product $U \rtimes R$), acting linearly on a complex projective variety Y with respect to an ample line bundle L. Then $\mathrm{Proj}(\hat{\mathcal{O}}_L(Y)^H)$ is not in general well-defined as a projective variety, since the ring of invariants

$$\hat{\mathcal{O}}_L(Y)^H = \bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^H$$

is not necessarily finitely generated as a graded complex algebra, and so it is not obvious how GIT might be generalised to this situation (cf. [**9, 11, 10, 14, 15, 18**]). However in some cases it is known that $\hat{\mathcal{O}}_L(Y)^U$ is finitely generated, which implies that

$$\hat{\mathcal{O}}_L(Y)^H = \left(\bigoplus_{k \geq 0} H^0(Y, L^{\otimes k})^U \right)^{H/U}$$

is finitely generated and hence the *enveloping quotient* in the sense of [**9**] is given by the associated projective variety

$$Y/\!/H = \mathrm{Proj}(\hat{\mathcal{O}}_L(Y)^H).$$

Similarly if Y is affine and H acts linearly on Y with $\mathcal{O}(Y)^H$ finitely generated, then we have the enveloping quotient

$$Y/\!/H = \mathrm{Spec}(\mathcal{O}(Y)^H).$$

There is a morphism

$$q : Y^{ss} \to Y//H,$$

from an open subset Y^{ss} of Y (where $Y^{ss} = Y$ when Y is affine), which restricts to a geometric quotient

$$q : Y^s \to Y^s/H$$

for an open subset $Y^s \subset Y^{ss}$. However in contrast with the reductive case, the morphism $q : Y^{ss} \to Y//H$ is not in general surjective; indeed the image of q is not in general a subvariety of $Y//H$, but is only a constructible subset.

If there is a complex reductive group G containing the unipotent radical U of H such that the algebra $\mathcal{O}(G)^U$ is finitely generated and the action of U on Y extends to a linear action of G, then

$$\mathcal{O}(Y)^U \cong (\mathcal{O}(Y) \otimes \mathcal{O}(G)^U)^G$$

is finitely generated and hence so is

$$\mathcal{O}(Y)^H = (\mathcal{O}(Y)^U)^{H/U}$$

(or if Y is projective with an ample linearisation L then $\hat{\mathcal{O}}_L(Y)^U$ is finitely generated and hence so is $\hat{\mathcal{O}}_L(Y)^H$). In this situation we say that U is a Grosshans subgroup of G (cf. [**16, 17**]). Then geometrically G/U is a quasi-affine variety with $\mathcal{O}(G/U) \cong \mathcal{O}(G)^U$, and it has a canonical affine embedding as an open subvariety of the affine variety

$$G//U = \mathrm{Spec}(\mathcal{O}(G)^U)$$

with complement of codimension at least two. Moreover if a linear action of U on an affine variety Y extends to a linear action of G then

$$Y//U \cong (Y \times G//U)//G$$

(and a corresponding result is true if Y is projective). Conversely if we can find an embedding of G/U as an open subvariety of an affine variety Z with complement of codimension at least two, then

$$\mathcal{O}(G)^U \cong \mathcal{O}(Z)$$

is finitely generated and $G//U \cong Z$.

Suppose that U is a unipotent group with a reductive group R of automorphisms of U given by a homomorphism $\phi : R \to \mathrm{Aut}(U)$ such that R contains a central one-parameter subgroup $\lambda : \mathbb{C}^* \to R$ for which the weights of the induced $\mathbb{C}^*$ action on the Lie algebra $\mathfrak{u}$ of U are all nonzero. Then we can form the semi-direct product

$$\hat{U} = \mathbb{C}^* \ltimes U \subseteq R \ltimes U$$

given by $\mathbb{C}^* \times U$ with group multiplication

$$(z_1, u_1).(z_2, u_2) = (z_1 z_2, (\lambda(z_2^{-1})(u_1))u_2).$$

The groups $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$ and $\mathbb{G}_{k,p} = \mathbb{U}_{k,p} \rtimes \mathrm{GL}(p)$ which act on the fibres of the jet bundles J_k and $J_{k,p}$ are of this form. We will use this structure to study the Demailly-Semple algebras of invariant jet differentials E_k^n and $E_{k,p}^n$ and prove

THEOREM 3.2. *The fibres $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ and $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ of the bundles E_k^n and $E_{k,p}^n$ are finitely generated graded complex algebras.*

Thus we have non-reductive GIT quotients

$$(J_k)_x//\mathbb{U}_k = \mathrm{Spec}(\mathcal{O}((J_k)_x)^{\mathbb{U}_k})$$

and

$$(J_{k,p})_x//\mathbb{G}'_{k,p} = \mathrm{Spec}(\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}})$$

and we would like to understand them geometrically. There is a crucial difference here from the case of reductive group actions, even though the invariants are finitely generated: when H is a non-reductive group we cannot describe $Y//H$ geometrically as Y^{ss} modulo some equivalence relation. Instead our aim is to use methods inspired by [**2**] to study these geometric invariant theoretic quotients and the associated algebras of invariants.

Here a crucial ingredient would be to find an open subset W of $(J_{k,p})_x$ with a geometric quotient $W/\mathbb{G}'_{k,p}$ embedded as an open subset of an affine variety Z such that the complement of $W/\mathbb{G}'_{k,p}$ in Z has (complex) codimension at least two, and the complement of W in $(J_{k,p})_x$ has codimension at least two. For then we would have

$$\mathcal{O}((J_{k,p})_x) = \mathcal{O}(W)$$

and

$$\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}} = \mathcal{O}(W)^{\mathbb{G}'_{k,p}} = \mathcal{O}(W/\mathbb{G}'_{k,p}) = \mathcal{O}(Z),$$

and it follows that $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ is finitely generated since Z is affine, and that

$$Z = \mathrm{Spec}(\mathcal{O}(Z)) = \mathrm{Spec}(\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}) = ((J_{k,p})_x)//\mathbb{G}'_{k,p}.$$

Similarly if we can find a complex reductive group G containing $\mathbb{G}'_{k,p}$ as a subgroup, and an embedding of $G/\mathbb{G}'_{k,p}$ as an open subset of an affine variety Z with complement of codimension at least two, then $\mathcal{O}(G)^{\mathbb{G}'_{k,p}}$ is finitely generated. It follows as above that if Y is any affine variety on which G acts linearly then

$$\mathcal{O}(Y)^{\mathbb{G}'_{k,p}} \cong (\mathcal{O}(Y) \otimes \mathcal{O}(G)^{\mathbb{G}'_{k,p}})^G$$

is finitely generated, and hence so is $\mathcal{O}(Y)^{\mathbb{G}_{k,p}} = (\mathcal{O}(Y)^{\mathbb{G}'_{k,p}})^{\mathbb{C}^*}$, and similarly $\hat{\mathcal{O}}_L(Y)^{\mathbb{G}'_{k,p}}$ and $\hat{\mathcal{O}}_L(Y)^{\mathbb{G}_{k,p}}$ are finitely generated if Y is any projective variety wtih an ample line bundle L on which G acts linearly.

We can use the ideas of [**2**] to look for suitable affine varieties Z as above, and in particular to prove

THEOREM 3.3. *$\mathbb{G}'_{k,p}$ is a subgroup of the special linear group* $\mathrm{SL}(\mathrm{sym}^{\leq k}p)$ *where*

$$\mathrm{sym}^{\leq k}p = \sum_{i=1}^{k} \dim \mathrm{Sym}^i \mathbb{C}^p = \begin{pmatrix} k+p-1 \\ k-1 \end{pmatrix}$$

such that the algebra of invariants $\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k}p))^{\mathbb{G}'_{k,p}}$ is finitely generated, and every linear action of $\mathbb{G}'_{k,p}$ or $\mathbb{G}_{k,p}$ on an affine or projective variety (with an ample linearisation) which extends to a linear action of $\mathrm{GL}(\mathrm{sym}^{\leq k}p)$ *has finitely generated invariants.*

Theorem 3.2 is an immediate consequence of this theorem, since the action of $\mathbb{G}_{k,p}$ on $(J_{k,p})_x$ extends to an action of the general linear group $\mathrm{GL}(\mathrm{sym}^{\leq k}p)$. Moreover we will find a geometric description of

$$\mathrm{SL}(\mathrm{sym}^{\leq k}p)//\mathbb{G}'_{k,p} \cong \mathrm{Spec}(\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k}p))^{\mathbb{G}'_{k,p}})$$

and thus a geometric description of

$$(J_{k,p})_x//\mathbb{G}'_{k,p} \cong ((J_{k,p})_x \times \mathrm{SL}(\mathrm{sym}^{\leq k}p)//\mathbb{G}'_{k,p})//\mathrm{SL}(\mathrm{sym}^{\leq k}p).$$

4. A description via test curves

In [**2**] the action of $\mathbb{G}_k$ on jet bundles is studied using an idea coming from global singularity theory. The construction goes as follows.

If u, v are positive integers, let $J_k(u,v)$ denote the vector space of k-jets of holomorphic maps $(\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$ at the origin; that is, the set of equivalence classes of maps $f : (\mathbb{C}^u, 0) \to (\mathbb{C}^v, 0)$, where $f \sim g$ if and only if $f^{(j)}(0) = g^{(j)}(0)$ for all $j = 1, \ldots, k$.

With this notation, the fibres of J_k are isomorphic to $J_k(1,n)$, and the group $\mathbb{G}_k$ is simply $J_k(1,1)$ with the composition action on itself.

If we fix local coordinates $z_1, \ldots, z_u$ at $0 \in \mathbb{C}^u$ we can again identify the k-jet of f, using derivatives at the origin, with $(f'(0), f''(0)/2!, \ldots, f^{(k)}(0)/k!)$, where $f^{(j)}(0) \in \mathrm{Hom}(\mathrm{Sym}^j \mathbb{C}^u, \mathbb{C}^v)$. This way we get an identification

$$J_k(u,v) = \oplus_{j=1}^{k} \mathrm{Hom}(\mathrm{Sym}^j \mathbb{C}^u, \mathbb{C}^v).$$

We can compose map-jets via substitution and elimination of terms of degree greater than k; this leads to the composition maps

$$\begin{aligned} &J_k(v,w) \times J_k(u,v) \to J_k(u,w), \\ &(\Psi_2, \Psi_1) \mapsto \Psi_2 \circ \Psi_1 \text{ modulo terms of degree } > k. \end{aligned} \tag{8}$$

When $k = 1$, $J_1(u,v)$ may be identified with u-by-v matrices, and (8) reduces to multiplication of matrices.

The k-jet of a curve $(\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ is simply an element of $J_k(1,n)$. We call such a curve φ *regular* if $\varphi'(0) \neq 0$. Let us introduce the notation $J_k^{\mathrm{reg}}(1,n)$ for the set of regular curves:

$$J_k^{\mathrm{reg}}(1,n) = \left\{ \gamma \in J_k(1,n); \gamma'(0) \neq 0 \right\}.$$

Note that if $n>1$ then the complement of $J_k^{\mathrm{reg}}(1,n)$ in $J_k(1,n)$ has codimension at least two. Let $N \geq n$ be any integer and define

$$\Upsilon_k = \left\{\Psi \in J_k(n,N) : \exists \gamma \in J_k^{\mathrm{reg}}(1,n) : \Psi \circ \gamma = 0\right\}$$

to be the set of those k-jets which take at least one regular curve to zero. By definition, Υ_k is the image of the closed subvariety of $J_k(n,N) \times J_k^{\mathrm{reg}}(1,n)$ defined by the algebraic equations $\Psi \circ \gamma = 0$, under the projection to the first factor. If $\Psi \circ \gamma = 0$, we call γ a *test curve* of Ψ.

This term originally comes from global singularity theory, where this is called the test curve model of A_k-singularities. In global singularity theory singularities of polynomial maps $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ are classified by their local algebras, and

$$\Sigma_k = \{f \in J_k(n,m) : \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle \simeq \mathbb{C}[t]/t^{k+1}\}$$

is called a Morin singularity, or A_k-singularity. The test curve model of Gaffney [**12**] tells us that

$$\overline{\Sigma_k} = \overline{\Upsilon_k}$$

in $J_k(n,m)$.

A basic but crucial observation is the following. If γ is a test curve of $\Psi \in \Upsilon_k$, and $\varphi \in J_k^{\mathrm{reg}}(1,1) = G_k$ is a holomorphic reparametrization of $\mathbb{C}$, then $\gamma \circ \varphi$ is, again, a test curve of Ψ:

$$\mathbb{C} \xrightarrow{\quad\varphi\quad} \mathbb{C} \xrightarrow{\quad\gamma\quad} \mathbb{C}^n \xrightarrow{\quad\Psi\quad} \mathbb{C}^N \tag{9}$$

$$\Psi \circ \gamma = 0 \;\;\Rightarrow\;\; \Psi \circ (\gamma \circ \varphi) = 0.$$

In fact, we get all test curves of Ψ in this way from a single γ if the following open dense property holds: the linear part of Ψ has 1-dimensional kernel. Before stating this more precisely in Proposition 4.3 below, let us write down the equation $\Psi \circ \gamma = 0$ in coordinates in an illustrative case. Let $\gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ and $\Psi = (\Psi', \Psi'', \ldots, \Psi^{(k)}) \in J_k(n,N)$ be the k-jets. Using the chain rule, the equation $\Psi \circ \gamma = 0$ reads as follows for $k=4$:

$$\begin{aligned} &\Psi'(\gamma') = 0, \\ &\tfrac{1}{2!}\Psi'(\gamma'') + \Psi''(\gamma', \gamma') = 0, \\ &\tfrac{1}{3!}\Psi'(\gamma''') + \tfrac{2}{2!}\Psi''(\gamma', \gamma'') + \Psi'''(\gamma', \gamma', \gamma') = 0, \\ &\tfrac{1}{4!}\Psi'(\gamma'''') + \tfrac{2}{3!}\Psi''(\gamma', \gamma''') + \tfrac{1}{2!2!}\Psi''(\gamma'', \gamma'') + \tfrac{3}{2!}\Psi'''(\gamma', \gamma', \gamma'') \\ &\quad + \Psi''''(\gamma', \gamma', \gamma', \gamma') = 0. \end{aligned} \tag{10}$$

Definition 4.1. *To simplify our formulas we introduce the following notation for a partition $\tau = [i_1 \ldots i_l]$ of the integer $i_1 + \cdots + i_l$:*

- *the length: $|\tau| = l$,*
- *the sum: $\sum \tau = i_1 + \cdots + i_l$,*

- *the number of permutations:* $\mathrm{perm}(\tau)$ *is the number of different sequences consisting of the numbers* $i_1, \dots, i_l$ *(e.g.* $\mathrm{perm}([1,1,1,3]) = 4$*),*
- $\gamma_\tau = \prod_{j=1}^{l} \gamma^{(i_j)} \in \mathrm{Sym}^l \mathbb{C}^n$ *and* $\Psi(\gamma_\tau) = \Psi^l(\gamma^{(i_1)}, \dots, \gamma^{(i_l)}) \in \mathbb{C}^N$.

LEMMA 4.2. *Let* $\gamma = (\gamma', \gamma'', \dots, \gamma^{(k)}) \in J_k^{\mathrm{reg}}(1,n)$ *and* $\Psi = (\Psi', \Psi'', \dots, \Psi^{(k)}) \in J_k(n,N)$ *be* k*-jets. Then the equation* $\Psi \circ \gamma = 0$ *is equivalent to the following system of* k *linear equations with values in* $\mathbb{C}^N$*:*

$$\sum_{\tau \in \Pi[m]} \frac{\mathrm{perm}(\tau)}{\prod_{i \in \tau} i!} \Psi(\gamma_\tau) = 0, \quad m = 1, 2, \dots, k, \tag{11}$$

where $\Pi[m]$ *denotes the set of all partitions of* m*.*

For a given $\gamma \in J_k^{\mathrm{reg}}(1,n)$ let $\mathcal{S}_\gamma$ denote the set of solutions of (11); that is,

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n,N); \Psi \circ \gamma = 0\}.$$

The equations (11) are linear in Ψ, hence

$$\mathcal{S}_\gamma \subset J_k(n,N)$$

is a linear subspace of codimension kN. Moreover, the following holds:

PROPOSITION 4.3. *([**2**], Proposition 4.4)*

(i) *For* $\gamma \in J_k^{\mathrm{reg}}(1,n)$*, the set of solutions* $\mathcal{S}_\gamma \subset J_k(n,N)$ *is a linear subspace of codimension* kN*.*

(ii) *Set*

$$J_k^o(n,N) = \left\{\Psi \in J_k(n,N) | \dim \ker(\Psi') = 1\right\}.$$

For any $\gamma \in J_k^{\mathrm{reg}}(1,n)$*, the subset* $\mathcal{S}_\gamma \cap J_k^o(n,N)$ *of* $\mathcal{S}_\gamma$ *is dense.*

(iii) *If* $\Psi \in J_k^o(n,N)$*, then* Ψ *belongs to at most one of the spaces* $\mathcal{S}_\gamma$*. More precisely,*

$$\text{if } \gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n), \quad \Psi \in J_k^o(n,N) \text{ and } \Psi \circ \gamma_1 = \Psi \circ \gamma_2 = 0,$$

then there exists $\varphi \in J_k^{\mathrm{reg}}(1,1)$ *such that* $\gamma_1 = \gamma_2 \circ \varphi$*.*

(iv) *Given* $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n)$*, we have* $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *if and only if there is some* $\varphi \in J_k^{\mathrm{reg}}(1,1)$ *such that* $\gamma_1 = \gamma_2 \circ \varphi$*.*

By the second part of Proposition 4.3 we have a well-defined map

$$\nu : J_k^{\mathrm{reg}}(1,n) \to \mathrm{Grass}(\mathrm{codim} = kN, J_k(n,N)), \quad \gamma \mapsto \mathcal{S}_\gamma$$

to the Grassmannian of codimension-kN subspaces in $J_k(n,N)$. From the last part of Proposition 4.3 it follows that:

Proposition 4.4. *([**2**]) ν is $\mathbb{G}_k$-invariant on the $J_k^{\text{reg}}(1,1)$-orbits, and the induced map on the orbits*

$$\bar{\nu}: J_k^{\text{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \text{Grass}(\text{codim} = kN, J_k(n,N)) \tag{12}$$

is injective.

5. Embedding into the flag of equations

In this section we will recast the embedding (12) of $J_k^{\text{reg}}(1,n)/\mathbb{G}_k$ given by Proposition 4.4 into a more useful form, still following [**2**]. Let us rewrite the linear system $\Psi \circ \gamma = 0$ associated to $\gamma \in J_k^{\text{reg}}(1,n)$ in a dual form. The system is based on the standard composition map (8):

$$J_k(n,N) \times J_k(1,n) \longrightarrow J_k(1,N),$$

which, via the identification $J_k(n,N) = J_k(n,1) \otimes \mathbb{C}^N$, is derived from the map

$$J_k(n,1) \times J_k(1,n) \longrightarrow J_k(1,1)$$

via tensoring with $\mathbb{C}^N$. Observing that composition is linear in its first argument, and passing to linear duals, we may rewrite this correspondence in the form

$$\phi : J_k(1,n) \longrightarrow \text{Hom}\,(J_k(1,1)^*, J_k(n,1)^*). \tag{13}$$

If $\gamma = (\gamma', \gamma'', \ldots, \gamma^{(k)}) \in J_k(1,n) = (\mathbb{C}^n)^k$ is the k-jet of a curve, we can put $\gamma^{(j)} \in \mathbb{C}^n$ into the jth column of an $n \times k$ matrix, and

- identify $J_k(1,n)$ with $\text{Hom}\,(\mathbb{C}^k, \mathbb{C}^n)$;
- identify $J_k(n,1)^*$ with $\text{Sym}^{\leq k}\mathbb{C}^n = \oplus_{l=1}^k \text{Sym}^{\,l}\mathbb{C}^n$;
- identify $J_k(1,1)^*$ with $\mathbb{C}^k$.

Using these identifications, we can recast the map ϕ in (13) as

$$\phi_k : \text{Hom}\,(\mathbb{C}^k, \mathbb{C}^n) \longrightarrow \text{Hom}\,(\mathbb{C}^k, \text{Sym}^{\leq k}\mathbb{C}^n), \tag{14}$$

which may be written out explicitly as follows

$$(\gamma', \gamma'', \ldots, \gamma^{(k)}) \longmapsto \left(\gamma', \gamma'' + (\gamma')^2, \ldots, \sum_{i_1+i_2+\cdots+i_s=d} \frac{1}{i_1! \ldots i_s!} \gamma^{(i_1)}\gamma^{(i_2)} \ldots \gamma^{(i_s)} \right).$$

The set of solutions $\mathcal{S}_\gamma$ is the linear subspace orthogonal to the image of $\phi_k(\gamma', \ldots \gamma^{(k)})$ tensored by $\mathbb{C}^N$; that is,

$$\mathcal{S}_\gamma = \text{im}(\phi_k(\gamma))^\perp \otimes \mathbb{C}^N \subset J_k(n,N).$$

Consequently, it is straightforward to take $N = 1$ and define

$$\mathcal{S}_\gamma = \operatorname{im}(\phi_k(\gamma)) \in \operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n). \tag{15}$$

Moreover, let $B_k \subset GL(k)$ denote the Borel subgroup consisting of upper triangular matrices and let

$$\begin{aligned}\operatorname{Flag}_k(\mathbb{C}^n) &= \operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^{\leq k}\mathbb{C}^n)/B_k \\ &= \{0 = F_0 \subset F_1 \subset \cdots \subset F_k \subset \mathbb{C}^n, \ \dim F_l = l\}\end{aligned}$$

denote the full flag of k-dimensional subspaces of $\operatorname{Sym}^{\leq k}\mathbb{C}^n$. In addition to (15) we can analogously define

$$\mathcal{F}_\gamma = (\operatorname{im}(\phi(\gamma^1)) \subset \operatorname{im}(\phi(\gamma^2)) \subset \cdots \subset \operatorname{im}(\phi(\gamma^k))) \in \operatorname{Flag}_k(\operatorname{Sym}^{\leq k}\mathbb{C}^n). \tag{16}$$

Using these definitions Proposition 4.3 implies the the following version of Proposition 4.4, which does not contain the parameter N.

Proposition 5.1. *The map ϕ in* (14) *is a $\mathbb{G}_k$-invariant algebraic morphism*

$$\phi : J_k^{\mathrm{reg}}(1,n) \to \operatorname{Hom}(\mathbb{C}^k, \operatorname{Sym}^{\leq k}\mathbb{C}^n),$$

which induces

- *an injective map on the $\mathbb{G}_k$-orbits to the Grassmannian:*
$$\phi^{Gr} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n)$$
defined by $\phi^{Gr}(\gamma) = \mathcal{S}_\gamma$;
- *an injective map on the $\mathbb{G}_k$-orbits to the flag manifold:*
$$\phi^{Flag} : J_k^{\mathrm{reg}}(1,n)/G_k \hookrightarrow \operatorname{Flag}_k(\operatorname{Sym}^{\leq k}\mathbb{C}^n)$$
defined by $\phi^{Flag}(\gamma) = \mathcal{F}_\gamma$.

In addition,

$$\phi^{Gr} = \phi^{Flag} \circ \pi_k$$

where $\pi_k : \operatorname{Flag}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n) \to \operatorname{Grass}_k(\operatorname{Sym}^{\leq k}\mathbb{C}^n)$ is the projection to the k-dimensional subspace.

Composing ϕ^{Gr} with the Plücker embedding

$$\operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n) \hookrightarrow \mathbb{P}(\wedge^k \operatorname{Sym}^{\leq k}\mathbb{C}^n)$$

we get an embedding

$$\phi^{\mathrm{Proj}} : J_k^{\mathrm{reg}}(1,n)/\mathbb{G}_k \hookrightarrow \mathbb{P}(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^n)). \tag{17}$$

The image

$$\phi^{Gr}(J_k^{\mathrm{reg}}(1,n))/\mathbb{G}_k \ \subset \ \operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n)$$

is a $GL(n)$-orbit in $\operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n)$, and therefore a nonsingular quasi-projective variety. Its closure is, however, a highly singular subvariety of $\operatorname{Grass}(k, \operatorname{Sym}^{\leq k}\mathbb{C}^n)$, which when $k \leq n$ is a finite union of $GL(n)$ orbits.

Definition 5.2. *Recall that we can identify* $J_k(1,n)$ *with* $\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)$ *and then*

$$J_k^{\mathrm{reg}}(1,n)=\{\rho\in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n):\rho(e_1)\neq 0\}.$$

Let

$$J_k^{\mathrm{nondeg}}(1,n)=\{\rho\in \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n):\mathrm{rank}\rho=\max\{k,n\}\}$$

and let

$$X_{n,k}=\phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(1,n)),\;\; Y_{n,k}=\phi^{\mathrm{Proj}}(J_k^{\mathrm{reg}}(1,n)),$$

so that if $n\leq k$ *then*

$$X_{n,k}\subset Y_{n,k}\subset \mathrm{Grass}(n,\mathrm{Sym}^{\leq k}\mathbb{C}^n)\subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n)).$$

It is clear that $J_k^{\mathrm{nondeg}}(1,n)$ is an open subset of $J_k^{\mathrm{reg}}(1,n)$. If we identify the elements of $J_k(1,n)$ with $n\times k$ matrices whose columns are the derivatives of the map germs $f=(f',\dots,f^{(n)}):\mathbb{C}\to\mathbb{C}^n$, then $J_k^{\mathrm{nondeg}}(1,n)$ is the set of such matrices of maximal rank and $J_k^{\mathrm{reg}}(1,n)$ consists of the matrices with nonzero first column.

Definition 5.3. *Let* $e_1,\dots,e_n$ *be the standard basis of* $\mathbb{C}^n$*; then*

$$\{e_{i_1,i_2,\dots,i_s}=e_{i_1}\dots e_{i_s}:1\leq i_1\leq\dots\leq i_s\leq n, 1\leq s\leq k\}$$

is a basis of $\mathrm{Sym}^{\leq k}\mathbb{C}^n$*, and*

$$\{e_{\varepsilon_1}\wedge\dots\wedge e_{\varepsilon_n}:\varepsilon_l\in\Pi_{\leq n}\}$$

is a basis of $\mathbb{P}(\wedge^n(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$*, where*

$$\Pi_{\leq n}=\{(i_1,i_2,\dots,i_s):1\leq i_1\leq\dots\leq i_s\leq n, 1\leq s\leq k\}.$$

The corresponding coordinates of $x\in \mathrm{Sym}^{\leq k}\mathbb{C}^n$ *will be denoted by* $x_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_d}$*. Let* $A_{n,k}\subset\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ *consist of the points whose projection to* $\wedge^k(\mathbb{C}^n)$ *is nonzero. This is the subset where* $x_{i_1,i_2,\dots,i_k}\neq 0$ *for some* $1\leq i_1\leq\dots\leq i_k\leq n$*.*

Remark 5.4. If $n=k$ then $A_{n,n}\subset\mathbb{P}(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ is the affine chart where $x_{1,2,\dots,n}\neq 0$.

Let us take a closer look at the space $\mathrm{Grass}(n,\mathrm{Sym}^{\leq k}\mathbb{C}^n)$, which has an induced $\mathrm{GL}(n)$ action coming from the $\mathrm{GL}(n)$ action on $\mathrm{Sym}^{\leq k}\mathbb{C}^n$. Since ϕ^{Proj} is a $\mathrm{GL}(n)$-equivariant embedding, we conclude that

Lemma 5.5. *(i) For* $k\leq n$ $X_{n,k}$ *is the* $\mathrm{GL}(n)$ *orbit of*

$$\mathbf{z}=\phi^{\mathrm{Proj}}(e_1,\dots,e_k)=\left[e_1\wedge(e_2+e_1^2)\wedge\dots\wedge\left(\sum_{i_1+\dots i_s=k}e_{i_1}\dots e_{i_s}\right)\right] \tag{18}$$

in $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$. *For arbitrary* $g \in \mathrm{GL}(n)$ *with column vectors* $v_1, \dots, v_n$ *the action is given by*

$$\begin{aligned} g \cdot \mathbf{z} &= \phi^{\mathrm{Proj}}(g) = \phi^{\mathrm{Proj}}(v_1, \dots, v_n) \\ &= \left[v_1 \wedge (v_2 + v_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=n} v_{i_1} \dots v_{i_s} \right) \right]. \end{aligned}$$

(ii) *For* $k \le n$ $Y_{n,k}$ *is a finite union of* $\mathrm{GL}(n)$ *orbits.*
(iii) *For* $k > n$ *the images* $X_{n,k}$ *and* $Y_{n,k}$ *are* $\mathrm{GL}(n)$*-invariant quasi-projective varieties with no dense* $\mathrm{GL}(n)$ *orbit.*

Lemma 5.6. *If* $k \le n$ *then*

(i) $A_{n,k}$ *is invariant under the* $\mathrm{GL}(n)$ *action on* $\mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^n))$.
(ii) $X_{n,k} \subset A_{n,k}$*; however,* $Y_{n,k} \not\subseteq A_{n,k}$.

Proof. To prove the first part take a lift

$$\tilde{z} = \tilde{z}^1 \oplus \tilde{z}^2 \in \mathrm{Hom}\,(\mathbb{C}^n, \mathrm{Sym}^{\le k}\mathbb{C}^n)$$

of $z \in \mathrm{Grass}(n, \mathrm{Sym}^{\le k}\mathbb{C}^n)$, where

$$z^1 \in \mathrm{Hom}\,(\mathbb{C}^n, \mathbb{C}^n) \quad \text{and} \quad z^2 \in \mathrm{Hom}\,(\mathbb{C}^n, \oplus_{i=2}^n \mathrm{Sym}^{\,i}(\mathbb{C}^n))$$

Then $z \in A_{n,k}$ if and only if $x_{1,2,\dots,n}(z) = \det(\tilde{z}^1) \ne 0$, which is preserved by the $\mathrm{GL}(n)$ action. For the second part note that for $(v_1, \dots, v_k) \in J_k^{\mathrm{nondeg}}(1,n)$ we have $v_1 \wedge \cdots \wedge v_k \ne 0$ so by definition $\phi^{\mathrm{Proj}}(v_1, \dots, v_k) \in A_{n,k}$. On the other hand

$$\phi^{\mathrm{Proj}}(e_1, 0, \dots, 0) = e_1 \wedge e_1^2 \wedge \cdots \wedge e_1^k \in Y_{n,k} \setminus A_{n,k}. \qquad \square$$

When $k = n$ we have

Lemma 5.7. $X_{k,k} \cong \mathrm{GL}(k)/\mathbb{G}_k$ *is embedded in the affine space* $A_{k,k} \subset \mathbb{P}(\wedge^k \mathrm{Sym}^{\le k}\mathbb{C}^k)$ *as the* $\mathrm{GL}(k)$ *orbit of* $[e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge (\sum_{i_1+\cdots+i_s=k} e_{i_1} \dots e_{i_s})]$.

6. Affine embeddings of $\mathrm{SL}(k)/\mathbb{U}_k$

In the last section we embedded $\mathrm{GL}(k)/\mathbb{G}_k$ in the affine space $A_{k,k} \subset \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^k))$ as the $\mathrm{GL}(k)$ orbit of

$$\left[e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} e_{i_1} \dots e_{i_s} \right) \right] \in \mathbb{P}(\wedge^k(\mathrm{Sym}^{\le k}\mathbb{C}^k)).$$

Equivalently we have

$$\mathrm{SL}(k)/\mathrm{SL}(k) \cap \mathbb{G}_k = \mathrm{SL}(k)/\mathbb{U}_k \rtimes F_k$$

embedded in $\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)$ as the SL(k) orbit of

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} e_{i_1} \dots e_{i_s} \right),$$

where $\mathrm{SL}(k) \cap \mathbb{G}_k$ is the semi-direct product $\mathbb{U}_k \rtimes F_k$ of $\mathbb{U}_k$ by the finite group F_k of ℓ_kth roots of unity in $\mathbb{C}$ for $\ell_k = 1+\cdots+k = \binom{k+1}{2}$, embedded in SL$(k)$ as

$$\epsilon \mapsto \begin{pmatrix} \epsilon & 0 & \dots & 0 \\ 0 & \epsilon^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \epsilon^k \end{pmatrix} \in \mathrm{SL}(k).$$

In this section we will look for affine embeddings of $\mathrm{SL}(k)/\mathbb{U}_k$ in spaces of the form

$$W_{k,K} = \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

for suitable K and study their closures.

Lemma 6.1. *Let $K = M(1+2+\cdots+k)+1 = \binom{k+1}{2}M+1$ where $M \in \mathbb{N}$. Then the point*

$$p_k \otimes e_1^{\otimes K} \in \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

where

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} e_{i_1} \dots e_{i_s} \right) \in \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)$$

has stabiliser $\mathbb{U}_k$ in SL(k).

Proof. By Proposition 5.1 the stabiliser of

$$[p_k] \in \mathbb{P}(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)) \cong \mathbb{P}(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}e_1)^{\otimes K}) \subseteq \mathbb{P}(W_{k,K})$$

in $GL(k)$ is $\mathbb{G}_k = \mathbb{U}_k \rtimes \mathbb{C}^*$, so the stabiliser of

$$p_k \otimes e_1^{\otimes K} \in \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

is contained in $\mathbb{G}_k$. Moreover by the proof of Proposition 5.1 the stabiliser of $p_k \otimes e_1^{\otimes K}$ contains $\mathbb{U}_k$. Finally

$$\begin{pmatrix} z & 0 & \dots & 0 \\ 0 & z^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & z^k \end{pmatrix} \in \mathbb{C}^* \subseteq \mathbb{G}_k$$

acts on $p_k \otimes e_1^{\otimes K}$ as multiplication by

$$z^{1+2+\cdots+k+K} = z^{(M+1)(1+2+\cdots+k)+1}$$

and has determinant 1 if and only if $z^{1+2+\cdots+k}=1$, so it lies in $\mathrm{SL}(k)$ and fixes $p_k \otimes e_1^{\otimes K}$ if and only if $z=1$. $\square$

We will prove

THEOREM 6.2. *If $k \geq 4$ and $K = M(1+2+\cdots+k)+1$ where $M \in \mathbb{N}$ is sufficiently large, then the orbit of $p_k \otimes e_1^{\otimes K}$ where*

$$p_k = e_1 \wedge (e_2 + e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} e_{i_1} \ldots e_{i_s} \right) \in \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)$$

under the natural action of $\mathrm{SL}(k)$ on

$$W_{k,K} = \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$$

is isomorphic to $\mathrm{SL}(k)/\mathbb{U}_k$, and its complement in its closure $\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ in $W_{k,K}$ has codimension at least two.

This theorem has an immediate corollary.

COROLLARY 6.3. *If $k \geq 2$ then $\mathbb{U}_k$ is a Grosshans subgroup of $\mathrm{SL}(k)$, so that every linear action of $\mathbb{U}_k$ which extends to a linear action of $\mathrm{SL}(k)$ has finitely generated invariants.*

PROOF. This follows directly from Theorem 6.2 when $k \geq 4$. When $k=2$ and $k=3$ it is already known (cf. [**27**]). $\square$

The remainder of this section will be devoted to proving Theorem 6.2.

It follows directly from Lemma 6.1 that the $\mathrm{SL}(k)$-orbit of $p_k \otimes e_1^{\otimes K}$ in $W_{k,K} = \wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k) \otimes (\mathbb{C}^k)^{\otimes K}$ is isomorphic to $\mathrm{SL}(k)/\mathbb{U}_k$.

Recall that

$$\mathbb{U}_k = \left\{ \begin{pmatrix} 1 & \alpha_2 & \alpha_3 & \cdots & \alpha_k \\ 0 & 1 & 2\alpha_2 & \cdots & 2\alpha_{k-1}+\cdots \\ 0 & 0 & 1 & \cdots & 3\alpha_{k-2}+\cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & (k-1)\alpha_2 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} : \alpha_2, \ldots, \alpha_k \in \mathbb{C} \right\}$$

so that $\mathbb{U}_k$ is generated along its last column as well as along its first row.

Let $B_k \subset \mathrm{SL}(k)$ denote the standard Borel subgroup of $\mathrm{SL}(k)$ which stabilises the filtration $\mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \cdots \mathbb{C}^k$. Then $B_k = B_{k-1} \cdot \mathbb{U}_k$ where the Borel subgroup B_{k-1} of $\mathrm{GL}(k-1) = \mathrm{GL}(\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{k-1})$ is embedded diagonally in $\mathrm{SL}(k)$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix}.$$

Since $\mathbb{U}_k$ stabilises p_k and e_1 we have

$$\overline{B_k(p_k \otimes e_1^{\otimes K})} = \overline{B_{k-1}(p_k \otimes e_1^{\otimes K})},$$

and since $\mathrm{SL}(k)/B_k$ is projective we have

$$\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})} = \mathrm{SL}(k)\overline{B_k(p_k \otimes e_1^{\otimes K})} = \mathrm{SL}(k)\overline{B_{k-1}(p_k \otimes e_1^{\otimes K})}.$$

Since the closure $\overline{\mathrm{SL}(k)(p_k \otimes e_1^{\otimes K})}$ of the $\mathrm{SL}(k)$-orbit of $p_k \otimes e_1^{\otimes K}$ in $W_{k,K}$ is the union of finitely many $\mathrm{SL}(k)$-orbits, to prove Theorem 6.2 it suffices to prove

Lemma 6.4. *Suppose that $k \geq 4$ and a and b are strictly positive integers with b/a large enough and that x lies in the closure in*

$$(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the Borel subgroup B_k of $\mathrm{SL}(k)$. Then either $x \in B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k+1$.

We will split the proof of this lemma into two parts. Let T_k denote the standard maximal torus of $\mathrm{SL}(k)$ consisting of the diagonal matrices in $\mathrm{SL}(k)$. Lemma 6.4 follows immediately from Lemmas 6.5 and 6.6 below.

Lemma 6.5. *Suppose that $k \geq 4$ and a and b are strictly positive integers with b/a large enough and that x lies in the closure $\overline{T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})}$ in*

$$(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the maximal torus T_k of $\mathrm{SL}(k)$. Then either $x \in T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k+1$.

Lemma 6.6. *Suppose that $k \geq 2$ and a and b are strictly positive integers and that x lies in the closure in*

$$(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b}$$

of the orbit $B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})$ of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ under the natural action of the Borel subgroup B_k of $\mathrm{SL}(k)$. Then either $x \in B_k\overline{T_k(p_k^{\otimes a} \otimes e_1^{\otimes b})}$ or the stabiliser of x in $\mathrm{SL}(k)$ has dimension at least $k+1$.

We will start with the proof of Lemma 6.6.

Proof. We have

$$x \in \overline{B_k(p_k^{\otimes a} \otimes e_1^{\otimes b})} = \overline{B_{k-1}(p_k^{\otimes a} \otimes e_1^{\otimes b})}$$

as above, so there is a sequence of matrices

$$b^{(m)} = \begin{pmatrix} b_{11}^{(m)} & b_{12}^{(m)} & \dots & b_{1k-1}^{(m)} & 0 \\ 0 & b_{22}^{(m)} & \dots & b_{2k-1}^{(m)} & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 0 & b_{kk}^{(m)} \end{pmatrix} \in B_{k-1} \subset \mathrm{SL}(k)$$

such that $b^{(m)}(p_k^{\otimes a} \otimes e_1^{\otimes b}) \to x$ as $m \to \infty$. Now expanding the wedge product in the definition of p_k we get

$$b^{(m)}(p_k^{\otimes a}) = \left(e_1 \wedge \dots \wedge e_n + \dots + (b_{11}^{(m)})^{1+2+\dots+k} e_1 \otimes e_1^2 \otimes \dots \otimes e_1^k\right)^{\otimes a}$$

while

$$b^{(m)}(e_1^{\otimes b}) = (b_{11}^{(m)})^b e_1^{\otimes b},$$

so by considering the coefficient of $(e_1 \wedge \dots \wedge e_n)^{\otimes a} \otimes e_1^{\otimes b}$ we see that $(b_{11}^{(m)})^b$ tends to a limit in $\mathbb{C}$ as $m \to \infty$. Thus, by replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$b_{11}^{(m)} \to b_{11}^{(\infty)} \in \mathbb{C}$$

as $m \to \infty$.

First suppose that $k = 2$. Then $\mathrm{Sym}^{\leq k}\mathbb{C}^k = \mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2$ and

$$(\wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k))^{\otimes a} \otimes (\mathbb{C}^k)^{\otimes b} = (\wedge^2(\mathbb{C}^2 \oplus \mathrm{Sym}^2\mathbb{C}^2))^{\otimes a} \otimes (\mathbb{C}^2)^{\otimes b}$$

and

$$p_k = e_1 \wedge (e_2 + e_1^2),$$

so if

$$b^{(m)} = \begin{pmatrix} b_{11}^{(m)} & b_{12}^{(m)} \\ 0 & b_{22}^{(m)} \end{pmatrix} \in \mathrm{SL}(2)$$

then $b_{11}^{(m)} b_{22}^{(m)} = 1$ and

$$b^{(m)}(p_2^{\otimes a} \otimes e_1^{\otimes b}) = (b_{11}^{(m)})^b (e_1 \wedge (e_2 + (b_{11}^{(m)})^3 e_1^2)))^{\otimes a} \otimes e_1^{\otimes b}$$

$$\to x = (b_{11}^{(\infty)})^b (e_1 \wedge (e_2 + (b_{11}^{(\infty)})^3 e_1^2)))^{\otimes a} \otimes e_1^{\otimes b}$$

as $m \to \infty$. If $b_{11}^{(\infty)} \neq 0$ then $x \in \mathrm{SL}(2)((p_2^{\otimes a} \otimes e_1^{\otimes b})$, while if $b_{11}^{(\infty)} = 0$ then $x = 0$ is fixed by $\mathrm{SL}(2)$ which has dimension $3 = k + 1$.

Now suppose that $k > 2$, and assume first that $b_{11}^{(\infty)} \neq 0$. We have that

$$b^{(m)}(p_k^{\otimes a} \otimes e_1^{\otimes b}) = (b_{11}^{(m)})^b (b^{(m)} p_k)^{\otimes a}) \otimes e_1^{\otimes b} \to x$$

and $b_{11}^{(m)} \to b_{11}^{(\infty)} \in \mathbb{C} \setminus \{0\}$ as $m \to \infty$, so by replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$(b_{11}^{(m)})^{b/a} b^{(m)} p_k \to p_k^{\infty} \in \wedge^k(\mathrm{Sym}^{\leq k}\mathbb{C}^k)$$

as $m \to \infty$, where

$$b^{(m)}p_k = b_{11}^{(m)}e_1 \wedge (b_{22}^{(m)}e_2 + (b_{11}^{(m)})^2 e_1^2) \wedge \cdots \wedge (b_{ii}^{(m)}e_i + b_{i-1i}^{(m)}e_{i-1} + \cdots + b_{1i}^{(m)}e_1 + \sum_{s=2}^{i-1} \sum_{i_1+\cdots+i_s=i} (b_{i_1i_1}^{(m)}e_{i_1} + \cdots + b_{1i_1}^{(m)}e_1)\cdots \times (b_{i_si_s}^{(m)}e_{i_s} + \cdots + b_{1i_s}^{(m)}e_1) + (b_{11}^{(m)})^i e_1^i) \wedge \cdots \tag{19}$$

Looking at the coefficient of

$$e_1 \wedge e_1^2 \wedge \cdots \wedge e_1^{i-1} \wedge e_j \wedge e_1^{i+1} \wedge \cdots \wedge e_1^k$$

when $1 \le j \le i \le k$, we see that

$$(b_{11}^{(m)})^{1+2+\cdots+(i-1)+(i+1)+\cdots+k} b_{ji}^{(m)}$$

tends to a limit in $\mathbb{C}$ as $m \to \infty$, and so since $b_{11}^{(\infty)} \neq 0$

$$b_{ji}^{(m)} \to b_{ji}^{(\infty)} \in \mathbb{C}.$$

Also $b_{11}^{(m)}b_{22}^{(m)}\cdots b_{kk}^{(m)} = 1$ for all m, so $b_{11}^{(\infty)}b_{22}^{(\infty)}\cdots b_{kk}^{(\infty)} = 1$, so $b^{(m)} \to b^{(\infty)} \in \mathrm{SL}(k)$. Therefore

$$x = b^{(\infty)}(p_k^{\otimes a} \otimes e_1^{\otimes b})$$

lies in the orbit of $p_k^{\otimes a} \otimes e_1^{\otimes b}$ as required.

So it remains to consider the case when $b_{11}^{(\infty)} = 0$. If $p_k^{\infty} = 0$ then its stabiliser is $\mathrm{SL}(k)$ which has dimension $k^2 - 1 \ge k+1$, so we can assume that $p_k^{\infty} \neq 0$. Recall that then

$$(b_{11}^{(m)})^{b/a} b^{(m)}p_k \to p_k^{\infty} \in \wedge^k(\operatorname{Sym}^{\le k}\mathbb{C}^k)$$

and

$$[b^{(m)}p_k] \to [p_k^{\infty}] \in \mathbb{P}(\wedge^k(\operatorname{Sym}^{\le k}\mathbb{C}^k))$$

as $m \to \infty$, where

$$b^{(m)}p_k = b_{11}^{(m)}e_1 \wedge (b_{22}^{(m)}e_2 + (b_{11}^{(m)})^2 e_1^2) \wedge \cdots \wedge (b_{ii}^{(m)}e_i + b_{i-1i}^{(m)}e_{i-1} + \cdots + b_{1i}^{(m)}e_1 + \sum_{s=2}^{i-1} \sum_{i_1+\cdots+i_s=i} (b_{i_1i_1}^{(m)}e_{i_1} + \cdots + b_{1i_1}^{(m)}e_1)\cdots \times (b_{i_si_s}^{(m)}e_{i_s} + \cdots + b_{1i_s}^{(m)}e_1) + (b_{11}^{(m)})^i e_1^i) \wedge \cdots$$

By replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$[b_{ii}^{(m)}e_i + b_{i-1i}^{(m)}e_{i-1} + \cdots + b_{1i}^{(m)}e_1] \to [c_{ii}^{(\infty)}e_i + c_{i-1i}^{(\infty)}e_{i-1} + \cdots + c_{1i}^{(\infty)}e_1] \in \mathbb{P}(\mathbb{C}^k)$$

as $m \to \infty$ for $2 \le i \le k$, which implies that

$$[(b^{(m)}_{i_1 i_1} e_{i_1} + \cdots + b^{(m)}_{1 i_1} e_1) \ldots (b^{(m)}_{i_s i_s} e_{i_s} + \cdots + b^{(m)}_{1 i_s} e_1)] \\ \to [(c^{(\infty)}_{i_1 i_1} e_{i_1} + \cdots + c^{(\infty)}_{1 i_1} e_1) \ldots (c^{(\infty)}_{i_s i_s} e_{i_s} + \cdots + c^{(\infty)}_{1 i_s} e_1)] \ \in \ \mathbb{P}(\mathrm{Sym}^i \mathbb{C}^k)$$

whenever $i_1 + \cdots + i_s = i \in \{2, \ldots, k\}$, and hence that

$$p_k^\infty \in \wedge^k (\mathrm{Sym}^{\le k} D)$$

where D is the span in $\mathbb{C}^k$ of

$$\{e_1\} \cup \{c^{(\infty)}_{ii} e_i + c^{(\infty)}_{i-1 i} e_{i-1} + \cdots + c^{(\infty)}_{1i} e_1 : 2 \le i \le k\}.$$

Moreover since $b^{(m)} \in B_{k-1}$ we have $b^{(m)}_{jk} = 0$ if $j < k$ so

$$[c^{(\infty)}_{kk} e_k + c^{(\infty)}_{k-1 k} e_{k-1} + \cdots + c^{(\infty)}_{1k} e_1] = [e_k]$$

so $e_k \in D$.

Note that $b^{(m)} \in B_{k-1}$ and B_{k-1} normalises the maximal unipotent subgroup U_k of B_k which contains the stabiliser $\mathbb{U}_k$ of p_k. Therefore for each m there is a $(k-1)$-dimensional subgroup of U_k which stabilises $b^{(m)} p_k$, and it follows that there is a $(k-1)$-dimensional subgroup $\mathbb{U}_k^\infty$ of U_k which stabilises p_k^∞. In addition by [**3**] Theorem 6.4 if p_k^∞ does not lie in $\mathrm{SL}(k) p_k$ then it is stabilised by a nontrivial one-parameter subgroup $\lambda^\infty : \mathbb{C}^* \to \mathrm{SL}(k)$ of $\mathrm{SL}(k)$. Moreover if $D \ne \mathbb{C}^k$ then there is some $j \in \{2, \ldots, k-1\}$ such that e_j is not in D, and then there is an automorphism of $\mathbb{C}^k$ which fixes every element of D and sends e_j to $e_j + e_k$. This automorphism is independent of $\mathbb{U}_k^\infty$ (since $\mathbb{U}_k^\infty \subseteq U_k$) and the one-parameter subgroup λ^∞ of $\mathrm{SL}(k)$ fixing p_k^∞, so the stabiliser of p_k^∞ in $\mathrm{SL}(k)$ has dimension at least

$$\dim \mathbb{U}_k^\infty + 2 = k + 1.$$

Thus we can assume that $D = \mathbb{C}^k$, and hence $c^{(\infty)}_{ii} \ne 0$ for $2 \le i \le k$, so that

$$\frac{b^{(m)}_{ji}}{b^{(m)}_{ii}} \to \frac{c^{(\infty)}_{ji}}{c^{(\infty)}_{ii}} \in \mathbb{C}$$

as $m \to \infty$. Then by applying an element of B_{k-1} to p_k^∞ we can assume that

$$[c^{(\infty)}_{ii} e_i + c^{(\infty)}_{i-1 i} e_{i-1} + \cdots + c^{(\infty)}_{1i} e_1] = [e_i]$$

or equivalently that

$$[b^{(m)}_{ii} e_i + b^{(m)}_{i-1 i} e_{i-1} + \cdots + b^{(m)}_{1i} e_1] \to [e_i]$$

as $m \to \infty$ for $2 \le i \le k$, and hence that

$$[(b^{(m)}_{i_1 i_1} e_{i_1} + \cdots + b^{(m)}_{1 i_1} e_1) \ldots (b^{(m)}_{i_s i_s} e_{i_s} + \cdots + b^{(m)}_{1 i_s} e_1)] \to [e_{i_1} \cdots e_{i_s}] \in \mathbb{P}(\mathrm{Sym}^i \mathbb{C}^k)$$

whenever $i_1+\cdots+i_s=i\in\{2,\ldots,k\}$. Now by again replacing the sequence $(b^{(m)})$ with a subsequence if necessary, we can assume that

$$[b_{ii}^{(m)}e_i+b_{i-1i}^{(m)}e_{i-1}+\cdots+b_{1i}^{(m)}e_1+\sum_{s=2}^{i-1}\sum_{i_1+\cdots+i_s=i}(b_{i_1i_1}^{(m)}e_{i_1}+\cdots+b_{1i_1}^{(m)}e_1]$$
$$\to[d_i^\infty]\in\mathbb{P}(\operatorname{Sym}^{\leq k}\mathbb{C}^k)$$

where

$$d_i^\infty=\gamma_i^{(\infty)}e_i+\sum_{s=2}^{i}\sum_{i_1+\cdots+i_s=i}\gamma_{i_1\ldots i_s}^{(\infty)}e_{i_1}\cdots e_{i_s}\in\operatorname{Sym}^{\leq k}\mathbb{C}^k\setminus\{0\}$$

for some $\gamma_{i_1\ldots i_s}^{(\infty)}\in\mathbb{C}$. In addition $\{d_i^\infty:1\leq i\leq k\}$ is linearly independent so that

$$[p_k^\infty]=[d_i^\infty\wedge\cdots\wedge d_k^\infty]\in\mathbb{P}(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k))$$

and $p_k^\infty=\lim_{m\to\infty}t^{(m)}p_k$ where $t^{(m)}$ is the diagonal matrix with entries $b_{11}^{(m)},\ldots,b_{kk}^{(m)}$.

Thus we can assume that $p_k^\infty\in\overline{T_kp_k}$ where T_k is the standard maximal torus in $\mathrm{SL}(k)$, which completes the proof of Lemma 6.6. □

It therefore remains to prove Lemma 6.5. We can continue with the notation above and use the following standard result:

LEMMA 6.7. *Let T be an algebraic torus acting on the projective variety Z, and $z\in Z$. Then $y\in\overline{Tz}$ if and only if there is $\tau\in T$, and a one-parameter subgroup $\lambda:\mathbb{C}^*\to T$ such that $\tau y\in\overline{\lambda(\mathbb{C}^*)z}$.*

Hence we may assume without loss of generality that there is a one-parameter subgroup

$$t\mapsto\lambda(t)=\begin{pmatrix} t^{\lambda_1} & 0 & \cdots & & 0\\ 0 & t^{\lambda_2} & 0 & \cdots & 0\\ & & \cdots & & \\ 0 & \cdots & & 0 & t^{\lambda_k}\end{pmatrix}$$

of $\mathrm{SL}(k)$ such that $\lambda_1>0$ and $t^{\lambda_1 b/a}\lambda(t)p_k\to p_k^\infty$ as $t\to 0$. Therefore

$$p_k^\infty=\lim_{t\to 0}t^{\lambda_1 b/a}e_1\wedge(e_2+t^{2\lambda_1-\lambda_2}e_1^2)\wedge\cdots\wedge$$
$$\times\left(e_k+\sum_{s=2}^{k}\sum i_1+\cdots+i_s=kt^{\lambda_{i_1}+\cdots+\lambda_{i_s}-\lambda_k}e_{i_1}\cdots e_{i_s}\right)$$

where $\lambda_1 + \cdots + \lambda_k = 0$. We are assuming that $p_k^\infty \neq 0$ so

$$[p_k^\infty] = \lim_{t\to 0} \Bigg[e_1 \wedge (e_2 + t^{2\lambda_1 - \lambda_2} e_1^2) \wedge \cdots \wedge \\ \times \left(e_k + \sum_{s=2}^{k} \sum i_1 + \cdots + i_s = k t^{\lambda_{i_1} + \cdots + \lambda_{i_s} - \lambda_k} e_{i_1} \cdots e_{i_s} \right) \Bigg].$$

If $\lambda_{i_1} + \cdots + \lambda_{i_s} < \lambda_j$ for some $j \in \{2, \ldots, k-1\}$ and $s \geq 2$ and $i_1, \ldots, i_s \geq 1$ such that $i_1 + \cdots + i_s = j$, then $[p_k^\infty]$ is independent of e_j and so as above the stabiliser of p_k^∞ in $\mathrm{SL}(k)$ has dimension at least $k+1$. So we can assume that

$$\lambda_{i_1} + \cdots + \lambda_{i_s} \geq \lambda_j \tag{20}$$

for any $j \in \{2, \ldots, k-1\}$ and $s \geq 2$ and $i_1, \ldots, i_s \geq 1$ such that $i_1 + \cdots + i_s = j$, and in particular that $\lambda_j \leq j\lambda_1$ for each $j \in \{2, \ldots, k-1\}$. Let

$$\rho_j = j\lambda_1 - \lambda_j \tag{21}$$

for $j \in \{1, \ldots, k-1\}$; then $\rho_1 = 0$ and $\rho_j \geq 0$ and

$$\rho_{i_1} + \cdots + \rho_{i_s} \leq \rho_j$$

for any $j \in \{2, \ldots, k-1\}$ and $s \geq 2$ and $i_1, \ldots, i_s \geq 1$ such that $i_1 + \cdots + i_s = j$. In addition looking at the coefficient of

$$e_1 \wedge e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{i_1} \cdots e_{i_s}$$

where $i_1 + \cdots + i_s = k$, we find that

$$\begin{aligned} 0 &\leq \lambda_1 b/a + \lambda_{i_1} + \cdots + \lambda_{i_s} - \lambda_k \\ &= \lambda_1(b/a + k(k+1)/2) - (\rho_{i_1} + \cdots + \rho_{i_s} + \rho_2 + \cdots + \rho_{k-1}), \end{aligned}$$

and since $p_k^\infty \neq 0$ there is some $i_1, \ldots, i_s$ with $i_1 + \cdots + i_s = k$ and

$$\lambda_1 b/a + \lambda_{i_1} + \cdots + \lambda_{i_s} = \lambda_k \tag{22}$$

or equivalently

$$\lambda_1(b/a + k(k+1)/2) = \rho_{i_1} + \cdots + \rho_{i_s} + \rho_2 + \cdots + \rho_{k-1}.$$

Thus

$$p_k^\infty = \lim_{t\to 0} e_1 \wedge (e_2 + t^{2\lambda_1-\lambda_2}e_1^2) \wedge \cdots \wedge \tag{23}$$
$$\times \left(e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1+\cdots+i_s=k-1} t^{\lambda_{i_1}+\cdots+\lambda_{i_s}-\lambda_{k-1}} e_{i_1}\cdots e_{i_s}\right)$$
$$\wedge \left(t^{\lambda_1 b/a} \sum_{s=2}^{k} \sum_{i_1+\cdots+i_s=k} t^{\lambda_{i_1}+\cdots+r_{i_s}-r_k} e_{i_1}\cdots e_{i_s}\right.$$
$$= e_1 \wedge \cdots \wedge \left(e_{k-1} + \sum_{s=2}^{k-1} \sum_{i_1+\cdots+i_s=k-1:\rho_{i_1}+\cdots+\rho_{i_s}=\rho_{k-1}} e_{i_1}\cdots e_{i_s}\right)$$
$$\wedge \left(\sum_{s=2}^{k} \sum_{\substack{i_1+\cdots+i_s=k \\ \lambda_1(b/a+k(k+1)/2)=\rho_{i_1}+\cdots+\rho_{i_s}+\rho_2+\cdots+\rho_{k-1}}} e_{i_1}\cdots e_{i_s}\right)$$

is independent of e_k and hence is fixed by the automorphisms of $\mathbb{C}^k$ which fix $e_1,\ldots,e_{k-1}$ and send e_k to e_k+e_j for $j\in\{1,\ldots,k-1\}$, as well as by the one-parameter subgroup

$$\lambda(t) = \begin{pmatrix} t^{\lambda_1} & 0 & \cdots & & 0 \\ 0 & t^{\lambda_2} & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & & 0 & t^{\lambda_k} \end{pmatrix}$$

of T_k. Thus to complete the proof of Lemma 6.5 and hence of Theorem 6.2, it suffices to find an additional one-dimensional stabiliser, which will be done in the rest of this section.

Letting

$$\mathbf{z} = [p_k] = \left[e_1 \wedge (e_2+e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} e_{i_1}\ldots e_{i_s}\right)\right]$$

as at (18) we have

$$\lambda(t)\mathbf{z} = \left[t^{\lambda_1}e_1 \wedge (t^{\lambda_2}e_2 + t^{2\lambda_1}e_1^2) \wedge \cdots \wedge \left(\sum_{i_1+\cdots+i_s=k} t^{\lambda_{i_1}+\cdots+\lambda_{i_s}} e_{i_1}\ldots e_{i_s}\right)\right]$$
$$= [t^{\lambda_1+\cdots+\lambda_k}(e_1\wedge\cdots\wedge e_k) + t^{\lambda_1+2\lambda_1+\lambda_3+\cdots+\lambda_k}$$
$$\times (e_1 \wedge e_1^2 \wedge e_3 \wedge \cdots \wedge e_k) + \cdots].$$

The generic term in this expression is

$$t^{\lambda_{\varepsilon_1}+\lambda_{\varepsilon_2}+\cdots\lambda_{\varepsilon_k}}(\mathbf{e}_{\varepsilon_1}\wedge\cdots\wedge\mathbf{e}_{\varepsilon_k}), \quad \Sigma(\varepsilon_i)=i$$

where

$$\lambda_\tau = \sum_{i\in\tau} \lambda_i \quad \text{and} \quad \mathbf{e}_\tau = \Pi_{i\in\tau} e_i \quad \text{if } \tau = (i_1, \dots, i_s). \tag{24}$$

Definition 6.8. *For any one-parameter subgroup λ as above let*

- $m_\lambda = \min_{\substack{(\varepsilon_1,\dots\varepsilon_k)\\ 1\le\Sigma(\varepsilon_i)\le k}} (\lambda_{\varepsilon_1} + \lambda_{\varepsilon_2} + \cdots \lambda_{\varepsilon_k})$,
- $\mathbf{z}_\lambda = [\sum_{1\le\Sigma\varepsilon\le k,\lambda_\varepsilon=m_\lambda} \mathbf{e}_\varepsilon]$,
- $m_\lambda[i] = \min_{\Sigma(\varepsilon)=i} \lambda_\varepsilon$ *for* $1 \le i \le k$,
- $\mathbf{z}_\lambda[i] = [\sum_{\Sigma\varepsilon=i,\lambda_\varepsilon=m_\lambda[i]} \mathbf{e}_\varepsilon]$.

Let $\mathcal{O}_\lambda$ denote the $\mathrm{SL}(k)$*-orbit of* $\mathbf{z}_\lambda$.

It is clear that the one-parameter subgroup $\tilde\lambda(t) = (t, t^2, \dots, t^k)$ stabilises $\mathbf{z}$, where $\mathbf{z}$ is defined as at (18), and therefore $\mathbf{z} = \mathbf{z}_{\tilde\lambda}$ and its $\mathrm{SL}(k)$-orbit is equal to its $\mathrm{GL}(k)$-orbit.

We need a more precise description of the orbit structure of the closure of the orbit $\mathcal{O}_0 = \mathcal{O}_{\tilde\lambda}$. Since $\tilde\lambda_i = i\tilde\lambda_1$ for $i = 1, \dots, k$, for $\lambda \ne \tilde\lambda$ we have a smallest index $\sigma \in \{2, \dots, k\}$ with $\lambda_\sigma \ne \sigma\lambda_1$.

Definition 6.9. *We call $\sigma = Head(\lambda)$ the head of $\lambda = (\lambda_1, \dots, \lambda_n)$ if*

$$\lambda_i = i\lambda_1 \quad \textit{for } i < \sigma \quad \textit{and} \quad \lambda_\sigma \ne \sigma\lambda_1.$$

If $\lambda_\sigma < \sigma\lambda_1$ then we call λ regular *; otherwise we call λ* degenerate*.*

We will say that a one-parameter subgroup λ is *maximal* if the closure of the orbit $\mathrm{GL}(k) \cdot \mathbf{z}_\lambda$ is a maximal boundary component of the closure of the orbit of $\mathbf{z}$.

Definition 6.10. *Fix $0 < \varepsilon < 1$ and $2 \le \sigma \le k$. Let $\lambda^\sigma = (\lambda_1^\sigma, \dots, \lambda_k^\sigma)$ and $\mu^\sigma = (\mu_1^\sigma, \dots, \mu_k^\sigma)$ be the following one-parameter subgroups of* $\mathrm{GL}(k)$*:*

$$\lambda_i^\sigma = i - \left\lfloor \frac{i}{\sigma} \right\rfloor \varepsilon \quad \textit{for } 1 \le i \le k, \tag{25}$$

$$\mu_i^\sigma = \begin{cases} i & \textit{for } i \ne \sigma, i \le k, \\ \sigma + \varepsilon & \textit{for } i = \sigma. \end{cases} \tag{26}$$

It is easy to see that $\mathrm{Head}(\lambda^\sigma) = \mathrm{Head}(\mu^\sigma) = \sigma$, and λ^σ is regular, whereas μ^σ is degenerate.

Definition 6.11. *Let λ be a 1-parameter subgroup. We call*

$$\sharp\{i : \mathbf{z}_\lambda[i] = e_i\}$$

the toral dimension of the limit point $\mathbf{z}_\lambda$.

Lemma 6.12. *If the* $\mathrm{SL}(k)$*-orbit of p_k^∞ has codimension one in $\overline{\mathrm{SL}(k)p_k}$, then $[p_k^\infty]$ lies in the orbit of one of $\mathbf{z}_{\lambda^2}, \dots, \mathbf{z}_{\lambda^k}$ or $\mathbf{z}_{\mu^2}, \dots, \mathbf{z}_{\mu^{k-1}}$.*

Proof. We can assume that $[p_k^\infty]=\mathbf{z}_\lambda$ for some one-parameter subgroup λ. First suppose that λ is a regular one-parameter subgroup with $\mathrm{Head}(\lambda)=\sigma$ and $[p_k^\infty]=\mathbf{z}_\lambda$. Without loss of generality we can assume that

$$\lambda_i = i \quad \text{for } i<\sigma \quad \text{and} \quad \lambda_\sigma=\sigma-\varepsilon.$$

We will call $d(i)=\lfloor \frac{i}{\sigma}\rfloor$ the defect of i and $d(\tau)=d(i_1)+\cdots+d(i_s)$ the defect of $\tau=(i_1,\ldots,i_s)$, so that when $i\le\sigma$ we have $d(i)\epsilon=\rho_i$ as defined at (21). Since

$$\lambda_{(j,\underbrace{\sigma,\ldots,\sigma}_{m})}=j+m(\sigma-\varepsilon) \quad \text{for } 1\le j\le \sigma-1,\ m\ge 0,$$

we have

$$m_\lambda[i]\le i-d(i)\varepsilon \quad \text{for } 1\le i\le k. \tag{27}$$

If $\lambda_s<s-d(s)\varepsilon$ for $s>i$ and s is the smallest index with this property then $m_\lambda[s]=\lambda_s$ and $\mathbf{z}_\lambda[s]=e_s$, so

$$\mathbf{z}_\lambda[1]=e_1,\ \ \mathbf{z}_\lambda[\sigma]=e_\sigma,\ \ \mathbf{z}_\lambda[s]=e_s,$$

while $\mathbf{z}_\lambda$ is independent of e_k by (23), so $[p_k^\infty]$ is fixed by a three-dimensional torus in $\mathrm{SL}(k)$ and thus p_k^∞ is fixed by a two-dimensional torus in $\mathrm{SL}(k)$ as well as a unipotent subgroup of dimension $k-1$. So we can assume that $\lambda_i\ge i-d(i)\varepsilon$ for $1\le i\le k$, and therefore

$$m_\lambda[i]=i-d(i)\varepsilon \quad \text{for } 1\le i\le k.$$

So

$$\mathbf{e}_\tau\notin \mathbf{z}_\lambda[i] \quad \text{if } d(\tau)>d(i). \tag{28}$$

On the other hand the distinguished 1-parameter subgroup λ^σ is defined as $\lambda^\sigma_i=i-d(i)\varepsilon$, and therefore

$$\mathbf{z}_{\lambda^\sigma}[i]=\sum_{\Sigma(\tau)=i,d(\tau)=d(i)}\mathbf{e}_\tau. \tag{29}$$

Comparing (28) and (29) we conclude

$$\mathbf{z}_\lambda[i]\subset \mathbf{z}_{\lambda^\sigma}[i] \quad \text{for } 1\le i\le k.$$

Now let μ be a degenerate 1-parameter subgroup with $\mathrm{Head}(\mu)=\sigma$. Without loss of generality we can assume again that

$$\mu_i=i \quad \text{for } i<\sigma \quad \text{and} \quad \mu_\sigma=\sigma+\varepsilon.$$

Since

$$\mu_{(\underbrace{1,\ldots 1}_{i})}=i \quad \text{for } 1\le i\le k$$

we have

$$m_\mu[i]\le i. \tag{30}$$

Again, $\mu_s < s$ cannot happen for $s > \sigma$ since in that case $\mathbf{z}_\mu[s] = e_s$ would hold and the codimension of $\mathrm{SL}(k)p_k^\infty$ would be at least two. So $\mu_s \geq s$ and therefore $\mu_\tau \geq \Sigma(\tau)$ with strict inequality if $\sigma \in \tau$. Therefore

$$\mathbf{e}_\tau \notin \mathbf{z}_\mu[i] \quad \text{if } \sigma \in \tau. \tag{31}$$

On the other hand μ^σ satisfies equality in (30), and

$$\mathbf{z}_{\mu^\sigma}[i] = \sum_{\Sigma(\tau)=i, \sigma \notin \tau} \mathbf{e}_\tau. \tag{32}$$

Comparing (31) and (32) we get

$$\mathbf{z}_\mu[i] \subset \mathbf{z}_{\mu^\sigma}[i] \quad \text{for } 1 \leq i \leq k,$$

and so it remains to consider the possibility that $[p_k^\infty] = \mathbf{z}_{\mu^k}$. But by (22) there is some $i_1, \ldots, i_s$ with $i_1 + \cdots + i_s = k$ and

$$\lambda_1 b/a + \lambda_{i_1} + \cdots + \lambda_{i_s} = \lambda_k$$

and hence $\lambda_k > \lambda_{i_1} + \cdots + \lambda_{i_s}$. Thus $[p_k^\infty]$ cannot be equal to $\mathbf{z}_{\mu^k}$ because the coefficient of $e_1 \wedge e_1^2 \ldots \wedge e_1^k$ is nonzero for $\mathbf{z}_{\mu^k}$ but zero for $[p_k^\infty]$, and the result follows. □

We summarize our information about the maximal boundary components in

PROPOSITION 6.13. *We have* $\mathbf{z}_{\lambda^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\lambda^\sigma}[i]$, *where* $\mathbf{z}_{\lambda^\sigma}[i] = \oplus_{\Sigma(\tau)=i, d(\tau)=d(i)} \mathbf{e}_\tau$, *and* $\mathbf{z}_{\mu^\sigma} = \wedge_{i=1}^k \mathbf{z}_{\mu^\sigma}[i]$ *where* $\mathbf{z}_{\mu^\sigma}[i] = \oplus_{\Sigma(\tau)=i, \sigma \notin \tau} \mathbf{e}_\tau$.

REMARK 6.14. Since the one-parameter subgroup $\tilde{\lambda}(t) = (t, t^2, \ldots, t^k)$ of $\mathrm{GL}(k)$ stabilises $T_k\mathbf{z}$, it follows from Lemma 6.12 that it is enough to prove the codimension-at-least-two property we require only for the one-parameter subgroups $\tilde{\lambda}^\sigma$ (for $2 \leq s \leq k$) and $\tilde{\mu}^\sigma$ (for $2 \leq s \leq k-1$) of $\mathrm{SL}(k)$ given by

$$\tilde{\lambda}^\sigma(t) = (\lambda^\sigma(t)\tilde{\lambda}(t)^{q_\sigma})^{n_\sigma}$$

and

$$\tilde{\mu}^\sigma(t) = (\mu^\sigma(t)\tilde{\lambda}(t)^{r_\sigma})^{m_\sigma}$$

for suitable $q_\sigma, r_\sigma \in \mathbb{Q}$ and $n_\sigma, m_\sigma \in \mathbb{Z}$. But we observed at (20) that the property is satisfied by a one-parameter subgroup λ of $\mathrm{SL}(k)$ if $\lambda_{i_1} + \cdots + \lambda_{i_s} < \lambda_j$ for any $j \in \{2, \ldots, k-1\}$ such that $i_1 + \cdots + i_s = j$, so it is enough to consider the one-parameter subgroups $\tilde{\lambda}^\sigma$ for $2 \leq s \leq k$.

6.1. The limit of the stabilisers. In order to prove Lemma 6.5, it now suffices by Remark 6.14 to find a k-dimensional unipotent subgroup of the stabiliser $G_{\mathbf{z}_{\lambda^\sigma}}$ of $\mathbf{z}_{\lambda^\sigma}$ in $\mathrm{GL}(k)$ for each σ when $\mathbf{z}_{\lambda^\sigma} = [p_k^\infty]$, since we know that p_k^∞ is fixed by a one-parameter subgroup of the maximal torus T_k of $\mathrm{SL}(k)$, and any unipotent group which stabilises $\mathbf{z}_{\lambda^\sigma} = [p_k^\infty]$ also stabilises p_k^∞.

In this subsection we will study the limits $\lim G_{\lambda^\sigma(t)\mathbf{z}}$ of the stabiliser groups for the one-parameter subgroups λ^σ for $2 \le \sigma \le k$, and use this to prove Lemma 6.5, which together with Lemma 6.6 will complete the proof of Theorem 6.2.

Proposition 6.15. *$G^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}} \subset GL(k)$ is a k-dimensional subgroup of $G_{\mathbf{z}_{\lambda^\sigma}}$ which contains a $k-1$-dimensional subgroup of the maximal unipotent subgroup U_k of $\mathrm{SL}(k)$.*

Proof. Consider the stabilizer

$$G_{\lambda^\sigma(t)\mathbf{z}} = \lambda^\sigma(t)^{-1} G_{\mathbf{z}} \lambda^\sigma(t).$$

Recall that

$$G_{\mathbf{z}} = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_k \\ 0 & \alpha_1^2 & 2\alpha_1\alpha_2 & \dots & 2\alpha_1\alpha_{n-1} + \cdots \\ 0 & 0 & \alpha_1^3 & \dots & 3\alpha_1^2\alpha_{k-2} + \cdots \\ 0 & 0 & 0 & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \alpha_1^d \end{pmatrix} \right\}$$

where the polynomial in the (i,j) entry is

$$p_{i,j}(\alpha) = \sum_{a_1+a_2+\cdots+a_i=j} \alpha_{a_1}\alpha_{a_2}\dots\alpha_{a_i}.$$

Therefore, the (i,j) entry of the stabilizer of $\lambda^s(t)\mathbf{z}$ is

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma - \lambda_j^\sigma} p_{i,j}(\alpha) \tag{33}$$

If ε is small enough then $\lambda_1^\sigma < \lambda_2^\sigma < \cdots < \lambda_k^\sigma$, and we define the positive number

$$n_i^\sigma = \max_{1\le j\le n-i+1} (\lambda_{j+i-1}^\sigma - \lambda_j^\sigma), \quad i = 1, \dots, k. \tag{34}$$

Note that by definition $n_1^\sigma = 0$ for all σ.

Lemma 6.16. *Under the substitution*

$$\beta_i^\sigma = t^{-n_i^\sigma} \alpha_i^\sigma$$

we have

$$G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1, \dots, \beta_k) \in GL(\mathbb{C}[\beta_1, \dots, \beta_k][t]),$$

so the entries are polynomials in t with coefficients in $\mathbb{C}[\beta_1, \dots, \beta_k]$.

Proof. Compute the substitution as follows:

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = t^{\lambda_i^\sigma - \lambda_j^\sigma} \sum_{a_1+a_2+\cdots+a_i=j} \alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_i} = \tag{35}$$

$$= \sum_{a_1+\cdots a_i=j} t^{\lambda_i^\sigma - \lambda_j^\sigma} t^{n^\sigma_{a_1}+n^\sigma_{a_2}+\cdots+n^\sigma_{a_i}} \beta_{a_1}\beta_{a_2}\ldots\beta_{a_i}. \tag{36}$$

By definition

$$n^\sigma_{a_1} \geq \lambda^\sigma_{i+a_1-1} - \lambda^\sigma_i;\ n^\sigma_{a_2} \geq \lambda^\sigma_{i+a_1+a_2-2} - \lambda^\sigma_{i+a_1-1};\ \ldots\ ;$$
$$n^\sigma_{a_j} \geq \lambda^\sigma_{i+a_1+\cdots+a_i-i} - \lambda^\sigma_{i+a_1+\cdots+a_{i-1}-(i-1)}.$$

Adding up these inequalites and using $a_1+\cdots+a_i=j$ we get an alternating sum on the left cancelling up to

$$n^\sigma_{a_1}+\cdots+n^\sigma_{a_i} \geq \lambda^\sigma_j - \lambda^\sigma_i.$$

Substituting this into (35) we get

$$(G_{\lambda^\sigma(t)\mathbf{z}})_{i,j} = \sum_{a_1+\cdots a_i=j} t^{\lambda_i^\sigma - \lambda_j^\sigma} t^{n^\sigma_{a_1}+n^\sigma_{a_2}+\cdots+n^\sigma_{a_i}} \beta_{a_1}\beta_{a_2}\ldots\beta_{a_i} \in \mathbb{C}[\beta_1,\ldots,\beta_k][t]. \tag{37}$$

This proves Lemma 6.16. □

As a corollary we get the existence of

$$G^\sigma = \lim_{t\to 0} G_{\lambda^\sigma(t)\mathbf{z}}(\beta_1,\ldots,\beta_k) \in GL(\mathbb{C}[\beta_1,\ldots,\beta_k]).$$

To prove that $\dim G^\sigma = k$ and complete the proof of Proposition 6.15, for $1 \leq i \leq k$ choose $\theta(i)$ such that

$$n^\sigma_i = \lambda_{\theta(i)+i-1} - \lambda_{\theta(i)} \tag{38}$$

holds. Then

$$p_{\theta(i),\theta(i)+i-1}(\beta_1,\ldots,\beta_k) = \sum_{a_1+\cdots+a_{\theta(i)}=\theta(i)+i-1} t^{n^\sigma_{a_1}+\cdots+n^\sigma_{a_{\theta(i)}}} \beta_{a_1}\ldots\beta_{a_{\theta(i)}} \tag{39}$$

so

$$(G^\sigma)_{\theta(i),\theta(i)+i-1} = \lim_{t\to 0} t^{-n^\sigma_i} p_{\theta(i),\theta(i)+i-1}(\beta_1,\ldots,\beta_k) = \lim_{t\to 0}(t^{n_i^\sigma}\beta_1^{\theta(i)-1}\beta_i+\cdots)$$

$$= \beta_1^{\theta(i)-1}\beta_i + q_{\theta(i),\theta(i)+i-1} \tag{40}$$

where

$$q_{\theta(i),\theta(i)+i-1} \in \mathbb{C}[\beta_1,\ldots,\beta_k][t].$$

It follows that the elements $\frac{d}{dt}A^\sigma(t(e_1+e_i)1) \in \mathrm{Lie}(G^\sigma)$ are independent, where $t(e_1+e_i) = (t,0,\ldots,0,t,0,\ldots,0)$ with the t's are in the 1st and ith position if $i>1$ but interpreted as $(2t,0,\ldots,0)$ if $i=1$. This completes the proof of Proposition 6.15. □

In order to prove Lemma 6.5, it now suffices to find an extra one-dimensional unipotent subgroup of the stabiliser $G_{\mathbf{z}_{\lambda^\sigma}}$ of $\mathbf{z}_{\lambda^\sigma}$ for each σ when $\mathbf{z}_{\lambda^\sigma}=[p_k^\infty]$, since we know that p_k^∞ is fixed by a one-parameter subgroup of the maximal torus T_k of $\mathrm{SL}(k)$ and by a $k-1$-dimensional unipotent subgroup of $G^\sigma=\lim_{\theta\to 0}G_{\lambda^\sigma(t)\mathbf{z}}$ which is contained in the standard maximal unipotent subgroup U_k of $\mathrm{SL}(k)$. It turns out that we have to distinguish three cases here.

Case 1: $\sigma=k$.

PROOF. Let $T_\zeta\in GL(k)$ denote the transformation

$$T_\zeta(e_i)=e_i \text{ for } i\neq k-1;\quad T_\zeta(e_{k-1})=e_{k-1}+\zeta e_k \text{ for } \zeta\in\mathbb{C}.$$

Since e_{k-1} does not occur just in $\mathbf{z}_{\lambda^\sigma}[k-1]$, T_ζ stabilises p_k^∞. This gives us a subgroup of $\mathrm{SL}(k)$ of dimension at least $k+1$ which stabilises p_k^∞, because T_ζ is unipotent but not upper triangular if $\zeta\neq 0$. □

Case 2: $\sigma<k$ and $k\neq -1 \bmod \sigma$.

PROOF. Let T be the transformation

$$T(e_i)=e_i \quad \text{for } i\neq k;\quad T(e_k)=e_k+\zeta e_\sigma. \tag{41}$$

Since e_k occurs only in $\mathbf{z}_{\lambda^\sigma}[k]$, and $\mathbf{z}_{\lambda^\sigma}[\sigma]=\sigma$, we have

$$\begin{aligned} T\cdot\mathbf{z}_{\lambda^\sigma}&=\mathbf{z}_{\lambda^\sigma}(e_1,\dots,e_{k-1},e_k+\zeta e_\sigma)\\ &=\mathbf{z}_{\lambda^\sigma}[1]\wedge\cdots\wedge\mathbf{z}_{\lambda^\sigma}[\sigma-1]\wedge e_\sigma\wedge\mathbf{z}_{\lambda^\sigma}[\sigma+1]\wedge\cdots\wedge\mathbf{z}_{\lambda^\sigma}[k])\\ &\quad+\zeta\cdot\mathbf{z}_{\lambda^\sigma}[1]\wedge\cdots\wedge\mathbf{z}_{\lambda^\sigma}[\sigma-1]\\ &\quad\wedge e_\sigma\wedge\mathbf{z}_{\lambda^\sigma}[\sigma+1]\wedge\cdots\wedge\mathbf{z}_{\lambda^\sigma}[k-1]\wedge e_\sigma=\mathbf{z}_{\lambda^\sigma}, \end{aligned}\tag{42}$$

so $T\in G_{\mathbf{z}_{\lambda^\sigma}}$.

It is slightly harder task to show that $T\notin G^\sigma=\lim_{\theta\to 0}G_{\lambda^\sigma(t)\mathbf{z}}$. First, we compute n_i for $i=k-\sigma$. We claim that for $k\neq -1 \bmod \sigma$

$$n_{k-\sigma+1}=\lambda_k^\sigma-\lambda_\sigma^\sigma=\lambda_{k-\sigma+1}^\sigma-\lambda_1^\sigma. \tag{43}$$

Indeed,

$$\lambda_{j+k-\sigma-1}-\lambda_j=\cdots\leq\lambda_k^\sigma-\lambda_\sigma^\sigma=\lambda_{k-\sigma+1}^\sigma-\lambda_1^\sigma$$

This means that we can choose $\theta(k-\sigma+1)=\sigma$ in (38) and substitute into (40)

$$(G^\sigma)_{\sigma,k}=\beta_1^{\sigma-1}\beta_{k-\sigma+1}+q_{\sigma,k}(\beta_1,\dots,\beta_k), \tag{44}$$

where $q_{\sigma,k}(\beta_1,\dots,\beta_k)$ is a polynomial, whose monomials $\beta_{i_1}^{b_1}\dots\beta_{i_\sigma}^{b_\sigma}$ satisfy

$$i_1b_1+\cdots+i_\sigma b_\sigma=k. \tag{45}$$

Moreover, we can also choose $\theta(k-\sigma+1)=1$, by (43), and then (40) gives us

$$(G^\sigma)_{1,k-\sigma+1}=\beta_{k-\sigma+1}. \tag{46}$$

Suppose now that $T \in G^\sigma$, that is

$$T = G^\sigma(\beta_1, \dots, \beta_k) \quad \text{for some } \beta_1 \in \mathbb{C}^*, \beta_2, \dots, b_k \in \mathbb{C}. \tag{47}$$

Let $(T)_{i,j}$ denote the (i,j) entry of T. Then

$$(T)_{\sigma,k} = \zeta, \;\; (T)_{i,j} = 0 \quad \text{for } i \neq j, \;\; (T)_{i,i} = 1.$$

Comparing the $(1,1)$ and $(1, k-\sigma+1)$ entries of T and G^σ we get

$$\beta_1 = 1, \quad \beta_{\delta-\sigma+1} = 0. \tag{48}$$

Choose $\theta(i)$ for $i = 2, \dots, k$ as in (38) and let $\theta(k-\sigma+1) = \sigma$. Since all off-diagonal entries of T but the (σ, k) are zero, (47) forces the following equations

$$\beta_i + q_{\theta(i),\theta(i)+i-1} = 0 \quad \text{for } i \neq k - \sigma + 1, \tag{49}$$

$$\beta_{k-\sigma+1} + q_{\sigma,k} = \zeta. \tag{50}$$

By (48), these are $k-1$ polynomial equations in $k-2$ variables, and the Jacobian at 0 is the origin, so we have finitely many solutions near the origin. Therefore, for some ζ, it follows that T is not in G^σ. □

Case 3: $\sigma < k$ and $d = -1 \bmod \sigma$.

Proof. This case works very similarly to the previous one. Suppose $k - 1 > \sigma$, that is, if $k = c\sigma - 1$ where $c \geq 2$ (this holds because $k \geq \sigma$), the condition is that $c\sigma - 2 > \sigma$, which is true for all $k \geq 4$.

Let T be the transformation

$$T(e_i) = e_i \text{ for } i \neq k, k-1 \; ; \; T(e_{k-1}) = e_{k-1} + \zeta e_\sigma \; ; \; T(e_k) = e_k + \zeta e_\sigma \tag{51}$$

First we check again that $T \in G_{\mathbf{z}_{\lambda^\sigma}}$. We have

$$\begin{aligned}
\mathbf{z}_{\lambda^\sigma}[\sigma] &= e_\sigma \; ; \\
\mathbf{z}_{\lambda^\sigma}[\sigma+1] &= e_{\sigma+1} + e_1 e_\sigma \; ; \\
\mathbf{z}_{\lambda^\sigma}[k] &= e_k + \sum_{i=1}^{k-1} e_i e_{k-i} \; .
\end{aligned}$$

An easy computation shows that

$$\begin{aligned}
T \cdot \mathbf{z}_{\lambda^\sigma} &= \mathbf{z}_{\lambda^\sigma}(e_1, \dots, e_{k-2}, e_{k-1} + \zeta e_\sigma, e_k + \zeta e_{\sigma+1}) \\
&= \mathbf{z}_{\lambda^\sigma}[1] \wedge \cdots \wedge \mathbf{z}_{\lambda^\sigma}[k-2] \wedge (\mathbf{z}_{\lambda^\sigma}[k-1] \\
&\quad + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma]) \wedge (\mathbf{z}_{\lambda^\sigma}[k] + \zeta \mathbf{z}_{\lambda^\sigma}[\sigma+1] \\
&= \mathbf{z}_{\lambda^\sigma}[1] \wedge \cdots \wedge \mathbf{z}_{\lambda^\sigma}[k] = \mathbf{z}_{\lambda^\sigma}.
\end{aligned} \tag{52}$$

Now we prove that $T \notin G^\sigma$ in a similar way to the second case above. Since $k - 1 \neq -1 \bmod \sigma$ we can substitute $k-1$ instead of k in (43):

$$n_{k-\sigma} = \lambda^\sigma_{k-1} - \lambda^\sigma_\sigma = \lambda^\sigma_{k-\sigma} - \lambda^\sigma_1. \tag{53}$$

Moreover, we also get the extra equation

$$n_{k-\sigma} = \lambda_k^\sigma - \lambda_{\sigma+1}^\sigma, \tag{54}$$

and similarly to (44) and (46) it follows that

$$(G^\sigma)_{\sigma,k-1} = \beta_1^{\sigma-1}\beta_{k-\sigma} + q_{\sigma,k-1}(\beta_1,\dots,\beta_k); \tag{55}$$

$$(G^\sigma)_{\sigma+1,k} = \beta_1^{\sigma}\beta_{k-\sigma} + q_{\sigma+1,k}(\beta_1,\dots,\beta_k); \tag{56}$$

$$(G^\sigma)_{1,k-\sigma} = \beta_{k-\sigma}. \tag{57}$$

Since T differs from the identity matrix only by the entries

$$(T)_{\sigma,k-1} = (T)_{\sigma+1,k} = \zeta,$$

the equality

$$T = G^\sigma(\beta_1,\dots,\beta_k)$$

forces $\beta_{k-\sigma} = 0, \beta_1 = 1$ and the analogue of (49) ,(50):

$$\beta_i + q_{\theta(i),\theta(i)+i-1} = 0 \quad \text{for } i \neq k-\sigma \tag{58}$$

$$\beta_{k-\sigma} + q_{\sigma,k-1} = \zeta \tag{59}$$

$$\beta_{k-\sigma} + q_{\sigma+1,k} = \zeta \tag{60}$$

which are, again, $k+1$ nondegenerate polynomial equations in $k-1$ variables, such that for some ζ there is no solution. ☐

We have now proved Lemma 6.5, which together with Lemma 6.6 completes the proof of Theorem 6.2.

7. Geometric description of Demailly-Semple invariants

As an immediate consequence of Corollary 6.3, we can now prove Theorem 3.3 in the case when $p = 1$.

THEOREM 7.1. *If $k \geq 2$ then $\mathbb{G}_k' = \mathbb{U}_k$ is a Grosshans subgroup of the special linear group $SL(k)$, so that $\mathcal{O}(\mathrm{SL}(k)^{\mathbb{U}_k})^{\mathrm{SL}(k)}$ is a finitely generated complex algebra and moreover every linear action of $\mathbb{U}_k$ or $\mathbb{G}_k$ on an affine or projective variety Y (with respect to an ample linearisation) which extends to a linear action of $GL(k)$ has finitely generated invariants.*

In particular we have the special case of Theorem 3.2 when $p = 1$.

THEOREM 7.2. *The fibre $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ of the bundle E_k^n is a finitely generated graded complex algebra.*

PROOF. We have

$$\mathcal{O}((J_k)_x)^{\mathbb{U}_k} \cong (\mathcal{O}((J_k)_x) \otimes \mathcal{O}(\mathrm{SL}(k)^{\mathbb{U}_k})^{\mathrm{SL}(k)}$$

which is finitely generated because $\mathcal{O}(\mathrm{SL}(k)^{\mathbb{U}_k})^{\mathrm{SL}(k)}$ is finitely generated and $\mathrm{SL}(k)$ is reductive. ☐

Theorem 6.2 also allows us to describe the algebra $\mathcal{O}(SL(k))^{\mathbb{U}_k}$. In §6 we constructed an embedding of $SL(k)/\mathbb{U}_k$ in the affine space $\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)\otimes(\mathbb{C}^k)^{\otimes K}$ for suitable large K, and in Theorem 6.2 we proved that the boundary components of the closure $\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})}$ of its image have codimension at least two. Thus we obtain the following corollary of Theorem 6.2:

THEOREM 7.3. *(i) If $k\geq 4$ then the canonical affine completion*

$$\mathrm{SL}(k)//\mathbb{U}_k=\operatorname{Spec}(\mathcal{O}(\mathrm{SL}(k))^{\mathbb{U}_k})$$

of $\mathrm{SL}(k)/\mathbb{U}_k$ is isomorphic to the closure $\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})}$ of the orbit $\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})\cong\mathrm{SL}(k)/\mathbb{U}_k$ of $p_k\otimes e_1^{\otimes K}$ in $\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)\otimes(\mathbb{C}^k)^{\otimes K}$ where $K=M(1+2+\cdots+k)+1$ for any strictly positive integer M;

(ii) The algebra

$$\mathcal{O}(SL(k))^{\mathbb{U}_k}$$

is generated by the Plücker coordinates on $\mathbb{P}(\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k))$, which can be expressed as

$$\{\Delta_{\mathbf{i}_1,\dots,\mathbf{i}_s}:s\leq k\},$$

where $\mathbf{i}_j$ denotes a multi-index corresponding to basis elements of $\operatorname{Sym}^{\leq k}(\mathbb{C}^k)$, and $\Delta_{\mathbf{i}_1,\dots,\mathbf{i}_s}$ is the corresponding minor of $\phi(f'\dots,f^{(k)})\in\operatorname{Hom}(\mathbb{C}^k,\operatorname{Sym}^{\leq k}(\mathbb{C}^k))$, together with the coordinates $f_1',\dots,f_k'$ of f'.

It follows immediately from this theorem that the non-reductive GIT quotient

$$(J_k)_x//\mathbb{U}_k=\operatorname{Spec}(\mathcal{O}((J_k)_x)^{\mathbb{U}_k})$$

is isomorphic to the reductive GIT quotient

$$((J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})})//\mathrm{SL}(k).$$

This can be identified with the quotient of the open subset $((J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})})^{ss}$ of $\mathrm{SL}(k)$-semistable points of $(J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})}$ by the equivalence relation $\sim$ such that $y\sim z$ if and only if the closures of the $\mathrm{SL}(k)$-orbits of y and z intersect in $((J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})})^{ss}$. Equivalently it can be identified with the closed $\mathrm{SL}(k)$-orbits in $((J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})})^{ss}$. Since $\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})}$ is the union of finitely many $\mathrm{SL}(k)$-orbits, with stabilisers $H_1=\mathbb{U}_k,H_2,\dots,H_s$, say, we can stratify $(J_k)_x//\mathbb{U}_k$ so that the stratum corresponding to H_j is identified with the H_j-orbits in $(J_k)_x$ such that the corresponding $\mathrm{SL}(k)$-orbit in $(J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})}$ is semistable and closed in $((J_k)_x\times\overline{\mathrm{SL}(k)(p_k\otimes e_1^{\otimes K})})^{ss}$.

EXAMPLE 7.4. *When $k=2$ we have*

$$J_2^{\mathrm{reg}}(1,2)=\{(f_1',f_2',f_1'',f_2'')\in(\mathbb{C}^2)^2;(f_1',f_2')\neq(0,0)\},$$

and fixing a basis $\{e_1,e_2\}$ *of* $\mathbb{C}^2$ *and the induced basis* $\{e_1,e_2,e_1^2,e_1e_2,e_2^2\}$ *of* $\mathbb{C}^2\oplus \mathrm{Sym}^2\mathbb{C}^2$, *the map* $\phi: J_2(1,2)=\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2)\to \mathrm{Hom}(\mathbb{C}^2,\mathrm{Sym}^{\le 2}\mathbb{C}^2)$ *of* (14) *is given by*

$$(f_1',f_2',f_1'',f_2'')\mapsto \begin{pmatrix} f_1' & f_2' & 0 & 0 & 0\\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & (f_1')^2 & f_1'f_2' & (f_2')^2\end{pmatrix}.$$

The 2×2 *minors of this* 2×5 *matrix are* $(f_1')^3$, $(f_1')^2f_2'$, $f_1'(f_2')^2, (f_2')^3$ *and*

$$\Delta_{[1,2]}=f_1'f_2''-f_1''f_2'.$$

On $SL(2)$ *we have* $\Delta_{[1,2]}=1$ *and the algebra of invariants* $\mathcal{O}(SL(2))^{\mathbb{U}_2}$ *is generated by* f_1' *and* f_2', *as expected since* $\mathrm{SL}(2)/\mathbb{U}_2\cong\mathbb{C}^2\setminus\{0\}$ *and its canonical affine completion* $\mathrm{SL}(2)/\!/\mathbb{U}_2$ *is* $\mathbb{C}^2$.

Example 7.5. *When* $k=3$ *the finite generation of the Demailly-Semple algebra* $\mathcal{O}((J_k)_x)^{\mathbb{U}_k}$ *was proved by Rousseau in* [**27**]. *We have*

$$\begin{aligned}J_3^{\mathrm{reg}}(1,3)=\{&(f_1',f_2',f_3',f_1'',f_2'',f_3'',f_1''',f_2''',f_3''')\\ &\in(\mathbb{C}^3)^3;(f_1',f_2',f_3')\ne(0,0,0)\},\end{aligned}$$

and if we fix a basis $\{e_1,e_2,e_3\}$ *of* $\mathbb{C}^3$ *and the induced basis*

$$\{e_1,e_2,e_3,e_1^2,e_1e_2,e_2^2,e_1e_3,e_2e_3,e_3^2,e_1^3,e_1^2e_2,\ldots,e_3^3\}$$

of $\mathbb{C}^3\oplus\mathrm{Sym}^2\mathbb{C}^3\oplus\mathrm{Sym}^3\mathbb{C}^3$, *the map* $\phi:\mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^3)\to\mathrm{Hom}(\mathbb{C}^3,\mathrm{Sym}^{\le 3}\mathbb{C}^3)$ *in* (14) *sends*

$$(f_1',f_2',f_3',f_1'',f_2'',f_3'',f_1''',f_2''',f_3''')$$

to a 3×19 *matrix, whose first* 9 *columns (corresponding to* $\mathrm{Sym}^{\le 2}\mathbb{C}^3$*) are*

$$\begin{pmatrix} f_1' & f_2' & f_3' & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{1}{2!}f_1'' & \frac{1}{2!}f_2'' & \frac{1}{2!}f_3'' & (f_1')^2 & f_1'f_2' & (f_2')^2 & f_1'f_3' & f_2'f_3' & (f_3')^2\\ \frac{1}{3!}f_1''' & \frac{1}{3!}f_2''' & \frac{1}{3!}f_3''' & f_1'f_1'' & f_1'f_2''+f_1''f_2' & f_2'f_2'' & f_1'f_3''+f_3'f_1'' & f_2'f_3''+f_2''f_3' & f_3'f_3''\end{pmatrix},$$

and the remaining 10 *columns (corresponding to* $\mathrm{Sym}^3\mathbb{C}^3$*) are*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ (f_1')^3 & (f_1')^2f_2' & f_1'(f_2')^2 & (f_2')^3 & f_1'(f_3')^2 & (f_1')^2f_3' & (f_2')^2f_3' & f_2'(f_3')^2 & (f_3')^3 & f_1'f_2'f_3'\end{pmatrix}.$$

The 3×3 *minors of this matrix together with* f_1',f_2',f_3' *generate the algebra of invariants* $\mathcal{O}(SL(3))^{\mathbb{U}_3}$.

8. Generalized Demailly-Semple jet bundles

The aim of this section is to extend the earlier constructions for $p=1$ to generalized Demailly-Semple invariant jet differentials when $p>1$.

Let X be a compact, complex manifold of dimension n. We fix a parameter $1\le p\le n$, and study the maps $\mathbb{C}^p\to X$. Recall that as before we fix the degree k of the map, and introduce the bundle $J_{k,p}\to X$ of k-jets of maps $\mathbb{C}^p\to X$, so that the fibre over $x\in X$ is the set of equivalence classes of germs of holomorphic maps $f:(\mathbb{C}^p,0)\to(X,x)$, with the equivalence relation $f\sim g$ if and only if all derivatives $f^{(j)}(0)=g^{(j)}(0)$ are equal for $0\le j\le k$. Recall also that $\mathbb{G}_{k,p}$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p,0)$, which has a natural fibrewise right action on $J_{k,p}$ with the matrix representation given by

$$G_{k,p}=\begin{pmatrix} \Phi_1 & \Phi_2 & \Phi_3 & \dots & \Phi_k \\ 0 & \Phi_1^2 & \Phi_1\Phi_2 & \dots & \\ 0 & 0 & \Phi_1^3 & \dots & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \Phi_1^k \end{pmatrix}, \tag{61}$$

for $G_{k,p}\in\mathbb{G}_{p,k}$ where $\Phi_i\in \operatorname{Hom}(\operatorname{Sym}^i\mathbb{C}^p,\mathbb{C}^p)$ and $\det\Phi_1\neq 0$. Recall also that $\mathbb{G}_{k,p}$ is generated along its first p rows, in the sense that the parameters in the first p rows are independent, and all the remaining entries are polynomials in these parameters. The parameters in the $(1,m)$ block are indexed by a basis of $\operatorname{Sym}^m(\mathbb{C}^p)\times\mathbb{C}^p$, so they are of the form α_ν^l where $\nu\in\binom{p+m-1}{m-1}$ is an m-tuple and $1\le l\le p$, and the polynomial in the (l,m) block and entry indexed by $\tau=(\tau[1],\dots,\tau[l])\in\binom{p+l-1}{l-1}$ and $\nu\in\binom{p+m-1}{m-1}$ is given by

$$(G_{k,p})_{\tau,\nu}=\sum_{\nu_1+\cdots+\nu_l=\nu}\alpha_{\nu_1}^{\tau[1]}\alpha_{\nu_2}^{\tau[2]}\dots\alpha_{\nu_l}^{\tau[l]}. \tag{62}$$

Recall also that $\mathbb{G}_{k,p}=\mathbb{U}_{k,p}\rtimes \mathrm{GL}(p)$ is an extension of its unipotent radical $\mathbb{U}_{k,p}$ by $\mathrm{GL}(p)$, and that the generalized Demailly-Semple jet bundle $E_{k,p,m}\to X$ of invariant jet differentials of order k and weighted degree $(m,\dots,m)$ consists of the jet differentials which transform under any reparametrization $\phi\in\mathbb{G}_{k,p}$ of $(\mathbb{C}^p,0)$ as

$$Q(f\circ\phi)=(J_\phi)^mQ(f)\circ\phi,$$

where $J_\phi=\det\Phi_1$ denotes the Jacobian of ϕ, so that $E_{k,p}=\oplus_{m\ge 0}E_{k,p,m}$ is the graded algebra of $\mathbb{G}'_{k,p}$-invariants where $\mathbb{G}'_{k,p}=\mathbb{U}_{k,p}\rtimes\mathrm{SL}(p)$.

8.1. Geometric description for $p>1$. As in the case when $p=1$ our goal is to prove that $\mathbb{G}'_{k,p}$ is a Grosshans subgroup of $\mathrm{SL}(\mathrm{sym}^{\le k}(p))$ where $\mathrm{sym}^{\le k}(p)=\sum_{i=1}^k\dim\operatorname{Sym}^i\mathbb{C}^p$ by finding a suitable embedding of the quotient $\mathrm{SL}(\mathrm{sym}^{\le k}(p))/\mathbb{G}'_{k,p}$.

REMARK 8.1. In [**25**] Pacienza and Rousseau generalize the inductive process given in [**5**] of constructing a smooth compactification of the Demailly-Semple jet bundles. Using the concept of a directed manifold, they define a bundle $X_{k,p} \to X$ with smooth fibres, and the effective locus $Z_{k,p} \subset X_{k,p}$, and a holomorphic embedding $J_{k,p}^{reg}/\mathbb{G}_{k,p} \hookrightarrow Z_{k,p}$ which identifies $J_{k,p}^{reg}/\mathbb{G}_{k,p}$ with $Z_{k,p}^{reg} = X_{k,p}^{reg} \cap Z_{k,p}$, so that $Z_{k,p}$ is a relative compactification of $J_{k,p}/\mathbb{G}_{k,p}$. We choose a different approach, generalizing the test curve model, resulting in a holomorphic embedding of $J_{k,p}/\mathbb{G}_{k,p}$ into a partial flag manifold and a different compactification, which is a singular subvariety of the partial flag manifold, such that the invariant jet differentials of degree divisible by $\mathrm{sym}^{\leq k}(p)$ are given by polynomial expressions in the Plücker coordinates.

Fix $x \in X$ and an identification of T_xX with $\mathbb{C}^n$; then let $J_k(p,n) = J_{k,p,x}$ as defined in §2. Let

$$J_k^{\mathrm{reg}}(p,n) = \{\gamma \in J_k(p,n) : \Gamma_1 \text{ is non-degenerate}\}$$

where γ is represented by

$$\mathbf{u} \mapsto \gamma(\mathbf{u}) = \Gamma_1\mathbf{u} + \Gamma_2\mathbf{u}^2 + \cdots + \Gamma_k\mathbf{u}^k$$

with $\Gamma_i \in \mathrm{Hom}\,(\,\mathrm{Sym}^{\,i}\mathbb{C}^p, \mathbb{C}^p)$. Let $N \geq n$ be any integer and define

$$\Upsilon_{k,p} = \left\{\Psi \in J_k(n,N) : \exists\gamma \in J_k^{\mathrm{reg}}(p,n) : \Psi \circ \gamma = 0\right\}.$$

REMARK 8.2. The global singularity theory description of $\Upsilon_{k,p}$ is

$$\begin{aligned}\Upsilon_{k,p} \doteq \{&p = (p_1,\ldots,p_N) \in J_k(n,N) : \mathbb{C}[z_1,\ldots,z_n]/\langle p_1,\ldots,p_N\rangle \\ &\cong \mathbb{C}[x,y]/\langle z_1,\ldots,z_n\rangle^{k+1}\}.\end{aligned}$$

Note, again, as in the $p = 1$ case, that if $\gamma \in J_k^{\mathrm{reg}}(p,n)$ is a test surface of $\Psi \in \Upsilon_{k,p}$, and $\varphi \in \mathbb{G}_k$ is a holomorphic reparametrization of $\mathbb{C}^p$, then $\gamma \circ \varphi$ is, again, a test surface of Ψ:

$$\mathbb{C}^p \xrightarrow{\quad\varphi\quad} \mathbb{C}^p \xrightarrow{\quad\gamma\quad} \mathbb{C}^n \xrightarrow{\quad\Psi\quad} \mathbb{C}^N \tag{63}$$

$$\Psi \circ \gamma = 0 \;\;\Rightarrow\;\; \Psi \circ (\gamma \circ \varphi) = 0$$

EXAMPLE 8.3. *Let* $k = 2, p = 2$ *and let* $\Psi(\mathbf{z}) = \Psi'\mathbf{z} + \Psi''\mathbf{z}^2$ *for* $\mathbf{z} \in \mathbb{C}^n$, *and*

$$\gamma(u_1,u_2) = \gamma_{10}u_1 + \gamma_{01}u_2 + \gamma_{20}u_1^2 + \gamma_{11}u_1u_2 + \gamma_{02}u_2^2, \quad \gamma_{ij} \in \mathbb{C}^n.$$

Then $\Psi \circ \gamma = 0$ *has the form*

$$\begin{aligned}&\Psi'(\gamma_{10}) = 0; \quad \Psi'(\gamma_{01}) = 0 \\ &\Psi'(\gamma_{20}) + \Psi''(\gamma_{10},\gamma_{10}) = 0,; \quad \Psi'(\gamma_{11}) + 2\Psi''(\gamma_{10},\gamma_{01}) = 0,; \\ &\Psi'(\gamma_{01}) + \Psi''(\gamma_{01},\gamma_{01}) = 0,\end{aligned} \tag{64}$$

We introduce

$$\mathcal{S}_\gamma = \{\Psi \in J_k(n,N) : \Psi \circ \gamma = 0\}$$

and the following analogue of $J_k^o(1,n)$:

$$J_k^o(n,N) = \{\Psi \in J_k(n,N) : \dim\ker\Psi = p\}\,.$$

The proof of the following proposition is analogous to that of Proposition 4.7 in [**2**], and we omit the details. We use the notation

$$\mathrm{sym}^i(p) = \dim(\mathrm{Sym}^i\mathbb{C}^p);$$

$$\mathrm{sym}^{\le k}(p) = \dim(\mathbb{C}^p \oplus \mathrm{Sym}^2\mathbb{C}^p \oplus \cdots \oplus \mathrm{Sym}^k\mathbb{C}^p) = \sum_{i=1}^k \mathrm{sym}^i p.$$

Proposition 8.4. *(i) If $\gamma \in J_k^{\mathrm{reg}}(p,n)$ then $\mathcal{S}_\gamma \subset J_k(n,N)$ is a linear subspace of codimension $N\mathrm{sym}^{\le k}(p)$.*

(ii) For any $\gamma \in J_k^{\mathrm{reg}}(p,n)$, the subset $\mathcal{S}_\gamma \cap J_k^o(n,N)$ of $\mathcal{S}_\gamma$ is dense.

(iii) If $\Psi \in J_k^o(n,N)$, then Ψ belongs to at most one of the spaces $\mathcal{S}_\gamma$. More precisely, if $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(p,n)$, $\Psi \in J_k^o(n,N)$ and $\Psi\circ\gamma_1 = \Psi\circ\gamma_2 = 0$, then there exists $\varphi \in J_k^{\mathrm{reg}}(p,p)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

(iv) Given $\gamma_1, \gamma_2 \in J_k^{\mathrm{reg}}(1,n)$, we have $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ if and only if there is some $\varphi \in J_k^{\mathrm{reg}}(1,1)$ such that $\gamma_1 = \gamma_2 \circ \varphi$.

With the notation

$$\Upsilon_{k,p} = \Upsilon_{k,p} \cap J_k^o(n,N),$$

we deduce from Proposition 8.4 the following

Corollary 8.5. *$\Upsilon_{k,p}^0$ is a dense subset of $\Upsilon_{k,p}$, and $\Upsilon_{k,p}^0$ has a fibration over the orbit space $J_k^{\mathrm{reg}}(p,n)/J_k^{\mathrm{reg}}(p,p) = J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p}$ with linear fibres.*

Remark 8.6. In fact, Proposition 8.4 says a bit more, namely that $\Upsilon_{k,p}^0$ is fibrewise dense in $\Upsilon_{k,p}$ over $J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p}$, but we will not use this stronger statement.

By the first part of Proposition 8.4 the assignment $\gamma \to \mathcal{S}_\gamma$ defines a map

$$\nu : J_k^{\mathrm{reg}}(p,n) \to \mathrm{Grass}(kN, J_k(n,N))$$

which, by the fourth part, descends to the quotient

$$\bar\nu : J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Grass}(kN, J_k(n,N)) \tag{65}$$

(cf. Proposition 4.4). Next, we want to rewrite this embedding in terms of the identifications introduced in §5. So we

- identify $J_k(p,n)$ with $\mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^1 p} \oplus \cdots \oplus \mathbb{C}^{\mathrm{sym}^k p}, \mathbb{C}^n) = \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\le k}(p)}, \mathbb{C}^n)$ where $\mathrm{sym}^j p = \dim\mathrm{Sym}^j\mathbb{C}^p$ and $\mathrm{sym}^{\le k}(p) = \sum_{j=1}^k \mathrm{sym}^j p$;
- identify $J_k(n,1)^*$ with $\mathrm{Sym}^{\le k}\mathbb{C}^n = \oplus_{l=1}^k \mathrm{Sym}^l\mathbb{C}^n$.

We think of an element v of $\mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n)$ as an $n \times \mathrm{sym}^{\leq k}(p)$ matrix, with column vectors in $\mathbb{C}^n$. These columns correspond to basis elements of $\mathbb{C}^{\mathrm{sym}^1 p} \oplus \cdots \oplus \mathbb{C}^{\mathrm{sym}^k p}$, and the columns in the ith component are indexed by i-tuples $1 \leq t_1 \leq t_2 \leq \cdots \leq t_i \leq p$, or equivalently by

$$(e_{t_1} + e_{t_2} + \cdots + e_{t_i}) \in \mathbb{Z}^p_{\geq 0}$$

where $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the jth place, giving us

$$v = (v_{10,\ldots 0}, v_{01\ldots 0}, \ldots, v_{0\ldots 0k}) \in \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n).$$

The elements of $J_k^{\mathrm{reg}}(p,n)$ correspond to matrices whose first p columns are linearly independent. When $n \geq \mathrm{sym}^{\leq k}(p)$ there is a smaller dense open subset $J_k^{\mathrm{nondeg}}(p,n) \subset J_k^{\mathrm{reg}}(p,n)$ consisting of the $n \times \mathrm{sym}^{\leq k}(p)$ matrices of rank $\mathrm{sym}^{\leq k}(p)$.

Define the following map, whose components correspond to the equations in (64):

$$\phi : \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n) \to \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n) \tag{66}$$
$$(v_{10,\ldots 0}, v_{01\ldots 0}, \ldots, v_{0\ldots 0k}) \mapsto (\ldots, \textstyle\sum_{\mathbf{s}_1+\mathbf{s}_2+\cdots+\mathbf{s}_j=\mathbf{s}} v_{\mathbf{s}_1} v_{\mathbf{s}_2} \ldots v_{\mathbf{s}_j}, \ldots),$$

where on the right hand side $\mathbf{s} \in \mathbb{Z}^p_{\geq 0}$.

Example 8.7. *If $k = p = 2$ then ϕ is given by*

$$\phi(v_{10}, v_{01}, v_{20}, v_{11}, v_{02}) = (v_{10}, v_{01}, v_{20} + v_{10}^2, v_{11} + 2v_{10}v_{01}, v_{02} + v_{01}^2).$$

Let $P_{k,p} \subset \mathrm{GL}_{\mathrm{sym}^{\leq k}(p)}$ denote the standard parabolic subgroup with Levi subgroup

$$\mathrm{GL}(\mathrm{sym}^1 p) \times \cdots \times \mathrm{GL}(\mathrm{sym}^k p),$$

where $\mathrm{sym}^j p = \dim \mathrm{Sym}^j \mathbb{C}^p$ and $\mathrm{sym}^{\leq k}(p) = \sum_{j=1}^k \mathrm{sym}^j p$. Then (65) has the following reformulation, analogous to Proposition 5.1.

Proposition 8.8. *The map ϕ in (66) is a $\mathbb{G}_{k,p}$-invariant algebraic morphism*

$$\phi : J_k^{\mathrm{reg}}(p,n) \to \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

which induces an injective map ϕ^{Grass} on the $\mathbb{G}_{k,p}$-orbits:

$$\phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Grass}_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

and

$$\phi^{\mathrm{Flag}} : J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p} \hookrightarrow \mathrm{Flag}_{\mathrm{sym}^1(p),\ldots,\mathrm{sym}^k(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$
$$\hookrightarrow \mathrm{Hom}\,(\mathbb{C}^{\mathrm{sym}(p)}, \mathrm{Sym}^{\leq k}\mathbb{C}^n)/P_{k,p}.$$

Composition with the Plücker embedding gives

$$\phi^{\mathrm{Proj}} = \mathrm{Pluck} \circ \phi^{\mathrm{Grass}} : J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)} \mathrm{Sym}^{\leq k}\mathbb{C}^n).$$

As in the case when $p=1$, we introduce the following notation

$$X_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{reg}}(p,n)),$$
$$Y_{n,k,p} = \phi^{\mathrm{Proj}}(J_k^{\mathrm{nondeg}}(p,n)) \subset \mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)).$$

Definition 8.9. *Let $n \geq \mathrm{sym}^{\leq k}(p) = \mathrm{sym}^1(p) + \cdots + \mathrm{sym}^k(p)$. Then the open subset of $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$ where the projection to $\wedge^{\mathrm{sym}^{\leq k}(p)}\mathbb{C}^n$ is nonzero is denoted by $A_{n,k,p}$.*

Since ϕ^{Grass} and ϕ^{Proj} are $\mathrm{GL}(n)$-equivariant, and for $n \geq \mathrm{sym}^{\leq k}(p)$ the action of $\mathrm{GL}(n)$ is transitive on $\mathrm{Hom}^{\mathrm{nondeg}}(\mathbb{C}^{\mathrm{sym}^{\leq k}(p)}, \mathbb{C}^n)$, we have

Lemma 8.10. (*i*) *If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ is the $\mathrm{GL}(n)$ orbit of*

(67)
$$\mathbf{z} = \phi^{\mathrm{Proj}}(e_1, \ldots, e_{\mathrm{Sym}^{\leq k}(p)}) = \left[\wedge_{j_1+\cdots+j_p \leq k} \sum_{\mathbf{i}_1+\cdots+\mathbf{i}_s=(j_1,\ldots,j_p)} e_{\mathbf{i}_1} \ldots e_{\mathbf{i}_s}\right]$$

in $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$.

- (*ii*) *If $n \geq \mathrm{sym}^{\leq k}(p)$ then $X_{n,k,p}$ and $Y_{n,k,p}$ are finite unions of $\mathrm{GL}(n)$ orbits.*
- (*iii*) *For $k > n$ the images $X_{n,k,p}$ and $Y_{n,k,p}$ are $\mathrm{GL}(n)$-invariant quasi-projective varieties, though they have no dense $\mathrm{GL}(n)$ orbit.*

Similar statements hold for the closure of the image in the Grassmannian

$$\mathrm{Grass}_{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

(or equivalently in the projective space $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$).

Lemma 8.11. *Let $n \geq \mathrm{sym}^{\leq k}(\mathbb{C}^n)$; then*

- (*i*) *$A_{n,k,p}$ is invariant under the $GL(n)$ action on $\mathbb{P}(\wedge^{\mathrm{sym}^{\leq k}(p)}(\mathrm{Sym}^{\leq k}\mathbb{C}^n))$;*
- (*ii*) *$X_{n,k,p} \subset A_{n,k,p}$, although $Y_{n,k,p} \not\subseteq A_{n,k,p}$;*
- (*iii*) *$\overline{X}_{n,k,p}$ is the union of finitely many $GL(n)$-orbits.*

9. Affine embeddings of $\mathrm{SL}(\mathrm{sym}^{\leq k}p)/\mathbb{G}_{k,p}$

In this section we study the case when $n = \mathrm{sym}^{\leq k}p$ and so $\mathrm{GL}(n) \subset J_k^{\mathrm{reg}}(p,n)$. In the previous section we embedded $J_k^{\mathrm{reg}}(p,n)/\mathbb{G}_{k,p}$ in the affine space $A_{n,k,p} \subset \mathbb{P}(\wedge^n \mathrm{Sym}^{\leq k}\mathbb{C}^n)$, which can be restricted to $\mathrm{GL}(n)$ to give us an embedding

$$\mathrm{GL}(n)/\mathbb{G}_{k,p} \hookrightarrow \mathbb{P}(\wedge^n \mathrm{Sym}^{\leq k}\mathbb{C}^n)$$

as the $\mathrm{GL}(n)$ orbit of

$$\left[\cdots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\cdots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \ldots e_{\mathbf{s}_j} \wedge \cdots \right].$$

Equivalently we have $\mathrm{SL}(n)/(\mathrm{SL}(n) \cap \mathbb{G}_{k,p}) = \mathrm{SL}(n)/\mathbb{G}'_{k,p} \rtimes F_{k,p}$ embedded in $\wedge^k(\operatorname{Sym}^{\leq k}\mathbb{C}^k)$ as the $\mathrm{SL}(k)$ orbit of

$$p_{k,p} = \cdots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\cdots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \ldots e_{\mathbf{s}_j} \wedge \cdots ,$$

where $\mathrm{SL}(n) \cap \mathbb{G}_{k,p}$ is the semi-direct product $\mathbb{G}'_{k,p} \rtimes F_{k,p}$ of $\mathbb{G}'_{k,p}$ by the finite group $F_{k,p}$ of $l_{k,p}$th roots of unity in $\mathbb{C}$ for $l_{k,p} = \sum_{i=1}^k i\mathrm{sym}^i p$. In analogy with §6 we can consider an embedding of $\mathrm{SL}(n)/\mathbb{G}'_{k,p}$ in

$$\wedge^n(\operatorname{Sym}^{\leq k}\mathbb{C}^n) \otimes (\wedge^p(\mathbb{C}^n))^{\otimes K}$$

for suitable K and its closure in this affine space. We expect the following result generalising Theorem 6.2.

Conjecture 9.1. *Let $K = M(\sum_{i=1}^k i\mathrm{sym}^i p)+1$ where $M \in \mathbb{N}$. Then the point*

$$p_{k,p} \otimes (e_1 \wedge \cdots \wedge e_p)^{\otimes K} \in \wedge^n(\operatorname{Sym}^{\leq k}\mathbb{C}^n) \otimes (\wedge^p(\mathbb{C}^n))^{\otimes K}$$

where

$$p_{k,p} = \cdots \wedge \sum_{|\mathbf{s}|=j} \sum_{\mathbf{s}_1+\mathbf{s}_2+\cdots+\mathbf{s}_j=\mathbf{s}} e_{\mathbf{s}_1} e_{\mathbf{s}_2} \ldots e_{\mathbf{s}_j} \wedge \cdots$$

has stabiliser $\mathbb{G}'_{k,p}$ in $\mathrm{SL}(n)$, and the closure of its $\mathrm{SL}(n)$ orbit

$$\overline{\mathrm{SL}(n)(p_{k,p} \otimes (e_1 \wedge \cdots \wedge e_p)^{\otimes K})}$$

is the union of the orbit of $p_{k,p} \otimes (e_1 \wedge \cdots \wedge e_p)^{\otimes K}$ and finitely many other $\mathrm{SL}(n)$-orbits, all of which have codimension at least two if k is large enough (depending on p) and M is sufficiently large (depending on k and p).

The proof of Conjecture 9.1 should be similar to that of Theorem 6.2, with the rôle of the Borel subgroup B_k of $\mathrm{SL}(k)$ played by the standard parabolic subgroup $P \subset \mathrm{SL}(n)$ which stabilises the filtration

$$\begin{aligned} 0 \subset \mathbb{C}^p = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_p \subset \mathbb{C}^p \oplus \operatorname{Sym}^2\mathbb{C}^p \subset \cdots \subset \mathbb{C}^p \oplus \operatorname{Sym}^2\mathbb{C}^p \\ \oplus \cdots \oplus \operatorname{Sym}^k\mathbb{C}^p = \mathbb{C}^n. \end{aligned}$$

It follows immediately from Conjecture 9.1 that we would have

COROLLARY 9.2. *If $p \geq 1$ and k is large enough (depending on p) then the reparametrisation group $\mathbb{G}'_{k,p}$ is a subgroup of the special linear group* $\mathrm{SL}(\mathrm{sym}^{\leq k} p)$, *where*

$$\mathrm{sym}^{\leq k} p = \sum_{i=1}^{k} \dim \mathrm{Sym}^i \mathbb{C}^p = \left(\begin{array}{c} k+p-1 \\ k-1 \end{array} \right),$$

such that the algebra of invariants

$$\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}}$$

is finitely generated, so that every linear action of $\mathbb{G}_{k,p}$ or $\mathbb{G}'_{k,p}$ on an affine or projective variety (with respect to an ample linearisation) which extends to a linear action of $GL(\mathrm{sym}^{\leq k} p)$ has finitely generated invariants.

In particular we would have

COROLLARY 9.3. *If $p \geq 1$ and k is large enough (depending on p) then the fibres $\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}}$ of the bundle $E^n_{k,p}$ are finitely generated graded complex algebras.*

We would also obtain geometric descriptions of the associated affine varieties

$$\mathrm{Spec}(\mathcal{O}(\mathrm{SL}(\mathrm{sym}^{\leq k} p))^{\mathbb{G}'_{k,p}})$$

and $\mathrm{Spec}(\mathcal{O}((J_{k,p})_x)^{\mathbb{G}'_{k,p}})$ generalising those in §7.

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References

[1] G. Bérczi, Thom polynomials and the Green-Griffiths conjecture, arxiv:1011.4710.

[2] G. Bérczi, A. Szenes, Thom polynomials of Morin singularities, arXiv:math/0608285, *Annals of Math.*, to appear.

[3] D. Birkes, Orbits of linear algebraic groups, *Annals of Math.* 93 (1971) 459–475.

[4] A. Bloch, Sur les systmes de fonctions uniformes satisfaisant l'équation dune variété algébrique dont l'irrégularité dépasse la dimension, *J. de Math.* 5 (1926), 19–66.

[5] J.-P. Demailly, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, *Proc. Sympos. Pure Math. 62 (1982), Amer. Math. Soc., Providence, RI*, 1997, 285–360.

[6] J.-P. Demailly, J. El-Goul, Hyperbolicity of generic surfaces of high degree in projective 3-space, *Amer. J. Math.* 122 (2000), 515–546.

[7] S. Diverio, J. Merker, E. Rousseau, Effective algebraic degeneracy, *Invent. Math.* 180(2010) 161–223.

[8] I. Dolgachev, Lectures on invariant theory, *London Mathematical Society Lecture Note Series 296, Cambridge University Press*, 2003.

[9] B. Doran and F. Kirwan, Towards non-reductive geometric invariant theory, *Pure Appl. Math. Q.* 3 (2007), 61–105.

[10] A. Fauntleroy, Categorical quotients of certain algebraic group actions, *Illinois Journal Math.* 27 (1983), 115–124.

[11] A. Fauntleroy, Geometric invariant theory for general algebraic groups, *Compositio Mathematica* 55 (1985), 63–87.

[12] T. Gaffney, The Thom polynomial of P^{1111}, *Singularities, Part 1, Proc. Sympos. Pure Math.*, 40, (1983), 399–408.

[13] M. Green, P. Griffiths, Two applications of algebraic geometry to entire holomorphic mappings, *The Chern Symposium 1979. (Proc. Intern. Sympos., Berkeley, California, 1979)* 41–74, Springer, New York, 1980.

[14] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions, *Proc. London Math. Soc.* (3) 67 (1993) 75–105.

[15] G.-M. Greuel and G. Pfister, Geometric quotients of unipotent group actions II, *Singularities (Oberwolfach 1996), 27–36, Progress in Math. 162, Birkhauser, Basel* 1998.

[16] F. Grosshans, Algebraic homogeneous spaces and invariant theory, *Lecture Notes in Mathematics, 1673, Springer-Verlag, Berlin*, 1997.

[17] F. Grosshans, The invariants of unipotent radicals of parabolic subgroups, *Invent. Math.* 73 (1983), 1–9.

[18] F. Kirwan, Quotients by non-reductive algebraic group actions, 'Moduli Spaces and Vector Bundles', L. Brambila-Paz, S. Bradlow, O. Garcia-Prada, S. Ramanan (editors), London Mathematical Society Lecture Note Series **359**, Cambridge University Press 2009.

[19] S. Kobayashi, Hyperbolic complex spaces, *Grundlehren der Mathematischen Wissenschaften* 318, Springer Verlag, Berlin, 1998.

[20] J. Merker, Applications of computational invariant theory to Kobayashi hyperbolicity and to Green-Griffiths algebraic degeneracy, *Journal of Symbolic Computations*, 45 (2010), 986–1074.

[21] J. Merker, Jets de Demailly-Semple dordres 4 et 5 en dimension 2, *Int. J. Contemp. Math. Sciences*, 3 (2008) no. 18. 861–933.

[22] S. Mukai, An introduction to invariants and moduli, *Cambridge University Press* 2003.

[23] D. Mumford, J. Fogarty and F. Kirwan, Geometric invariant theory, *3rd edition, Springer*, 1994.

[24] P. E. Newstead, Introduction to moduli problems and orbit spaces, *Tata Institute Lecture Notes, Springer*, 1978.

[25] G. Pacienza, E. Rousseau, Generalized Demailly-Semple jet bundles and holomorphic mappings into complex manifolds, arXiv:0810.4911.

[26] V. Popov, E. Vinberg, Invariant theory, Algebraic geometry IV, *Encyclopedia of Mathematical Sciences v. 55*, 1994.

[27] E. Rousseau, Etude des jets de Demailly-Semple en dimension 3, *Ann. Inst. Fourier (Grenoble)* 56 (2006), no. 2, 397–421.

[28] Y.-T. Siu, Some recent transcendental techniques in algebraic and complex geometry. *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), 439–448, Higher Ed. Press, Beijing*, 2002.

[29] Y.-T. Siu, Hyperbolicity in complex geometry, *The legacy of Niels Henrik Abel, Springer, Berlin*, 2004, 543–566.
[30] Y.-T. Siu, S.-K. Yeung, Hyperbolicity of the complement of a generic smooth curve of high degree in the complex projective plane, *Invent. Math.* 124, (1996), 573–618.

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