A survey of geometric structure in geometric analysis

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The subject of geometric analysis evolves according to our understanding of geometry and analysis. However, one should say that ideas of algebraic geometry and representation theory have been extremely powerful in both global and local geometry.

In fact, the spectacular idea of using geometry to understand Diophantine problem has already widened our concept of space. The desire to find suitable geometry to accommodate unified field theory in physics would certainly drastically change the scope of geometry in the near future.

In the following lectures, we shall focus on an important branch of geometric analysis: the construction of geometric structures over a given topological space.

1. Part I

There are many kinds of geometric structures; most of them can be classified through the theory of groups and their representations. Some of their structures are motivated by physical science.

The idea of classifying geometric structures through group theory dated back to the famous Erlangen Program of Felix Klein and the later work of E. Cartan.

Most geometric structures are defined by a family of special coordinate charts such that the coordinate transformations or the Jacobian of the coordinate transformations respect some algebraic structure, such as a complex structure, an affine structure, a projective structure or a foliated structure.

Special coordinate systems give connections on natural bundles such as the tangent bundles or some bundles construct from tangent bundles. (Projective structure is related to tangent bundle plus the trivial line bundle, for example.) Connections provide ways to covariantly differentiate vector fields along any curve. For any closed loop at a fixed point, parallel transportation

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along the loop gives rise to a linear transformation of the tangent space at the point to itself.

The totality of such transformations forms a group called the holonomy group of the connection. This group reflects the algebraic aspects of the geometric structure. Therefore, a necessary condition for the geometric structure to exist is the existence of a connection with a special holonomy group on some natural bundle.

On the other hand, connections give rise to a torsion tensor. In order for the existence of a connection with special holonomy group to become sufficient condition for existence of preferred coordinate systems, we usually require the torsion tensor of the connection to be trivial.

In fact, Cartan-Kähler developed an extensive theory of exterior differential systems to provide proofs that, in the real analytic category, existence of a torsion-free connection with special holonomy group is indeed sufficient for the existence of local coordinate systems for most geometric structures.

The smooth version of Cartan-Kähler theory has not been established in general. The most spectacular work to date was due to Newlander-Nirenberg [**49**] on the existence of an integrable complex structure, assuming the complex Frobenius condition. In accordance with our previous discussion, Newlander-Nirenberg proved that if the tangent bundle admits a connection with holonomy group $U(n)$ and the torsion form equal to zero, then the manifold admits a complex structure.

The paper of Newlander-Nirenberg is the first application of nonlinear partial differential equations to constructing geometric structures. Cartan-Kähler theory and Newlander-Nirenberg theory are key contributions to the local theory of geometry.

The global theory of geometric structures is quite complicated and is far from being completed. Deformation theory of global structure was initiated by Kodaira-Spencer [**32**]. Calabi-Vesentini [**6**] and Kuranishi [**33**] studied deformations of complex structures based on Hodge theory. Calabi, Weil, Borel, Matsushima and others studied deformations of geometric structures on deformation of discrete group which eventually lead to the global rigidity theorems of Mostow [**47**] and Margulis [**45**] for locally symmetric spaces.

The approaches of using periods of holomorphic forms (Torelli) and geometric invariant theory (Mumford) to study global algebraic structures are very powerful. Geometric invariant theory has modern interpretations in terms of moment maps of group actions on symplectic manifold. Moment maps and symplectic reductions have important consequence on the theory of nonlinear differential equations.

The idea is that stability arose from actions of noncompact groups and based on this, I proposed the following point of view. If there is a noncompact group acting behind a system of nonlinear differential equations, the existence question of such system will be related to the question of the stability of some algebraic structure that defines this system of nonlinear equations.

An important example is the existence of the Kähler-Einstein metric on Fano manifolds where I conjectured [**68**] to be equivalent to the stability of the algebraic manifolds in the sense of geometric invariant theory.

I shall only touch on the part of geometric structures that can be studied by nonlinear differential equations. They are questions that I am fond of.

The basic idea is to use nonlinear differential equations to build geometric structures which in turn can be used to solve problems in topology or algebraic geometry.

Historically the first global question on geometric structure is the uniformization of conformal structure for domains in the plane. This question dates back to Riemann. It is still an important problem. For instance, we are still trying to understand the structure of moduli space of complex structures over manifolds.

For two dimensional domains, the uniformization theorem of conformal structure gives a description of canonical domains which are bounded by circular arcs. Any finitely connected domain must be conformal to such canonical domains. (The moduli space of such canonical domains can be described easily.)

On the other hand, we can say that any finitely connected domain admits a conformal metric which is flat and whose boundary has constant geodesic curvature. The question of uniformization is then reduced to proving existence and classifying such conformal metrics. Such differential geometric interpretations of problems in conformal geometry is the approach that we shall follow.

For surfaces with higher genus, there are natural conformal metrics that have constant negative curvature. Poincaré was the first to demonstrate that every metric can be conformally deformed to a unique metric with curvature equal to -1. The construction of the Poincaré metric has been fundamental in the understanding of the moduli space of Riemann surfaces.

The cotangent space of the moduli space are represented by holomorphic quadratic differentials. Using the Poincaré metric, one can define an inner product among such quadratic differentials and integrate the product over the surface. The resulting metric can be proved to be a Kähler metric called the Weil-Petersson metric.

On the Riemann surface, there are simple closed geodesics that will decompose the Riemann surface into a planar domain. The function defined by minus log of the sum of the length of these geodesic defines a convex function along geodesics of the Weil-Petersson geometry. This was observed by Scott Wolpert [**66**] who used this to re-prove the fact that the universal cover of the moduli space is contractible and is a Stein manifold.

However, the moduli space of curves are such important object that their global geometry need to be studied in depth. The recent works of Mumford conjecture due to I. Madsen and M. Weiss [**44**] is an important example. There are also works on intersection theory of Chern classes of various bundles over the moduli space which has deep algebraic geometric meaning. Many of these works such as Witten conjecture, Faber conjecture, etc. are all exciting developments.

Holomorphic quadratic differential is very important in classical surface theory. For example, if a map from a Riemann surface into another manifold is harmonic (the map is a critical map of the energy), the pulled back metric $\sum h_{ij} dx^i dx^j$ gives rise to a holomorphic quadratic differential

$$h_{11} - h_{22} + 2\sqrt{-1} h_{12}.$$

This well known statement allows one to apply harmonic map to study the geometry of Teichmüller space. Michael Wolf [**65**] made use of them to give a compactification which is equivalent to the Thurston compactification of the Teichmüller space, which depends on the theory of measured foliation.

Another interesting application of holomorphic quadratic differential is to solve the vacuum Einstein equation for spacetime with dimensional two plus one. Given a conformal structure on a Riemann surface and a holomorphic quadratic differential, the Einstein equation gives a path in the cotangent space of the Teichmüller space of the Riemann surface.

The Weil-Petersson metric is not complete in general. However, the negative of its Ricci tensor is complete. Liu, Sun and myself [**40, 41**] proved that it is equivalent to the Teichmüller metric which is obtained by considering extremal quasiconformal maps between Riemann surfaces. It is also equivalent to the canonical Kähler-Einstein metric that I shall discuss later.

There has been attempts to find a good representation of Teichmüller space or the moduli space of Riemann surfaces. For genus greater than 23, Harris-Mumford [**27**] proved that moduli space is of general type. Hence there is no good parametrization of moduli space.

Teichmüller space has an embedding into $\mathbb{C}^{3g-3}$ due to Bers [**3**]. However, it is not explicit and it is not known how smooth the boundary is. If a bounded domain is smooth, the curvature of the canonical Kähler-Einstein metric must be asymptotic to constant negative curvature in a neighborhood of the point where the domain is convex. This was observed by Cheng-Yau [**9**].

Since the moduli space of Riemann surfaces have a compactification where the divisor at infinity cannot be blown down to a point, the Kähler-Einstein metric cannot be asymptotic to constant negative curvature in any neighborhood. Hence there is no representation of the Teichmüller space as a smooth domain.

The question of how to represent a conformal structure on a Riemann surface is quite interesting. Of course one can compute periods of holomorphic differentials over cycles and Torelli theorem asserts that they can determine the conformal structure of a generic surface. However, how to construct the Riemann surface explicitly from the period is not clear. This is especially true if we want to recognize it in $\mathbb{R}^3$. Can we find canonical surfaces in three space that represent different conformal structures of the surface?

There is another important geometric structure over a two dimensional surface with higher genus. This is the projective structure. They are defined by coordinate neighborhoods whose coordinate transformations are given by projective transformations. There is a map from the universal cover of the surface to $\mathbb{R}P^2$ which preserves the projective structures. If the image is a convex domain, we call the projective structure convex. It turns out that convex projective structures are classified by Riemann surfaces with a cubic holomorphic differentials.

Since this classification is a good illustration of how we construct geometric structures, I shall discuss the construction little more detail.

Convex projective structure on a manifold has an invariant metric obtained in the following way:

The structure is obtained by the quotient of a bounded convex domain Ω in $\mathbb{R}^n$ quotiented by a discrete group of projective transformations. A projectively invariant metric on Ω is obtained by solving the following equation

$$\begin{cases} \det\left(\frac{\partial u}{\partial x_i \partial x_j}\right) = \left(-\frac{1}{u}\right)^{n+2} \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The following metric

$$\sum \left(-\frac{1}{u}\right) \frac{\partial^2 u}{\partial x_i \partial x_j} dx^i \wedge dx^j$$

is observed by Loewner and Nirenberg [**42**] to be invariant under projective transformation. It generalizes the Hilbert model of the Poincaré disk.

Loewner-Nirenberg proved the existence and completeness of the metric for $n = 2$. The general case was proved by Cheng-Yau [**8**]. The Ricci curvature of the metric can be proved to be negative [**5**].

A Legendre transformation will transform the graph $(x, u(x))$ to a new convex surface which is an affine sphere $\sum$. (Affine sphere is a hypersurface where all affine normals converge to a point. Affine normal is a vector transversal to the tangent space invariant under the affine group.) The discrete group of projective transformation become affine group of $\mathbb{R}^3$ acting on $\sum$. (This construction was observed by Calabi [**5**].)

The affine metric can be written as $e^v ds^2$ where $ds^2 = e^\phi \mid dz \mid^2$ is the hyperbolic metric on a Riemann surface.

Using the structure equation for affine sphere, C.P. Wang observed [**64**] that the Pick cubic form in affine geometry is an holomorphic cubic differential Ψdz^3 on the Riemann surface defined by the affine metric so that

$$\triangle v + 4\exp(-2v) \parallel \Psi \parallel^2 - 2\exp(v) - 2K = 0,$$

where K is the Gauss curvature of the conformal metric. (This formulation is essentially due to Tzitzeica [**62**] in 1908.) Conversely, given a holomorphic cubic differential on a Riemann surface, a solution v of the above equation can be used to define an affine sphere in $\mathbb{R}^3$ which in turn gives rise to the projective structure.

The projective connection is in fact given by

$$\begin{pmatrix} \partial v + \partial \phi & \Psi \exp(-v - \phi) d\bar{z} \\ \Psi \exp(-v - \phi) dz & \bar{\partial} v + \bar{\partial} \phi \end{pmatrix}$$

with respect to the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$.

Hence we have a good classification of convex projective structure over a Riemann surface. In general, there are projective structures which are not convex.

Choi has proved that projective structures on surfaces can be uniquely decomposed into several pieces [**11, 12**]. However, we do not have good understanding of the nonconvex part of the projective structure. The study of the moduli space of convex projective structure on surfaces was due to Hitchin [**29**], Goldman [**18**], Labourie [**34**] and Loftin [**43**] using different approaches. The above approach relating it to affine spheres was due to Loftin.

Compact Riemann surface with higher genus cannot admit affine structures. But open surfaces may admit such a structure. In general, we are interested in affine structures over a compact manifold which may be singular along a codimensional two complex. The coordinate transformations are linear whose Jacobian has determinant equal to one.

Motivated by our study of real Monge-Ampère equations and Kähler geometry, S.Y. Cheng and I [**10**] considered in 1979 affine manifolds which may support a metric which we called affine Kähler metric. This is a Riemannian metric which has the property that in each affine chart, there is a convex potential V_α where the metric can be written as

$$\sum \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j} dx_i dx_j.$$

Note that the potentials are well defined up to a linear function.

The equation

$$\det\left(\frac{\partial^2 u_\alpha}{\partial x_i \partial x_j}\right) = 1$$

is well-defined and can be considered as an analogue of the corresponding equation for Calabi-Yau Kähler metrics. In fact, one simply introduces coordinates y_i and define $z_i = x_i + \sqrt{-1} y_i$. Then we can extend u_α to be a function on z_i and obtain a Kähler metric with zero Ricci curvature.

Note that the equation and the affine structures are defined only on the complement of a codimensional two complex. There is a monodromy associated to the equation. The study of existence for the equation with a given monodromy can be considered as an nonlinear analogue of the Riemann-Hilbert correspondence.

If the monodromy preserves some lattice structure, we can define a torus bundle over the affine manifold where the total space is a Ricci flat Kähler manifold. Strominger, Zaslow and myself [**58**] conjectured that for those Calabi-Yau manifolds that admit mirror partner, the total space can be deformed to a complex manifold admitting a (singular) fibration structure, whose fibers are special Lagrangian torus. These are minimal Lagrangian submanifolds and were studied by Harvey and Lawson [**28**] from different point of view.

There is another geometric structure that is of importance in surface theory. This is the line field structure (with singularity) on a surface. An important case is the line field defined by holomorphic quadratic differential and a polynomial vector field. The former case is used by Thurston to form a compactification of the Teichmüller space and the later case is related to the famous Hilbert sixteenth problem which asked the number of limit cycles associated to the vector field. The behavior of the singular points of the line field has practical importance also, e.g., in the study of finger print.

The attempts to generalize these structures on Riemann surfaces to higher dimensional manifolds have occupied the activities of geometric analysts in the past thirty years. The fact that there are much more freedom in higher dimensional manifolds mean that there are many different varieties of geometric structures.

2. Part II

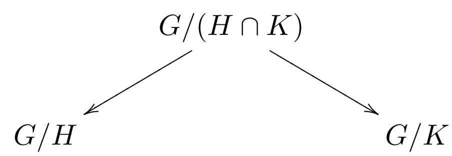
The concept of geometric structure has been enriched continuously. It has been found that metrics with special holonomy group may not be enough to describe the structure. In order to explain this, I will motivate the idea through the concept of duality in string theory. Let us start with some classical examples.

The theory of Lie groups and their discrete subgroups gives rise to Cartan's theory of locally symmetric and homogeneous spaces. They provide examples with rich properties for geometers and analysts. many important properties of these spaces were obtained when we consider them to be moduli space of other geometric objects.

For example, the Siegel upper space can be considered as moduli space of abelian varieties. Occasionally, moduli space of some algebraic manifolds can be locally Hermitian symmetric: Such manifolds include $K3$-surfaces, Calabi-Yau manifolds obtained by taking branched cover over $\mathbb{C}P^3$ along eight hyperplanes or cubic surfaces. Many hyperKähler manifolds such as symmetric products of $K3$ surfaces can be considered as moduli space of semi-stable vector bundles over hyperKähler manifolds.

On the other hand, to understand geometric structures, it is important to understand nonlinear transformations between these spaces that are of geometric importance. For example, if H and K are two closed subgroup in a Lie group G, one can construct a natural map from sheaves or cohomology classes of the space G/H to the space G/K by pulling back the objects from

G/H to $G/(H\cap K)$. After twisting by some universal object on $G/(H\cap K)$, one can push the product to objects on G/K:

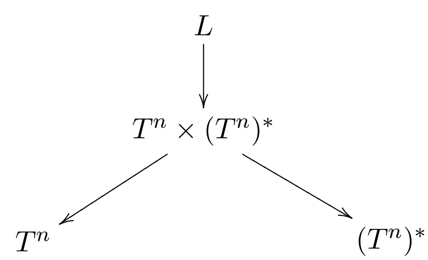


As was observed by Chern, the classical Kinematic Formulae of Poincaré, Santalo and Blaschke can be formulated in terms of the above transformation by taking G to be the group of motions on the homogeneous space where incidence relations of submanifolds are considered.

This kind of transformations also appeared in many places. A very important one is the case of four dimensional manifold M and we consider the moduli space $\mathcal{M}$ of rank two bundles over M whose curvature is self-dual. On the product space $M\times\mathcal{M}$, there is a rank two universal bundle V and we can use the second Chern class of V to transform second cohomology of M to $\mathcal{M}$ and obtain the Donaldson polynomials.

Another important case is the T-duality that has played an important role in number theory and algebraic geometry.

Let $T^n=\mathbb{R}^n/\mathbb{Z}_n$ be a torus and $(T^n)^*=\mathbb{R}^n/(\mathbb{Z}_n)^*$ be the dual torus, which can be considered as the moduli space of complex flat line bundles over T^n. Then we have the following diagram



There is a universal complex line bundle L over $T^n\times(T^n)^*$ so that L restricted to $T^n\times\{q\}$ is isomorphic to q. We can pull back cohomology classes from T^n to $T^n\times(T^n)^*$ where we multiply the class by $\exp(c_1(L))$. Then we can push the product class to $(T^n)^*$. Such a transform can be considered as a nonlinear transform between the torus T^n and its dual $(T^n)^*$. It is called $T-$duality in the recent developments in string theory.

Note that when $n=1$, this is the duality between circle of radius r to circle of radius $\frac{1}{r}$.

Strominger-Yau-Zaslow [**58**] found that a certain algebraic manifold M (Calabi-Yau) admits a T^3 fiber structure over S^3 where generic fibers are T^3. By replacing T^3 by $(T^3)^*$, we obtain another algebraic manifold M^* which is also Calabi-Yau.

By performing a family version of the above $T-$duality and a Legendre transformation on some affine structures on the base, one obtains a transform that maps one geometric structure over the algebraic manifold M to another geometric structure over M^*. (The affine structure on the base space is the one described previously where we have a potential for the metric and the Legendre transform acts on those potentials.)

Note M and M^* may be topologically distinct. This transformation has many important properties. For example, holomorphic bundles V over M are supposed to be mapped to special Lagrangian cycles C in M^*.

In terms of cohomology, the class $Ch(V)\sqrt{Tod(M)}$ in $H^0(M)\oplus H^2(M)\oplus H^4(M)\oplus H^6(M)$ is mapped to cohomology class of $[C]$ in $H^3(M^*)$. The fact that the algebraic bundles of M are mapped to $H^3(M^*)$ raise the following question:

If M and M^* are defined over some number field, will the Frobenius action on the Étale cohomology of $H^3(M)$ be mapped to certain action on the K groups defined by algebraic vector bundles? Will the Adams operation play a role?

SYZ argued that the above nonlinear transform is the same as the mysterious mirror symmetry that was initiated by Greene-Plesser [**19**], Candelas-de la Ossa-Green-Parkes [**7**] based on speculations of conformal field theory.

Both the analytic and algebraic properties of the mirror transform are spectacular. However, they are not yet well-understood. It would be very useful to understand the above construction to map even dim cohomology of M to odd dim cohomology of M^*.

On the other hand, it has already produced a powerful method in geometry. For example, it allows algebraic geometers to calculate the number of algebraic curves in a Calabi-Yau manifold. This was a major classical problem in algebraic geometry. It was solved by Candelas et. al., in that they found the right formula. The rigorous mathematical proof came from the works of Liu-Lian-Yau [**38**] and Givental [**17**].

In principle, we can extend the above T-duality to a more general situation. For example, T^n can be replaced by a $K3$-surface or other algebraic manifold and $(T^n)^*$ can be replaced by the moduli space of semi-stable holomorphic bundles over that manifold. In this case, L can be replaced by the universal bundle. Gukov-Yau-Zaslow [**22**] observed that certain manifolds with holonomy group G_2 have a fiber structure with fiber given by $K3$ surfaces and they are dual to algebraic manifolds which are Calabi-Yau.

The arguments of SYZ and GYZ are based on brane theory, a quantized version of string theory. The belief that the transformation should work well for fibrations with singularities comes from intuition that arose from physics. Mirror symmetry gives rise to many conjectures in geometry which were proved later by rigorous mathematics. The mathematical proof in turn justifies the intuition of the physicists.

Let us now examine how submanifolds can help the construction of geometric structures. It has been an open problem in geometry to construct an explicit metric on a $K3$-surface with holonomy group $SU(2)$. Greene, Shapere, Vafa, and I [**20**] found an explicit metric (with $SU(2)$ holonomy) on the $K3$-surface fibered over the two sphere with torus fiber. All the fibers have flat metric.

However, our metric is singular along the singular fiber. One can perturb this singular metric to be a smooth one with $SU(2)$ holonomy. The perturbation series is believed to be expressible in terms of areas of holomorphic disks with boundary specified to be a subset of the fiber torus. The motivation comes from the interpretation of our metric as a semi-classical approximation to the quantum theory based on the $K3$ surface. The holomorphic disks are instanton corrections.

There is a similar picture for three-dimensional Calabi-Yau manifolds.

In the process of performing the mirror transform, the metric and the complex structure is perturbed by quantities that come from holomorphic cycles or bundles. Hence, it is reasonable to believe that a good geometric structure should include a metric with a certain holonomy group, a space of bundles that have special holonomy group, and a space of cycles such as holomorphic cycles or special Lagrangian cycles. (The Lagrangian that appeared in low energy string theory includes all these quantities and some scalar functions.)

Philosophically, we know that certain subspace of functions can determine the space where they are defined. In fact, algebraic geometers use the rings of algebraic functions to determine the algebraic structure of the manifold. Analytically, we can use solutions of differential equations constructed from the metric to determine the geometric structure.

Obvious functions are harmonic functions, eigenfunctions, eigenforms or spinors. But there are many naturally defined nonlinear differential operators such as the Monge-Ampère operator. Solutions of these nonlinear operators can be directly related to the construction of the metric.

The moduli space of self-dual Yang-Mills bundles or Seiberg-Witten equations have been used by Donaldson et. al. to detect the topological structure of the manifold. One expects that more refined properties of geometric structures can be determined by special bundles or special cycles.

Intuitions from physics have been very useful. In fact, an ultimate goal of geometry is to find a geometric structure that can describe quantum physics when distance is small and general relativity when distance is large. For such a picture, the classical view of spacetime is expected to be changed drastically.

Classical relativity has been verified successfully. The large scale structure of spacetime is therefore in reasonable good shape. However, curvature (or gravity) can drive spacetime to form singularities, which may have to be understood and resolved by quantum physics. The famous conjecture of Penrose says that generic singularity in classical relativity has to be of black hole type.

Singularities are places where physical laws do not hold. What it means is that classical concept of spacetime is not adequate to describe physics at small scale. For small scale structure of spacetime, quantum field theory has to be brought in and it is likely that all the quantities such as bundles and cycles will contribute.

Let me now discuss the approach from the point of view of geometric analysis to construct geometric structures. Two major ways had been developed: one is by gluing structures together and the other one is by calculus of variation or deformation by parabolic equations.

Given a smooth manifold, how does one construct geometric structures over such a manifold? Ideally we would like to find necessary and sufficient conditions in terms of algebraic topological data such as homology classes, homotopy groups and characteristic classes of the manifold.

This is indeed possible for questions such as the existence of almost complex structure by studying the classifying map of the manifold into the classifying space $BU(n)$. The question is reduced to study the lifting of the map to $B(SO(2n))$ which classifies the tangent bundle to a map into $BU(n)$. It is a homotopic question and is completely understood when $n \leq 4$.

$$\begin{array}{ccc} & & BU(n) \\ & \nearrow & \downarrow \\ M & \longrightarrow & BSO(2n) \end{array}$$

In principle, we can replace $U(n)$ by other Lie subgroups of the orthogonal group in the above discussion.

It would be useful to find a necessary and sufficient condition for the existence of G_2-structure on a seven dimensional manifold where the associated three form is closed.

The question of existence of geometric structures is very much related to uniqueness. One can of course relax uniqueness to finite dimensionality of the geometric structures. Only in such cases, techniques of elliptic or parabolic theory of differential equations can be useful. Fortunately, most of the geometric structures have this finite dimensionality property.

However, it should be pointed out that there can be infinitely many distinct components of complex structures on a fixed compact manifold. It will be useful to classify all the possible Chern classes of such complex structures. Similarly, there may be infinite number of components of symplectic structures on a given compact manifold, all of whose symplectic forms belong to the same cohomology class.

The most direct way to construct geometric structures is to perform surgery on manifolds: replacing one handlebody by another handlebody. In the process, one needs to make sure that the new handlebody has compatible geometric structure and the gluing is smooth. The detail of the geometric structure on a manifold with boundary is thus important.

A beautiful example is Thurston's approach to constructing hyperbolic metrics on atoroidal irreducible three-manifolds. Thurston found an important generalization of the rigidity theorem of Mostow on hyperbolic manifolds to three-manifolds with geodesic boundary. The hyperbolic structure is determined by its fundamental group and the conformal structure on the boundary. The possibility of gluing two such manifolds is obtained by a fixed point formula on the Teichmüller space.

Another example is given by the work of Schoen-Yau [**54**] and Gromov-Lawson [**21**] on the classification of manifolds with positive scalar curvature. They prove that surgery on embedded spheres with codimension ≥ 3 preserves the existence of metrics with positive scalar curvature. (Recently some question was raised on the formula given in [**21**].)

Construction of geometric structures on a manifold by surgery can be powerful, as many tools of algebraic topology can be brought in. However, the gluing procedure usually involves some question of convexity. For example, a ball is convex for most geometric structures, and in order to glue it to another manifold along the boundary, the boundary of the other manifold has to be concave in a suitable manner. However, in conformal geometry, inversion turns the ball inside out. Therefore, one can prove that the connected sum of conformally flat manifolds is still conformally flat.

It is much more difficult to glue complex manifolds along a complex submanifold unless the normal bundle of the submanifold is trivial. Even in such cases, it remains to find obstructions to constructing an integrable complex structure on the connected sum of two complex manifolds along the complex submanifold. (If the normal bundle of the complex submanifold is negative, one can perform a contraction and a suitable surgery can be carried out.)

The idea of combining methods from geometric analysis and gluing a geometric structure to a given manifold was initiated by the pioneering work of Taubes. He was the first one to construct anti-self dual bundles on four manifolds by gluing the instantons from four spheres to a given four dimensional manifold. This eventually leads to the Donaldson theory, which is the major tool in four manifold theory.

In 1992, Taubes [**59**] was able to perform similar procedure to construct anti-self dual metrics on any four dimensional manifold as long as we glue in enough copies of $\mathbb{C}P^2$. The twistor space of these manifolds are complex three dimensional manifold fibered over the four manifold with S^2 fibers.

Similar technique was later used by Joyce [**30, 31**] in 1996 to construct seven dimensional manifolds with holonomy group equal to G_2 and eight dimensional manifolds with holonomy group equal to $Spin(7)$.

The works are all based on singular perturbation method and are very powerful.

Unfortunately the perturbation method is not powerful enough to provide the information of the full moduli space of the corresponding structures. And this is the most basic question in order to apply G_2 manifolds to M-theory that appeared in string theory.

Existence and moduli space of affine, projective flat and conformally flat structure in higher dimension is much more difficult than two dimension, for example, it is not known whether hyperbolic three manifolds admit affine structures.

A well-known question whether compact affine manifolds have zero Euler number is not solved. It is known to be true if the connection is complete.

Many of the questions are related to developing map from the universal cover of the manifold to $\mathbb{R}^n$, $\mathbb{R}P^n$ or S^n. In general, the map need not be injective. If the map is injective, the manifold with such geometric structure will be equivalent to study of discrete group acting on a domain. Only in one case, we know the developing map is injective. Schoen-Yau [**55**] proved in 1986, that any conformal map from a complete conformal flat manifold with positive scalar curvature into S^n is injective.

This property is false without assuming positivity of scalar curvature. Conformally flat manifolds with positive scalar curvature are then quotients of domains in S^n by a discrete group of Mobius transformations. The domain is dense in S^n with large codimension.

When symmetry is imposed, we have much better understanding of the spacetime. In the past twenty years, the most fruitful results have been found for spacetime with supersymmetries. The concept of supersymmetry may not be acceptable to some physicists, but it does provide a beautiful and elegant playing ground for geometers. Many classical questions in geometry were resolved by supersymmetric considerations.

A good example is the Seiberg-Witten theory which was motivated by supersymmetric Yang-Mills theory.

The invariant created by Seiberg-Witten theory has been very powerful for the study of four manifolds: especially for those four dimensional symplectic four manifolds.

In the later case, Taubes proved the deep theorem that creates existence of pseudo-holomorphic curves based on the topological data of Seiberg-Witten invariants. As a corollary, he proved that there is only one symplectic structure on $\mathbb{C}P^2$.

A.K. Liu [**39**] was also able to classify all four dimensional symplectic manifolds that support a metric with positive scalar curvature.

3. Part III

When the topological method of surgery and gluing fails, we have to find a method that does not depend on the detailed topological information of the manifold. The best example is the proof of the Severi conjecture and the Poincaré conjecture and the geometrization conjecture.

The beauty of the method of nonlinear differential equation is that we can keep on deforming some unknown structure until we can recognize them eventually. The control on this process of deformation depends on careful a priori estimate of the nonlinear equation. However, if the structure is to be changed in a large scale, standard energy estimate usually cannot be used as the underlying Sobolov inequality depends on the geometric structure and need not hold in general. Hence maximum principle is used in most cases.

A fruitful idea to construct geometric structure is to construct metrics that satisfy the Einstein equation. We demand that the Ricci tensor of the metric be proportional to the metric itself. This can be considered as a generalization of the Poincaré metric to higher dimensions. This is an elliptic system, if we identify metrics up to diffeomorphisms. The problem of existence of an Einstein metric is really a very difficult but central problem in geometry.

One can obtain such metrics by a variational principle: After normalization of the metrics by setting their volume equal to one, we minimize the total scalar curvature in each fixed conformal class; then we vary the conformal class and maximize the (constant) scalar curvature. The first part is called the Yamabe problem and was settled by the works of Trudinger [**60**], Aubin [**2**] and Schoen [**53**].

The most subtle part was the case when the manifold is conformally flat, where Schoen made use of the positive mass conjecture to control the Green's function of the conformally invariant operator and hence settle this famous analytic problem. The relation of this problem with general relativity is a pleasant surprise and should be considered as an important development in geometric analysis. The second part of maximization among all conformal structure is much more difficult. Schoen and his students, and also M. Anderson have made contributions towards this approach.

Let me now discuss the other two major general approaches to constructing Einstein-metrics. The first one is to solve the equation on a space with certain internal symmetries. For such manifolds, the ability to choose a special gauge, such as holomorphic coordinates is very helpful. The space with internal symmetry can be a Kähler manifold or a manifold with special holonomy group.

A very important example is given by the Calabi conjecture, where one asked whether the necessary condition for the first Chern class to have definite sign is also sufficient for the existence of Kähler-Einstein metric.

Algebraic varieties are classified according to the map of the manifold into the complex projective space by powers of the canonical line bundle. If the map is an immersion at generic point, the manifold is called an algebraic manifold of general type. This class of manifolds comprises the majority of algebraic manifolds, and these manifolds can be considered as generalizations of algebraic curves of higher genus.

In general, the above canonical map may have a "base point" and hence be singular. However, the minimal model theory of the Italian and Japanese school (Castelnuovo, Fano, Enriques, Severi, Bombieri, Kodaira, Mori, Kawamata, Miyaoka, Inoue) showed that an algebraic manifold of general type can be contracted to a certain minimal model, where the canonical map has no base point. In this case, the first Chern class of the minimal model is non-positive and negative in a Zariski open set.

Most algebraic manifolds of general type have negative first Chern class. In this case, Aubin and I independently proved the existence and uniqueness of a Kähler-Einstein metric.

For the general case of minimal models of manifolds of general type, the first Chern class is not negative everywhere. Hence it does not admit a regular Kähler-Einstein metric.

However, it admits a canonical Kähler-Einstein metric which may have singularities. This statement was observed by me right after I wrote my paper on the Calabi conjecture, where I also discussed the regularity of degenerate Kähler-Einstein metrics. (Tsuji [**61**] later reproved this theorem in 1985 using Hamilton's Ricci flow.)

The singularity of this canonical Kähler-Einstein metric that I constructed on manifolds of general type is not so easy to handle. By making some assumption on the divisors, Cheng-Yau and later Tian-Yau contributed to understanding of the structure of these metrics. These metrics give important algebraic geometric informations of the manifolds.

In 1976, I [**67**] observed that the Kähler-Einstein metric can be used to settle important questions in algebraic geometry. An important contribution is the algebraic-geometric characterization of Shimura varieties: quotients of Hermitian symmetric domains by discrete groups. They are characterized by the statement that certain natural bundle, constructed from tensor product of tangent bundles, has nontrivial holomorphic section.

The other important assertions are the inequalities between Chern numbers for algebraic manifolds. For an algebraic surface, I proved $3C_2(M) \geq C_1^2(M)$, an inequality which was independently proved by Miyaoka by algebraic means. I [**67**] proved further that equality holds only if M has constant holomorphic sectional curvature. My inequality holds in arbitrary dimension.

It is the last assertion that enabled me to prove that there is only one complex structure on the complex projective plane. This statement was a famous conjecture of Severi.

The construction of Ricci flat Kähler metric has been used extensively in both algebraic geometry and string theory, such as Torelli theorem for $K3$ surfaces and deformation of complex structure.

The construction of Kähler-Einstein metric with positive scalar curvature has been a very active field. In early eighties, I proposed its existence in relation to stability of the manifolds.

In the hands of Donaldson, and others, we see that my proposal is close to be realized. It gives new information about the algebraic geometric stability of manifolds.

In general, there should be an interesting program to study Kähler-Einstein metrics on the moduli space of either complex structures or stable bundles. It should provide some informations for the moduli space. For example, recently, using this metric, Liu-Sun-Yau [**41**] proved the Mumford stability of the logarithmic cotangent bundle of the moduli spaces of Riemann surfaces.

Hence we see that by constructing new geometric structure through nonlinear partial differential equation, one can solve problems in algebraic geometry that are a priori independent of this new geometric structure.

A holomorphic coordinate system is a very nice gauge and a Kähler metric is a beautiful metric as it depends only on one function. When we come to the space of Riemannian metrics, we need to understand a large system of nonlinear equations invariant under the group of diffeomorphism. The choice of gauge causes difficulty.

The Severi conjecture can be considered as a complex analog of the Poincaré conjecture. The fact that Einstein metrics were useful in settling the Severi conjecture indicates that these metrics should also be useful for the geometrization conjecture and hence the Poincaré conjecture. This was what we believed in the late seventies.

Many methods motivated by the calculus of variation were proposed. The most promising method was due to Hamilton who proposed to deform any metric along the negative of its Ricci curvature. The development of the Ricci flow has gone through several important stages of development.

The first decisive one was Hamilton's demonstration of the global convergence of the Ricci flow [**23**] when the initial metric has positive Ricci curvature. This is a fundamental contribution that give confidence on the importance of the equation.

To move further, it was immediately clear that one needs to control the singularities of the flow. This was studied extensively by Hamilton. The necessary a priori estimate was based on Hamilton's spectacular generalization of the works of Li-Yau [**37**].

Li-Yau introduced a distance function on spacetime to control the precise behavior of the parabolic system near the singularity. The concept appears naturally from the point of view of a priori estimate. For example, if the equation is

$$\frac{\partial u}{\partial t} = \triangle u - Vu.$$

The distance introduced by Li-Yau is given by

$$d((x,t_1),(y,t_2)) = \inf_r \Big\{ \frac{1}{4(t_2-t_1)} \int_0^1 \mid \dot{r} \mid^2 + (t_2-t_1) \int_0^1 V\big(r(s),(1-s)t_2+st_1\big) \Big\}.$$

where V are paths joining (x,t_1) to (y,t_2).

The kernel of the parabolic equation can then be estimated by this distance function.

The potential V is naturally replaced by the scalar curvature in the case of Ricci flow as it appears in the action of gravity. This is what Perelman did later. The idea of Li-Yau-Hamilton come from the careful study of maximum principle. The basic philosophy of LYH is to study the extreme situation. In the case of Ricci flow, one looks at the soliton solution and verify some equality holds along the soliton and such equality can be turned to be estimates for general solutions of the parabolic system, via maximum principle.

In the nineties, Hamilton [**24, 25, 26**] was able to classify singularities of the Ricci flow in three dimension and prove the geometrization conjecture if the curvature of the flow is uniformly bounded. These are very deep works both from the point of view of geometry and analysis. Many ideas in geometric analysis were used. This includes the proof of the positive mass conjecture, the injectivity radius estimate and an improved version of the Mostow rigidity theorem. In particular, he introduced the concept of Ricci flow with surgery.

In his classification of singularities, Hamilton could not determine the existence or nonexistence of one type of singularity which he called cigar. This type of singularity was proved to be non-existent by Perelman [**50**] in 2002 in an elegant manner. Perelman [**51**] then extended the work of Hamilton on flows with surgery. Among many creative ideas, he found a priori estimates for the gradient of the scalar curvature, the concept of reduced volume and a new way to perform surgery with control.

The accumulated works of Hamilton-Perelman are spectacular. Today, 5 years after the first preprint of Perelman was available, several groups of mathematicians have put forward their manuscripts explaining their understandings on how Hamilton-Perelman's ideas can be put together to prove the Poincaré conjecture; at the same time, other experts are still working diligently on the proof of this century old conjecture.

Besides the Poincaré conjecture, Ricci flow has many other applications: A very important one is the contribution due to Chau, Chen, Ni, Tam and Zhu, towards the proof of the conjecture that every complete noncompact Kähler manifold with positive bisectional curvature is bi-holomorphic to $\mathbb{C}^n$. (I made this conjecture in 1972 as a generalization to higher dimension of the uniformization theorem. Proceeding to the conjecture, there were important works of Greene-Wu to proving Steinness of the complete noncompact Kähler manifold with positive sectional curvature.)

More recently, several old problems were solved by using the classical results of Hamilton that were published in 1983, 1986 and 1997.

The most outstanding one is the recent result of Brendle-Schoen [**4**]. They proved that manifolds with pointwise quarter-pinching curvature are diffeomorphic to manifolds with constant positive curvature.

This question has puzzled mathematicians for more than half a century.

It has been studied by many experts in differential geometry.

Back in 1950s, Rauch was the first one who introduce the concept of pinching condition. Berger and Klingenberg proved such a manifold to be homeomorphic to a sphere when it is simply connected.

The diffeomorphic type of the manifolds is far more difficult to understand. For example, Gromoll's thesis achieved a partial result toward settling such a result: he assumed a much stronger pinching condition.

The result of Brendle-Schoen achieved optimal pinching condition. More remarkably, they only need pointwise pinching condition and do not have to assume simply-connectivity. Both of these conditions are not accessible by the older methods of comparison theorems.

This result partially builds on fundamental work by Böhm and Wilking who proved that a manifold with positive curvature operator is diffeomorphic to a spherical space form.

Therefore, the program on Ricci flow laid down by Hamilton in 1983 has opened a new era for geometric analysts to build geometric structures.

Other obvious problems are to construct geometric structure on other low dimensional manifolds, especially four-dimensional manifolds. Besides the fundamental works based of Donaldson and Seiberg-Witten, we know very little about the geometry of four manifolds. The most fundamental structures on four manifold are complex structures and metrics with anti-self-dual curvature. (Most four manifolds are obtained by some simple surgery on complex surfaces. An important operation called log transformation was introduced by Kodaira. It can change the diffeomorphisim type of the four manifold.)

The Atiyah-Singer index formula gives very important obstructions for the existence of integrable complex structures on surfaces, as was found by Kodaira. The moduli spaces of holomorphic vector bundles have been a major source for Donaldson to provide invariants for smooth structures. On the other hand, the existence of pseudoholomorphic curves based on Seiberg-Witten invariant constructed by Taubes is a powerful tool for symplectic topology. It seems natural that one should build geometric structures over a smooth manifold that include all these types of information.

The integrability condition derived from Atiyah-Singer formula for almost complex structures in $\dim_{\mathbb{C}} \geq 3$ is not powerful enough to rule out the following conjecture:

For $\dim_{\mathbb{C}} \geq 3$, every almost complex manifold admits an integrable complex structure.

If this conjecture is true, we need to build geometry over such nonKähler complex manifolds. This is especially interesting in higher dimension. It is possible to deform an algebraic manifold to another one with different topology by tunneling through nonKähler structures. A good example is related to the Clemens-Friedman construction that one can collapse rational curves in a Calabi-Yau three manifold to conifold singularity. Then by smoothing the singularity, one obtains nonsingular nonKähler manifold. Reversing the procedure, one may get another Calabi-Yau manifold.

Reid [**52**] proposed that this procedure may connect all Calabi-Yau manifolds in three dimension. There is perhaps no reason to restrict ourselves only to Calabi-Yau manifolds, but to more general algebraic manifolds on a fixed topological manifold, is there other general construction to deform one algebraic structure from one component of the moduli space of a complex structure to other component through nonKähler complex structures?

Non-Kähler complex structures are difficult to handle geometrically. However, there is an interesting concept of Hermitian structure that can be useful. This is the class of balanced structure.

An Hermitian metric ω is called balanced iff

$$d(\omega^{n-1}) = 0.$$

It was first studied by M. Michelsohn [**46**] and Alessandrini-Bassanelli [**1**], who observed that twistor space admits a balanced metric and that existence of balanced metric is invariant under birational transformation. Recently, it came up in the theory of Heterotic string, based on a warped product compactification.

Strominger [**57**] suggests that there should be a holomorphic vector bundle that should admit a Hermitian Yang-Mills connection and that there should be Hermitian metric that is conformally balanced. To be more precise, there should be a holomorphic 3-form Ω so that

$$d(\| \Omega \|_\omega \, \omega^2) = 0,$$

where ω is the Hermitian form. An important link between the bundle and the metric is that connections on both structures give trivial first Chern form and the difference between their second Chern forms can be written as $\sqrt{-1}\partial\bar{\partial}\omega$.

This geometric structure constructed for Heterotic string theory is based on construction of parallel spinors and the anomaly equation required by quantization of string theory.

On the other hand, general existence theorem for the Strominger system is still not known.

An interesting mathematical question is to construct a balanced complex three manifold with a nonvanishing holomorphic 3-form. Then we like to construct a stable holomorphic vector bundle that satisfies all of the above equations of Strominger.

Jun Li and I [**36**] proved the existence of Strominger system by perturbing around the Calabi-Yau metric.

The first example on a nonKähler manifold is due to Fu-Yau [**16**]. It is obtained by forming a torus fiber bundle over $K3$ surface (due to Dasgupta-Rajesh-Sethi, Becker-Becker-Dasgupta-Green and Goldstein-Prokushkin).

The construction of Strominger system over this manifold can be achieved if we can solve the following complex Monge-Ampère equation:

$$\Delta(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

where f and μ are given functions on $K3$ surface S so that $f \geq 0$ and $\int_S \mu = 0$. This was achieved by Fu-Yau based on a priori estimates of u, which is more complicated than those used in Calabi conjecture.

There are nontrivial interpretation of the Fu-Yau example through conformal field theory.

The supersymmetric heterotic string gives an $SU(3)$ Hermitian connection on the tangent bundle. But the connection has torsion (which is trace free).

If we are interested in G-structure on the tangent bundle with $G \subseteq U(n)$, it can be accomplished by considering the work of Donaldson-Uhlenbeck-Yau [**14, 63**] on stable holomorphic bundles over Kähler manifold. The work generalizes the work of Narasimhan-Seshadri [**48**] for algebraic curves.

It was extended to nonKähler manifolds by Li-Yau [**35**] where the base complex manifold admit a Gauduchon metric ω with

$$\partial\bar{\partial}(\omega^{n-1}) = 0.$$

If the tangent bundle T is stable and if some irreducible subbundles constructed from tensor product of T admits nontrivial holomorphic section, the structure group can be reduced. The major question is how to control the torsion of this connection by choosing ω suitably.

In the other direction, one should mention that Smith-Thomas-Yau [**56**] succeeded to construct symplectic manifold mirror to the Clemens-Friedman construction. While the Clemens-Friedman [**13, 15**] construction leads to nonKähler complex structures over connected sums $S^3 \times S^3$, the Smith-Thomas-Yau construction lead to symplectic non-complex structure over connected sums of $\mathbb{C}P^3$ (which may not admit any integrable complex structure).

We expect a mirror structure for the Strominger system in symplectic geometry, where we hope to build an almost complex structure compatible to the symplectic form. They should satisfy a good system of equations. We expect that special Lagrangian cycles and pseudoholomorphic curves will play roles in such a new structure which is dual to the above system of equations of Strominger.

The inspirations from string theory has given amazingly deep insight into the structure of Calabi-Yau manifolds which are manifolds with holonomy group $SU(n)$.

Constructions of geometric structures by coupling metrics with vector bundles and submanifolds should give a new direction in geometry, as they may exhibit supersymmetry. An important idea provided by string theory is that duality exists between supersymmetric manifolds. Duality allows us to compute difficult geometric information by perturbation methods on the dual objects.

General relativity and string theory have inspired a great deal of geometric ideas and it has been very fruitful. Nature also tells us everything vibrates and there should be intrinsic frequency associated to our geometric structure. In the classical geometry, we have an elliptic operator associated to deformation of the structure. For space of Einstein metrics, it is called the Lichnerowicz operator. It will be interesting to study the spectrum of this operator.

Quantum gravity may provide a deeper concept. A successful construction of quantum geometry will change our scope of geometric structures.

> *Pure logical thinking cannot yield us any knowledge of the empirical world. All knowledge of reality starts from experience and ends in it. Propositions arrived at by purely logical means are completely empty as regards reality.*
>
> —**Einstein** (Herbert Spencer lecture at Oxford in 1933)

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