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Generalized Donaldson–Thomas invariants

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ABSTRACT. This is a survey of the book [**16**] with Yinan Song. Donaldson–Thomas invariants $DT^\alpha(\tau) \in \mathbb{Z}$ 'count' τ-(semi)stable coherent sheaves with Chern character α on a Calabi–Yau 3-fold X. They are unchanged under deformations of X. The conventional definition works only for classes α with no strictly τ-semistable sheaves. Behrend showed that $DT^\alpha(\tau)$ can be written as a weighted Euler characteristic $\chi\big(\mathcal{M}_{\rm st}^\alpha(\tau), \nu_{\mathcal{M}_{\rm st}^\alpha(\tau)}\big)$ of the stable moduli scheme $\mathcal{M}_{\rm st}^\alpha(\tau)$ by a constructible function $\nu_{\mathcal{M}_{\rm st}^\alpha(\tau)}$ we call the 'Behrend function'.

We discuss *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$. These are defined for all classes α, and are equal to $DT^\alpha(\tau)$ when it is defined. They are unchanged under deformations of X, and transform according to a known wall-crossing formula under change of stability condition τ. We conjecture that they can be written in terms of integral *BPS invariants* $\hat{DT}^\alpha(\tau) \in \mathbb{Z}$ when the stability condition τ is 'generic'.

We extend the theory to abelian categories mod-$\mathbb{C}Q/I$ of representations of a quiver Q with relations I coming from a superpotential W on Q, and connect our ideas with Szendrői's noncommutative Donaldson–Thomas invariants, and work by Reineke and others on invariants counting quiver representations. The book [**16**] has significant overlap with a recent, independent paper of Kontsevich and Soibelman [**18**].

1. Introduction

This is a survey of the book [**16**] by the author and Yinan Song. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathcal{O}_X(1)$ a very ample line bundle on X. Our definition of Calabi–Yau 3-fold requires X to be projective, with $H^1(\mathcal{O}_X) = 0$. Write coh(X) for the abelian category of coherent sheaves on X, and $K(X)$ for the numerical Grothendieck group of coh(X). Let τ denote Gieseker stability of coherent sheaves w.r.t. $\mathcal{O}_X(1)$. If E is a coherent sheaf on X then $[E] \in K(X)$ is in effect the Chern character $\mathrm{ch}(E)$ of E in $H^{\mathrm{even}}(X;\mathbb{Q})$.

For $\alpha \in K(X)$ we can form the coarse moduli schemes $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$, $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ of τ-(semi)stable sheaves E with $[E]=\alpha$. Then $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ is a projective $\mathbb{C}$-scheme whose points correspond to S-equivalence classes of τ-semistable sheaves, and $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ is an open subscheme of $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)$ whose points correspond to isomorphism classes of τ-stable sheaves.

For Chern characters α with $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)=\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$, following Donaldson and Thomas [**4**, §3], Thomas [**33**] constructed a symmetric obstruction theory on $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ and defined the *Donaldson–Thomas invariant* to be the virtual class

$$DT^{\alpha}(\tau)=\textstyle\int_{[\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)]^{\mathrm{vir}}} 1 \in \mathbb{Z},$$

an integer which 'counts' τ-semistable sheaves in class α. Thomas' main result [**33**, §3] is that $DT^{\alpha}(\tau)$ is unchanged under deformations of the underlying Calabi–Yau 3-fold X. Later, Behrend [**1**] showed that Donaldson–Thomas invariants can be written as a weighted Euler characteristic

$$DT^{\alpha}(\tau)=\chi\big(\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau),\nu_{\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)}\big),$$

where $\nu_{\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)}$ is the *Behrend function*, a constructible function on $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ depending only on $\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ as a $\mathbb{C}$-scheme.

Conventional Donaldson–Thomas invariants $DT^{\alpha}(\tau)$ are only defined for classes α with $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)=\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$, that is, when there are no strictly τ-semistable sheaves. Also, although $DT^{\alpha}(\tau)$ depends on the stability condition τ, that is, on the choice of very ample line bundle $\mathcal{O}_X(1)$ on X, this dependence was not understood until now. The main goal of [**16**] is to address these two issues.

For a Calabi–Yau 3-fold X over $\mathbb{C}$ we will define *generalized Donaldson–Thomas invariants* $\bar{DT}^{\alpha}(\tau) \in \mathbb{Q}$ for all $\alpha \in K(X)$, which 'count' τ-semistable sheaves in class α. These have the following important properties:

- $\bar{DT}^{\alpha}(\tau) \in \mathbb{Q}$ is unchanged by deformations of the Calabi–Yau 3-fold X.
- If $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)=\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ then $\bar{DT}^{\alpha}(\tau)$ lies in $\mathbb{Z}$ and equals the conventional Donaldson–Thomas invariant $DT^{\alpha}(\tau)$ defined by Thomas [**33**].
- If $\mathcal{M}_{\mathrm{ss}}^{\alpha}(\tau)\neq\mathcal{M}_{\mathrm{st}}^{\alpha}(\tau)$ then conventional Donaldson–Thomas invariants $DT^{\alpha}(\tau)$ are not defined for class α. Our generalized invariant $\bar{DT}^{\alpha}(\tau)$ may lie in $\mathbb{Q}$ because strictly semistable sheaves E make (complicated) $\mathbb{Q}$-valued contributions to $\bar{DT}^{\alpha}(\tau)$. For 'generic' τ we have a conjecture that writes the $\bar{DT}^{\alpha}(\tau)$ in terms of other, integer-valued invariants $\hat{DT}^{\alpha}(\tau)$.
- If $\tau,\tilde{\tau}$ are two stability conditions on $\mathrm{coh}(X)$, there is an explicit change of stability condition formula giving $\bar{DT}^{\alpha}(\tilde{\tau})$ in terms of the $\bar{DT}^{\beta}(\tau)$.

These invariants are a continuation of the author's programme [**9**–**15**].

We begin in §2 with some background material on *constructible functions* and *stack functions* on Artin stacks, taken from [**9**,**10**]. Then §3 summarizes ideas from [**11**–**14**] on Euler-characteristic type invariants $J^\alpha(\tau)$ counting sheaves on Calabi–Yau 3-folds and their wall-crossing under change of stability condition, and facts on Donaldson–Thomas invariants from Thomas [**33**] and Behrend [**1**].

Section 4 summarizes [**16**, §5–§6], and is the heart of the paper. Let X be a Calabi–Yau 3-fold, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. Write $\bar\chi : K(X)\times K(X)\to\mathbb{Z}$ for the Euler form of $\mathrm{coh}(X)$. We will explain that the Behrend function $\nu_{\mathfrak{M}}$ of $\mathfrak{M}$ satisfies two important identities

$$\nu_{\mathfrak{M}}(E_1\oplus E_2)=(-1)^{\bar\chi([E_1],[E_2])}\nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2),$$

$$\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda\Leftrightarrow 0\to E_1\to F\to E_2\to 0}}\nu_{\mathfrak{M}}(F)\mathrm{d}\chi-\int_{\substack{[\lambda']\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \lambda'\Leftrightarrow 0\to E_2\to F'\to E_1\to 0}}\nu_{\mathfrak{M}}(F')\mathrm{d}\chi$$
$$=\bigl(\dim\mathrm{Ext}^1(E_2,E_1)-\dim\mathrm{Ext}^1(E_1,E_2)\bigr)\nu_{\mathfrak{M}}(E_1\oplus E_2).$$

We use these to define a Lie algebra morphism $\tilde\Psi:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})\to\tilde L(X)$, where $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ is a special Lie subalgebra of the *Ringel–Hall algebra* $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ of X, a large algebra with a universal construction, and $\tilde L(X)$ is a much smaller explicit Lie algebra, the $\mathbb{Q}$-vector space with basis $\tilde\lambda^\alpha$ for $\alpha\in K(X)$, and Lie bracket

$$[\tilde\lambda^\alpha,\tilde\lambda^\beta]=(-1)^{\bar\chi(\alpha,\beta)}\bar\chi(\alpha,\beta)\tilde\lambda^{\alpha+\beta}.$$

If τ is Gieseker stability in $\mathrm{coh}(X)$ and $\alpha\in K(X)$, we define an element $\bar\epsilon^\alpha(\tau)$ in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ which 'counts' τ-semistable sheaves in class α in a special way. We define the *generalized Donaldson–Thomas invariant* $\bar{DT}^\alpha(\tau)\in\mathbb{Q}$ by

$$\tilde\Psi\bigl(\bar\epsilon^\alpha(\tau)\bigr)=-\bar{DT}^\alpha(\tau)\tilde\lambda^\alpha.$$

By results in [**14**], the $\bar\epsilon^\alpha(\tau)$ transform according to a universal transformation law in the Lie algebra $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ under change of stability condition. Applying $\tilde\Psi$ shows that $-\bar{DT}^\alpha(\tau)\tilde\lambda^\alpha$ transform according to the same law in $\tilde L(X)$. This yields a wall-crossing formula for two stability conditions $\tau,\tilde\tau$ on $\mathrm{coh}(X)$:

$$\bar{DT}^\alpha(\tilde\tau)=\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}}\;\sum_{\substack{\kappa:I\to C(X):\\ \sum_{i\in I}\kappa(i)=\alpha}}\;\sum_{\substack{\text{connected,}\\ \text{simply-}\\ \text{connected}\\ \text{digraphs }\Gamma,\\ \text{vertices } I}}\begin{aligned}[t]&(-1)^{|I|-1}V(I,\Gamma,\kappa;\tau,\tilde\tau)\cdot\textstyle\prod_{i\in I}\bar{DT}^{\kappa(i)}(\tau)\\ &\cdot(-1)^{\frac12\sum_{i,j\in I}|\bar\chi(\kappa(i),\kappa(j))|}\cdot\prod_{\text{edges } \overset{i}{\bullet}\to\overset{j}{\bullet}\text{ in }\Gamma}\bar\chi(\kappa(i),\kappa(j)),\end{aligned}\tag{1}$$

where $V(I,\Gamma,\kappa;\tau,\tilde\tau)\in\mathbb{Q}$ are combinatorial coefficients, and there are only finitely many nonzero terms.

To prove that $\bar{DT}^\alpha(\tau)$ is unchanged under deformations of X, we introduce auxiliary invariants $PI^{\alpha,n}(\tau')\in\mathbb{Z}$ counting 'stable pairs' $s:\mathcal{O}(-n)\to E$, for $n\gg 0$ and $E\in\mathrm{coh}(X)$ τ-semistable in class $\alpha\in K(X)$. The moduli space $\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')$ of such stable pairs is a proper fine moduli $\mathbb{C}$-scheme with a symmetric obstruction theory, so by the same proof as for Donaldson–Thomas invariants [**33**], the virtual count $PI^{\alpha,n}(\tau')$ of $\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')$ is deformation-invariant. By a wall-crossing proof similar to that for (1) we find that

$$PI^{\alpha,n}(\tau')=\sum_{\substack{\alpha_1,\ldots,\alpha_l\in C(X),\\ l\geqslant 1:\,\alpha_1+\cdots+\alpha_l=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\text{ all } i}}\frac{(-1)^l}{l!}\prod_{i=1}^{l}\bigl[(-1)^{\bar\chi([\mathcal{O}_X(-n)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)}\,\bar\chi\bigl([\mathcal{O}_X(-n)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i\bigr)\bar{DT}^{\alpha_i}(\tau)\bigr].$$

Using deformation-invariance of the $PI^{\alpha,n}(\tau')$ and induction on $\mathrm{rank}\,\alpha$ we find that $\bar{DT}^\alpha(\tau)$ is deformation-invariant.

Examples show that in general the $\bar{DT}^\alpha(\tau)$ lie in $\mathbb{Q}$ rather than $\mathbb{Z}$. So it is an interesting question whether we can rewrite the $\bar{DT}^\alpha(\tau)$ in terms of some system of $\mathbb{Z}$-valued invariants, just as $\mathbb{Q}$-valued Gromov–Witten invariants of Calabi–Yau 3-folds are (conjecturally) written in terms of $\mathbb{Z}$-valued Gopakumar–Vafa invariants [**7**]. We define new *BPS invariants* $\hat{DT}^\alpha(\tau)$ for $\alpha\in C(X)$ to satisfy

$$\bar{DT}^\alpha(\tau)=\sum_{m\geqslant 1,\,m|\alpha}\frac{1}{m^2}\,\hat{DT}^{\alpha/m}(\tau),$$

and we conjecture that $\hat{DT}^\alpha(\tau)\in\mathbb{Z}$ for all α if the stability condition τ is 'generic'. Evidence for this conjecture is given in [**16**, §6.1–§6.5 & §7.6].

Section 5 summarizes [**16**, §7], which develops an analogue of Donaldson–Thomas theory for representations of quivers with relations coming from a superpotential. This provides a kind of toy model for Donaldson–Thomas invariants using only polynomials and finite-dimensional algebra, and is a source of many simple, explicit examples. Counting invariants for quivers with superpotential have been studied by Nakajima, Reineke, Szendrői and other authors for some years [**5**,**24**–**27**,**29**,**30**,**32**], under the general name of 'noncommutative Donaldson–Thomas invariants'. Curiously, the invariants studied so far are the analogues of our pair invariants $PI^{\alpha,n}(\tau')$, and the analogues of $\bar{DT}^\alpha(\tau),\hat{DT}^\alpha(\tau)$ seem to have received no attention, although they appear to the author to be more fundamental.

A recent paper by Kontsevich and Soibelman [**18**], summarized in [**19**], has considerable overlap with both [**16**] and the already published [**9**–**15**]. The two were completed largely independently, and the first versions of [**16**, **18**] appeared on the arXiv within a few days of each other. Kontsevich and Soibelman are far more ambitious than us, working in triangulated categories rather than abelian categories, over general fields $\mathbb{K}$ rather than $\mathbb{C}$, and with general motivic invariants rather than the Euler characteristic. But for this reason, almost every major result in [**18**] depends explicitly or implicitly on conjectures. The author would like to acknowledge the contribution of [**18**] to the ideas on $\hat{DT}^\alpha(\tau)$ and integrality in §4.4 below, and to the material on quivers with superpotential in §5. The relationship between [**16**] and [**18**] is discussed in detail in [**16**, §1.6].

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2. Constructible functions and stack functions

We begin with some background material on Artin stacks, constructible functions, and stack functions, drawn from [**9**, **10**]. We restrict to the field $\mathbb{K} = \mathbb{C}$.

2.1. Artin stacks and constructible functions.

Artin stacks are a class of geometric spaces, generalizing schemes and algebraic spaces. For a good introduction to Artin stacks see Gómez [**6**], and for a thorough treatment see Laumon and Moret-Bailly [**20**]. We work throughout over the field $\mathbb{C}$. We make the convention that all Artin stacks in this paper are *locally of finite type*, with *affine geometric stabilizers*, that is, all stabilizer groups $\mathrm{Iso}_{\mathfrak{F}}(x)$ are affine algebraic $\mathbb{C}$-groups, and substacks are *locally closed*.

Artin $\mathbb{C}$-stacks form a 2-*category*. That is, we have *objects* which are $\mathbb{C}$-stacks $\mathfrak{F}, \mathfrak{G}$, and also two kinds of morphisms, 1-*morphisms* $\phi, \psi : \mathfrak{F} \to \mathfrak{G}$ between $\mathbb{C}$-stacks, and 2-*morphisms* $A : \phi \to \psi$ between 1-morphisms.

Definition 2.1. Let $\mathfrak{F}$ be a $\mathbb{C}$-stack. Write $\mathfrak{F}(\mathbb{C})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{C} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{C})$ are called $\mathbb{C}$-*points* of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{C}) \to \mathfrak{G}(\mathbb{C})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{C} \to \mathfrak{F}$, the *stabilizer group* $\mathrm{Iso}_{\mathfrak{F}}(x)$ is the group of 2-morphisms $A : x \to x$. When $\mathfrak{F}$ is an Artin $\mathbb{C}$-stack, $\mathrm{Iso}_{\mathfrak{F}}(x)$ is an *algebraic* $\mathbb{C}$-*group*, which we assume is affine. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of $\mathbb{C}$-groups $\phi_* : \mathrm{Iso}_{\mathfrak{F}}([x]) \to \mathrm{Iso}_{\mathfrak{G}}\big(\phi_*([x])\big)$, for $[x] \in \mathfrak{F}(\mathbb{C})$.

We discuss *constructible functions* on $\mathbb{C}$-stacks, following [**9**].

Definition 2.2. Let $\mathfrak{F}$ be an Artin $\mathbb{C}$-stack. We call $C \subseteq \mathfrak{F}(\mathbb{C})$ *constructible* if $C = \bigcup_{i\in I} \mathfrak{F}_i(\mathbb{C})$, where $\{\mathfrak{F}_i : i \in I\}$ is a finite collection of finite type Artin $\mathbb{C}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S \subseteq \mathfrak{F}(\mathbb{C})$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{C})$. A function $f : \mathfrak{F}(\mathbb{C}) \to \mathbb{Q}$ is called *constructible* if $f(\mathfrak{F}(\mathbb{C}))$ is finite and $f^{-1}(c)$ is a constructible set in $\mathfrak{F}(\mathbb{C})$ for each $c \in f(\mathfrak{F}(\mathbb{C})) \setminus \{0\}$. A function $f : \mathfrak{F}(\mathbb{C}) \to \mathbb{Q}$ is called *locally constructible* if $f \cdot \delta_C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{C})$, where δ_C is the characteristic function of C. Write $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{LCF}(\mathfrak{F})$ for the

$\mathbb{Q}$-vector spaces of $\mathbb{Q}$-valued constructible and locally constructible functions on $\mathfrak{F}$.

Following [**9**, §4–§5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms.

Definition 2.3. Let $\mathfrak{F}, \mathfrak{G}$ be Artin $\mathbb{C}$-stacks and $\phi : \mathfrak{F} \to \mathfrak{G}$ a representable 1-morphism. For $f \in \mathrm{CF}(\mathfrak{F})$, define $\mathrm{CF}^{\mathrm{stk}}(\phi)f : \mathfrak{G}(\mathbb{C}) \to \mathbb{Q}$ by

$$\mathrm{CF}^{\mathrm{stk}}(\phi)f(y) = \chi\bigl(\mathfrak{F} \times_{\phi,\mathfrak{G},y} \mathrm{Spec}\,\mathbb{C}, \pi_{\mathfrak{F}}^*(f)\bigr) \quad \text{for } y \in \mathfrak{G}(\mathbb{C}),$$

where $\mathfrak{F} \times_{\phi,\mathfrak{G},y} \mathrm{Spec}\,\mathbb{C}$ is a $\mathbb{C}$-scheme (or algebraic space) as ϕ is representable, and $\chi(\cdots)$ is the Euler characteristic of this $\mathbb{C}$-scheme weighted by $\pi_{\mathfrak{F}}^*(f)$. Then $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ is a $\mathbb{Q}$-linear map called the *stack pushforward.*

Let $\theta : \mathfrak{F} \to \mathfrak{G}$ be a finite type 1-morphism. The *pullback* $\theta^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ is given by $\theta^*(f) = f \circ \theta_*$. It is a $\mathbb{Q}$-linear map.

Here [**9**, §4–§5] are some properties of these.

Theorem 2.4. *Let $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be Artin $\mathbb{C}$-stacks and $\beta : \mathfrak{F} \to \mathfrak{G}$, $\gamma : \mathfrak{G} \to \mathfrak{H}$ be 1-morphisms. Then*

$$\mathrm{CF}^{\mathrm{stk}}(\gamma \circ \beta) = \mathrm{CF}^{\mathrm{stk}}(\gamma) \circ \mathrm{CF}^{\mathrm{stk}}(\beta) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{H}), \tag{2}$$

$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{CF}(\mathfrak{H}) \to \mathrm{CF}(\mathfrak{F}), \tag{3}$$

supposing β, γ representable in (2), *and of finite type in* (3). *If*

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array}$$

is a Cartesian square with η, ϕ representable and θ, ψ of finite type, then the following commutes:

$$\begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow[\mathrm{CF}^{\mathrm{stk}}(\eta)]{} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & \uparrow{\scriptstyle\psi^*} \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}). \end{array}$$

2.2. Stack functions. *Stack functions* are a universal generalization of constructible functions introduced in [**10**, §3]. Here [**10**, Def. 3.1] is the basic definition.

Definition 2.5. Let $\mathfrak{F}$ be an Artin $\mathbb{C}$-stack. Consider pairs $(\mathfrak{R}, \rho)$, where $\mathfrak{R}$ is a finite type Artin $\mathbb{C}$-stack and $\rho : \mathfrak{R} \to \mathfrak{F}$ is a representable 1-morphism. We call two pairs $(\mathfrak{R}, \rho)$, $(\mathfrak{R}', \rho')$ *equivalent* if there exists a 1-isomorphism $\iota : \mathfrak{R} \to \mathfrak{R}'$ such that $\rho' \circ \iota$ and ρ are 2-isomorphic 1-morphisms $\mathfrak{R} \to \mathfrak{F}$. Write $[(\mathfrak{R}, \rho)]$ for the equivalence class of $(\mathfrak{R}, \rho)$. If $(\mathfrak{R}, \rho)$ is such a pair and $\mathfrak{S}$ is a closed $\mathbb{C}$-substack of $\mathfrak{R}$ then $(\mathfrak{S}, \rho|_{\mathfrak{S}})$, $(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})$ are pairs of the same kind.

Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R}, \rho)]$ as above, with for each closed $\mathbb{C}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation

$$[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})]. \tag{4}$$

Elements of $\mathrm{SF}(\mathfrak{F})$ will be called *stack functions*. We relate $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{SF}(\mathfrak{F})$.

Definition 2.6. Let $\mathfrak{F}$ be an Artin $\mathbb{C}$-stack and $C \subseteq \mathfrak{F}(\mathbb{C})$ be constructible. Then $C = \coprod_{i=1}^n \mathfrak{R}_i(\mathbb{C})$, for $\mathfrak{R}_1, \ldots, \mathfrak{R}_n$ finite type $\mathbb{C}$-substacks of $\mathfrak{F}$. Let $\rho_i : \mathfrak{R}_i \to \mathfrak{F}$ be the inclusion 1-morphism. Then $[(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$. Define $\bar{\delta}_C = \sum_{i=1}^n [(\mathfrak{R}_i, \rho_i)] \in \mathrm{SF}(\mathfrak{F})$. We think of this as the analogue of the characteristic function $\delta_C \in \mathrm{CF}(\mathfrak{F})$ of C. Define a $\mathbb{Q}$-linear map $\iota_{\mathfrak{F}} : \mathrm{CF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{F})$ by $\iota_{\mathfrak{F}}(f) = \sum_{0 \neq c \in f(\mathfrak{F}(\mathbb{C}))} c \cdot \bar{\delta}_{f^{-1}(c)}$. Define $\mathbb{Q}$-linear $\pi_{\mathfrak{F}}^{\mathrm{stk}} : \mathrm{SF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{F})$ by

$$\pi_{\mathfrak{F}}^{\mathrm{stk}}\bigl(\textstyle\sum_{i=1}^n c_i [(\mathfrak{R}_i, \rho_i)]\bigr) = \textstyle\sum_{i=1}^n c_i \, \mathrm{CF}^{\mathrm{stk}}(\rho_i) 1_{\mathfrak{R}_i},$$

where $1_{\mathfrak{R}_i}$ is the function $1 \in \mathrm{CF}(\mathfrak{R}_i)$. Then $\pi_{\mathfrak{F}}^{\mathrm{stk}} \circ \iota_{\mathfrak{F}}$ is the identity on $\mathrm{CF}(\mathfrak{F})$.

The operations on constructible functions in §2.1 extend to stack functions.

Definition 2.7. Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a representable 1-morphism of Artin $\mathbb{C}$-stacks. Define the *pushforward* $\phi_* : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{G})$ by

$$\phi_* : \textstyle\sum_{i=1}^m c_i [(\mathfrak{R}_i, \rho_i)] \longmapsto \textstyle\sum_{i=1}^m c_i [(\mathfrak{R}_i, \phi \circ \rho_i)]. \tag{5}$$

Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be of finite type. Define the *pullback* $\phi^* : \mathrm{SF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F})$ by

$$\phi^* : \textstyle\sum_{i=1}^m c_i [(\mathfrak{R}_i, \rho_i)] \longmapsto \textstyle\sum_{i=1}^m c_i [(\mathfrak{R}_i \times_{\rho_i, \mathfrak{G}, \phi} \mathfrak{F}, \pi_{\mathfrak{F}})]. \tag{6}$$

The *tensor product* $\otimes : \mathrm{SF}(\mathfrak{F}) \times \mathrm{SF}(\mathfrak{G}) \to \mathrm{SF}(\mathfrak{F} \times \mathfrak{G})$ is

$$\bigl(\textstyle\sum_{i=1}^m c_i [(\mathfrak{R}_i, \rho_i)]\bigr) \otimes \bigl(\textstyle\sum_{j=1}^n d_j [(\mathfrak{S}_j, \sigma_j)]\bigr) = \textstyle\sum_{i,j} c_i d_j [(\mathfrak{R}_i \times \mathfrak{S}_j, \rho_i \times \sigma_j)]. \tag{7}$$

Here [**10**, Th. 3.5] is the analogue of Theorem 2.4.

Theorem 2.8. *Let $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be Artin $\mathbb{C}$-stacks and $\beta : \mathfrak{F} \to \mathfrak{G}$, $\gamma : \mathfrak{G} \to \mathfrak{H}$ be 1-morphisms. Then*

$$(\gamma \circ \beta)_* = \gamma_* \circ \beta_* : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{H}), \quad (\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \mathrm{SF}(\mathfrak{H}) \to \mathrm{SF}(\mathfrak{F}),$$

for β, γ representable in the first equation, and of finite type in the second. If

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow \theta & & \psi \downarrow \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array}$$

is a Cartesian square with θ, ψ of finite type and η, ϕ representable, then the following commutes:

$$\begin{array}{ccc} \mathrm{SF}(\mathfrak{E}) & \xrightarrow{\eta_*} & \mathrm{SF}(\mathfrak{G}) \\ \uparrow \theta^* & & \psi^* \uparrow \\ \mathrm{SF}(\mathfrak{F}) & \xrightarrow{\phi_*} & \mathrm{SF}(\mathfrak{H}). \end{array}$$

In [**10**, §3] we relate pushforwards and pullbacks of stack and constructible functions using $\iota_{\mathfrak{F}}, \pi^{\rm stk}_{\mathfrak{F}}$.

THEOREM 2.9. *Let $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ be a 1-morphism of Artin $\mathbb{C}$-stacks. Then*

(a) $\phi^*\circ\iota_{\mathfrak{G}}=\iota_{\mathfrak{F}}\circ\phi^*:{\rm CF}(\mathfrak{G})\rightarrow{\rm SF}(\mathfrak{F})$ *if ϕ is of finite type;*

(b) $\pi^{\rm stk}_{\mathfrak{G}}\circ\phi_*={\rm CF}^{\rm stk}(\phi)\circ\pi^{\rm stk}_{\mathfrak{F}}:{\rm SF}(\mathfrak{F})\rightarrow{\rm CF}(\mathfrak{G})$ *if ϕ is representable; and*

(c) $\pi^{\rm stk}_{\mathfrak{F}}\circ\phi^*=\phi^*\circ\pi^{\rm stk}_{\mathfrak{G}}:{\rm SF}(\mathfrak{G})\rightarrow{\rm CF}(\mathfrak{F})$ *if ϕ is of finite type.*

We define some projections $\Pi^{\rm vi}_n:{\rm SF}(\mathfrak{F})\rightarrow{\rm SF}(\mathfrak{F})$, [**10**, §5].

DEFINITION 2.10. For any Artin $\mathbb{C}$-stack $\mathfrak{F}$ we will define linear maps $\Pi^{\rm vi}_n:{\rm SF}(\mathfrak{F})\rightarrow{\rm SF}(\mathfrak{F})$ for $n\geqslant 0$. Now ${\rm SF}(\mathfrak{F})$ is generated by $[(\mathfrak{R},\rho)]$ with $\mathfrak{R}$ 1-isomorphic to a quotient $[X/G]$, for X a quasiprojective $\mathbb{C}$-variety and G a special algebraic $\mathbb{C}$-group, with maximal torus T^G.

Let $\mathcal{S}(T^G)$ be the set of subsets of T^G defined by Boolean operations upon closed $\mathbb{C}$-subgroups L of T^G. Define a measure ${\rm d}\mu_n:\mathcal{S}(T^G)\rightarrow\mathbb{Z}$ to be additive upon disjoint unions of sets in $\mathcal{S}(T^G)$, and to satisfy ${\rm d}\mu_n(L)=1$ if $\dim L=n$ and ${\rm d}\mu_n(L)=0$ if $\dim L\neq 0$ for all algebraic $\mathbb{C}$-subgroups L of T^G. Define

$$\Pi^{\rm vi}_n\bigl([(\mathfrak{R},\rho)]\bigr)= \int_{t\in T^G}\frac{|\{w\in W(G,T^G):w\cdot t=t\}|}{|W(G,T^G)|}\,\bigl[\bigl([X^{\{t\}}/C_G(\{t\})],\rho\circ\iota^{\{t\}}\bigr)\bigr]{\rm d}\mu_n. \tag{8}$$

Here $X^{\{t\}}$ is the subscheme of X fixed by t, and $C_G(\{t\})$ is the centralizer of t in G, and $\iota^{\{t\}}:[X^{\{t\}}/C_G(\{t\})]\rightarrow[X/G]$ is the obvious 1-morphism.

The integrand in (8), regarded as a function of $t\in T^G$, is a constructible function taking only finitely many values. The level sets of the function lie in $\mathcal{S}(T^G)$, so they are measurable w.r.t. ${\rm d}\mu_n$, and the integral is well-defined. In [**10**, §5] we show (8) induces a unique linear map $\Pi^{\rm vi}_n:{\rm SF}(\mathfrak{F})\rightarrow{\rm SF}(\mathfrak{F})$.

Here [**10**, §5] are some properties of the $\Pi^{\rm vi}_n$.

THEOREM 2.11. *In the situation above, we have:*

(i) $(\Pi^{\rm vi}_n)^2=\Pi^{\rm vi}_n$, *so that $\Pi^{\rm vi}_n$ is a projection, and $\Pi^{\rm vi}_m\circ\Pi^{\rm vi}_n=0$ for $m\neq n$.*

(ii) *For all $f\in{\rm SF}(\mathfrak{F})$ we have $f=\sum_{n\geqslant 0}\Pi^{\rm vi}_n(f)$, where the sum makes sense as $\Pi^{\rm vi}_n(f)=0$ for $n\gg 0$.*

(iii) *If $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ is a 1-morphism of Artin $\mathbb{C}$-stacks then $\Pi^{\rm vi}_n\circ\phi_*=\phi_*\circ\Pi^{\rm vi}_n:{\rm SF}(\mathfrak{F})\rightarrow{\rm SF}(\mathfrak{G})$.*

(iv) *If $f\in{\rm SF}(\mathfrak{F})$, $g\in{\rm SF}(\mathfrak{G})$ then $\Pi^{\rm vi}_n(f\otimes g)=\sum_{m=0}^n\Pi^{\rm vi}_m(f)\otimes\Pi^{\rm vi}_{n-m}(g)$.*

Roughly speaking, $\Pi^{\rm vi}_n$ projects $[(\mathfrak{R},\rho)]\in{\rm SF}(\mathfrak{F})$ to $[(\mathfrak{R}_n,\rho)]$, where $\mathfrak{R}_n$ is the substack of points $r\in\mathfrak{R}(\mathbb{C})$ whose stabilizer groups ${\rm Iso}_{\mathfrak{R}}(r)$ have rank n.

2.3. Stack function spaces $\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$. We will also need another family of spaces $\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$, from [**10**, §5–§6].

Definition 2.12. Let $\mathfrak{F}$ be an Artin $\mathbb{C}$-stack. Consider pairs $(\mathfrak{R},\rho)$, where $\mathfrak{R}$ is a finite type Artin $\mathbb{C}$-stack and $\rho:\mathfrak{R}\to\mathfrak{F}$ is a representable 1-morphism, with equivalence as in Definition 2.5. Define $\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R},\rho)]$, with the following relations:

(i) Given $[(\mathfrak{R},\rho)]$ as above and $\mathfrak{S}$ a closed $\mathbb{C}$-substack of $\mathfrak{R}$ we have $[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho|_{\mathfrak{S}})]+[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$, as in (4).

(ii) Let $\mathfrak{R}$ be a finite type Artin $\mathbb{C}$-stack, U a quasiprojective $\mathbb{C}$-variety, $\pi_{\mathfrak{R}}:\mathfrak{R}\times U\to\mathfrak{R}$ the natural projection, and $\rho:\mathfrak{R}\to\mathfrak{F}$ a 1-morphism. Then $[(\mathfrak{R}\times U,\rho\circ\pi_{\mathfrak{R}})]=\chi([U])[(\mathfrak{R},\rho)]$.
Here $\chi(U)\in\mathbb{Z}$ is the *Euler characteristic* of U. It is a *motivic invariant* of $\mathbb{C}$-schemes, that is, $\chi(U)=\chi(V)+\chi(U\setminus V)$ for $V\subset U$ closed.

(iii) Given $[(\mathfrak{R},\rho)]$ as above and a 1-isomorphism $\mathfrak{R}\cong[X/G]$ for X a quasiprojective $\mathbb{C}$-variety and G a very special algebraic $\mathbb{C}$-group acting on X with maximal torus T^G, we have

$$[(\mathfrak{R},\rho)]=\textstyle\sum_{Q\in\mathcal{Q}(G,T^G)}F(G,T^G,Q)\big[\big([X/Q],\rho\circ\iota^Q\big)\big],$$

where $\iota^Q:[X/Q]\to\mathfrak{R}\cong[X/G]$ is the natural projection 1-morphism.

Here $\mathcal{Q}(G,T^G)$ is a certain finite set of $\mathbb{C}$-subgroups of T^G, and $F(G,T^G,Q)\in\mathbb{Q}$ are a system of rational coefficients defined in [**10**, §6.2]. Define $\bar{\Pi}^{\chi,\mathbb{Q}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\to\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$ by $\bar{\Pi}^{\chi,\mathbb{Q}}_{\mathfrak{F}}:\sum_{i\in I}c_i[(\mathfrak{R}_i,\rho_i)]\mapsto\sum_{i\in I}c_i[(\mathfrak{R}_i,\rho_i)]$. Define *pushforwards* ϕ_*, *pullbacks* ϕ^*, *tensor products* $\otimes$ and *projections* Π^{vi}_n on the spaces $\bar{\mathrm{SF}}(*,\chi,\mathbb{Q})$ as in §2.2. The important point is that (5)–(8) are compatible with the relations defining $\bar{\mathrm{SF}}(*,\chi,\mathbb{Q})$, or they would not be well-defined. The analogues of Theorems 2.8, 2.9 and 2.11 hold for $\bar{\mathrm{SF}}(*,\chi,\mathbb{Q})$.

Here [**10**, §5–§6] is a useful way to represent these spaces. It means that by working in $\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$, we can treat all stabilizer groups as if they are abelian.

Proposition 2.13. *$\bar{\mathrm{SF}}(\mathfrak{F},\chi,\mathbb{Q})$ is spanned over $\mathbb{Q}$ by $[(U\times[\operatorname{Spec}\mathbb{C}/T],\rho)]$, for U a quasiprojective $\mathbb{C}$-variety and T an algebraic $\mathbb{C}$-group isomorphic to $\mathbb{G}_m^k\times K$ for $k\geqslant 0$ and K finite abelian. Moreover*

$$\Pi^{\mathrm{vi}}_n\big([(U\times[\operatorname{Spec}\mathbb{C}/T],\rho)]\big)=\begin{cases}[(U\times[\operatorname{Spec}\mathbb{C}/T],\rho)], & \dim T=n,\\ 0, & \textit{otherwise.}\end{cases}$$

3. Background material on Calabi–Yau 3-folds

We now summarize some facts on Donaldson–Thomas invariants and other sheaf-counting invariants on Calabi–Yau 3-folds prior to our book [**16**]. Sections 3.1–3.3 review material from the author's series of papers [**11**–**14**], and §3.4 explains results on Donaldson–Thomas theory from Thomas [**33**] and Behrend [**1**]. For simplicity we restrict to Calabi–Yau 3-folds and to the field $\mathbb{K}=\mathbb{C}$, although much of [**1**, **11**–**14**, **33**] works in greater generality.

3.1. The Ringel–Hall algebra of a Calabi–Yau 3-fold. We will use the following notation for the rest of the paper.

Definition 3.1. A *Calabi–Yau* 3*-fold* is a smooth projective 3-fold X over $\mathbb{C}$, with trivial canonical bundle K_X. In §4 we will also assume that $H^1(\mathcal{O}_X)=0$. The *Grothendieck group* $K_0(X)$ of $\mathrm{coh}(X)$ is the abelian group generated by all isomorphism classes $[E]$ of objects E in $\mathrm{coh}(X)$, with the relations $[E]+[G]=[F]$ for each short exact sequence $0\to E\to F\to G\to 0$. The *Euler form* $\bar\chi:K_0(X)\times K_0(X)\to\mathbb{Z}$ is a biadditive map satisfying

$$\bar\chi\bigl([E],[F]\bigr)=\textstyle\sum_{i\geqslant 0}(-1)^i\dim\mathrm{Ext}^i(E,F) \tag{9}$$

for all $E,F\in\mathrm{coh}(X)$. As X is a Calabi–Yau 3-fold, Serre duality gives $\mathrm{Ext}^i(F,E)\cong\mathrm{Ext}^{3-i}(E,F)^*$, so $\dim\mathrm{Ext}^i(F,E)=\dim\mathrm{Ext}^{3-i}(E,F)$ for all $E,F\in\mathrm{coh}(X)$. Therefore $\bar\chi$ is also given by

$$\begin{aligned}\bar\chi\bigl([E],[F]\bigr)=&\bigl(\dim\mathrm{Hom}(E,F)-\dim\mathrm{Ext}^1(E,F)\bigr)\\&-\bigl(\dim\mathrm{Hom}(F,E)-\dim\mathrm{Ext}^1(F,E)\bigr).\end{aligned} \tag{10}$$

Thus the Euler form $\bar\chi$ on $K_0(X)$ is *antisymmetric*.

The *numerical Grothendieck group* $K(X)$ is the quotient of $K_0(X)$ by the kernel of $\bar\chi$. Then $\bar\chi$ on $K_0(X)$ descends to a nondegenerate, biadditive Euler form $\bar\chi:K(X)\times K(X)\to\mathbb{Z}$.

Define the 'positive cone' $C(X)$ in $K(X)$ to be

$$C(X)=\bigl\{[E]\in K(X):0\not\cong E\in\mathrm{coh}(X)\bigr\}\subset K(X).$$

Write $\mathfrak{M}$ for the moduli stack of objects in $\mathrm{coh}(X)$. It is an Artin $\mathbb{C}$-stack, locally of finite type. Points of $\mathfrak{M}(\mathbb{C})$ correspond to isomorphism classes $[E]$ of objects E in $\mathrm{coh}(X)$, and the stabilizer group $\mathrm{Iso}_{\mathfrak{M}}([E])$ in $\mathfrak{M}$ is isomorphic as an algebraic $\mathbb{C}$-group to the automorphism group $\mathrm{Aut}(E)$. For $\alpha\in C(X)$, write $\mathfrak{M}^\alpha$ for the substack of objects $E\in\mathrm{coh}(X)$ in class α in $K(X)$. It is an open and closed $\mathbb{C}$-substack of $\mathfrak{M}$.

Write $\mathfrak{Exact}$ for the moduli stack of short exact sequences $0\to E_1\to E_2\to E_3\to 0$ in $\mathrm{coh}(X)$. It is an Artin $\mathbb{C}$-stack, locally of finite type. For $j=1,2,3$ write $\pi_j:\mathfrak{Exact}\to\mathfrak{M}$ for the 1-morphism projecting $0\to E_1\to E_2\to E_3\to 0$ to E_j. Then π_2 is *representable*, and $\pi_1\times\pi_3:\mathfrak{Exact}\to\mathfrak{M}\times\mathfrak{M}$ is of *finite type.*

In [**12**] we define *Ringel–Hall algebras*, using stack functions.

Definition 3.2. Define bilinear operations $*$ on $\mathrm{SF}(\mathfrak{M}), \bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$ by

$$f * g = (\pi_2)_*\big((\pi_1 \times \pi_3)^*(f \otimes g)\big),$$

using pushforwards, pullbacks and tensor products in Definition 2.7. They are well-defined as π_2 is representable, and $\pi_1 \times \pi_3$ is of finite type. By [**12**, Th. 5.2], whose proof uses Theorem 2.8, this * is *associative*, and makes $\mathrm{SF}(\mathfrak{M})$, $\bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$ into noncommutative $\mathbb{Q}$-algebras, called *Ringel–Hall algebras*, with identity $\bar{\delta}_{[0]}$, where $[0] \in \mathfrak{M}$ is the zero object. The projection $\bar{\Pi}^{\chi,\mathbb{Q}}_{\mathfrak{M}} : \mathrm{SF}(\mathfrak{M}) \to \bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$ is an algebra morphism.

As these algebras are inconveniently large for some purposes, in [**12**, Def. 5.5] we define subalgebras $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ using the algebra structure on stabilizer groups in $\mathfrak{M}$. Suppose $[(\mathfrak{R}, \rho)]$ is a generator of $\mathrm{SF}(\mathfrak{M})$. Let $r \in \mathfrak{R}(\mathbb{C})$ with $\rho_*(r) = [E] \in \mathfrak{M}(\mathbb{C})$, for some $E \in \mathrm{coh}(X)$. Then ρ induces a morphism of stabilizer $\mathbb{C}$-groups $\rho_* : \mathrm{Iso}_{\mathfrak{R}}(r) \to \mathrm{Iso}_{\mathfrak{M}}([E]) \cong \mathrm{Aut}(E)$. As ρ is representable this is *injective*, and induces an isomorphism of $\mathrm{Iso}_{\mathfrak{R}}(r)$ with a $\mathbb{C}$-subgroup of $\mathrm{Aut}(E)$. Now $\mathrm{Aut}(E) = \mathrm{End}(E)^\times$ is the $\mathbb{C}$-group of invertible elements in a *finite-dimensional* $\mathbb{C}$-*algebra* $\mathrm{End}(E) = \mathrm{Hom}(E, E)$. We say that $[(\mathfrak{R}, \rho)]$ *has algebra stabilizers* if whenever $r \in \mathfrak{R}(\mathbb{C})$ with $\rho_*(r) = [E]$, the $\mathbb{C}$-subgroup $\rho_*\big(\mathrm{Iso}_{\mathfrak{R}}(r)\big)$ in $\mathrm{Aut}(E)$ is the $\mathbb{C}$-group $A^\times$ of invertible elements in a $\mathbb{C}$-subalgebra A in $\mathrm{End}(E)$. Write $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ for the subspaces of $\mathrm{SF}(\mathfrak{M}), \bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$ spanned over $\mathbb{Q}$ by $[(\mathfrak{R}, \rho)]$ with algebra stabilizers. Then [**12**, Prop. 5.7] shows that $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ are *subalgebras* of the Ringel–Hall algebras $\mathrm{SF}(\mathfrak{M})$, $\bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$.

Now [**12**, Cor. 5.10] shows that $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ are closed under the operators Π^{vi}_n on $\mathrm{SF}(\mathfrak{M}), \bar{\mathrm{SF}}(\mathfrak{M}, \chi, \mathbb{Q})$ defined in §2.2. In [**12**, Def. 5.14] we define $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ to be the subspaces of f in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ and $\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ with $\Pi^{\mathrm{vi}}_1(f) = f$. We think of $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ as stack functions *'supported on virtual indecomposables'*.

In [**12**, Th. 5.18] we show that $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ are closed under the Lie bracket $[f, g] = f * g - g * f$ on $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$. Thus, $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ are *Lie subalgebras* of $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M}), \bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$.

As in [**12**, Cor. 5.11], Proposition 2.13 simplifies to give:

Proposition 3.3. *$\bar{\mathrm{SF}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ is spanned over $\mathbb{Q}$ by elements of the form $[(U \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m^k], \rho)]$ with algebra stabilizers, for U a quasiprojective $\mathbb{C}$-variety and $k \geqslant 0$. Also $\bar{\mathrm{SF}}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ is spanned over $\mathbb{Q}$ by $[(U \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m], \rho)]$ with algebra stabilizers, for U a quasiprojective $\mathbb{C}$-variety.*

All the above except (10) works for X an arbitrary smooth projective $\mathbb{C}$-scheme, but our next result uses the Calabi–Yau 3-fold assumption on X in an essential way. We follow [**12**, §6.5–§6.6], but use the notation of [**16**, §3.4].

Definition 3.4. Define an explicit Lie algebra $L(X)$ over $\mathbb{Q}$ to be the $\mathbb{Q}$-vector space with basis of symbols λ^α for $\alpha \in K(X)$, with Lie bracket

$$[\lambda^\alpha, \lambda^\beta] = \bar{\chi}(\alpha, \beta)\lambda^{\alpha+\beta} \tag{11}$$

for $\alpha, \beta \in K(X)$. As $\bar{\chi}$ is antisymmetric, (11) satisfies the Jacobi identity and makes $L(X)$ into an infinite-dimensional Lie algebra over $\mathbb{Q}$.

Define a $\mathbb{Q}$-linear map $\Psi^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q}) \to L(X)$ by

$$\Psi^{\chi,\mathbb{Q}}(f) = \textstyle\sum_{\alpha \in K(X)} \gamma^\alpha \lambda^\alpha, \tag{12}$$

where $\gamma^\alpha \in \mathbb{Q}$ is defined as follows. Proposition 3.3 says $\bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q})$ is spanned by elements $[(U \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m], \rho)]$. We may write

$$f|_{\mathfrak{M}^\alpha} = \textstyle\sum_{i=1}^n \delta_i[(U_i \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m], \rho_i)], \tag{13}$$

where $\delta_i \in \mathbb{Q}$ and U_i is a quasiprojective $\mathbb{C}$-variety. We set

$$\gamma^\alpha = \textstyle\sum_{i=1}^n \delta_i \chi(U_i).$$

This is independent of the choices in (13). Now define $\Psi : \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}) \to L(X)$ by $\Psi = \Psi^{\chi,\mathbb{Q}} \circ \bar{\Pi}^{\chi,\mathbb{Q}}_{\mathfrak{M}}$.

In [**12**, Th. 6.12], using equation (10), we prove:

Theorem 3.5. *$\Psi : \mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}) \to L(X)$ and $\Psi^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}{}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M}, \chi, \mathbb{Q}) \to L(X)$ are Lie algebra morphisms.*

3.2. Stability conditions on coh(X) and invariants $J^\alpha(\tau)$. Next we discuss material in [**13**] on *stability conditions*. We continue to use the notation of §3.1, with X a Calabi–Yau 3-fold.

Definition 3.6. Suppose $(T, \leqslant)$ is a totally ordered set, and $\tau : C(X) \to T$ a map. We call $(\tau, T, \leqslant)$ a *stability condition* on coh(X) if whenever $\alpha, \beta, \gamma \in C(X)$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) < \tau(\beta) < \tau(\gamma)$, or $\tau(\alpha) > \tau(\beta) > \tau(\gamma)$, or $\tau(\alpha) = \tau(\beta) = \tau(\gamma)$. We call $(\tau, T, \leqslant)$ a *weak stability condition* on coh(X) if whenever $\alpha, \beta, \gamma \in C(X)$ with $\beta = \alpha + \gamma$ then either $\tau(\alpha) \leqslant \tau(\beta) \leqslant \tau(\gamma)$, or $\tau(\alpha) \geqslant \tau(\beta) \geqslant \tau(\gamma)$. For such $(\tau, T, \leqslant)$, we call a nonzero sheaf E in coh(X)

(i) τ-*stable* if for all $S \subset E$ with $S \ncong 0, E$ we have $\tau([S]) < \tau([E/S])$; and
(ii) τ-*semistable* if for all $S \subset E$ with $S \ncong 0, E$ we have $\tau([S]) \leqslant \tau([E/S])$.

For $\alpha \in C(X)$, write $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau), \mathfrak{M}^\alpha_{\mathrm{st}}(\tau)$ for the moduli stacks of τ-(semi) stable $E \in \mathcal{A}$ with class $[E] = \alpha$ in $K(X)$. They are open $\mathbb{C}$-substacks of $\mathfrak{M}^\alpha$. We call $(\tau, T, \leqslant)$ *permissible* if:

(a) coh(X) is τ-*artinian*, that is, there exist no infinite chains of subobjects $\cdots \subsetneq E_2 \subsetneq E_1 \subsetneq E_0 = X$ in $\mathcal{A}$ and $\tau([E_{n+1}]) \geqslant \tau([E_n/E_{n+1}])$ for all n; and
(b) $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau)$ is a *finite type* substack of $\mathfrak{M}^\alpha$ for all $\alpha \in C(X)$.

Here are two important examples:

EXAMPLE 3.7. Define G to be the set of monic rational polynomials in t of degree at most 3:

$$G = \{p(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_0 : d = 0, 1, 2, 3, \ a_0, \ldots, a_{d-1} \in \mathbb{Q}\}.$$

Define a *total order* '$\leqslant$' on G by $p \leqslant p'$ for $p, p' \in G$ if either

(a) $\deg p > \deg p'$, or
(b) $\deg p = \deg p'$ and $p(t) \leqslant p'(t)$ for all $t \gg 0$.

We write $p < q$ if $p \leqslant q$ and $p \neq q$.

Fix a very ample line bundle $\mathcal{O}_X(1)$ on X. For $E \in \mathrm{coh}(X)$, the *Hilbert polynomial* P_E is the unique polynomial in $\mathbb{Q}[t]$ such that $P_E(n) = \dim H^0(E(n))$ for all $n \gg 0$. Equivalently, $P_E(n) = \bar\chi([\mathcal{O}_X(-n)], [E])$ for all $n \in \mathbb{Z}$. Thus, P_E depends only on the class $\alpha \in K(X)$ of E, and we may write P_α instead of P_E. Define $\tau : C(X) \to G$ by $\tau(\alpha) = P_\alpha / r_\alpha$, where P_α is the Hilbert polynomial of α, and r_α is the (positive) leading coefficient of P_α. Then $(\tau, G, \leqslant)$ is a *permissible stability condition* on $\mathrm{coh}(X)$ [**13**, Ex. 4.16], called *Gieseker stability*.

Gieseker stability is studied in [**8**, §1.2]. Write $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau), \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ for the coarse moduli schemes of τ-(semi)stable sheaves E with class $[E] = \alpha$ in $K(X)$. By [**8**, Th. 4.3.4], $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$ is a projective $\mathbb{C}$-scheme whose $\mathbb{C}$-points correspond to S-equivalence classes of Gieseker semistable sheaves in class α, and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is an open $\mathbb{C}$-subscheme whose $\mathbb{C}$-points correspond to isomorphism classes of Gieseker stable sheaves in class α.

EXAMPLE 3.8. In the situation of Example 3.7, define

$$M = \{p(t) = t^d + a_{d-1}t^{d-1} : d = 0, 1, 2, 3, \ a_{d-1} \in \mathbb{Q}, \ a_{-1} = 0\} \subset G$$

and restrict the total order $\leqslant$ on G to M. Define $\mu : C(X) \to M$ by $\mu(\alpha) = t^d + a_{d-1}t^{d-1}$ when $\tau(\alpha) = P_\alpha / r_\alpha = t^d + a_{d-1}t^{d-1} + \cdots + a_0$, that is, $\mu(\alpha)$ is the truncation of the polynomial $\tau(\alpha)$ in Example 3.7 at its second term. Then as in [**13**, Ex. 4.17], $(\mu, M, \leqslant)$ is a *permissible weak stability condition* on $\mathrm{coh}(X)$. It is called *μ-stability*, and is studied in [**8**, §1.6].

In [**13**, §8] we define interesting stack functions $\bar\delta_{\mathrm{ss}}^\alpha(\tau), \bar\epsilon^\alpha(\tau)$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$.

DEFINITION 3.9. Let $(\tau, T, \leqslant)$ be a permissible weak stability condition on $\mathrm{coh}(X)$. Define stack functions $\bar\delta_{\mathrm{ss}}^\alpha(\tau) = \bar\delta_{\mathfrak{M}_{\mathrm{ss}}^\alpha(\tau)}$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ for $\alpha \in C(X)$. That is, $\bar\delta_{\mathrm{ss}}^\alpha(\tau)$ is the characteristic function, in the sense of Definition 2.6, of the moduli substack $\mathfrak{M}_{\mathrm{ss}}^\alpha(\tau)$ of τ-semistable sheaves in $\mathfrak{M}$. In [**13**, Def. 8.1] we define elements $\bar\epsilon^\alpha(\tau)$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$ by

$$\bar\epsilon^\alpha(\tau) = \sum_{\substack{n \geqslant 1,\ \alpha_1, \ldots, \alpha_n \in C(X):\\ \alpha_1 + \cdots + \alpha_n = \alpha,\ \tau(\alpha_i) = \tau(\alpha),\ \text{all } i}} \frac{(-1)^{n-1}}{n} \ \bar\delta_{\mathrm{ss}}^{\alpha_1}(\tau) * \bar\delta_{\mathrm{ss}}^{\alpha_2}(\tau) * \cdots * \bar\delta_{\mathrm{ss}}^{\alpha_n}(\tau), \tag{14}$$

where $*$ is the Ringel–Hall multiplication in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$. Then [**13**, Th. 8.2] proves

$$(15)\quad \bar\delta^\alpha_{\mathrm{ss}}(\tau)=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha,\ \tau(\alpha_i)=\tau(\alpha),\text{ all } i}}\frac{1}{n!}\,\bar\epsilon^{\alpha_1}(\tau)*\bar\epsilon^{\alpha_2}(\tau)*\cdots*\bar\epsilon^{\alpha_n}(\tau).$$

There are only finitely many nonzero terms in (14)–(15).

Equations (14) and (15) are inverse, so that knowing the $\bar\epsilon^\alpha(\tau)$ is equivalent to knowing the $\bar\delta^\alpha_{\mathrm{ss}}(\tau)$. If $\mathfrak{M}^\alpha_{\mathrm{ss}}(\tau)=\mathfrak{M}^\alpha_{\mathrm{st}}(\tau)$ then $\bar\epsilon^\alpha(\tau)=\bar\delta^\alpha_{\mathrm{ss}}(\tau)$. The difference between $\bar\epsilon^\alpha(\tau)$ and $\bar\delta^\alpha_{\mathrm{ss}}(\tau)$ is that $\bar\epsilon^\alpha(\tau)$ 'counts' strictly semistable sheaves in a special, complicated way. Here [**13**, Th. 8.7] is an important property of the $\bar\epsilon^\alpha(\tau)$, which does not hold for the $\bar\delta^\alpha_{\mathrm{ss}}(\tau)$. The proof is highly nontrivial, using the full power of the configurations formalism of [**11**–**14**].

Theorem 3.10. *$\bar\epsilon^\alpha(\tau)$ lies in the Lie subalgebra $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{M})$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{M})$.*

In [**14**, §6.6] we define *invariants* $J^\alpha(\tau)\in\mathbb{Q}$ for all $\alpha\in C(X)$ by

$$(16)\qquad \Psi\bigl(\bar\epsilon^\alpha(\tau)\bigr)=J^\alpha(\tau)\lambda^\alpha.$$

This is valid by Theorem 3.10. These $J^\alpha(\tau)$ are rational numbers 'counting' τ-semistable sheaves E in class α. When $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau)=\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ we have

$$(17)\qquad J^\alpha(\tau)=\chi\bigl(\mathcal{M}^\alpha_{\mathrm{st}}(\tau)\bigr),$$

that is, $J^\alpha(\tau)$ is the Euler characteristic of the moduli space $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$. In the notation of §3.4, this is not weighted by the Behrend function $\nu_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}$, and so is not the Donaldson–Thomas invariant $DT^\alpha(\tau)$. As in [**16**, Ex. 6.9], the $J^\alpha(\tau)$ are in general not unchanged under deformations of X.

3.3. Changing stability conditions and algebra identities. In [**14**] we prove transformation laws for the $\bar\delta^\alpha_{\mathrm{ss}}(\tau),\bar\epsilon^\alpha(\tau)$ under change of stability condition. These involve combinatorial coefficients $S(*;\tau,\tilde\tau)\in\mathbb{Z}$ and $U(*;\tau,\tilde\tau)\in\mathbb{Q}$ defined in [**14**, §4.1].

Definition 3.11. Let $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be weak stability conditions on $\mathrm{coh}(X)$. Let $n\geqslant 1$ and $\alpha_1,\ldots,\alpha_n\in C(X)$. If for all $i=1,\ldots,n-1$ we have either

(a) $\tau(\alpha_i)\leqslant\tau(\alpha_{i+1})$ and $\tilde\tau(\alpha_1+\cdots+\alpha_i)>\tilde\tau(\alpha_{i+1}+\cdots+\alpha_n)$ or
(b) $\tau(\alpha_i)>\tau(\alpha_{i+1})$ and $\tilde\tau(\alpha_1+\cdots+\alpha_i)\leqslant\tilde\tau(\alpha_{i+1}+\cdots+\alpha_n)$,

then define $S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)=(-1)^r$, where r is the number of $i=1,\ldots,n-1$ satisfying (a). Otherwise define $S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)=0$. Now define

$$U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)= \sum_{\substack{1\leqslant l\leqslant m\leqslant n,\ 0=a_0<a_1<\cdots<a_m=n,\ 0=b_0<b_1<\cdots<b_l=m:\\ \text{Define } \beta_1,\ldots,\beta_m\in C(X) \text{ by } \beta_i=\alpha_{a_{i-1}+1}+\cdots+\alpha_{a_i}.\\ \text{Define } \gamma_1,\ldots,\gamma_l\in C(X) \text{ by } \gamma_i=\beta_{b_{i-1}+1}+\cdots+\beta_{b_i}.\\ \text{Then } \tau(\beta_i)=\tau(\alpha_j),\ i=1,\ldots,m,\ a_{i-1}<j\leqslant a_i,\\ \text{and } \tilde\tau(\gamma_i)=\tilde\tau(\alpha_1+\cdots+\alpha_n),\ i=1,\ldots,l}} \frac{(-1)^{l-1}}{l}\cdot\prod_{i=1}^{l} S(\beta_{b_{i-1}+1},\beta_{b_{i-1}+2},\ldots,\beta_{b_i};\tau,\tilde\tau) \cdot\prod_{i=1}^{m}\frac{1}{(a_i-a_{i-1})!}.$$

Then in [**14**, §5] we derive wall-crossing formulae for the $\bar\delta^\alpha_{\rm ss}(\tau),\bar\epsilon^\alpha(\tau)$ under change of stability condition from $(\tau,T,\leqslant)$ to $(\tilde\tau,\tilde T,\leqslant)$:

THEOREM 3.12. *Let $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ be permissible weak stability conditions on* $\mathrm{coh}(X)$. *Then under some mild extra conditions, for all $\alpha\in C(X)$ we have*

$$\bar\delta^\alpha_{\rm ss}(\tilde\tau)=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha}} S(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot \bar\delta^{\alpha_1}_{\rm ss}(\tau)*\bar\delta^{\alpha_2}_{\rm ss}(\tau)*\cdots*\bar\delta^{\alpha_n}_{\rm ss}(\tau), \tag{18}$$

$$\bar\epsilon^\alpha(\tilde\tau)=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha}} U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot \bar\epsilon^{\alpha_1}(\tau)*\bar\epsilon^{\alpha_2}(\tau)*\cdots*\bar\epsilon^{\alpha_n}(\tau), \tag{19}$$

where there are only finitely many nonzero terms in (18)–(19).

The 'mild extra conditions' in the theorem are required to ensure that there are only finitely many nonzero terms in (18)–(19). In fact the author expects that this always holds when $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ are of Gieseker or μ-stability type, but for irritating technical reasons has not been able to prove this. As in [**14**, §5.1], the author can show that one can go between any two (weak) stability conditions on $\mathrm{coh}(X)$ of Gieseker or μ-stability type by finitely many applications of Theorem 3.12. In [**14**, Th. 5.4] we prove:

THEOREM 3.13. *Equation* (19) *may be rewritten as an equation in* $\mathrm{SF}^{\rm ind}_{\rm al}(\mathfrak{M})$ *using the Lie bracket* $[\,,\,]$ *on* $\mathrm{SF}^{\rm ind}_{\rm al}(\mathfrak{M})$, *rather than as an equation in* $\mathrm{SF}_{\rm al}(\mathfrak{M})$ *using the Ringel–Hall product* $*$.

Therefore we may apply the Lie algebra morphism Ψ of §3.1 to equation (19). As (19) is not expressed explicitly in terms of Lie brackets, it is helpful to write this in the *universal enveloping algebra* $U(L(X))$. This gives

$$J^\alpha(\tilde\tau)\lambda^\alpha=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha}} U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot\prod_{i=1}^n J^{\alpha_i}(\tau)\cdot \lambda^{\alpha_1}\star\lambda^{\alpha_2}\star\cdots\star\lambda^{\alpha_n}, \tag{20}$$

where $\star$ is the product in $U(L(X))$.

Now in [**12**, §6.5], a basis is given for $U(L(X))$ in terms of symbols $\lambda_{[I,\kappa]}$, and multiplication $\star$ in $U(L(X))$ is written in terms of the $\lambda_{[I,\kappa]}$ as a sum over graphs. Here I is a finite set, κ maps $I\rightarrow C(X)$, and when $|I|=1$, so that $I=\{i\}$, we have $\lambda_{[I,\kappa]}=\lambda^{\kappa(i)}$. Then [**14**, eq. (127)] gives an expression for $\lambda^{\alpha_1}\star\cdots\star\lambda^{\alpha_n}$ in $U(L(X))$, in terms of sums over *directed graphs* (*digraphs*):

$$\lambda^{\alpha_1}\star\cdots\star\lambda^{\alpha_n}=\text{ terms in }\lambda_{[I,\kappa]},\ |I|>1, \\ +\Biggl[\frac{1}{2^{n-1}}\sum_{\substack{\text{connected, simply-connected digraphs }\Gamma:\\ \text{vertices }\{1,\ldots,n\},\text{ edge }\overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\text{ implies }i<j}}\prod_{\substack{\text{edges}\\ \overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\\ \text{in }\Gamma}}\bar\chi(\alpha_i,\alpha_j)\Biggr]\lambda^{\alpha_1+\cdots+\alpha_n}. \tag{21}$$

Substitute (21) into (20). The terms in $\lambda_{[I,\kappa]}$ for $|I|>1$ all cancel, as (20) lies in $L(X)\subset U(L(X))$. So equating coefficients of λ^α yields

$$J^\alpha(\tilde\tau)=\sum_{\substack{n\geqslant 1,\ \alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha}}\ \sum_{\substack{\text{connected, simply-connected digraphs }\Gamma:\\ \text{vertices }\{1,\ldots,n\},\text{ edge }\overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\text{ implies }i<j}} \\ \frac{1}{2^{n-1}}U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\prod_{\text{edges }\overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\text{ in }\Gamma}\bar\chi(\alpha_i,\alpha_j)\prod_{i=1}^n J^{\alpha_i}(\tau). \tag{22}$$

Following [**14**, Def. 6.27], we define combinatorial coefficients $V(I,\Gamma,\kappa;\tau,\tilde\tau)$:

Definition 3.14. In the situation above, let Γ be a connected, simply-connected digraph with finite vertex set I, where $|I|=n$, and $\kappa:I\rightarrow C(X)$ be a map. Define $V(I,\Gamma,\kappa;\tau,\tilde\tau)\in\mathbb{Q}$ by

$$V(I,\Gamma,\kappa;\tau,\tilde\tau)=\frac{1}{2^{n-1}n!}\sum_{\substack{\text{orderings }i_1,\ldots,i_n\text{ of }I:\\ \text{edge }\overset{i_a}{\bullet}\rightarrow\overset{i_b}{\bullet}\text{ in }\Gamma\text{ implies }a<b}}U(\kappa(i_1),\kappa(i_2),\ldots,\kappa(i_n);\tau,\tilde\tau). \tag{23}$$

Then as in [**14**, Th. 6.28], using (23) to rewrite (22) yields a transformation law for the $J^\alpha(\tau)$ under change of stability condition:

$$J^\alpha(\tilde\tau)=\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets }I}}\ \sum_{\substack{\kappa:I\rightarrow C(X):\\ \sum_{i\in I}\kappa(i)=\alpha}}\ \sum_{\substack{\text{connected,}\\ \text{simply-connected}\\ \text{digraphs }\Gamma,\\ \text{vertices }I}}V(I,\Gamma,\kappa;\tau,\tilde\tau)\cdot\prod_{\text{edges }\overset{i}{\bullet}\rightarrow\overset{j}{\bullet}\text{ in }\Gamma}\bar\chi(\kappa(i),\kappa(j)) \\ \cdot\prod_{i\in I}J^{\kappa(i)}(\tau). \tag{24}$$

3.4. Donaldson–Thomas invariants of Calabi–Yau 3-folds.
Donaldson–Thomas invariants $DT^\alpha(\tau)$ were defined by Richard Thomas [**33**], following a proposal of Donaldson and Thomas [**4**, §3].

DEFINITION 3.15. Let X be a Calabi–Yau 3-fold. Fix a very ample line bundle $\mathcal{O}_X(1)$ on X, and let $(\tau, G, \leqslant)$ be Gieseker stability on $\mathrm{coh}(X)$ w.r.t. $\mathcal{O}_X(1)$, as in Example 3.7. For $\alpha \in K(X)$, write $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau), \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ for the coarse moduli schemes of τ-(semi)stable sheaves E with class $[E]=\alpha$. Then $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$ is a projective $\mathbb{C}$-scheme, and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ an open subscheme.

Thomas [**33**] constructs a symmetric obstruction theory on $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. Suppose that $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau) = \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. Then $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is proper, so using the obstruction theory Behrend and Fantechi [**2**] define a virtual class $[\mathcal{M}_{\mathrm{st}}^\alpha(\tau)]^{\mathrm{vir}} \in A_0(\mathcal{M}_{\mathrm{st}}^\alpha(\tau))$. The *Donaldson–Thomas invariant* [**33**] is defined to be

$$DT^\alpha(\tau) = \int_{[\mathcal{M}_{\mathrm{st}}^\alpha(\tau)]^{\mathrm{vir}}} 1. \tag{25}$$

Note that $DT^\alpha(\tau)$ *is defined only when* $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau) = \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$*, that is, there are no strictly semistable sheaves* E *in class* α. One of our main goals is to extend the definition to all $\alpha \in K(X)$. Thomas' main result [**33**, §3] is that

THEOREM 3.16. *$DT^\alpha(\tau)$ is unchanged by continuous deformations of the underlying Calabi–Yau 3-fold X.*

An important advance in Donaldson–Thomas theory was made by Behrend [**1**], who found a way to rewrite the definition (25) of Donaldson–Thomas invariants as a weighted Euler characteristic. Let $\mathfrak{F}$ be an Artin $\mathbb{C}$-stack, locally of finite type. Then $\mathfrak{F}$ has a unique *Behrend function* $\nu_{\mathfrak{F}} : \mathfrak{F}(\mathbb{C}) \to \mathbb{Z}$, a $\mathbb{Z}$-valued locally constructible function on $\mathfrak{F}$. The definition, which we do not give, can be found in [**1**, §1] when $\mathfrak{F}$ is a finite type $\mathbb{C}$-scheme, and in [**16**, §4.1] in the general case. Here are some important properties of Behrend functions from [**1**, **16**].

THEOREM 3.17. *Let $\mathfrak{F}, \mathfrak{G}$ be Artin $\mathbb{C}$-stacks locally of finite type. Then:*

(i) *If $\mathfrak{F}$ is a smooth of dimension n then $\nu_{\mathfrak{F}} \equiv (-1)^n$.*
(ii) *If $\varphi : \mathfrak{F} \to \mathfrak{G}$ is smooth with relative dimension n then $\nu_{\mathfrak{F}} \equiv (-1)^n \varphi^*(\nu_{\mathfrak{G}})$.*
(iii) *$\nu_{\mathfrak{F}\times\mathfrak{G}} = \nu_{\mathfrak{F}} \boxdot \nu_{\mathfrak{G}}$ in $\mathrm{LCF}(\mathfrak{F}\times\mathfrak{G})$, where $(\nu_{\mathfrak{F}} \boxdot \nu_{\mathfrak{G}})(x,y) = \nu_{\mathfrak{F}}(x)\nu_{\mathfrak{G}}(y)$.*
(iv) *Suppose $\mathcal{M}$ is a proper $\mathbb{C}$-scheme and has a symmetric obstruction theory, and $[\mathcal{M}]^{\mathrm{vir}} \in A_0(\mathcal{M})$ is the corresponding virtual class from* [**2**]. *Then*
$$\int_{[\mathcal{M}]^{\mathrm{vir}}} 1 = \chi(\mathcal{M}, \nu_{\mathcal{M}}) \in \mathbb{Z},$$
*where $\chi(\mathcal{M}, \nu_{\mathcal{M}}) = \int_{\mathcal{M}(\mathbb{C})} \nu_{\mathcal{M}} \mathrm{d}\chi$ is the **weighted Euler characteristic** of $\mathcal{M}$, weighted by the constructible function $\nu_{\mathcal{M}}$. In particular, $\int_{[\mathcal{M}]^{\mathrm{vir}}} 1$ depends only on the $\mathbb{C}$-scheme structure of $\mathcal{M}$, not on the choice of symmetric obstruction theory.*
(v) *Let $\mathcal{M}$ be a $\mathbb{C}$-scheme, let $x \in \mathcal{M}(\mathbb{C})$, and suppose there exist a complex manifold U, a holomorphic function $f : U \to \mathbb{C}$, and a point $u \in \mathrm{Crit}(f) \subseteq U$ such that locally in the analytic topology, $\mathcal{M}(\mathbb{C})$*

near x is isomorphic as a complex analytic space to $\mathrm{Crit}(f)$ *near u. Then*

$$\nu_{\mathcal{M}}(x) = (-1)^{\dim U}\bigl(1 - \chi(MF_f(u))\bigr),$$

where $\chi(MF_f(u))$ is the Euler characteristic of the Milnor fibre $MF_f(u)$.

Here the *Milnor fibre* in (v) is defined as follows:

DEFINITION 3.18. Let U be a complex analytic space, locally of finite type, $f : U \to \mathbb{C}$ a holomorphic function, and $u \in U$. Let $d(\, , \,)$ be a metric on U near u induced by a local embedding of U in some $\mathbb{C}^N$. For $u \in U$ and $\delta, \epsilon > 0$, consider the holomorphic map

$$\Phi_{f,u} : \bigl\{v \in U : d(u,v) < \delta,\ 0 < |f(v) - f(u)| < \epsilon\bigr\} \longrightarrow \bigl\{z \in \mathbb{C} : 0 < |z| < \epsilon\bigr\}$$

given by $\Phi_{f,u}(v) = f(v) - f(u)$. Then $\Phi_{f,u}$ is a smooth locally trivial fibration provided $0 < \epsilon \ll \delta \ll 1$. The *Milnor fibre* $MF_f(u)$ is the fibre of $\Phi_{f,u}$. It is independent of the choice of $0 < \epsilon \ll \delta \ll 1$.

Theorem 3.17(iv) implies that $DT^\alpha(\tau)$ in (25) is given by

$$DT^\alpha(\tau) = \chi\bigl(\mathcal{M}^\alpha_{\mathrm{st}}(\tau), \nu_{\mathcal{M}^\alpha_{\mathrm{st}}(\tau)}\bigr). \tag{26}$$

This is similar to the expression (17) for $J^\alpha(\tau)$ when $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau) = \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$. There is a big difference between the two equations (25) and (26) defining Donaldson–Thomas invariants. Equation (25) is non-local, and non-motivic, and makes sense only if $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ is a proper $\mathbb{C}$-scheme. But (26) is local, and (in a sense) motivic, and makes sense for arbitrary finite type $\mathbb{C}$-schemes $\mathcal{M}^\alpha_{\mathrm{st}}(\tau)$. It is tempting to take (26) to be the definition of Donaldson–Thomas invariants even when $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau) \neq \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$, but in [**16**, §6.5] we show that this is not a good idea, as then $DT^\alpha(\tau)$ would not be unchanged under deformations of X.

Equation (26) was the inspiration for [**16**]. It shows that Donaldson–Thomas invariants $DT^\alpha(\tau)$ can be written as *motivic* invariants, like those studied in [**11**–**15**], and suggests extending the results of [**11**–**15**] to Donaldson–Thomas invariants by including Behrend functions as weights.

4. Generalized Donaldson–Thomas invariants

We now summarize [**16**, §5–§6]. All this section is joint work with Yinan Song. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathcal{O}_X(1)$ a very ample line bundle over X. We now assume that $H^1(\mathcal{O}_X) = 0$, which was not needed in §3. We use the notation of §3, with $\mathfrak{M}$ the moduli stack of coherent sheaves on X, and so on.

4.1. Local description of the moduli of coherent sheaves. In [**16**, Th. 5.5] we give a local characterization of an atlas for the moduli stack $\mathfrak{M}$ as the critical points of a holomorphic function on a complex manifold.

THEOREM 4.1. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. Suppose E is a coherent sheaf on X, so that $[E]\in\mathfrak{M}(\mathbb{C})$. Let G be a maximal compact subgroup in $\mathrm{Aut}(E)$, and $G^{\mathbb{C}}$ its complexification. Then $G^{\mathbb{C}}$ is an algebraic $\mathbb{C}$-subgroup of $\mathrm{Aut}(E)$, a maximal reductive subgroup, and $G^{\mathbb{C}}=\mathrm{Aut}(E)$ if and only if $\mathrm{Aut}(E)$ is reductive.*

There exists a quasiprojective $\mathbb{C}$-scheme S, an action of $G^{\mathbb{C}}$ on S, a point $s\in S(\mathbb{C})$ fixed by $G^{\mathbb{C}}$, and a 1-morphism of Artin $\mathbb{C}$-stacks $\Phi:[S/G^{\mathbb{C}}]\to\mathfrak{M}$, which is smooth of relative dimension $\dim\mathrm{Aut}(E)-\dim G^{\mathbb{C}}$, where $[S/G^{\mathbb{C}}]$ is the quotient stack, such that $\Phi(s\,G^{\mathbb{C}})=[E]$, the induced morphism on stabilizer groups $\Phi_*:\mathrm{Iso}_{[S/G^{\mathbb{C}}]}(s\,G^{\mathbb{C}})\to\mathrm{Iso}_{\mathfrak{M}}([E])$ is the natural morphism $G^{\mathbb{C}}\hookrightarrow\mathrm{Aut}(E)\cong\mathrm{Iso}_{\mathfrak{M}}([E])$, and $\mathrm{d}\Phi|_{s\,G^{\mathbb{C}}}:T_sS\cong T_{s\,G^{\mathbb{C}}}[S/G^{\mathbb{C}}]\to T_{[E]}\mathfrak{M}\cong\mathrm{Ext}^1(E,E)$ is an isomorphism. Furthermore, S parametrizes a formally versal family $(S,\mathcal{D})$ of coherent sheaves on X, equivariant under the action of $G^{\mathbb{C}}$ on S, with fibre $\mathcal{D}_s\cong E$ at s. If $\mathrm{Aut}(E)$ is reductive then Φ is étale.

Write S_{an} for the complex analytic space underlying the $\mathbb{C}$-scheme S. Then there exists an open neighbourhood U of 0 in $\mathrm{Ext}^1(E,E)$ in the analytic topology, a holomorphic function $f:U\to\mathbb{C}$ with $f(0)=\mathrm{d}f|_0=0$, an open neighbourhood V of s in S_{an}, and an isomorphism of complex analytic spaces $\Xi:\mathrm{Crit}(f)\to V$, such that $\Xi(0)=s$ and $\mathrm{d}\Xi|_0:T_0\,\mathrm{Crit}(f)\to T_sV$ is the inverse of $\mathrm{d}\Phi|_{s\,G^{\mathbb{C}}}:T_sS\to\mathrm{Ext}^1(E,E)$. Moreover we can choose U,f,V to be $G^{\mathbb{C}}$-invariant, and Ξ to be $G^{\mathbb{C}}$-equivariant.

The proof of Theorem 4.1 comes in two parts. First we show in [**16**, §8] that $\mathfrak{M}$ near $[E]$ is locally isomorphic, as an Artin $\mathbb{C}$-stack, to the moduli stack $\mathfrak{Vect}$ of algebraic vector bundles on X near $[E']$ for some vector bundle $E'\to X$. The proof uses algebraic geometry, and is valid for X an Calabi–Yau m-fold for any $m>0$ over any algebraically closed field $\mathbb{K}$. The local morphism $\mathfrak{M}\to\mathfrak{Vect}$ is the composition of shifts and m Seidel–Thomas twists by $\mathcal{O}_X(-n)$ for $n\gg 0$.

Thus, it is enough to prove Theorem 4.1 with $\mathfrak{Vect}$ in place of $\mathfrak{M}$. We do this in [**16**, §9] using gauge theory on vector bundles over X, motivated by an idea of Donaldson and Thomas [**4**, §3], [**33**, §2], and results of Miyajima [**21**]. Let $E\to X$ be a fixed complex (not holomorphic) vector bundle over X. Write $\mathscr{A}$ for the infinite-dimensional affine space of smooth semiconnections ($\bar\partial$-operators) on E, and $\mathscr{G}$ for the infinite-dimensional Lie group of smooth gauge transformations of E. Then $\mathscr{G}$ acts on $\mathscr{A}$, and $\mathscr{B}=\mathscr{A}/\mathscr{G}$ is the space of gauge-equivalence classes of semiconnections on E.

We fix $\bar\partial_E$ in $\mathscr{A}$ coming from a holomorphic vector bundle structure on E. Then points in $\mathscr{A}$ are of the form $\bar\partial_E+A$ for $A\in C^\infty\big(\mathrm{End}(E)\otimes_{\mathbb{C}}\Lambda^{0,1}T^*X\big)$, and $\bar\partial_E+A$ makes E into a holomorphic vector bundle if $F_A^{0,2}=\bar\partial_EA+A\wedge A$ is zero in $C^\infty\big(\mathrm{End}(E)\otimes_{\mathbb{C}}\Lambda^{0,2}T^*X\big)$. Thus, the moduli space (stack) of holomorphic vector bundle structures on E is isomorphic to $\{\bar\partial_E + A \in \mathscr{A} : F_A^{0,2} = 0\}/\mathscr{G}$. Thomas observes that when X is a Calabi–Yau 3-fold, there is a natural holomorphic function $CS : \mathscr{A} \to \mathbb{C}$ called the *holomorphic Chern–Simons functional*, invariant under $\mathscr{G}$ up to addition of constants, such that $\{\bar\partial_E + A \in \mathscr{A} : F_A^{0,2} = 0\}$ is the critical locus of CS. Thus, $\mathfrak{Vect}$ is (informally) locally the critical points of a holomorphic function CS on an infinite-dimensional complex stack $\mathscr{B} = \mathscr{A}/\mathscr{G}$. To prove Theorem 4.1 we show that we can find a finite-dimensional complex submanifold U in $\mathscr{A}$ and a finite-dimensional complex Lie subgroup $G^{\mathbb{C}}$ in $\mathscr{G}$ preserving U such that the theorem holds with $f = CS|_U$.

In [**16**, Th. 5.11] we prove identities on the *Behrend function* of $\mathfrak{M}$, as in §3.4.

THEOREM 4.2. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $\mathfrak{M}$ the moduli stack of coherent sheaves on X. The **Behrend function** $\nu_{\mathfrak{M}} : \mathfrak{M}(\mathbb{C}) \to \mathbb{Z}$ is a natural locally constructible function on $\mathfrak{M}$. For all $E_1, E_2 \in \mathrm{coh}(X)$, it satisfies:*

$$\nu_{\mathfrak{M}}(E_1 \oplus E_2) = (-1)^{\bar\chi([E_1],[E_2])} \nu_{\mathfrak{M}}(E_1)\nu_{\mathfrak{M}}(E_2), \tag{27}$$

$$\begin{aligned} &\int_{\substack{[\lambda]\in\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)):\\ \lambda \Leftrightarrow 0\to E_1\to F\to E_2\to 0}} \nu_{\mathfrak{M}}(F)\,\mathrm{d}\chi - \int_{\substack{[\lambda']\in\mathbb{P}(\mathrm{Ext}^1(E_1,E_2)):\\ \lambda' \Leftrightarrow 0\to E_2\to F'\to E_1\to 0}} \nu_{\mathfrak{M}}(F')\,\mathrm{d}\chi \\ &\qquad = \big(\dim \mathrm{Ext}^1(E_2,E_1) - \dim \mathrm{Ext}^1(E_1,E_2)\big)\nu_{\mathfrak{M}}(E_1 \oplus E_2). \end{aligned} \tag{28}$$

Here $\bar\chi([E_1],[E_2])$ in (27) is defined in (9), and in (28) the correspondence between $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$ and $F \in \mathrm{coh}(X)$ is that $[\lambda] \in \mathbb{P}(\mathrm{Ext}^1(E_2,E_1))$ lifts to some $0 \neq \lambda \in \mathrm{Ext}^1(E_2,E_1)$, which corresponds to a short exact sequence $0 \to E_1 \to F \to E_2 \to 0$ in $\mathrm{coh}(X)$ in the usual way. The function $[\lambda] \mapsto \nu_{\mathfrak{M}}(F)$ is a constructible function $\mathbb{P}(\mathrm{Ext}^1(E_2,E_1)) \to \mathbb{Z}$, and the integrals in (28) are integrals of constructible functions using the Euler characteristic as measure.

We prove Theorem 4.2 using Theorem 4.1 and the Milnor fibre description of Behrend functions from Theorem 3.17(v). We apply Theorem 4.1 to $E = E_1 \oplus E_2$, and we take the maximal compact subgroup G of $\mathrm{Aut}(E)$ to contain the subgroup $\{\mathrm{id}_{E_1} + \lambda\,\mathrm{id}_{E_2} : \lambda \in \mathrm{U}(1)\}$, so that $G^{\mathbb{C}}$ contains $\{\mathrm{id}_{E_1} + \lambda\,\mathrm{id}_{E_2} : \lambda \in \mathbb{G}_m\}$. Equations (27) and (28) are proved by a kind of localization using this $\mathbb{G}_m$-action on $\mathrm{Ext}^1(E_1 \oplus E_2, E_1 \oplus E_2)$.

Note that Theorem 4.2 makes sense as a statement in algebraic geometry, for Calabi–Yau 3-folds over an algebraically closed field $\mathbb{K}$ of characteristic zero, and the author expects it to be true in this generality. However, our proof of Theorem 4.2 uses gauge theory, and transcendental complex analytic geometry methods, and is valid only over $\mathbb{K} = \mathbb{C}$.

4.2. A Lie algebra morphism $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{L}(X)$, and generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)$. In §3.1 we defined an explicit Lie algebra $L(X)$ and Lie algebra morphisms $\Psi : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \to L(X)$ and $\Psi^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \to L(X)$. We now define modified versions $\tilde{L}(X), \tilde{\Psi}, \tilde{\Psi}^{\chi,\mathbb{Q}}$, with $\tilde{\Psi}, \tilde{\Psi}^{\chi,\mathbb{Q}}$ weighted by the Behrend function $\nu_{\mathfrak{M}}$ of $\mathfrak{M}$. We continue to use the notation of §2–§3.

DEFINITION 4.3. Define a Lie algebra $\tilde{L}(X)$ to be the $\mathbb{Q}$-vector space with basis of symbols $\tilde{\lambda}^\alpha$ for $\alpha \in K(X)$, with Lie bracket

$$[\tilde{\lambda}^\alpha, \tilde{\lambda}^\beta] = (-1)^{\bar{\chi}(\alpha,\beta)} \bar{\chi}(\alpha, \beta) \tilde{\lambda}^{\alpha+\beta}, \tag{29}$$

which is (11) with a sign change. As $\bar{\chi}$ is antisymmetric, (29) satisfies the Jacobi identity, and makes $\tilde{L}(X)$ into an infinite-dimensional Lie algebra over $\mathbb{Q}$.

Define a $\mathbb{Q}$-linear map $\tilde{\Psi}^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \to \tilde{L}(X)$ by

$$\tilde{\Psi}^{\chi,\mathbb{Q}}(f) = \textstyle\sum_{\alpha \in K(X)} \gamma^\alpha \tilde{\lambda}^\alpha,$$

as in (12), where $\gamma^\alpha \in \mathbb{Q}$ is defined as follows. Write $f|_{\mathfrak{M}^\alpha}$ in terms of δ_i, U_i, ρ_i as in (13), and set

$$\gamma^\alpha = \textstyle\sum_{i=1}^n \delta_i \chi\big(U_i, \rho_i^*(\nu_{\mathfrak{M}})\big), \tag{30}$$

where $\rho_i^*(\nu_{\mathfrak{M}})$ is the pullback of the Behrend function $\nu_{\mathfrak{M}}$ to a constructible function on $U_i \times [\mathrm{Spec}\,\mathbb{C}/\mathbb{G}_m]$, or equivalently on U_i, and $\chi\big(U_i, \rho_i^*(\nu_{\mathfrak{M}})\big)$ is the Euler characteristic of U_i weighted by $\rho_i^*(\nu_{\mathfrak{M}})$. One can show that the map from (13) to (30) is compatible with the relations in $\bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}^\alpha, \chi, \mathbb{Q})$, and so $\tilde{\Psi}^{\chi,\mathbb{Q}}$ is well-defined. Define $\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{L}(X)$ by $\tilde{\Psi} = \tilde{\Psi}^{\chi,\mathbb{Q}} \circ \bar{\Pi}_{\mathfrak{M}}^{\chi,\mathbb{Q}}$.

The reason for the sign change between (11) and (29) is the signs involved in Behrend functions, in particular, the $(-1)^n$ in Theorem 3.17(ii), which is responsible for the factor $(-1)^{\bar{\chi}([E_1],[E_2])}$ in (27). Here [**16**, Th. 5.14] is the analogue of Theorem 3.5.

THEOREM 4.4. *$\tilde{\Psi} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}) \to \tilde{L}(X)$ and $\tilde{\Psi}^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}, \chi, \mathbb{Q}) \to \tilde{L}(X)$ are Lie algebra morphisms.*

We can now define generalized Donaldson–Thomas invariants.

DEFINITION 4.5. Let X be a projective Calabi–Yau 3-fold over $\mathbb{C}$, let $\mathcal{O}_X(1)$ be a very ample line bundle on X, and let $(\tau, G, \leqslant)$ be Gieseker stability and $(\mu, M, \leqslant)$ be μ-stability on $\mathrm{coh}(X)$ w.r.t. $\mathcal{O}_X(1)$, as in Examples 3.7 and 3.8. As in (16), define *generalized Donaldson–Thomas invariants* $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ and $\bar{DT}^\alpha(\mu) \in \mathbb{Q}$ for all $\alpha \in C(X)$ by

$$\tilde{\Psi}\big(\bar{\epsilon}^\alpha(\tau)\big) = -\bar{DT}^\alpha(\tau)\tilde{\lambda}^\alpha \quad \text{and} \quad \tilde{\Psi}\big(\bar{\epsilon}^\alpha(\mu)\big) = -\bar{DT}^\alpha(\mu)\tilde{\lambda}^\alpha. \tag{31}$$

Here $\bar{\epsilon}^\alpha(\tau), \bar{\epsilon}^\alpha(\mu)$ are defined in (14), and lie in $\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M})$ by Theorem 3.10, so $\bar{DT}^\alpha(\tau), \bar{DT}^\alpha(\mu)$ are well-defined. In [**16**, Prop. 5.17] we show that if $\mathcal{M}_{\rm ss}^\alpha(\tau)=\mathcal{M}_{\rm st}^\alpha(\tau)$ then $\bar{DT}{}^\alpha(\tau)=DT^\alpha(\tau)$. That is, our new generalized Donaldson–Thomas invariants $\bar{DT}{}^\alpha(\tau)$ are equal to the original Donaldson–Thomas invariants $DT^\alpha(\tau)$ of [**33**] whenever the $DT^\alpha(\tau)$ are defined.

We can now repeat the argument of §3.3 to deduce transformation laws for generalized Donaldson–Thomas invariants under change of stability condition. In the situation of Theorem 3.12, equation (19) is an identity in the Lie algebra $\mathrm{SF}_{\rm al}^{\rm ind}(\mathfrak{M})$, so we can apply the Lie algebra morphism $\tilde\Psi$ to transform (19) into an identity in the Lie algebra $\tilde L(X)$, and use (31) to write this in terms of generalized Donaldson–Thomas invariants. As for (20), this gives an equation in the universal enveloping algebra $U(\tilde L(X))$:

$$\bar{DT}{}^\alpha(\tilde\tau)\tilde\lambda^\alpha=\sum_{\substack{n\geqslant 1,\,\alpha_1,\ldots,\alpha_n\in C(X):\\ \alpha_1+\cdots+\alpha_n=\alpha}}\begin{aligned}[t] &U(\alpha_1,\ldots,\alpha_n;\tau,\tilde\tau)\cdot(-1)^{n-1}\textstyle\prod_{i=1}^n\bar{DT}{}^{\alpha_i}(\tau)\cdot\\ &\qquad\qquad \tilde\lambda^{\alpha_1}\star\tilde\lambda^{\alpha_2}\star\cdots\star\tilde\lambda^{\alpha_n}.\end{aligned}$$

Following the proof of (24) in §3.3 with sign changes, in [**16**, Th. 5.18] we obtain:

THEOREM 4.6. *In the situation of Theorem* 3.12, *for all* $\alpha\in C(X)$ *we have*

$$\bar{DT}{}^\alpha(\tilde\tau)=\sum_{\substack{\text{iso.}\\ \text{classes}\\ \text{of finite}\\ \text{sets } I}}\;\sum_{\substack{\kappa:I\to C(X):\\ \sum_{i\in I}\kappa(i)=\alpha}}\;\sum_{\substack{\text{connected,}\\ \text{simply-}\\ \text{connected}\\ \text{digraphs }\Gamma,\\ \text{vertices } I}}\begin{aligned}[t]&(-1)^{|I|-1}V(I,\Gamma,\kappa;\tau,\tilde\tau)\cdot\textstyle\prod_{i\in I}\bar{DT}{}^{\kappa(i)}(\tau)\\ &\cdot(-1)^{\frac12\sum_{i,j\in I}|\bar\chi(\kappa(i),\kappa(j))|}\cdot\prod_{\text{edges }\overset{i}{\bullet}\to\overset{j}{\bullet}\text{ in }\Gamma}\bar\chi(\kappa(i),\kappa(j)),\end{aligned}\tag{32}$$

with only finitely many nonzero terms.

The discussion after Theorem 3.12 implies [**16**, Cor. 5.19]:

COROLLARY 4.7. *Let* $(\tau,T,\leqslant),(\tilde\tau,\tilde T,\leqslant)$ *be two permissible weak stability conditions on* $\mathrm{coh}(X)$ *of Gieseker or* μ*-stability type, as in Examples* 3.7 *and* 3.8. *Then the* $\bar{DT}{}^\alpha(\tau)$ *for all* $\alpha\in C(X)$ *completely determine the* $\bar{DT}{}^\alpha(\tilde\tau)$ *for all* $\alpha\in C(X)$, *and vice versa, through finitely many applications of* (32).

4.3. Invariants $PI^{\alpha,n}(\tau')$ counting stable pairs, and deformation-invariance of the $\bar{DT}{}^\alpha(\tau)$. We wish to prove that our invariants $\bar{DT}{}^\alpha(\tau)$ are unchanged under deformations of X. We do this indirectly: we first define another family of auxiliary invariants $PI^{\alpha,n}(\tau')$ counting *stable pairs* on X, and show that $PI^{\alpha,n}(\tau')$ are unchanged under deformations of X. Then we prove an identity (35) expressing $PI^{\alpha,n}(\tau')$ in terms of the $\bar{DT}{}^\beta(\tau)$, and use it to show $\bar{DT}{}^\alpha(\tau)$ is deformation-invariant. This approach was inspired by Pandharipande and Thomas [**28**], who use invariants counting pairs to study curve counting in Calabi–Yau 3-folds.

Definition 4.8. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, with $H^1(\mathcal{O}_X)=0$. Choose a very ample line bundle $\mathcal{O}_X(1)$ on X, and write $(\tau,G,\leqslant)$ for Gieseker stability w.r.t. $\mathcal{O}_X(1)$, as in Example 3.7.

Fix $n\gg 0$ in $\mathbb{Z}$. A *pair* is a nonzero morphism of sheaves $s:\mathcal{O}_X(-n)\to E$, where E is a nonzero sheaf. A *morphism* between two pairs $s:\mathcal{O}_X(-n)\to E$ and $t:\mathcal{O}_X(-n)\to F$ is a morphism of $\mathcal{O}_X$-modules $f:E\to F$, with $f\circ s=t$. A pair $s:\mathcal{O}_X(-n)\to E$ is called *stable* if:

(i) $\tau([E'])\leqslant\tau([E])$ for all subsheaves E' of E with $0\neq E'\neq E$; and
(ii) If also s factors through E', then $\tau([E'])<\tau([E])$.

Note that (i) implies that if $s:\mathcal{O}_X(-n)\to E$ is stable then E is τ-semistable. The *class* of a pair $s:\mathcal{O}_X(-n)\to E$ is the numerical class $[E]$ in $K(X)$. We will use τ' *to denote stability of pairs*, defined using $\mathcal{O}_X(1)$.

In [**16**, Th.s 5.22 & 5.23] we use results of Le Potier to prove:

Theorem 4.9. *If n is sufficiently large then the moduli functor of stable pairs has a fine moduli scheme, a projective $\mathbb{C}$-scheme $\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')$, with a symmetric obstruction theory.*

Definition 4.10. In the situation above, for $\alpha\in K(X)$ and $n\gg 0$, define *stable pair invariants $PI^{\alpha,n}(\tau')$* in $\mathbb{Z}$ by

$$PI^{\alpha,n}(\tau')=\int_{[\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')]^{\rm vir}}1, \tag{33}$$

where $[\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')]^{\rm vir}\in A_0(\mathcal{M}_{\rm stp}^{\alpha,n}(\tau'))$ is the virtual class constructed by Behrend and Fantechi [**2**] using the symmetric obstruction theory from Theorem 4.9. Theorem 3.17(iv) implies that the stable pair invariants may also be written

$$PI^{\alpha,n}(\tau')=\chi\bigl(\mathcal{M}_{\rm stp}^{\alpha,n}(\tau'),\nu_{\mathcal{M}_{\rm stp}^{\alpha,n}(\tau')}\bigr). \tag{34}$$

In [**16**, Cor. 5.26] we prove an analogue of Theorem 3.16:

Theorem 4.11. *$PI^{\alpha,n}(\tau')$ is unchanged by continuous deformations of the underlying Calabi–Yau 3-fold X.*

In [**16**, Th. 5.27] we express the pair invariants $PI^{\alpha,n}(\tau')$ above in terms of the generalized Donaldson–Thomas invariants $\bar{DT}^\beta(\tau)$ of §4.2. Equation (35) is a wall-crossing formula similar to (32), and we prove it by change of stability condition in an auxiliary abelian category.

Theorem 4.12. *For $\alpha\in C(X)$ and $n\gg 0$ we have*

$$PI^{\alpha,n}(\tau')=\sum_{\substack{\alpha_1,\ldots,\alpha_l\in C(X),\\ l\geqslant 1:\,\alpha_1+\cdots+\alpha_l=\alpha,\\ \tau(\alpha_i)=\tau(\alpha),\text{ all } i}}\frac{(-1)^l}{l!}\prod_{i=1}^{l}\Bigl[(-1)^{\bar\chi([\mathcal{O}_X(-n)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)}\,\bar\chi\bigl([\mathcal{O}_X(-n)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i\bigr)\bar{DT}^{\alpha_i}(\tau)\Bigr], \tag{35}$$

where there are only finitely many nonzero terms in the sum.

Equation (35) is useful for computing invariants $\bar{DT}^\alpha(\tau)$ in examples. By combining Theorems 4.11 and 4.12 and using induction on the leading coefficient of the Hilbert polynomial of α, we deduce [**16**, Cor. 5.28]:

Corollary 4.13. *The generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau)$ defined in §4.2 are unchanged under continuous deformations of the underlying Calabi–Yau 3-fold X.*

4.4. Integrality properties of the $\bar{DT}^\alpha(\tau)$. This subsection is based on ideas in Kontsevich and Soibelman [**18**, §2.5 & §7.1]. The following example is taken from [**16**, Ex.s 6.1 & 6.2].

Example 4.14. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$ equipped with a very ample line bundle $\mathcal{O}_X(1)$. Suppose $\alpha \in C(X)$, and that $E \in \mathrm{coh}(X)$ with $[E]=\alpha$ is τ-stable and rigid, so that $\mathrm{Ext}^1(E,E)=0$. Then $mE = \overbrace{E\oplus\cdots\oplus E}^{m \text{ copies}}$ for $m \geqslant 2$ is a strictly τ-semistable sheaf of class $m\alpha$, which is also rigid. For simplicity, assume that mE is the only τ-semistable sheaf of class $m\alpha$ for all $m \geqslant 1$, up to isomorphism, so that $\mathcal{M}_{\mathrm{ss}}^{m\alpha}(\tau)=\{[mE]\}$.

A pair $s:\mathcal{O}(-n)\rightarrow mE$ may be regarded as m elements $s^1,\ldots,s^m$ of $H^0(E(n))\cong\mathbb{C}^{P_\alpha(n)}$, where P_α is the Hilbert polynomial of E. Such a pair turns out to be stable if and only if $s^1,\ldots,s^m$ are linearly independent in $H^0(E(n))$. Two such pairs are equivalent if they are identified under the action of $\mathrm{Aut}(mE)\cong\mathrm{GL}(m,\mathbb{C})$, acting in the obvious way on $(s^1,\ldots,s^m)$. Thus, equivalence classes of stable pairs correspond to linear subspaces of dimension m in $H^0(E(n))$, so the moduli space $\mathcal{M}_{\mathrm{stp}}^{m\alpha,n}(\tau')$ is isomorphic as a $\mathbb{C}$-scheme to the Grassmannian $\mathrm{Gr}(\mathbb{C}^m,\mathbb{C}^{P_\alpha(n)})$. This is smooth of dimension $m(P_\alpha(n)-m)$, so that $\nu_{\mathcal{M}_{\mathrm{stp}}^{m\alpha,n}(\tau')}\equiv(-1)^{m(P_\alpha(n)-m)}$ by Theorem 3.17(i). Also $\mathrm{Gr}(\mathbb{C}^m,\mathbb{C}^{P_\alpha(n)})$ has Euler characteristic the binomial coefficient $\binom{P_\alpha(n)}{m}$. Therefore (34) gives

$$PI^{m\alpha,n}(\tau')=(-1)^{m(P_\alpha(n)-m)}\binom{P_\alpha(n)}{m}. \tag{36}$$

Consider (35) with $m\alpha$ in place of α. If $\alpha_1,\ldots,\alpha_l$ give a nonzero term on the right hand side of (35) then $m\alpha=\alpha_1+\cdots+\alpha_l$, and $\bar{DT}^{\alpha_i}(\tau)\neq 0$, so there exists a τ-semistable E_i in class α_i. Thus $E_1\oplus\cdots\oplus E_l$ lies in class $m\alpha$, and is τ-semistable as $\tau(\alpha_i)=\tau(\alpha)$ for all i. Hence $E_1\oplus\cdots\oplus E_l\cong mE$, which implies that $E_i\cong k_iE$ for some $k_1,\ldots,k_l\geqslant 1$ with $k_1+\cdots+k_l=m$, and $\alpha_i=k_i\alpha$.

Setting $\alpha_i=k_i\alpha$, we see that $\bar\chi(\alpha_j,\alpha_i)=0$ and $\bar\chi([\mathcal{O}_X(-n)],\alpha_i)=k_iP_\alpha(n)$, where P_α is the Hilbert polynomial of E. Thus in (35) we have $\bar\chi([\mathcal{O}_X(-n)]-\alpha_1-\cdots-\alpha_{i-1},\alpha_i)=k_iP_\alpha(n)$. Combining (36), and (35) with these substitutions, and cancelling a factor of $(-1)^{mP_\alpha(n)}$ on both sides, yields

$$(-1)^m \binom{P_\alpha(n)}{m} = \sum_{\substack{l,k_1,\dots,k_l\geqslant 1:\\ k_1+\dots+k_l=m}} \frac{(-1)^l}{l!} \prod_{i=1}^{l} k_i P_\alpha(n) \bar{DT}^{k_i\alpha}(\tau).$$

Regarding each side as a polynomial in $P_\alpha(n)$ and taking the linear term in $P_\alpha(n)$ we see that

$$\bar{DT}^{m\alpha}(\tau) = \frac{1}{m^2} \quad \text{for all } m \geqslant 1.$$

Example 4.14 shows that given a rigid τ-stable sheaf E in class α, the sheaves mE contribute $1/m^2$ to $\bar{DT}^{m\alpha}(\tau)$ for all $m \geqslant 1$. We can regard this as a kind of 'multiple cover formula', analogous to the well known Aspinwall–Morrison computation for a Calabi–Yau 3-fold X that a rigid embedded $\mathbb{CP}^1$ in class $\alpha \in H_2(X;\mathbb{Z})$ contributes $1/m^3$ to the genus zero Gromov–Witten invariant of X in class $m\alpha$ for all $m \geqslant 1$. So we can define new invariants $\hat{DT}^\alpha(\tau)$ which subtract out these contributions from mE for $m > 1$.

Definition 4.15. Let X be a projective Calabi–Yau 3-fold over $\mathbb{C}$, let $\mathcal{O}_X(1)$ be a very ample line bundle on X, and let $(\tau, T, \leqslant)$ be a weak stability condition on $\mathrm{coh}(X)$ of Gieseker or μ-stability type. Then Definition 4.5 defines generalized Donaldson–Thomas invariants $\bar{DT}^\alpha(\tau) \in \mathbb{Q}$ for $\alpha \in C(X)$.

Let us define new invariants $\hat{DT}^\alpha(\tau)$ for $\alpha \in C(X)$ to satisfy

$$\bar{DT}^\alpha(\tau) = \sum_{m\geqslant 1,\, m\mid\alpha} \frac{1}{m^2}\, \hat{DT}^{\alpha/m}(\tau). \tag{37}$$

By the Möbius inversion formula, the inverse of (37) is

$$\hat{DT}^\alpha(\tau) = \sum_{m\geqslant 1,\, m\mid\alpha} \frac{\text{Mö}(m)}{m^2}\, \bar{DT}^{\alpha/m}(\tau), \tag{38}$$

where the *Möbius function* $\text{Mö} : \mathbb{N} \to \{-1, 0, 1\}$ is $\text{Mö}(n) = (-1)^d$ if $n = 1, 2, \dots$ is square-free and has d prime factors, and $\text{Mö}(n) = 0$ otherwise.

We take (38) to be the definition of $\hat{DT}^\alpha(\tau)$, and then reversing the argument shows that (37) holds. We call $\hat{DT}^\alpha(\tau)$ the *BPS invariants* of X, as Kontsevich and Soibelman suggest their analogous invariants $\Omega(\alpha)$ count BPS states.

If $\mathcal{M}^\alpha_{\mathrm{ss}}(\tau) = \mathcal{M}^\alpha_{\mathrm{st}}(\tau)$ then $\mathcal{M}^{\alpha/m}_{\mathrm{ss}}(\tau) = \emptyset$ for all $m \geqslant 2$ dividing α, and so $\hat{DT}^\alpha(\tau) = \bar{DT}^\alpha(\tau) = DT^\alpha(\tau)$, as in Definition 4.5.

We make a conjecture [**16**, Conj. 6.12], based on [**18**, Conj. 6].

Conjecture 4.16. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $(\tau, T, \leqslant)$ a weak stability condition on $\mathrm{coh}(X)$ of Gieseker or μ-stability type. Call $(\tau, T, \leqslant)$* ***generic*** *if for all $\alpha, \beta \in C(X)$ with $\tau(\alpha) = \tau(\beta)$ we have $\bar{\chi}(\alpha, \beta) = 0$.*

If $(\tau, T, \leqslant)$ is generic, then $\hat{DT}^\alpha(\tau) \in \mathbb{Z}$ for all $\alpha \in C(X)$.

In [**16**, §6] we prove that Conjecture 4.16 holds in a number of examples, and give an example [**16**, Ex. 6.8] in which $(\tau,T,\leqslant)$ is not generic and $\hat{DT}^\alpha(\tau)\notin\mathbb{Z}$. In [**16**, Th. 7.29] we prove the analogue of Conjecture 4.16 for invariants counting representations of quivers without relations.

4.5. Counting dimension 0 and 1 sheaves. Let X be a Calabi–Yau 3-fold over $\mathbb{C}$ with $H^1(\mathcal{O}_X)=0$, let $\mathcal{O}_X(1)$ be a very ample line bundle on X, and $(\tau,G,\leqslant)$ the associated Gieseker stability condition on $\mathrm{coh}(X)$. The Chern character gives an injective group homomorphism $\mathrm{ch}:K(X)\rightarrow H^{\mathrm{even}}(X;\mathbb{Q})$. So we can regard $K(X)$ as a subgroup of $H^{\mathrm{even}}(X;\mathbb{Q})$, and write $\alpha\in K(X)$ as $(\alpha_0,\alpha_2,\alpha_4,\alpha_6)$ with $\alpha_{2j}\in H^{2j}(X;\mathbb{Q})$. If $E\rightarrow X$ is a vector bundle with $[E]=\alpha$ then $\alpha_0=\mathrm{rank}\,E\in\mathbb{Z}$.

We will consider invariants $\bar{DT}^\alpha(\tau),\hat{DT}^\alpha(\tau)$ counting pure sheaves E of dimensions 0 and 1 on X, following [**16**, §6.3–§6.4]. For sheaves E of dimension zero $\mathrm{ch}\,E=(0,0,0,d)$ where $d\geqslant 1$ is the length of E. In [**16**, §6.3] we observe that for dimension 0 sheaves the moduli scheme $\mathcal{M}_{\mathrm{stp}}^{(0,0,0,d),n}(\tau')$ is independent of n, and is isomorphic to the Hilbert scheme $\mathrm{Hilb}^d X$. Therefore (34) gives

$$PI^{(0,0,0,d),n}(\tau')=\chi\bigl(\mathrm{Hilb}^d X,\nu_{\mathrm{Hilb}^d X}\bigr),\quad \text{for all } n\in\mathbb{Z} \text{ and } d\geqslant 0.$$

Values for $\chi(\mathrm{Hilb}^d X,\nu_{\mathrm{Hilb}^d X})$ were conjectured by Maulik et al. [**22**, Conj. 1], and proved by Behrend and Fantechi [**3**, Th. 4.12] and others. These yield a generating function for the $PI^{(0,0,0,d),n}(\tau')$:

$$1+\textstyle\sum_{d\geqslant 1}PI^{(0,0,0,d),n}(\tau')s^d=\bigl[\prod_{k\geqslant 1}(1-s^k)^{-k}\bigr]^{\chi(X)}.$$

Computing using (35) then shows that

$$\bar{DT}^{(0,0,0,d)}(\tau)=-\chi(X)\sum_{l\geqslant 1,\,l|d}\frac{1}{l^2}.$$

So from (37)–(38) we deduce that

$$\hat{DT}^{(0,0,0,d)}(\tau)=-\chi(X),\quad \text{all } d\geqslant 1.$$

This confirms Conjecture 4.16 for dimension 0 sheaves. It is one of several examples in [**16**] in which the values of the $PI^{\alpha,n}(\tau')$ are complex, the values of the $\bar{DT}^\alpha(\tau)$ are simpler, and the values of the $\hat{DT}^\alpha(\tau)$ are simpler still, which suggests that of the three the invariants $\hat{DT}^\alpha(\tau)$ are the most fundamental.

Now let $\beta\in H^4(X;\mathbb{Z})$ and $k\in\mathbb{Z}$. In [**16**, §6.4] we study invariants $\bar{DT}^{(0,0,\beta,k)}(\tau),\hat{DT}^{(0,0,\beta,k)}(\tau)$ counting semistable dimension 1 sheaves, that is, sheaves E supported on curves C in X. One expects these to be related to curve-counting invariants like Gromov–Witten invariants, as in the MNOP Conjecture [**22**, **23**]. Here is a summary of our results:

(a) $\bar{DT}^{(0,0,\beta,k)}(\tau), \hat{DT}^{(0,0,\beta,k)}(\tau)$ are independent of the choice of $(\tau, T, \leqslant)$.
(b) Assume Conjecture 4.16 holds. Then $\hat{DT}^{(0,0,\beta,k)}(\tau) \in \mathbb{Z}$.
(c) For any $l \in \beta \cup H^2(X;\mathbb{Z}) \subseteq \mathbb{Z}$ we have $\bar{DT}^{(0,0,\beta,k)}(\tau) = \bar{DT}^{(0,0,\beta,k+l)}(\tau)$ and $\hat{DT}^{(0,0,\beta,k)}(\tau) = \hat{DT}^{(0,0,\beta,k+l)}(\tau)$.
(d) Let C be an embedded rational curve in X with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and $\beta \in H^4(X;\mathbb{Z})$ be Poincaré dual to $[C] \in H_2(X;\mathbb{Z})$. Then sheaves supported on C contribute $1/m^2$ to $\bar{DT}^{(0,0,m\beta,k)}(\tau)$ if $m \geqslant 1$ and $m \mid k$, and contribute 0 to $\bar{DT}^{(0,0,m\beta,k)}(\tau)$ if $m \geqslant 1$ and $m \nmid k$. They contribute 1 to $\hat{DT}^{(0,0,\beta,k)}(\tau)$, and 0 to $\hat{DT}^{(0,0,m\beta,k)}(\tau)$ if $m > 1$.
(e) Let C be a nonsingular embedded curve in X of genus $g \geqslant 1$, and let $\beta \in H^4(X;\mathbb{Z})$ be Poincaré dual to $[C] \in H_2(X;\mathbb{Z})$. Then sheaves supported on C contribute 0 to $\bar{DT}^{(0,0,m\beta,k)}(\tau), \hat{DT}^{(0,0,m\beta,k)}(\tau)$ for all $m \geqslant 1$ and $k \in \mathbb{Z}$.

Motivated by these and by Katz[**17**, Conj. 2.3], we conjecture[**16**, Conj. 6.20]:

Conjecture 4.17. *Let X be a Calabi–Yau 3-fold over $\mathbb{C}$, and $(\tau, T, \leqslant)$ a weak stability condition on $\mathrm{coh}(X)$ of Gieseker or μ-stability type. Then for $\gamma \in H_2(X;\mathbb{Z})$ with $\beta \in H^4(X;\mathbb{Z})$ Poincaré dual to γ and all $k \in \mathbb{Z}$ we have $\hat{DT}^{(0,0,\beta,k)}(\tau) = GV_0(\gamma)$. In particular, $\hat{DT}^{(0,0,\beta,k)}(\tau)$ is independent of k, τ.*

Here $GV_0(\gamma)$ is the *genus zero Gopakumar–Vafa invariant*, given in terms of the genus zero Gromov–Witten invariants $GW_0(\gamma) \in \mathbb{Q}$ of X by

$$GW_0(\gamma) = \sum_{m \mid \gamma} \frac{1}{m^3}\, GV_0(\gamma/m).$$

A priori we have $GV_0(\gamma) \in \mathbb{Q}$, but Gopakumar and Vafa [**7**] conjecture that the $GV_0(\gamma)$ are integers, and count something meaningful in String Theory.

5. Quivers with superpotentials

We now summarize [**16**, §7], which develops an analogue of the results of §4 for representations of a quiver Q with relations I coming from a superpotential W. In the quiver case we have no analogue of $\bar{DT}^{\alpha}(\tau), \hat{DT}^{\alpha}(\tau)$, $PI^{\alpha,n}(\tau)$ in §4 being *deformation-invariant*, since the proof of deformation-invariance uses the fact that the moduli scheme $\mathcal{M}_{\mathrm{stp}}^{\alpha,n}(\tau')$ is proper (i.e. compact), but the analogous moduli schemes $\mathcal{M}_{\mathrm{stf}\,Q,I}^{\boldsymbol{d},\boldsymbol{e}}(\mu')$ in the quiver case need not be proper. However, all the other important aspects of the sheaf case transfer to the quiver case.

5.1. Background on quivers. Here are the basic definitions in quiver theory.

Definition 5.1. A *quiver* Q is a finite directed graph. That is, Q is a quadruple (Q_0, Q_1, h, t), where Q_0 is a finite set of *vertices*, Q_1 is a finite set of *edges*, and $h, t : Q_1 \to Q_0$ are maps giving the *head* and *tail* of each edge.

The *path algebra* $\mathbb{C}Q$ is an associative algebra over $\mathbb{C}$ with basis all *paths of length* $k \geqslant 0$, that is, sequences of the form

$$v_0 \xrightarrow{e_1} v_1 \to \cdots \to v_{k-1} \xrightarrow{e_k} v_k,$$

where $v_0, \dots, v_k \in Q_0$, $e_1, \dots, e_k \in Q_1$, $t(a_i) = v_{i-1}$ and $h(a_i) = v_i$. Multiplication is given by composition of paths in reverse order.

For $n \geqslant 0$, write $\mathbb{C}Q_{(n)}$ for the vector subspace of $\mathbb{C}Q$ with basis all paths of length $k \geqslant n$. It is an ideal in $\mathbb{C}Q$. A *quiver with relations* (Q, I) is defined to be a quiver Q together with a two-sided ideal I in $\mathbb{C}Q$ with $I \subseteq \mathbb{C}Q_{(2)}$. Then $\mathbb{C}Q/I$ is an associative $\mathbb{C}$-algebra.

For $v \in Q_0$, write $i_v \in \mathbb{C}Q$ for the path of length 0 at v. The image of i_v in $\mathbb{C}Q/I$ is also written i_v. Then

$$(39)\quad i_v^2 = i_v,\ \ i_v i_w = 0 \text{ if } v \neq w \in Q_0, \quad \text{and} \quad \textstyle\sum_{v \in Q_0} i_v = 1 \text{ in } \mathbb{C}Q \text{ or } \mathbb{C}Q/I.$$

Write mod-$\mathbb{C}Q$ or mod-$\mathbb{C}Q/I$ for the abelian categories of finite-dimensional left $\mathbb{C}Q$ or $\mathbb{C}Q/I$-modules, respectively. If $E \in$ mod-$\mathbb{C}Q$ or mod-$\mathbb{C}Q/I$ then (39) implies a decomposition of complex vector spaces $E = \bigoplus_{v \in Q_0} i_v(E)$. Define the *dimension vector* $\mathbf{dim}\, E \in \mathbb{Z}_{\geqslant 0}^{Q_0} \subset \mathbb{Z}^{Q_0}$ by $\mathbf{dim}\, E : v \mapsto \dim_{\mathbb{C}}(i_v E)$. If $0 \to E \to F \to G \to 0$ is an exact sequence in mod-$\mathbb{C}Q$ or mod-$\mathbb{C}Q/I$ then $\mathbf{dim}\, F = \mathbf{dim}\, E + \mathbf{dim}\, G$. Hence $\mathbf{dim}$ induces surjective morphisms $\mathbf{dim} : K_0(\text{mod-}\mathbb{C}Q) \to \mathbb{Z}^{Q_0}$ and $\mathbf{dim} : K_0(\text{mod-}\mathbb{C}Q/I) \to \mathbb{Z}^{Q_0}$.

Write $K(\text{mod-}\mathbb{C}Q) = K(\text{mod-}\mathbb{C}Q/I) = \mathbb{Z}^{Q_0}$, regarded as quotients of the Grothendieck groups $K_0(\text{mod-}\mathbb{C}Q), K_0(\text{mod-}\mathbb{C}Q/I)$ induced by $\mathbf{dim}$. Write $C(\text{mod-}\mathbb{C}Q) = C(\text{mod-}\mathbb{C}Q/I) = \mathbb{Z}_{\geqslant 0}^{Q_0} \setminus \{0\}$, the subsets of classes in K $(\text{mod-}\mathbb{C}Q)$, $K(\text{mod-}\mathbb{C}Q/I)$ of nonzero objects in mod-$\mathbb{C}Q$, mod-$\mathbb{C}Q/I$. Here $K(\text{mod-}\mathbb{C}Q)$, $K(\text{mod-}\mathbb{C}Q/I)$ are our substitutes for $K(X) = K^{\text{num}}(\text{coh}(X))$ in §3–§4. We do not use the numerical Grothendieck groups $K^{\text{num}}(\text{mod-}\mathbb{C}Q)$, $K^{\text{num}}(\text{mod-}\mathbb{C}Q/I)$, as these may be zero in interesting cases.

Definition 5.2. Let Q be a quiver. A *superpotential* W for Q over $\mathbb{C}$ is an element of $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$. The cycles in Q up to cyclic permutation form a basis for $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ over $\mathbb{C}$, so we can think of W as a finite $\mathbb{C}$-linear combination of cycles up to cyclic permutation. We call W *minimal* if all cycles in W have length at least 3. We will consider only minimal superpotentials W.

Define I to be the two-sided ideal in $\mathbb{C}Q$ generated by $\partial_e W$ for all edges $e \in Q_1$, where if C is a cycle in Q, we define $\partial_e C$ to be the sum over all occurrences of the edge e in C of the path obtained by cyclically permuting

C until e is in first position, and then deleting it. Since W is minimal, $I \subseteq \mathbb{C}Q_{(2)}$, and (Q,I) is a quiver with relations. We allow $W \equiv 0$, so that $I=0$.

Here is [**16**, Th. 7.6], which gives an analogue of equation (10) for quivers with superpotentials. Now (10) depended crucially on X being a Calabi–Yau 3-fold, which implies that $\mathrm{coh}(X)$ has Serre duality in dimension 3. In general the categories mod-$\mathbb{C}Q/I$ coming from quivers with superpotentials do *not* have Serre duality in dimension 3. However, as explained in [**16**, §7.2], if (Q,I) comes from a quiver with superpotential then we can embed mod-$\mathbb{C}Q/I$ as the heart of a t-structure in a 3-Calabi–Yau triangulated category $\mathcal{T}$ (which is usually not D^bmod-$\mathbb{C}Q/I$), and Serre duality in dimension 3 holds in $\mathcal{T}$. This is why quivers with superpotentials are algebraic analogues of Calabi–Yau 3-folds, and have a version of Donaldson–Thomas theory.

Theorem 5.3. *Let $Q=(Q_0,Q_1,h,t)$ be a quiver with relations I coming from a minimal superpotential W on Q over $\mathbb{C}$. Define $\bar\chi:\mathbb{Z}^{Q_0}\times\mathbb{Z}^{Q_0}\rightarrow\mathbb{Z}$ by*

$$\bar\chi(\boldsymbol{d},\boldsymbol{e})=\textstyle\sum_{e\in Q_1}\bigl(\boldsymbol{d}(h(e))\boldsymbol{e}(t(e))-\boldsymbol{d}(t(e))\boldsymbol{e}(h(e))\bigr). \tag{40}$$

Then for any $D,E\in$ mod-$\mathbb{C}Q/I$ we have

$$\begin{aligned}\bar\chi\bigl(\mathbf{dim}\,D,\mathbf{dim}\,E\bigr)=&\bigl(\dim\mathrm{Hom}(D,E)-\dim\mathrm{Ext}^1(D,E)\bigr)\\&-\bigl(\dim\mathrm{Hom}(E,D)-\dim\mathrm{Ext}^1(E,D)\bigr).\end{aligned}$$

If Q is a quiver, the moduli stack $\mathfrak{M}_Q$ of objects E in mod-$\mathbb{C}Q$ is an Artin $\mathbb{C}$-stack. For $\boldsymbol{d}\in\mathbb{Z}_{\geqslant 0}^{Q_0}$, the open substack $\mathfrak{M}_Q^{\boldsymbol{d}}$ of E with $\mathbf{dim}\,E=\boldsymbol{d}$ has a very explicit description: as a quotient $\mathbb{C}$-stack we have

$$\mathfrak{M}_Q^{\boldsymbol{d}}\cong\bigl[\textstyle\prod_{e\in Q_1}\mathrm{Hom}(\mathbb{C}^{\boldsymbol{d}(t(e))},\mathbb{C}^{\boldsymbol{d}(h(e))})/\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v))\bigr].$$

If (Q,I) is a quiver with relations, the moduli stack $\mathfrak{M}_{Q,I}$ of objects E in mod-$\mathbb{C}Q/I$ is a substack of $\mathfrak{M}_Q$, and for $\boldsymbol{d}\in\mathbb{Z}_{\geqslant 0}^{Q_0}$ we may write

$$\mathfrak{M}_{Q,I}^{\boldsymbol{d}}\cong\bigl[V_{Q,I}^{\boldsymbol{d}}/\textstyle\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v))\bigr], \tag{41}$$

where $V_{Q,I}^{\boldsymbol{d}}$ is a closed $\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v))$-invariant $\mathbb{C}$-subscheme of $\prod_{e\in Q_1}\mathrm{Hom}(\mathbb{C}^{\boldsymbol{d}(t(e))},\mathbb{C}^{\boldsymbol{d}(h(e))})$ defined using the relations I.

When I comes from a superpotential W, we can improve the description (41) of the moduli stacks $\mathfrak{M}_{Q,I}^{\boldsymbol{d}}$. Define a $\prod_{v\in Q_0}\mathrm{GL}(\boldsymbol{d}(v))$-invariant polynomial

$$W^{\boldsymbol{d}}:\textstyle\prod_{e\in Q_1}\mathrm{Hom}\bigl(\mathbb{C}^{\boldsymbol{d}(t(e))},\mathbb{C}^{\boldsymbol{d}(h(e))}\bigr)\longrightarrow\mathbb{C}$$

as follows. Write W as a finite sum $\sum_i\gamma^iC^i$,where $\gamma^i\in\mathbb{C}$ and C^i is a cycle $v_0^i\xrightarrow{e_1^i}v_1^i\rightarrow\cdots\rightarrow v_{k^i-1}^i\xrightarrow{e_{k^i}^i}v_{k^i}^i=v_0^i$ in Q. Set

$$W^{\boldsymbol{d}}\bigl(A_e:e\in Q_1\bigr)=\textstyle\sum_i\gamma^i\,\mathrm{Tr}\bigl(A_{e^i_{k^i}}\circ A_{e^i_{k^i-1}}\circ\cdots\circ A_{e^i_1}\bigr).$$

Then $V_{Q,I}^{\boldsymbol{d}} = \mathrm{Crit}(W^{\boldsymbol{d}})$ in (41), so that

$$\mathfrak{M}_{Q,I}^{\boldsymbol{d}} \cong \left[\mathrm{Crit}(W^{\boldsymbol{d}}) / \prod_{v \in Q_0} \mathrm{GL}(\boldsymbol{d}(v))\right]. \tag{42}$$

Equation (42) is an analogue of Theorem 4.1 for categories mod-$\mathbb{C}Q/I$ coming from a superpotential W on Q.

We define a class of *stability conditions* on mod-$\mathbb{C}Q/I$, [**15**, Ex. 4.14].

EXAMPLE 5.4. Let (Q, I) be a quiver with relations. Let $c : Q_0 \to \mathbb{R}$ and $r : Q_0 \to (0, \infty)$ be maps. Define $\mu : C(\text{mod-}\mathbb{C}Q/I) \to \mathbb{R}$ by

$$\mu(\boldsymbol{d}) = \frac{\sum_{v \in Q_0} c(v)\boldsymbol{d}(v)}{\sum_{v \in Q_0} r(v)\boldsymbol{d}(v)}.$$

Note that $\sum_{v \in Q_0} r(v)\boldsymbol{d}(v) > 0$ as $r(v) > 0$ for all $v \in Q_0$, and $\boldsymbol{d}(v) \geqslant 0$ for all v with $\boldsymbol{d}(v) > 0$ for some v. Then [**15**, Ex. 4.14] shows that $(\mu, \mathbb{R}, \leqslant)$ is a *permissible stability condition* on mod-$\mathbb{C}Q$, which we call *slope stability.* Write $\mathfrak{M}_{\mathrm{ss}}^{\boldsymbol{d}}(\mu)$ for the open $\mathbb{C}$-substack of μ-semistable objects in class $\boldsymbol{d}$ in $\mathfrak{M}_{Q,I}^{\boldsymbol{d}}$.

A simple case is to take $c \equiv 0$ and $r \equiv 1$, so that $\mu \equiv 0$. Then $(0, \mathbb{R}, \leqslant)$ is a trivial stability condition on mod-$\mathbb{C}Q$ or mod-$\mathbb{C}Q/I$, and every nonzero object in mod-$\mathbb{C}Q$ or mod-$\mathbb{C}Q/I$ is 0-semistable, so that $\mathfrak{M}_{\mathrm{ss}}^{\boldsymbol{d}}(0) = \mathfrak{M}_{Q,I}^{\boldsymbol{d}}$.

5.2. Behrend function identities, Lie algebra morphisms, and Donaldson–Thomas type invariants. Let Q be a quiver with relations I coming from a minimal superpotential W on Q over $\mathbb{C}$. We now generalize §4 from $\mathrm{coh}(X)$ to mod-$\mathbb{C}Q/I$. The proof of Theorem 4.2 depends on two things: the description of $\mathfrak{M}$ in terms of $\mathrm{Crit}(f)$ in Theorem 4.1, and equation (10). For mod-$\mathbb{C}Q/I$ equation (42) provides an analogue of Theorem 4.1, and Theorem 5.3 an analogue of (10). Thus, the proof of Theorem 4.2 also yields [**16**, Th. 7.11]:

THEOREM 5.5. *In the situation above, with $\mathfrak{M}_{Q,I}$ the moduli stack of objects in a category* mod-$\mathbb{C}Q/I$ *coming from a quiver Q with superpotential W, and $\bar\chi$ defined in* (40), *the Behrend function $\nu_{\mathfrak{M}_{Q,I}}$ of $\mathfrak{M}_{Q,I}$ satisfies the identities* (27)–(28) *for all $E_1, E_2 \in$* mod-$\mathbb{C}Q/I$.

Here is the analogue of Definition 4.3.

DEFINITION 5.6. Define a Lie algebra $\tilde{L}(Q)$ to be the $\mathbb{Q}$-vector space with basis of symbols $\tilde\lambda^{\boldsymbol{d}}$ for $\boldsymbol{d} \in \mathbb{Z}^{Q_0}$, with Lie bracket

$$[\tilde\lambda^{\boldsymbol{d}}, \tilde\lambda^{\boldsymbol{e}}] = (-1)^{\bar\chi(\boldsymbol{d},\boldsymbol{e})} \bar\chi(\boldsymbol{d}, \boldsymbol{e}) \tilde\lambda^{\boldsymbol{d}+\boldsymbol{e}},$$

as for (29), with $\bar\chi$ given in (40). This makes $\tilde{L}(Q)$ into an infinite-dimensional Lie algebra over $\mathbb{Q}$. Define $\mathbb{Q}$-linear maps $\tilde\Psi_{Q,I}^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{Q,I}, \chi, \mathbb{Q}) \to \tilde{L}(Q)$ and $\tilde\Psi_{Q,I} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{Q,I}) \to \tilde{L}(Q)$ exactly as for $\tilde\Psi^{\chi,\mathbb{Q}}, \tilde\Psi$ in Definition 4.3.

The proof of Theorem 4.4 has two ingredients: equation (10) and Theorem 4.2. Theorems 5.3 and 5.5 are analogues of these for quivers with superpotentials. So the proof of Theorem 4.4 also yields [**16**, Th. 7.14]:

THEOREM 5.7. *$\tilde{\Psi}_{Q,I} : \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{Q,I}) \to \tilde{L}(Q)$ and $\tilde{\Psi}_{Q,I}^{\chi,\mathbb{Q}} : \bar{\mathrm{SF}}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{Q,I}, \chi, \mathbb{Q}) \to \tilde{L}(Q)$ are Lie algebra morphisms.*

Here is the analogue of Definitions 4.5 and 4.15.

DEFINITION 5.8. Let $(\mu, \mathbb{R}, \leqslant)$ be a slope stability condition on mod-$\mathbb{C}Q/I$ as in Example 5.4. As in §3.2 we have elements $\bar{\delta}_{\mathrm{ss}}^{\boldsymbol{d}}(\mu) \in \mathrm{SF}_{\mathrm{al}}(\mathfrak{M}_{Q,I})$ and $\bar{\epsilon}^{\boldsymbol{d}}(\mu) \in \mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{M}_{Q,I})$ for all $\boldsymbol{d} \in C(\text{mod-}\mathbb{C}Q/I)$. As in (31), define *quiver generalized Donaldson–Thomas invariants* $\bar{DT}_{Q,I}^{\boldsymbol{d}}(\mu) \in \mathbb{Q}$ for all $\boldsymbol{d} \in C(\text{mod-}\mathbb{C}Q/I)$ by

$$\tilde{\Psi}_{Q,I}\big(\bar{\epsilon}^{\boldsymbol{d}}(\mu)\big) = -\bar{DT}_{Q,I}^{\boldsymbol{d}}(\mu)\tilde{\lambda}^{\boldsymbol{d}}.$$

As in (38), define *quiver BPS invariants* $\hat{DT}_{Q,I}^{\boldsymbol{d}}(\mu) \in \mathbb{Q}$ by

$$\hat{DT}_{Q,I}^{\boldsymbol{d}}(\mu) = \sum_{m \geqslant 1,\, m | \boldsymbol{d}} \frac{\text{Mö}(m)}{m^2} \bar{DT}_{Q,I}^{\boldsymbol{d}/m}(\mu), \tag{43}$$

where $\text{Mö} : \mathbb{N} \to \mathbb{Q}$ is the Möbius function. As for (37), the inverse of (43) is

$$\bar{DT}_{Q,I}^{\boldsymbol{d}}(\mu) = \sum_{m \geqslant 1,\, m | \boldsymbol{d}} \frac{1}{m^2} \hat{DT}_{Q,I}^{\boldsymbol{d}/m}(\mu). \tag{44}$$

If $W \equiv 0$, so that mod-$\mathbb{C}Q/I =$ mod-$\mathbb{C}Q$, we write $\bar{DT}_{Q}^{\boldsymbol{d}}(\mu), \hat{DT}_{Q}^{\boldsymbol{d}}(\mu)$ for $\bar{DT}_{Q,I}^{\boldsymbol{d}}(\mu), \hat{DT}_{Q,I}^{\boldsymbol{d}}(\mu)$. Note that $\mu \equiv 0$ is allowed as a slope stability condition, with every object in mod-$\mathbb{C}Q/I$ 0-semistable, and is a natural choice. So we have invariants $\bar{DT}_{Q,I}^{\boldsymbol{d}}(0), \hat{DT}_{Q,I}^{\boldsymbol{d}}(0)$ and $\bar{DT}_{Q}^{\boldsymbol{d}}(0), \hat{DT}_{Q}^{\boldsymbol{d}}(0)$.

Here is the analogue of the integrality conjecture, Conjecture 4.16.

CONJECTURE 5.9. *Call $(\mu, \mathbb{R}, \leqslant)$ **generic** if for all $\boldsymbol{d}, \boldsymbol{e} \in C(\text{mod-}\mathbb{C}Q/I)$ with $\mu(\boldsymbol{d}) = \mu(\boldsymbol{e})$ we have $\bar{\chi}(\boldsymbol{d}, \boldsymbol{e}) = 0$. If $(\mu, \mathbb{R}, \leqslant)$ is generic, then $\hat{DT}_{Q,I}^{\boldsymbol{d}}(\mu) \in \mathbb{Z}$ for all $\boldsymbol{d} \in C(\text{mod-}\mathbb{C}Q/I)$.*

In [**16**, Th. 7.29] we prove Conjecture 5.9 when $W \equiv 0$, using results of Reineke [**31**]. That is, if μ is generic we show $\hat{DT}_{Q}^{\boldsymbol{d}}(\mu) \in \mathbb{Z}$ for all $\boldsymbol{d}$. In [**16**, Th. 7.17] we prove an analogue of Theorem 4.6. It holds for arbitrary $\mu, \tilde{\mu}$, without requiring extra technical conditions as in Theorem 3.12.

THEOREM 5.10. *Let $(\mu, \mathbb{R}, \leqslant)$ and $(\tilde{\mu}, \mathbb{R}, \leqslant)$ be any two slope stability conditions on* mod-$\mathbb{C}Q/I$, *and $\bar{\chi}$ be as in* (40). *Then for all $\boldsymbol{d} \in C$*

(mod-$\mathbb{C}Q/I$) *we have*

$$\bar{DT}{}^{\boldsymbol{d}}_{Q,I}(\tilde\mu)=\sum_{\substack{\text{iso. classes}\\ \text{of finite}\\ \text{sets } I}}\;\sum_{\substack{\kappa:I\to C(\text{mod-}\mathbb{C}Q/I):\\ \sum_{i\in I}\kappa(i)=\boldsymbol{d}}}\;\sum_{\substack{\text{connected,}\\ \text{simply-}\\ \text{connected}\\ \text{digraphs }\Gamma,\\ \text{vertices } I}} \begin{aligned}[t] &(-1)^{|I|-1}V(I,\Gamma,\kappa;\mu,\tilde\mu)\cdot\textstyle\prod_{i\in I}\bar{DT}{}^{\kappa(i)}_{Q,I}(\mu)\\ &\cdot(-1)^{\frac{1}{2}\sum_{i,j\in I}|\bar\chi(\kappa(i),\kappa(j))|}\cdot\prod_{\text{edges } \overset{i}{\bullet}\to\overset{j}{\bullet} \text{ in } \Gamma}\bar\chi(\kappa(i),\kappa(j)),\end{aligned}$$

with only finitely many nonzero terms.

5.3. Pair invariants for quivers. We now discuss analogues for quivers of the moduli spaces of stable pairs $\mathcal{M}^{\alpha,n}_{\rm stp}(\tau')$ and stable pair invariants $PI^{\alpha,n}(\tau')$ in §4.3, and the identity (35) in Theorem 4.12 relating $PI^{\alpha,n}(\tau')$ and the $\bar{DT}{}^{\beta}(\tau)$.

Definition 5.11. Let Q be a quiver with relations I coming from a superpotential W on Q over $\mathbb{C}$. Suppose $(\mu,\mathbb{R},\leqslant)$ is a slope stability condition on mod-$\mathbb{C}Q/I$, as in Example 5.4.

Let $\boldsymbol{d},\boldsymbol{e}\in\mathbb{Z}^{Q_0}_{\geqslant 0}$ be dimension vectors. A *framed representation* (E,σ) *of* (Q,I) *of type* $(\boldsymbol{d},\boldsymbol{e})$ consists of a representation E of mod-$\mathbb{C}Q/I$ with $\mathbf{dim}\,E=\boldsymbol{d}$, together with linear maps $\sigma_v:\mathbb{C}^{\boldsymbol{e}(v)}\to i_v(E)$ for all $v\in Q_0$. We call a framed representation (E,σ) *stable* if

(i) $\mu([E'])\leqslant\mu([E])$ for all subobjects $0\neq E'\subset E$ in mod-$\mathbb{C}Q/I$; and
(ii) If also σ factors through E', that is, $\sigma_v(\mathbb{C}^{e(v)})\subseteq i_v(E')\subseteq i_v(E)$ for all $v\in Q_0$, then $\mu([E'])<\mu([E])$.

We will use μ' *to denote stability of framed representations*, defined using μ.

Following Engel and Reineke [**5**, §3] or Szendrői [**32**, §1.2], we can in a standard way define moduli problems for all framed representations, and for stable framed representations. The moduli space of all framed representations of type $(\boldsymbol{d},\boldsymbol{e})$ is an Artin $\mathbb{C}$-stack $\mathfrak{M}^{\boldsymbol{d},\boldsymbol{e}}_{{\rm fr}\,Q,I}$ with an explicit description similar to (42), and the moduli space of stable framed representations of type $(\boldsymbol{d},\boldsymbol{e})$ is a fine moduli $\mathbb{C}$-scheme $\mathcal{M}^{\boldsymbol{d},\boldsymbol{e}}_{{\rm stf}\,Q,I}(\mu')$, an open $\mathbb{C}$-substack of $\mathfrak{M}^{\boldsymbol{d},\boldsymbol{e}}_{{\rm fr}\,Q,I}$.

We can now define our analogues of invariants $PI^{\alpha,n}(\tau')$ for quivers.

Definition 5.12. In the situation above, define

$$NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')=\chi\bigl(\mathcal{M}^{\boldsymbol{d},\boldsymbol{e}}_{{\rm stf}\,Q,I}(\mu'),\nu_{\mathcal{M}^{\boldsymbol{d},\boldsymbol{e}}_{{\rm stf}\,Q,I}(\mu')}\bigr). \tag{45}$$

When $W\equiv 0$, so that mod-$\mathbb{C}Q/I=$ mod-$\mathbb{C}Q$, we also write $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q}(\mu')=NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$. Following Szendrői [**32**] we call $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu'),NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q}(\mu')$ *noncommutative Donaldson–Thomas invariants.*

Here (45) is the analogue of (34) in the sheaf case. We have no analogue of (33), since in general $\mathcal{M}^{\boldsymbol{d},\boldsymbol{e}}_{\mathrm{stf}\,Q,I}(\mu')$ is not proper, and so does not have a fundamental class. These quiver analogues of $\mathcal{M}^{\alpha,n}_{\mathrm{stp}}(\tau'), PI^{\alpha,n}(\tau')$ are not new, similar things have been studied in quiver theory by Nakajima, Reineke, Szendrői and other authors for some years [**5**, **24**–**27**, **29**, **30**, **32**]. Here [**16**, Th. 7.23] is the analogue of Theorem 4.12 for quivers.

THEOREM 5.13. *Suppose Q is a quiver with relations I coming from a minimal superpotential W on Q over $\mathbb{C}$. Let $(\mu,\mathbb{R},\leqslant)$ be a slope stability condition on* mod-$\mathbb{C}Q/I$, *as in Example* 5.4, *and $\bar\chi$ be as in* (40). *Then for all $\boldsymbol{d},\boldsymbol{e}$ in $C(\text{mod-}\mathbb{C}Q/I)=\mathbb{Z}^{Q_0}_{\geqslant 0}\setminus\{0\}\subset\mathbb{Z}^{Q_0}$, we have*

$$NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')=\sum_{\substack{\boldsymbol{d}_1,\dots,\boldsymbol{d}_l\in C(\text{mod-}\mathbb{C}Q/I),\\ l\geqslant 1:\ \boldsymbol{d}_1+\cdots+\boldsymbol{d}_l=\boldsymbol{d},\\ \mu(\boldsymbol{d}_i)=\mu(\boldsymbol{d}),\ \text{all } i}}\frac{(-1)^l}{l!}\prod_{i=1}^{l}\bigl[(-1)^{\boldsymbol{e}\cdot\boldsymbol{d}_i-\bar\chi(\boldsymbol{d}_1+\cdots+\boldsymbol{d}_{i-1},\boldsymbol{d}_i)}\bigl(\boldsymbol{e}\cdot\boldsymbol{d}_i-\bar\chi(\boldsymbol{d}_1+\cdots+\boldsymbol{d}_{i-1},\boldsymbol{d}_i)\bigr)\bar{DT}^{\boldsymbol{d}_i}_{Q,I}(\mu)\bigr], \tag{46}$$

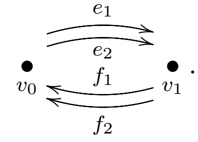
with $\boldsymbol{e}\cdot\boldsymbol{d}_i=\sum_{v\in Q_0}\boldsymbol{e}(v)\boldsymbol{d}_i(v)$, and $\bar{DT}^{\boldsymbol{d}_i}_{Q,I}(\mu), NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$ as in Definitions 5.8, 5.12. *When $W\equiv 0$, the same equation holds for $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q}(\mu'), \bar{DT}^{\boldsymbol{d}}_{Q}(\mu)$.*

For Donaldson–Thomas invariants in §4, we regarded the invariants $\bar{DT}^{\alpha}(\tau)$, $\hat{DT}^{\alpha}(\tau)$ as our primary objects of study, and the pair invariants $PI^{\alpha,n}(\tau')$ as secondary, not of that much interest in themselves. In contrast, in the quiver literature to date the invariants $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu), \bar{DT}^{\boldsymbol{d}}_{Q}(\mu)$ and $\hat{DT}^{\boldsymbol{d}}_{Q,I}(\mu), \hat{DT}^{\boldsymbol{d}}_{Q}(\mu)$ have not been seriously considered even in the stable = semistable case, and the analogues $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu'), NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q}(\mu')$ of pair invariants $PI^{\alpha,n}(\tau')$ have been the central object of study.

We argue that the invariants $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu),\dots,\hat{DT}^{\boldsymbol{d}}_{Q}(\mu)$ should actually be regarded as more fundamental and more interesting than the $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$. By (46) the $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$ can be written in terms of the $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$, and hence by (44) in terms of the $\hat{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$, so the pair invariants contain no more information. The $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$ are simpler than the $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$ as they depend only on $\boldsymbol{d}$ rather than on $\boldsymbol{d},\boldsymbol{e}$. In examples in [**16**, §7.5–§7.6] we find that the values of the $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$ and especially of the $\hat{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$ may be much simpler and more illuminating than the values of the $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$, as in (47)–(49) below.

Here is an example taken from [**16**, §7.5.2].

EXAMPLE 5.14. Following Szendrői [**32**, §2.1], let $Q=(Q_0,Q_1,h,t)$ have two vertices $Q_0=\{v_0,v_1\}$ and edges $e_1,e_2:v_0\to v_1$ and $f_1,f_2:v_1\to v_0$, as below:



Define a superpotential W on Q by $W = e_1 f_1 e_2 f_2 - e_1 f_2 e_2 f_1$, and let I be the associated relations. Then mod-$\mathbb{C}Q/I$ is a 3-Calabi–Yau abelian category. Theorem 5.3 shows that the Euler form $\bar{\chi}$ on mod-$\mathbb{C}Q/I$ is zero.

Write elements $\boldsymbol{d}$ of $C(\text{mod-}\mathbb{C}Q/I)$ as (d_0, d_1) where $d_j = \boldsymbol{d}(v_j)$. Szendrői [**32**, Th. 2.7.1] computed the noncommutative Donaldson–Thomas invariants $NDT_{Q,I}^{(d_0,d_1),(1,0)}(0')$ for mod-$\mathbb{C}Q/I$ as combinatorial sums, and using work of Young [**34**] wrote their generating function as a product [**32**, Th. 2.7.2], giving

$$\begin{aligned} 1 + \sum_{(0,0)\neq(d_0,d_1)\in\mathbb{N}^2} & NDT_{Q,I}^{(d_0,d_1),(1,0)}(0') q_0^{d_0} q_1^{d_1} \\ = \prod_{k\geqslant 1} & \bigl(1-(-q_0q_1)^k\bigr)^{-2k}\bigl(1-(-q_0)^k q_1^{k-1}\bigr)^k\bigl(1-(-q_0)^k q_1^{k+1}\bigr)^k. \end{aligned} \tag{47}$$

Computing using (46) and (47) shows that

$$\bar{DT}_{Q,I}^{(d_0,d_1)}(0) = \begin{cases} -2\sum_{l\geqslant 1,\, l|d} \frac{1}{l^2}, & d_0 = d_1 = d \geqslant 1, \\ \frac{1}{l^2}, & d_0 = kl,\ d_1 = (k-1)l,\ k,l \geqslant 1, \\ \frac{1}{l^2}, & d_0 = kl,\ d_1 = (k+1)l,\ k \geqslant 0,\ l \geqslant 1, \\ 0, & \text{otherwise.} \end{cases} \tag{48}$$

Combining (44) and (48) we see that

$$\hat{DT}_{Q,I}^{(d_0,d_1)}(0) = \begin{cases} -2, & (d_0,d_1) = (k,k),\ k \geqslant 1, \\ 1, & (d_0,d_1) = (k,k-1),\ k \geqslant 1, \\ 1, & (d_0,d_1) = (k-1,k),\ k \geqslant 1, \\ 0, & \text{otherwise.} \end{cases} \tag{49}$$

Note that the values of the $\hat{DT}_{Q,I}^{(d_0,d_1)}(0)$ in (49) lie in $\mathbb{Z}$, as in Conjecture 5.9, and are far simpler than those of the $NDT_{Q,I}^{(d_0,d_1),(1,0)}(0')$ in (47).

This example is connected to Donaldson–Thomas theory for (noncompact) Calabi–Yau 3-folds as follows. We have equivalences of derived categories

$$D^b(\text{mod-}\mathbb{C}Q/I) \sim D^b(\text{coh}_{\text{cs}}(X)) \sim D^b(\text{coh}_{\text{cs}}(X_+)), \tag{50}$$

where $\pi : X \to Y$ and $\pi_+ : X_+ \to Y$ are the two crepant resolutions of the conifold $Y = \bigl\{(z_1,z_2,z_3,z_4) \in \mathbb{C}^4 : z_1^2 + \cdots + z_4^2 = 0\bigr\}$, and X, X_+ are related by a flop. Here X, X_+ are regarded as 'commutative' crepant resolutions of Y, and mod-$\mathbb{C}Q/I$ as a 'noncommutative' resolution of Y, in the sense that mod-$\mathbb{C}Q/I$ can be regarded as the coherent sheaves on the 'noncommutative scheme' $\text{Spec}(\mathbb{C}Q/I)$ constructed from the noncommutative $\mathbb{C}$-algebra $\mathbb{C}Q/I$.

One idea in [**32**] is that counting invariants $NDT^{\boldsymbol{d},\boldsymbol{e}}_{Q,I}(\mu')$ for mod-$\mathbb{C}Q/I$ should be related to Donaldson–Thomas type invariants counting sheaves on X, X_+ by some kind of wall-crossing formula under change of stability condition in the derived categories, using the equivalences (50). This picture has been worked out further by Nagao and Nakajima [**25**,**26**]. In [**16**, §7.5.2] we show that in this case the situation for invariants $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu), \hat{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$ is actually much simpler, because they are unchanged by wall-crossing as $\bar\chi \equiv 0$, so we can identify the invariants $\bar{DT}^{\boldsymbol{d}}_{Q,I}(\mu), \hat{DT}^{\boldsymbol{d}}_{Q,I}(\mu)$ in (48)–(49) directly with Donaldson–Thomas invariants for X and X_+.

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