# Relativistic Teichmüller Theory – A Hamilton-Jacobi Approach to 2 + 1-Dimensional Einstein Gravity

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Abstract. We consider vacuum spacetimes in $2+1$ dimensions defined on manifolds of the form $M = \Sigma \times \mathbf{R}$ where $\Sigma$ is a compact, orientable surface of genus $> 1$. By exploiting the harmonicity properties of the Gauss map for an arbitrary constant mean curvature (CMC) slice in such a spacetime we relate the Hamiltonian dynamics of the corresponding reduced Einstein equations to some fundamental results in the Teichmüller theory of harmonic maps. In particular we show, expanding upon an argument sketched by Puzio, that a global complete solution to the Hamilton-Jacobi equation for the reduced Einstein equations can be expressed in terms of the Dirichlet energy for harmonic maps defined over the surface $\Sigma$. While in principle this complete solution to the Hamilton-Jacobi equation determines all the solution curves to the reduced Einstein equations, one can derive a more explicit characterization of these curves through the solution of an associated (parametrized) Monge-Ampêre equation. Using the latter we define a corresponding family of "ray structures" on the Teichmüller space of the chosen 2-manifold $\Sigma$. These ray structures are similar but complementary to a different family of such ray structures defined by M. Wolf and we herein derive a "relativistic interpretation" of both sets. We also use our Hamilton-Jacobi results to define complementary families of Lagrangian foliations of the cotangent bundle of the Teichmüller space of $\Sigma$ and to provide the corresponding "relativistic interpretation" of the leaves of these foliations.

## 1. Introduction

If Einstein's vacuum field equations are formulated for 3-dimensional Lorentzian metrics on manifolds of the form $M = \Sigma \times \mathbf{R}$, where $\Sigma$ is a compact surface, then it is hardly surprising to find the Teichmüller space of $\Sigma$

playing an important role in the analysis. Indeed, it can be shown from several independent points of view [**1, 2, 3**] that this Teichmüller space, $\mathcal{T}(\Sigma)$, serves as the natural reduced configuration space and its cotangent bundle $T^*\mathcal{T}(\Sigma)$ the natural reduced phase space for Einstein's equations treated as a Hamiltonian dynamical system. That this system is only finite dimensional, in contrast to the situation for higher dimensional spacetimes, is an immediate consequence of the fact that a vanishing Einstein tensor (i.e., the vacuum condition) implies a vanishing Riemann tensor in 3 dimensions and hence the absence of any local degrees of freedom for the gravitational field. Only certain global degrees of freedom remain and these can be identified with the Teichmüller parameters describing the conformal geometry induced on $\Sigma$ by the spacetime metric at any given "time". The evolution of these Teichmüller parameters as one sweeps through the leaves of a foliation of the spacetime by suitably chosen (hyper-) surfaces is the finite dimensional dynamical system that we are interested in.

In higher dimensions, the vanishing of the Einstein tensor leaves open the possibility of non-vanishing curvature and in fact, one can show that this latter tensor satisfies a hyperbolic equation whose (non-stationary) solutions can be thought of as describing gravitational waves. Here, too, it is possible to show that the conformal geometry induced on the leaves of a foliation by Cauchy hypersurfaces provides the natural reduced configuration degrees of freedom but, in contrast to the 3-dimensional case, the associated Teichmüller-like space of conformal structures is always infinite dimensional (to accommodate the gravitational waves) and the corresponding reduced field equations take the form of a hyperbolic/elliptic system of partial differential equations [**4, 5**]. Only in 3 dimensions (the lowest, nontrivial possibility) do the Einstein field equations reduce (after solution of the elliptic constraints and imposition of suitable coordinate gauge conditions) to ordinary differential equations unless some additional restriction, such as spatial homogeniety, is imposed upon the higher dimensional metrics under study.

In this article, we are primarily interested in studying the reduced Einstein equations in so-called CMCSH (constant-mean-curvature-spatially-harmonic) gauge which is defined by the requirements that the level surfaces of a suitably chosen time function, which are each Cauchy (hyper-) surfaces diffeomorphic to $\Sigma$, satisfy the CMC (constant-mean-curvature) condition and that the induced Riemannian metric $g$ on each such Cauchy surface is such that the identity map from $(\Sigma, g)$ to $(\Sigma, \tilde{g})$, for some conveniently chosen target metric $\tilde{g}$ is harmonic. This latter condition is well-known to depend only upon the conformal class of the domain metric $g$ and hence only upon the Teichmüller parameters of this slice dependent variable. The reduced Einstein equations, which can be expressed in Hamiltonian form, give the evolution of these Teichmüller parameters together with their canonically conjugate momenta (which, taken together, provide coordinates for $T^*\mathcal{T}(\Sigma)$).

One of our main results is to show that the foliation of such vacuum metrics on $\Sigma\times\mathbf{R}$ by CMC slices is globally determined through the solution of an associated (fully non-linear) elliptic equation of Monge-Ampêre type which depends parametrically upon the mean curvature variable $\tau$ (which plays the role of "time") and upon the choice of an arbitrary point of $T^*\mathcal{T}(\Sigma)$ which can be thought of as an asymptotic data point in the reduced phase space. By exploiting the method of continuity, we prove that every solution of this Monge-Ampêre equation (each of which is fixed by prescribing asymptotic data at $\tau=0$) extends globally to a solution for all $\tau$ in the interval $(-\infty,0]$. The limit $\tau\searrow-\infty$ corresponds to a big bang singularity at which the geometric area of $\Sigma$ tends to zero and for which, generically, the corresponding solution curve runs off-the-edge of Teichmüller space. The opposite limit $\tau\nearrow 0$ corresponds to that of infinite cosmological expansion for which the geometric area of $\Sigma$ blows up but for which the induced conformal geometry always asymptotes to an interior point of Teichmüller space (which together with an associated asymptotic "velocity" is determined by the chosen point of $T^*\mathcal{T}(\Sigma)$). It is known from earlier work that the range $(-\infty,0)$ always exhausts the maximal Cauchy development for each vacuum solution [**6**].

A closely related result that we shall derive shows how the Dirichlet energy for a suitably defined harmonic map (the Gauss Map for a CMC slice of the associated, flat spacetime) can be exploited to yield a global, complete solution to the Hamilton-Jacobi equation for the Hamiltonian system defined by the reduced field equations. A sketch of how to relate the Dirichlet energy for the Gauss map to a complete solution for the Hamilton-Jacobi equation was given earlier by Puzio [**7**]. To make his insight more precise, we fill in some of the details that were not provided in Puzio's argument and, in particular, show how the partial derivatives of the Dirichlet energy (with respect to the Teichmüller parameters) are related to the momenta of the reduced Hamiltonian formalism.

The complete solution to the reduced Einstein-Hamilton-Jacobi equation that we obtain, allows us to define a set of "ray-structures" on Teichmüller space that are similar but complementary to the well-known ray structures defined by Michael Wolf [**8**]. In our formulation each ray corresponds to (the projection of) a solution curve of the reduced Einstein equations and the collection of such curves yielding a particular ray structure corresponds to the collection of all those curves having the same asymptotic conformal geometry as $\tau\nearrow 0$. By varying this target point over the interior of Teichmüller space one obtains all of the ray structures defined by the complete solution to the Hamilton-Jacobi equation. By contrast to this, one can also give a relativistic interpretation to Wolf's rays but each such ray corresponds to the locus of endpoints defined by a one-parameter family of solutions to Einstein's equations defined by fixing (at say $\tau=-1$), the conformal geometry, but scaling up the traceless part of the second fundamental form (a holomorphic quadratic differential in Wolf's terminology) by a (spatially constant) multiplicative factor.

Wolf was able to prove that (holding the domain conformal geometry fixed) he could define a global chart for the (topologically trivial) Teichmüller space of $\Sigma$ by varying the holomorphic quadratic differential over the full vector space of such objects. In a complementary way here, we are able to exploit the Monge-Ampêre analysis mentioned above to define a family of global charts for Teichmüller space in each of which the target conformal geometry is held fixed and one varies a holomorphic quadratic differential defined relative to this fixed target. Wolf gets a family of such charts by varying the domain conformal geometry whereas we get a family by varying the target.

Our approach to this latter problem is quite different from that of B. Tabak [**9**]. She also holds the target (of a family of harmonic maps) fixed but exploits the properties of so-called subsonic $\varrho$-holomorphic quadratic differentials to develop a family of global charts for Teichmüller space. The concept of subsonic $\varrho$-holomorphic quadratic differentials was introduced by L.M. Sibner and R.J. Sibner in connection with a certain hydrodynamics problem wherein they showed that these objects were expressible in terms of a certain non-linear generalization of harmonic one-forms on a compact manifold [**10**]. Since the dimension of the space of such generalized harmonic one-forms coincides with that given by Hodge theory (the first Betti number of the manifold) when these objects are non-singular it is necessary to allow forms with certain well-defined singularities in order to match the correct dimension of Teichmüller space in the higher genus cases for surfaces. Tabak gives a careful study of such singularities that need be allowed. By contrast though, our approach only requires globally regular holomorphic quadratic differentials of the conventional type, in close parallel to Wolf's treatment.

Hamilton-Jacobi theory is closely connected to the construction of Lagrangian foliations of the associated phase space, in our case $T^*\mathcal{T}(\Sigma)$ the cotangent bundle of Teichmüller space. Since we are in the ideal situation of having a global, complete solution to the Hamilton-Jacobi equation, we can exploit this close connection to define two (one-parameter families of) global Lagrangian foliations of $T^*\mathcal{T}(\Sigma)$ and to give the leaves of these foliations a "natural" interpretation in terms of corresponding families of solutions to Einstein's equations.

## 2. Preliminary Computations

Let $\Sigma$ be a compact connected, orientable surface of genus $>1$ and set $M = \Sigma \times \mathbf{R}$. Relative to a "time" function $t$ defined on $M$ whose level surfaces are diffeomorphic to $\Sigma$ we can express Lorentzian metrics on $M$ in the Arnowitt, Deser and Misner (ADM) form

$$\begin{aligned} ds^2 &= {}^{(3)}g_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + g_{ab}(dx^a + X^a dt)(dx^b + X^b dt). \end{aligned} \tag{2.1}$$

Here $\mu, \nu \dots$ range over $\{0,1,2\}$, where $x^0 = t$ is the time, and $a, b, \dots$ range over $\{1,2\}$ where $\{x^1, x^2\}$ are the spatial coordinates. Induced upon each level surface of $t$ is a Riemannian metric $g_t$ (the first fundamental form) given by

$$g_t = g_{ab} dx^a \otimes dx^b \tag{2.2}$$

where here and below we suppress the spacetime coordinate dependence of component expressions such as $g_{ab}$ to simplify the notation. $N$ is a positive function on $M$ (the "lapse") and $X^a \frac{\partial}{\partial x^a}$ is an (in general $t$-dependent) vector field tangent to the level surfaces of $t$ (the "shift").

The covariant derivative of the unit normal field ("future" directed towards increasing $t$) to the surfaces of constant $t$ determines, in the usual way, another symmetric two-tensor $k_t$ (the second fundamental form) on each such surface which we shall write in component form as

$$k_t = k_{ab} dx^a \otimes dx^b. \tag{2.3}$$

Writing $\mu_g = \sqrt{\det g_{ab}}$ for the area element of $g_t$, we define the gravitational momentum $\pi_t$ (a symmetric, contravariant, two-tensor density with components $\pi^{ab}$) by

$$\pi^{ab} = -\mu_g (k^{ab} - g^{ab} tr_g k) \tag{2.4}$$

where $g^{ab} = (g_t^{-1})^{ab}$ are the components of the inverse metric to $g_t$, two-dimensional indices are raised and lowered using $g_t$ and $g_t^{-1}$ and where $tr_g k = g^{ab} k_{ab}$, the trace of $k_t$. This latter quantity, the mean curvature of the $t =$ constant hypersurfaces, will play an important role in what follows and we shall often designate it by the symbol $\tau$. Thus, from the formulas above

$$\tau := tr_g k = g^{ab} k_{ab} = \frac{tr_g \pi}{\mu_g} = \frac{g_{ab}\pi^{ab}}{\mu_g}. \tag{2.5}$$

The ADM action for Einstein's equations is given by

$$I_{ADM} = \int_{\mathcal{I}\times\Sigma} d^3x \{\pi^{ab} g_{ab,t} - N\mathcal{H} - X^a J_a\} \tag{2.6}$$

where $\mathcal{I} = [t_0, t_1] \subset \mathbf{R}$ is an arbitrary closed interval and where

$$\mathcal{H} = \mathcal{H}(g,\pi) = \frac{1}{\mu_g}(\pi^{ab}\pi_{ab} - (tr_g \pi)^2) - \mu_g^{(2)} R(g) \tag{2.7a}$$

$$J_a = J_a(g,\pi) = -2\pi_a^b|_b = -2^{(2)}\nabla_b \pi_a^b. \tag{2.7b}$$

Here ${}^{(2)}R(g)$ is the scalar curvature of the Riemannian metric $g$ and $|$ or ${}^{(2)}\nabla$ designates covariant differentiation with respect to this metric.

Variation of $I_{ADM}$ with respect to the lapse $N$ and shift $X^a$ yields the Einstein constraint equations

$$\mathcal{H}(g,\pi) = 0, \quad J_a(g,\pi) = 0 \tag{2.8}$$

whereas variation with respect to $g$ and $\pi$ yield evolution equations (in Hamiltonian form) for these "canonical" variables. There are no equations determining $N$ and $X^a$ and these quantities can be specified freely to determine a coordinate system on the developing spacetime. Two choices which we shall make use of are the gaussian normal choice $N=1, X^a=0$ (in which the spatial coordinates are held constant along the normal geodesics from an initial slice and $t$ is the metrically determined proper time along those geodesics) and another in which the hypersurfaces of constant $t$ are required to each have constant mean curvature and the spatial coordinates are required to satisfy a harmonic condition defined below. Imposing these conditions upon the spacetime coordinates leads to a system of linear elliptic equations for $N$ and $X^a$ which determines these quantities uniquely in terms of the remaining (canonical) data $\{g_t, \pi_t\}$. The Bianchi identities ensure that the constraints (2.8) are conserved by the evolution equations in essentially an arbitrary coordinate gauge.

The Einstein evolution equations for vacuum $2+1$ gravity simplify greatly when expressed in gaussian normal coordinates (gnc) and in fact reduce to a decoupled system of ordinary differential equations for $\{g_{ab}, \pi^{ab}\}$ along each normal geodesic in the evolving flat spacetime. Insofar as these geodesics are initially diverging the absence of spacetime curvature (which results from the fact that vanishing Einstein tensor implies vanishing Riemann tensor in 3 dimensions) ensures that these "straight lines" will never cross to the future of the initial surface and thus that the gnc coordinate system will never break down in this temporal direction (though in general it does break down in the opposite direction). As we shall show later by analyzing the constraint equations in detail, the future directed normals to a constant mean curvature hypersurface having $\tau =$ constant $<0$ (and where "future" designates the direction of increasing $\tau$) are always diverging and thus the gnc coordinate systems developed from such an initial surface cover the entire spacetime to the future of this surface. Indeed, by exploiting the simplified form of the evolution equations, one can construct the spacetime metric essentially explicitly in gaussian normal coordinates.

To see this, set $N=1$ and $X^a=0$ in the evolution equations and derive easily that

$$\begin{aligned} g_{ab,t} &= \frac{2}{\mu_g}(\pi_{ab} - g_{ab} tr_g \pi) \\ \partial_t \left(\pi_a^b - \frac{1}{2}\delta_a^b tr_g \pi\right) &= 0 \\ \partial_t (tr_g \pi) &= \frac{\pi_c^d \pi_d^c - (tr_g \pi)^2}{\mu_g} := \mathcal{K} \end{aligned} \tag{2.9}$$

from which follow

$$\partial_t \mu_g = -tr_g \pi, \tag{2.10}$$

$$\partial_t \mathcal{K} = \frac{\partial}{\partial t}\left(\frac{\pi^d_c \pi^c_d - (tr_g \pi)^2}{\mu_g}\right) = 0,$$

and, upon using the momentum constraint,

$$\partial_t (\mu_g {}^{(2)}R(g)) = 0. \tag{2.11}$$

Taking, with no real loss of generality $t = 0$ on the initial (CMC) surface, one sees immediately that $\mathcal{K} = \overset{o}{\mathcal{K}} = \mathcal{K}|_{t=0}$ and finds by straightforward integration that

$$\mu_g(t) = \overset{o}{\mu}_g - \left\{ \overset{o}{tr}_g \pi t + \frac{1}{2}\overset{o}{\mathcal{K}} t^2 \right\} \tag{2.12}$$

$$tr_g \pi(t) = \overset{o}{tr}_g \pi + \overset{o}{\mathcal{K}} t$$

where $\overset{o}{tr}_g \pi := \mathrm{tr}_g \pi|_{t=0}, \overset{o}{\mu}_g = \mu_g|_{t=0}$. The same evolution equations give $\frac{\partial}{\partial t} J_a(g, \pi) = 0$, and, if $J_a = 0$, that $\partial_t \mathcal{H}(g, \pi) = 0$ as well. In fact, as noted above, one has separately that $\partial_t(\mu_g {}^{(2)}R(g)) = 0$ and $\partial_t \mathcal{K} = 0$ where $\mathcal{H} = \mathcal{K} - \mu_g {}^{(2)}R(g)$. Clearly, Eq. (2.9b) also gives conservation of the traceless part of $\pi^b_a$ (i.e., that $\pi^b_a - \frac{1}{2}\delta^b_a tr_g \pi = (\pi^b_a - \frac{1}{2}\delta^b_a tr_g \pi)|_{t=0}$).

Knowing the solution for $\mu_g$, $tr_g \pi$ and $\pi^b_a - \frac{1}{2}\delta^b_a tr_g \pi$ we need only solve for the (conformally invariant) density $\mu_g g^{ab}$ which satisfies the differential equation

$$\partial_t(\mu_g g^{ab}) = -2(\mu_g g^{ac})\left(\frac{\pi^b_c - \frac{1}{2}\delta^b_c tr_g \pi}{\mu_g}\right). \tag{2.13}$$

The details of this solution are straightforward but uninteresting so we here give only the result that

(2.14)

$$\mu_g g^{ab}(t) = \cosh(R(t))(\mu_g g^{ab})\Big|_{t=0} + \sinh(R(t))\left(\frac{\pi^{ab} - \frac{1}{2}g^{ab} tr_g \pi}{\sqrt{\frac{1}{2}\frac{\lambda^d_c \lambda^c_d}{(\mu_g)^2}}}\right)\Bigg|_{t=0}$$

where

$$\lambda^b_a = \pi^b_a - \frac{1}{2}\delta^b_a tr_g \pi \tag{2.15}$$

and

$$e^{R(t)} = \left(\frac{2ct + b + \sqrt{-q}}{2ct + b - \sqrt{-q}}\right)\left(\frac{b - \sqrt{-q}}{b + \sqrt{-q}}\right) \tag{2.16}$$

with

$$b = -\left(\frac{tr_g\pi}{\mu_g}\right)\Bigg|_{t=0} = -\tau\Bigg|_{t=0} \tag{2.17}$$
$$c = -\frac{1}{2}\left(\frac{\mathcal{K}}{\mu_g}\right)\Bigg|_{t=0}$$
$$q = -2\left(\frac{\lambda_c^d\lambda_d^c}{(\mu_g)^2}\right)\Bigg|_{t=0}$$

and where, if we also assume that $\tau|_{t=0} = \text{constant}$ on the initial surface $\lambda_a^b$ has zero divergence with respect to $\overset{o}{g}_{ab} = g_{ab}|_{t=0}$ as well as zero trace.

It is straightforward to verify using the explicit formulas above, that $g_{ab}(t)$ is a smooth Riemannian metric $\forall\ t \geq 0$ provided that the initial data satisfies the conditions that

(i) $g_{ab}\big|_{t=0}$ is smooth and Riemannian on $\Sigma$,
(ii) $\tau\big|_{t=0}$ is smooth with $\tau\big|_{t=0} < 0$ on $\Sigma$,
(iii) $\mathcal{K}\big|_{t=0}$ is smooth with $\mathcal{K}|_{t=0} < 0$ on $\Sigma$.

We shall later impose the restriction that $\tau|_{t=0}$ be a negative constant on $\Sigma$ and find that the constraint equations force condition (iii) to hold provided the remaining Cauchy data $\lambda_a^b$ is also smooth. Note that the zeros of $\lambda_a^b$ do not disturb the smoothness of $\mu_g g^{ab}(t)$ since the factor $\sinh(R(t))$ vanishes whenever $\lambda_a^b$ does.

The gaussian normal slicing is in general not a CMC slicing (except when $\tau|_{t=0} = \text{constant}$ and $\lambda_a^b = 0$) but one nevertheless has

$$t\tau(t) = -2\frac{\left\{\frac{\mathcal{K}}{\mu_g}\Big|_{t=0} + \frac{1}{t}\tau\Big|_{t=0}\right\}}{\left\{\frac{\mathcal{K}}{\mu_g}\Big|_{t=0} + \frac{2}{t}\tau\Big|_{t=0} - \frac{2}{t^2}\right\}} \tag{2.18}$$
$$= -2 + O\left(\frac{1}{t}\right).$$

Of even more interest to us is the limiting behavior of the corresponding rescaled metric, $\frac{2}{t^2}g_{ab}(t)$. This metric also has a limit (as a Riemannian metric on $\Sigma$) as $t \to \infty$. Explicit computation gives

$$\rho_{ab} := \lim_{t\to\infty}\left(\frac{2}{t^2}g_{ab}\right) \tag{2.19}$$
$$= \left\{(k_c^d k_d^c)g_{ab} + 2(tr_g k)\left(k_{ab} - \frac{1}{2}g_{ab}(tr_g k)\right)\right\}\Big|_{t=0}$$

which is clearly smooth and symmetric; positive definiteness depends upon the condition that $\mathcal{K}\Big|_{t=0} < 0$ on $\Sigma$. As we have mentioned, this will be shown to follow from the constraint equations at least when the initial data surface is CMC.

In fact, when the constraint equations (2.8) hold for the initial data $\{g_{ab}, k_{ab}\}\big|_{t=0}$ the metric $\rho_{ab}$ actually has constant (negative) curvature, with ${}^{(2)}R(\rho) = -1$. This can be shown by an explicit calculation or, less directly, by the following argument. Conservation of the constraints, taken together with the gnc result that $\partial_t \mathcal{K} = 0$, yields

$$\begin{aligned} {}^{(2)}R\left(\frac{2}{t^2}g(t)\right) &= \frac{1}{2}t^{2\,(2)}R(g(t)) \\ &= \frac{\frac{1}{2}t^2\mathcal{K}\big|_{t=0}}{\left(\mu_g\big|_{t=0} - t(tr_g\pi)\big|_{t=0} - \frac{1}{2}t^2\mathcal{K}\big|_{t=0}\right)} \\ &\underset{t\to\infty}{\longrightarrow} -1 \end{aligned} \tag{2.20}$$

since, once again, $\overset{o}{\mathcal{K}} = \mathcal{K}\big|_{t=0} < 0$ on $\Sigma$. Thus ${}^{(2)}R(\rho) = \lim_{t\to\infty} {}^{(2)}R(\frac{2}{t^2}g(t)) = -1$.

Another remarkable property obtains if we restrict the initial surface to have constant mean curvature. The identity map from $(\Sigma, \overset{o}{g}_{ab}) = (\Sigma, g_{ab}\big|_{t=0})$ to $(\Sigma, \rho_{ab})$ is in fact a harmonic map. This can be shown by explicit computation of the quantity

$$V^c := \overset{o}{g}{}^{ab}(\Gamma^c_{ab}(\overset{o}{g}) - \tilde{\Gamma}^c_{ab}(\rho)) \tag{2.21}$$

where $\Gamma^c_{ab}(\overset{o}{g})$ and $\tilde{\Gamma}^c_{ab}(\rho)$ are the Christoffel connection components of $\overset{o}{g}_{ab}$ and $\rho_{ab}$ respectively. Harmonicity of the identity mapping corresponds precisely to the vanishing of the vector field $V^c$ and this in turn follows from imposition of the constraints and the additional condition that $\overset{o}{\tau} = \tau\big|_{t=0} =$ constant $< 0$ on $\Sigma$.

Again there is a less direct argument which shows why this should be true and which traces back to a well-known result of Ruh and Vilms [**11**] in the purely Riemannian case. One can show that the Gauss map from a CMC hypersurface in a flat spacetime is harmonic [**7**]. Our spacetime metric ${}^{(3)}g_{\mu\nu}$ is flat since the (vacuum) Einstein equations in three dimensions imply that ${}^{(3)}g_{\mu\nu}$ has vanishing curvature and we are now imposing the restriction that the initial slice be CMC. That the identity map from $(\Sigma, \overset{o}{g}_{ab})$ to $(\Sigma, \rho_{ab})$ realizes the Gauss map in this case follows from the construction using (appropriately enough) gaussian normal coordinates as we have done. To each (future directed) normal to the initial surface one assigns a point in the hyperbolic space $(\Sigma, \rho)$ by following the normal geodesic in that direction to its ideal endpoint. We use the starting point and ideal ending point of each such normal geodesic to identify the two copies of $\Sigma$ and appeal to the results given above to recognize that $(\Sigma, \rho)$ is indeed hyperbolic.

The data $\{g_{ab}, k_{ab}\}\big|_{t=0}$ are assumed to satisfy the constraints and to have $\tau\big|_{t=0} = tr_g k\big|_{t=0} =$ constant on $\Sigma$. We can remove this implicit restriction by appealing to the standard (conformal) method for solving the constraints as follows. First, it follows from the classical uniformization theorem that any metric $g_{ab}$ on a higher genus surface $\Sigma$ is uniquely and smoothly globally conformal to another metric $\gamma_{ab}$ which has constant curvature, ${}^{(2)}R(\gamma)=-1$. Writing $g_{ab}=e^{2\lambda}\gamma_{ab}$, for some smooth function $\lambda$ defined on $\Sigma$ and imposing the CMC condition that $\frac{tr_g\pi}{\mu_g}=tr_g k=\tau=$constant on $\Sigma$ one finds that the momentum constraint, $J_a(g,\pi)=-2^{(2)}\nabla_b\pi_a^b=0$, can now be expressed

$$(2.22)\qquad {}^{(2)}\tilde{\nabla}_b(\gamma)\left(\pi_a^b-\frac{1}{2}\delta_a^b tr_g\pi\right)={}^{(2)}\tilde{\nabla}_b(\gamma)\lambda_a^b=0$$

where ${}^{(2)}\tilde{\nabla}_b(\gamma)$ signifies covariant differentiation with respect to the conformal metric $\gamma_{ab}$. In other words, the traceless tensor density $\lambda_a^b$ should also be "transverse" (i.e. divergence free) with respect to the conformal metric $\gamma_{ab}$.

The Hamiltonian constraint, $\mathcal{H}(g,\pi)=0$, can now be expressed as a non-linear elliptic equation (the "Lichnerowicz equation" in relativity literature) for the conformal factor $\lambda$. In the notation above, this equation takes the form

$$(2.23)\qquad \begin{aligned}{}^{(2)}\Delta_\gamma\lambda &= \gamma^{ab(2)}\tilde{\nabla}_a(\gamma)^{(2)}\tilde{\nabla}_b(\gamma)\lambda\\ &=\frac{1}{4}\tau^2e^{2\lambda}-\frac{1}{2}\frac{\lambda_a^b\lambda_b^a}{(\mu_\gamma)^2}e^{-2\lambda}-\frac{1}{2}\end{aligned}$$

where $\mu_\gamma$ is the area element of the conformal metric $\gamma_{ab}$, $\lambda_a^b$ is transverse-traceless with respect to this metric (with $\gamma_{bc}\lambda_a^b$ symmetric) and ${}^{(2)}R(\gamma)=-1$.

Upon integration over $\Sigma$ it is easy to see that Eq. (2.23) has no solutions if $\tau=0$ on $\Sigma$. For any non-zero constant $\tau$ however, one can show (using for example the method of sub and super solutions [**12, 13**]) that Eq. (2.23) always has a unique, smooth, globally defined solution $\lambda$ on $\Sigma$. To summarize, the general solution to the constraint equations for a CMC slice (with $\tau=$constant $\neq 0$) can be expressed in terms of the free data

$$\{(\tau,\gamma_{ab},\lambda_a^b)\big|\tau=\text{const}\neq 0,\gamma_{ab}\text{ a hyperbolic metric on }\Sigma\text{ with }\\ {}^{(2)}R(\gamma)=-1,\lambda_a^b\text{ a }TT\text{ symmetric tensor density w.r.t. }\gamma\}$$

by setting

$$(2.24)\qquad g_{ab}=e^{2\lambda}\gamma_{ab},\ \pi_a^b=\lambda_a^b+\frac{1}{2}\tau\mu_g\delta_a^b$$

where $\lambda$ is the solution of Lichnerowicz's equation (2.23).

Since the constraint equations are naturally covariant with respect to diffeomorphisms of $\Sigma$ (which automatically conserve the constancy of $\tau$) one can, without any essential loss of generality, pass to the quotient, "Teichmüller," space

$$\mathcal{T}(\Sigma) \approx \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \approx \mathbf{R}^{6\ \mathrm{genus}(\Sigma)-6} \tag{2.25}$$

where $\mathcal{M}_{-1}(\Sigma)$ designates the space of Riemannian metrics $\gamma_{ab}$ on $\Sigma$ which have ${}^{(2)}R(\gamma) = -1$ and where $\mathcal{D}_0(\Sigma)$ signifies the group of diffeomorphisms of $\Sigma$ isotopic to the identity. As is known from the work of Eells, Earle and Sampson [**14**], which we shall recall in more detail below, one can represent Teichmüller spaces as a global cross section of the (trivial) $\mathcal{D}_0(\Sigma)$ bundle

$$\mathcal{M}_{-1}(\Sigma) \longrightarrow \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma). \tag{2.26}$$

Such cross sections can be constructed through the use of harmonic maps. Restricting the metric $\gamma_{ab}$ to lie in such a global cross section and recalling that $\lambda_a^b$ is TT (transverse-traceless) with respect to $\gamma_{ab}$ one can regard the space of pairs $\{\gamma_{ab}, \lambda_a^b\}$ as a representation of the cotangent bundle, $T^*\mathcal{T}(\Sigma)$, of the Teichmüller space of $\Sigma$ [**2**].

Noting that

$$\begin{aligned} \frac{g_{bc}}{\mu_g}\lambda_a^c &= \frac{g_{bc}}{\mu_g}\left(\pi_a^c - \frac{1}{2}\delta_a^c tr_g \pi\right) \\ &= -\left(k_{ab} - \frac{1}{2}g_{ab} tr_g k\right) \\ &= -\left(k_{ab} - \frac{1}{2}g_{ab}\tau\right) \end{aligned} \tag{2.27}$$

and that $\frac{g_{ab}}{\mu_g} = \frac{\gamma_{ab}}{\mu_\gamma}$ one sees that $(k_{ab} - \frac{1}{2}g_{ab}\tau)$ is transverse-traceless with respect to $\gamma_{ab}$ (or, in fact, to any metric conformal to $\gamma_{ab}$ such as $g_{ab}$). Writing $k_{ab}^{TT}$ for $(k_{ab} - \frac{1}{2}g_{ab}\tau)$ we can re-express Eq. (2.19) in the form

$$\rho_{ab} = \left\{\left(e^{-4\lambda}\gamma^{de}\gamma^{cf}k_{ce}^{TT}k_{df}^{TT} + \frac{1}{2}\tau^2\right)e^{2\lambda}\gamma_{ab} + 2\tau k_{ab}^{TT}\right\} \tag{2.28}$$

with now both $\rho_{ab}$ and $\gamma_{ab}$ hyperbolic (with unit negative scalar curvature) and with (since harmonicity depends only upon the conformal structure of the domain metric) the identity map from $(\Sigma, \gamma)$ to $(\Sigma, \rho)$ harmonic. Note that one can absorb $\tau$ into the remaining variables by defining

$$\tau^2 e^{2\lambda} = e^{2\tilde{\lambda}}, \tau k_{ab}^{TT} = \tilde{k}_{ab}^{TT} \tag{2.29}$$

so that

$$\rho_{ab} = \left\{\left(e^{-4\tilde{\lambda}}\gamma^{de}\gamma^{cf}\tilde{k}_{ce}^{TT}\tilde{k}_{df}^{TT} + \frac{1}{2}\right)e^{2\tilde{\lambda}}\gamma_{ab} + 2\tilde{k}_{ab}^{TT}\right\} \tag{2.30}$$

with $\tilde{\lambda}$ satisfying the ($\tau$-autonomous) equation

$$ {}^{(2)}\Delta_\gamma \tilde{\lambda} = \frac{1}{4} e^{2\tilde{\lambda}} - \frac{1}{2} \frac{\tilde{\lambda}^b_a \tilde{\lambda}^a_b}{(\mu_\gamma)^2} e^{-2\tilde{\lambda}} - \frac{1}{2} \tag{2.31} $$

where $\tilde{\lambda}^b_a := \tau \lambda^b_a$ (so that $\frac{g_{bc}}{\mu_g} \tilde{\lambda}^c_a = \frac{\gamma_{bc}}{\mu_\gamma} \tilde{\lambda}^c_a = -\tilde{k}^{TT}_{ab}$).

Equation (2.30) is equivalent to a remarkable, well-known formula in the theory of harmonic maps which allows one, if $\gamma_{ab}$ is held fixed, to parameterize target metrics $\rho_{ab}$ in terms of transverse-traceless symmetric tensors (holomorphic quadratic differentials in the mathematics literature) in such a way that the identity map from $(\Sigma, \gamma)$ to $(\Sigma, \rho)$ is automatically harmonic. Indeed this gives a particular means of constructing a global cross section of the bundle

$$ \mathcal{M}_{-1}(\Sigma) \longrightarrow \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \tag{2.32} $$

and thus a concrete model for the Teichmüller space $\mathcal{T}(\Sigma)$. We shall expand upon this connection to the conventional harmonic maps approach to Teichmüller theory in a subsequent section, but for now, return to the main thread of our discussion.

We conclude this section with the proof, promised above, that $\mathcal{K}$ defined by

$$ \begin{aligned} \mathcal{K} &:= \left( \frac{\pi^a_b \pi^b_a - (tr_g \pi)^2}{\mu_g} \right) \\ &= \mu_g (k_{ab} k^{ab} - (tr_g k)^2) \end{aligned} \tag{2.33} $$

satisfies $\mathcal{K} < 0$ on a CMC slice satisfying the initial value constraints. The momentum constraint is equivalent to

$$ k^b_{a|b} - (tr_g k)_{|a} = 0 \tag{2.34} $$

which may be re-expressed as the Codazzi condition

$$ k^a_{b|c} - k^a_{c|b} = 0. \tag{2.35} $$

Taking the divergence of this equation,

$$ k^a_{b|c}{}^{|c} - k^a_{c|b}{}^{|c} = 0, \tag{2.36} $$

commuting covariant derivatives in the second term and reexpressing the curvature of $g$ through

$$ {}^{(2)}R_{dabc} = \frac{1}{2}\, {}^{(2)}R(g)(g_{db} g_{ac} - g_{cd} g_{ab}) \tag{2.37} $$

one derives

$$ k_{ab|c}{}^{|c} - \left( k_{ac}{}^{|c} \right)_{|b} = {}^{(2)}R(g) \left( k_{ab} - \frac{1}{2} g_{ab} tr_g k \right). \tag{2.38} $$

Imposing the CMC condition $tr_g k = \tau =$ constant and the momentum constraint and decomposing $k_{ab}$ via

$$k_{ab} = k^{TT}_{ab} + \frac{1}{2} g_{ab} \tau \tag{2.39}$$

one arrives at

$$k^{TT\ \ |c}_{ab|c} = {}^{(2)}R(g) k^{TT}_{ab}. \tag{2.40}$$

It follows easily from Eq. (2.40), upon using now the Hamiltonian constraint

$$\mu_g \left( k^{TT}_{ab} k^{TT_{ab}} - \frac{1}{2}\tau^2 \right) = \mu_g \, {}^{(2)}R(g), \tag{2.41}$$

that

$$\begin{aligned} &{}^{(2)}\Delta_g \left( \left|k^{TT}\right|^2_g - \frac{1}{2}\tau^2 \right) - 2\left|k^{TT}\right|^2_g \left( \left|k^{TT}\right|^2_g - \frac{1}{2}\tau^2 \right) \\ &\quad = 2 k^{TT}_{ab|c} k^{TT_{ab|c}} \geq 0 \end{aligned} \tag{2.42}$$

where

$$\left|k^{TT}\right|^2_g := k^{TT}_{ab} k^{TT_{ab}}. \tag{2.43}$$

The strong maximum principle applies to this equation and implies that

$$\left|k^{TT}\right|^2_g - \frac{1}{2}\tau^2 < 0 \tag{2.44}$$

on the surface $\Sigma$ [**15**]. This strict inequality gives $\mathcal{K} < 0$ on $\Sigma$ and hence implies the global regularity of the gnc solutions presented above to the future of an initial CMC slice.

## 3. The Dirichlet energy of the Gauss map

Let any two Riemannian metrics $g$ and $\rho$, defined on $\Sigma$, be expressed in local coordinates $\{x^a\}$ and $\{\psi^A\}$ as

$$\begin{aligned} g &= g_{ab}(x) dx^a \otimes dx^b \\ \rho &= \rho_{AB}(\psi) d\psi^A \otimes d\psi^B \end{aligned} \tag{3.1}$$

and suppose that a mapping $\psi : (\Sigma, g) \to (\Sigma, \rho)$, expressible locally by giving $\psi^A(x^b)$, is to be harmonic. Then $\psi$ must satisfy the Euler-Lagrange equations which result from varying the "action" functional

$$\mathcal{A}(g, \rho, \psi) := \frac{1}{2} \int_\Sigma d\mu_g g^{ab} \psi^A_{,a} \psi^B_{,b} \rho_{AB} \tag{3.2}$$

with respect to $\psi$. These "harmonic map" equations take the form

$$^{(2)}\Delta_g \psi^C + g^{ab}\psi^A_{,a}\psi^B_{,b}\tilde{\Gamma}^C_{AB}(\psi) = 0 \tag{3.3}$$

where the $\tilde{\Gamma}^A_{BC}$ are the Christoffel symbols of $\rho_{AB}$ and $^{(2)}\Delta_g$ is the Laplacian of $g$. The action and its Euler-Lagrange equations are invariant with respect to conformal transformations of the metric $g$, where $g_{ab} \to e^{2\omega} g_{ab}$, and thus depend only upon the conformal class of $g$.

In the previous section we found that if $g$ and $\rho$ are related by Eq. (2.28) then the identity map from $(\Sigma, g)$ to $(\Sigma, \rho)$, expressible locally as $\psi^A(x) = x^A$, is harmonic. Evaluating the action $\mathcal{A}$ on this mapping yields

$$\begin{aligned} \mathcal{A}(g, \rho, Id) &= \frac{1}{2}\int_\Sigma d\mu_g g^{ab}\rho_{ab} \\ &= \int_\Sigma d\mu_g g^{ab}\left\{\frac{1}{2}k^d_c k^c_d g_{ab} + (tr_g k)\left(k_{ab} - \frac{1}{2}g_{ab}(tr_g k)\right)\right\} \\ &= \int_\Sigma d\mu_g k^d_c k^c_d. \end{aligned} \tag{3.4}$$

Recalling that the Hamiltonian constraint satisfied by the data $\{g, k\}$ takes the form

$$\begin{aligned} \mathcal{H} &= \mu_g\left[k^d_c k^c_d - (tr_g k)^2\right] - \mu_g{}^{(2)}R(g) \\ &= 0 \end{aligned} \tag{3.5}$$

one sees that Eq. (3.4) can also be written as

$$\begin{aligned} \mathcal{A}(g, \rho, Id) &= \int_\Sigma \tau^2 d\mu_g + \int_\Sigma d\mu_g{}^{(2)}R(g) \\ &= \int_\Sigma d\mu_g k^d_c k^c_d = \int_\Sigma d\mu_g\left[k^{TTd}_c k^{TTc}_d + \frac{1}{2}\tau^2\right] \end{aligned} \tag{3.6}$$

where $k^{TTd}_c = g^{de}k^{TT}_{ce}$ and $\tau = tr_g k$ as before. Thus one also has

$$\frac{1}{2}\int_\Sigma \tau^2 d\mu_g = \int_\Sigma d\mu_g k^{TTd}_c k^{TTc}_d - \int_\Sigma d\mu_g{}^{(2)}R(g) \tag{3.7}$$

where the second term on the right hand side is constant by the Gauss-Bonnet theorem. When we turn to the study of the (reduced) Einstein equations in CMC (as opposed to gnc) gauge the quantity $\int_\Sigma \tau^2 d\mu_g$, when re-expressed in terms of the variables $\{\tau, \gamma_{ab}, \lambda^b_a\}$ via $g_{ab} = e^{2\lambda}\gamma_{ab}$ (with $\lambda$ determined by the Lichnerowicz equation) will play the role of a (reduced) Hamiltonian for the Einstein "flow" on $T^*\mathcal{T}(\Sigma)\times\mathbf{R}$. Note that, from Eq. (3.7), the infimum of this quantity, which results from setting $k^{TTd}_c = 0$, is always given by the Gauss-Bonnet invariant (i.e., the Euler characteristic of $\Sigma$).

## 4. The reduced Hamiltonian

A local existence theorem for the vacuum Einstein equations in CMCSH (constant-mean-curvature-spatially-harmonic) gauge was proven in Ref. [**4**] for spatially compact spacetimes of dimension $n+1$ for arbitrary $n \geq 2$. The proof involved various higher order energy estimates to control the (Sobolev space) norms of solutions to the gauge-fixed field equations. The case $n=2$ is very special however, and can be treated by much simpler ODE methods once it is realized that the gauge-fixed field equations describe dynamics in a finite dimensional phase space - the cotangent bundle of the Teichmüller space $\mathcal{T}(\Sigma)$. The local existence result for this problem can be treated by the methods developed in Ref. [**2**] after only a slight modification to impose the CMCSH gauge conditions under consideration here.

The local result derived in [**2**] was subsequently extended to a global one in Ref. [**6**] wherein a non-zero cosmological constant was also allowed for. The main technique for this argument involved the use of the Dirichlet energy on Teichmüller space, exploiting its known properties as a proper function, to bound the motion to the interior of Teichmüller space for all values of mean curvature $\tau$ in the range $(-\infty, 0)$ and then to show that this motion captures the maximal Cauchy development of every solution. Except for a lower dimensional subset of "trivial" solutions which are known explicitly, all solutions run "off-the-edge" of Teichmüller space in the limit as $\tau \searrow -\infty$ which corresponds to the big-bang singularities of these (vacuum) $2+1$ dimensional cosmological models. The opposite limit, $\tau \nearrow 0$, corresponds to the limit of infinite cosmological expansion wherein, however, the motion remains confined to the interior of Teichmüller space. We shall show below that in fact every solution tends to a limit in $\mathcal{T}(\Sigma)$ as $\tau \nearrow 0$.

The arguments of Ref. [**6**] only exploited the Dirichlet energy in a rather crude way. In the present paper, we shall significantly refine the application of this energy by showing how it yields a complete solution to the Hamilton-Jacobi equation for the reduced Einstein equations in CMCSH gauge. As a first step in this direction, let us briefly recall some of the key results of Ref. [**2**] with the notation of that paper modified to conform to that used here and with the gauge conditions adjusted to agree with the CMCSH choice made here.

As we have already shown in Sec. 2 above, the general solution to the constraints, $\mathcal{H}(g,\pi)=J_a(g,\pi)=0$, in CMC gauge can be expressed (c.f. Eq. (2.24)) as:

$$g_{ab}=e^{2\lambda}\gamma_{ab},\ \ \pi^b_a=\lambda^b_a+\frac{1}{2}\tau\mu_g\delta^b_a \tag{4.1}$$

where ${}^{(2)}R(\gamma)=-1$, $\lambda^b_a$ is a symmetric $TT$ tensor density with respect to $\gamma_{ab}$, $\tau<0$ is constant on $\Sigma$ and $\lambda$ is the corresponding unique solution to the Lichnerowicz equation (2.23). Defining, as in Ref. [**2**],

$$p^{TTab}=\gamma^{ac}\lambda^b_c \tag{4.2}$$

we thus get

$$g_{ab} = e^{2\lambda}\gamma_{ab},\ \pi^{ab} = e^{-2\lambda}p^{TT_{ab}} + \frac{1}{2}\tau\mu_\gamma\gamma^{ab} \tag{4.3}$$

for this general solution.

When the constraints are satisfied along a differentiable curve (that need not be a solution of the remaining field equations), the ADM action (2.6) reduces to

$$I^*_{\text{ADM}} = \int_{\mathcal{I}\times\Sigma} d^3x\{\pi^{ab}g_{ab,t}\}. \tag{4.4}$$

We substitute the foregoing expressions for $\pi^{ab}$ and $g_{ab}$ into this formula but restrict the conformal metric to lie in global cross-section of the $\mathcal{D}_0(\Sigma)$ bundle $\mathcal{M}_{-1}(\Sigma) \to \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \approx \mathcal{T}(\Sigma) \approx \mathbf{R}^{6\text{genus}(\Sigma)-6}$ determined by the requirement that the identity map from $(\Sigma,\gamma)$ to $(\Sigma,\rho)$ be harmonic. Here $\rho$ is an arbitrary metric satisfying ${}^{(2)}R(\rho) = -1$. Later we shall allow $\rho$ itself to vary over another suitably chosen cross-section of this same bundle and thus consider a parametrized family of such reduced actions but, for now, simply regard $\rho$ as fixed. That the cross-sections defined by the requirements that Id : $(\Sigma,\gamma) \to (\Sigma,\rho)$ be harmonic are indeed global, was proven in Ref. [**14**].

By choosing global coordinates $\{q^\alpha \mid \alpha = 1,\dots,6\text{genus}(\Sigma)-6\}$ on the topologically trivial space $\mathcal{T}(\Sigma) \approx R^{6\text{genus}(\Sigma)-6}$ and lifting these up to the chosen cross section, one can express the metrics realizing this cross section as smooth functions of the $q^\alpha$'s and hence as $\gamma_{ab}(x^c,q^\alpha)$ relative to local coordinates $\{x^a\}$ on $\Sigma$. Along a differential curve of such metrics, one thus has

$$\frac{\partial\gamma_{ab}}{\partial t} = \frac{\partial\gamma_{ab}}{\partial q^\alpha}\dot{q}^\alpha \tag{4.5}$$

where, by construction, the tensor fields $\left\{\frac{\partial\gamma_{ab}}{\partial q^\alpha} \mid \alpha = 1,\dots,6\text{genus}(\Sigma)-6\right\}$ provide a basis to the tangent space to the cross section at any point thereof. As discussed in Ref. [**2**] (c.f. Eq. (2.22) and associated references) each such tangent vector has a unique $L^2$-orthogonal (relative to $\gamma_{ab}(x^c,q^a)$) decomposition of the form

$$\frac{\partial\gamma_{ab}}{\partial q^\alpha} = k^{TT}_{(\alpha)ab} + \left(\mathcal{L}_{{}^{(2)}X_{(\alpha)}}\gamma\right)_{ab} \tag{4.6}$$

where $k^{TT}_{(\alpha)}$ is a $TT$ symmetric tensor with respect to $\gamma$ and ${}^{(2)}X_{(\alpha)}$ a vector field on $\Sigma$ (with $\mathcal{L}_{{}^{(2)}X_{(\alpha)}}$ signifying its Lie derivative).

Exploiting this decomposition, it is straightforward to show that there exists a smoothly varying dual basis $\{m^{TTab(\beta)}(x^c,q^\gamma)\}$ of symmetric $TT$ tensor density fields defined on $\Sigma$ such that

$$\int_\Sigma m^{TTab(\beta)}k^{TT}_{(\alpha)ab}d^2x = \int_\Sigma m^{TTab(\beta)}\frac{\partial\gamma_{ab}}{\partial q^\alpha}d^2x = \delta^\beta_\alpha. \tag{4.7}$$

In terms of this natural dual basis for the cotanent space to $\mathcal{T}(\Sigma)$, one can express an arbitrary $TT$ tensor density $p^{TT}$ as

$$p^{TTab}(x^c, q^\alpha, p_\alpha) = p_\alpha m^{TTab(\alpha)}(x^c, q^\alpha) \tag{4.8}$$

for suitable coefficients $\{p^\alpha\}$. The coordinates $\{q^\alpha, p_\alpha\}$ may be regarded, as we shall see below, as a (global) canonical chart for $T^*\mathcal{T}(\Sigma)$.

Substituting the foregoing expressions into the reduced action and carrying out the steps displayed in Eq. (2.21) of Ref. [**2**], one arrives at

$$I^*_{ADM} = \int_{\mathcal{I}} dt \left\{ p_\alpha \frac{dq^\alpha}{dt} - \frac{d\tau}{dt} \int_\Sigma \mu_g d^2x \right\} + \int_\Sigma d^2x [\tau \mu_g] \mid_{t_0}^{t_1} \tag{4.9}$$

wherein we recognize the canonical character of the coordinates $\{q^\alpha, p^\alpha\}$. The boundary term on the right hand side of Eq. (4.9) makes no contribution to the equations of motion. We therefore drop it and define

$$I^*_{ADM} \mid_{\text{reduced}} = \int_{\mathcal{I}} dt \left\{ p_\alpha \dot{q}^\alpha - \frac{d\tau}{dt} \int_\Sigma \mu_g d^2x \right\}. \tag{4.10}$$

For our purposes the most natural choice of time function (which differs from that made in Ref. [**2**]) corresponds to setting $t = -\frac{1}{\tau}$, so that $\frac{d\tau}{dt} = \tau^2$, which thus yields a reduced Hamiltonian

$$H^*_{ADM|\text{reduced}} := H_{\text{reduced}} = \tau^2 \int_\Sigma \mu_g d^2x = \tau^2 \int_\Sigma e^{2\lambda} \mu_\gamma d^2x. \tag{4.11}$$

Here $H_{\text{reduced}}$ is regarded as a (globally defined) function on $T^*\mathcal{T}(\Sigma) \times \mathbf{R}^+$, $H(q^\alpha, p_\alpha, t)$, determined by expressing

$$\begin{aligned} \gamma_{ab} = \gamma_{ab}(x^c, q^\alpha),\ p^{TTab} &= p^{TTab}(x^c, q^\alpha, p_\alpha) \\ &= p_\alpha m^{TTab(\alpha)}(x^c, q^\alpha), \end{aligned} \tag{4.12}$$

solving the Lichnerowicz equation (2.23) for $\lambda = \lambda(x^c, q^\alpha, p_\alpha, t)$ and carrying out the integral over $\Sigma$.

As discussed in Ref. [**2**] the resulting Hamiltonian is independent of the choice of representative cross section used in its construction (e.g., independent of the metric $\rho$) and describes dynamics on the natural reduced phase space $T^*\mathcal{T}(\Sigma)$ in terms of canonical coordinates $\{q^\alpha, p_\alpha\}$ on that space.

On the other hand, since our ultimate aim is to reconstruct (from solution curves $\{q^\alpha(t), p_\alpha(t)\}$) vacuum metrics on $\Sigma \times \mathbf{R}$ we are more directly interested in the lifts of curves back up to the chosen cross section where they yield evolving sets of ADM data

$$\begin{aligned} g_{ab}(x^c, t) &= e^{2\lambda} \gamma_{ab}(x^c, q^\alpha(t)) \\ \pi^{ab}(x^c, t) &= e^{-2\lambda} p^{TTab}(x^c, q^\alpha(t), p_\alpha(t)) \\ &\quad + \frac{1}{2} \tau(t) (\mu_\gamma \gamma^{ab})(x^c, q^\alpha(t)) \end{aligned} \tag{4.13}$$

expressible in terms of the solution $\lambda = \lambda(x^c, t, q^\alpha(t), p_\alpha(t))$ to Lichnerowicz's equation.

To complete the formula for the spacetime metric we also need the lapse function $N$ and shift field $X^a$. As discussed in Ref. [**2**] and shown more explicitly in Ref. [**4**], these are uniquely fixed by the elliptic equations defined by the requirements that the gauge conditions be preserved in time. These latter correspond to

$$\tau = \frac{tr_g \pi}{\mu_g} = -\frac{1}{t} \tag{4.14}$$
$$V^c = g^{ab}(\Gamma^c_{ab}(g) - \tilde{\Gamma}^c_{ab}(\rho)) = 0$$

where $\Gamma^c_{ab}(g)$ and $\tilde{\Gamma}^c_{ab}(\rho)$ are the Christoffel connection components of the metrics $g$ and $\rho$ respectively. Computing the $t$-derivatives of $\tau$ and $V^c$ and setting these equal to $\tau^2$ and 0 respectively leads to

$$\tau^2 = \frac{\partial \tau}{\partial t} = -\Delta_g N + \frac{N}{(\mu_g)^2} \pi^b_a \pi^a_b \tag{4.15}$$

which can also be written as

$$e^{2\lambda}\tau^2 = -\Delta_\gamma N + N \left\{ \frac{e^{-2\lambda}}{(\mu_\gamma)^2} \left[ \gamma_{ab}\gamma_{cd} p^{TTac} p^{TTbd} + \frac{1}{2} e^{4\lambda} (\mu_\gamma)^2 \tau^2 \right] \right\} \tag{4.16}$$

and

$$0 = -g^{ad} g^{be} h_{de} \left( \Gamma^c_{ab}(g) - \tilde{\Gamma}^c_{ab}(\rho) \right) \tag{4.17}$$
$$+ \frac{1}{2} g^{ab} g^{ce} (h_{ae|b} + h_{be|a} - h_{ab|e})$$

where

$$h_{ab} = \frac{2N}{\mu_g}(\pi_{ab} - g_{ab} tr_g \pi) + X_{a|b} + X_{b|a}. \tag{4.18}$$

The existence and uniqueness of solutions to these equations was established in Ref. [**4**] together with the fact that imposing this choice for $\{N, X^a\}$ suffices to preserve the gauge conditions (4.14).

A remarkable feature of the reduced Hamiltonian, established in Ref. [**6**], is that it monotonically decays for all solutions except the trivial ones corresponding to $p^{TTab} = 0$ for which it stays constant. We shall show later that every solution tends, as $t \to \infty$ (or $\tau \nearrow 0$) to one such that $H_{\text{reduced}}$ always achieves its infinum (identified below Eq. (3.7)) in the limit of infinite cosmological expansion.

Given initial data ($\overset{o}{\gamma}_{ab}, \overset{o}{k}{}^{TT}_{ab}, \overset{o}{\tau} =$ constant $< 0$ with ${}^{(2)}R(\overset{o}{\gamma}) = -1$) we can set $\lambda^c_a = -\mu_{\overset{o}{\gamma}} \overset{o}{\gamma}{}^{cb} \overset{o}{k}{}^{TT}_{ab}$ (c.f. Eq. (2.27)) and solve the Lichnerowicz equation (2.23) for the function $\overset{o}{\lambda}$ and thereby compute the corresponding target hyperbolic metric $\overset{o}{\rho}_{ab}$ using equation (2.28). Since we know that the identity map from $(\Sigma, \overset{o}{\gamma})$ to $(\Sigma, \overset{o}{\rho})$ is harmonic, it seems natural to now fix the CMCSH gauge precisely by requiring that the evolving conformal metric $\gamma_\tau$ continue to lie in the (Eells, Earle) global cross section of the bundle

$$\mathcal{M}_{-1}(\Sigma) \to \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \approx \mathcal{T}(\Sigma) \tag{4.19}$$

defined by

$$\{\gamma \mid \ {}^{(2)}R(\gamma) = -1, \mathrm{Id} : (\Sigma, \gamma) \to (\Sigma, \overset{o}{\rho}) \text{ is harmonic}\} \tag{4.20}$$

during the subsequent evolution. Taking into account the conformal invariance of the harmonicity condition with respect to the domain metric, the results of Ref. [**4**] show that this gauge condition (combined with the CMC condition which fixes the lapse) uniquely determines the shift vector field and conversely, that when the shift is fixed by the associated elliptic equation, the CMCSH gauge conditions will continue to hold during the evolution.

However, we could now compute a (potentially) different, target hyperbolic metric $\rho_\tau$ on any CMC slice of the subsequent evolution by simply evaluating the right hand side of Eq. (2.19) for the geometric data $(g_\tau, k_\tau, \tau)$ induced on that slice. By construction $\gamma_\tau$ (which can always be recovered from $g_\tau$ by uniformization) will satisfy harmonicity of the mapping

$$\mathrm{Id} : (\Sigma, \gamma_\tau) \to (\Sigma, \rho_\tau) \tag{4.21}$$

as well as the CMCSH gauge condition which ensures harmonicity of

$$\mathrm{Id} : (\Sigma, \gamma_\tau) \to (\Sigma, \overset{o}{\rho}). \tag{4.22}$$

But what is the relationship between $\rho_\tau$ and $\overset{o}{\rho} := \rho_{\overset{o}{\tau}}$? We shall conclude this section by showing that $\rho_\tau = \overset{o}{\rho}$ and thus that the metric $\rho_\tau$, defined on each slice by Eq. (2.19), is in fact a constant of the motion for the particular CMCSH gauge under consideration.

First recall that, in a flat spacetime, the traces of the holonomics defined via the parallel propagation of vectors around incontractible loops are invariant with respect to arbitrary, continuous deformations of these loops within the spacetime. Thus if one evaluates any collection of such traces on say a given CMC slice and then deforms the chosen loops continuously to corresponding loops in another such slice, one will necessarily obtain the same values for the traces under study. We shall see however, that these same values for the traces may also be computed by a corresponding holonomy calculation carried out in a certain "reference spacetime" $(\Sigma \times \mathbb{R}^{+}, {}^{(3)}\eta_\tau)$ defined, for any fixed hyperbolic metric $\rho_\tau$ of the family described above, by the flat metric

$${}^{(3)}\eta_\tau = -dt \otimes dt + \frac{t^2}{2}(\rho_\tau(x^c))_{ab} dx^a \otimes dx^b. \tag{4.23}$$

But since these traces are independent of the value of $\tau$ used to compute $\rho_\mathcal{T}$ and since a complete, independent set of (6 genus($\Sigma$) $-$ 6) such traces determines the $\rho_\mathcal{T}$ appearing in Eq. (4.23) up to isometry it will follow that $\rho_\mathcal{T}$ can only have the form $\rho_\mathcal{T} = \varphi^*_\mathcal{T}\overset{o}{\rho} = \varphi^*_\mathcal{T}\rho_{\overset{o}{\mathcal{T}}}$ for some (possibly non-trivial) diffeomorphism $\varphi_\mathcal{T} : \Sigma \to \Sigma$ defined for each value of $\tau$ achieved during the (non-singular) CMCHS evolution. Finally, however, it will follow from the particular choice of CMCHS gauge that we have made to specify that evolution, that $\varphi_\mathcal{T} = \mathrm{Id}$ is the only possibility and thus that $\rho_\mathcal{T} = \overset{o}{\rho}$ for every allowed value of $\tau$.

For any given CMC slice of the spacetime under study, we can compute the future evolution from that slice in gnc coordinates via Eqs. (2.12)–(2.18). Parallel propagation of a vector $v$ around an arbitrary loop chosen (for convenience) to lie in a surface of constant gaussian time $t$ is determined by solving

$$\frac{dv^\mu}{d\lambda} + {}^{(3)}\Gamma^\mu_{\alpha\beta}\, v^\alpha \frac{dx^\beta}{d\lambda} = 0, \tag{4.24}$$

where ${}^{(3)}\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols of the spacetime metric ${}^{(3)}g_{\alpha\beta}$ expressed in the chosen gnc coordinates. Taking $t(\lambda) = t = \text{constant}$ and writing out this equation for the rescaled vector $\tilde{v}$ defined by

$$\tilde{v}^0 = v^0, \tilde{v}^a = t v^a \tag{4.25}$$

one gets

$$\begin{aligned} &\frac{d\tilde{v}^0}{d\lambda} + \frac{1}{2}\left(\frac{g_{ab,t}}{t}\right)\tilde{v}^a \frac{dx^b}{d\lambda} = 0, \\ &\frac{d\tilde{v}^a}{d\lambda} + {}^{(2)}\Gamma^a_{bc}(g)\tilde{v}^b \frac{dx^c}{d\lambda} + \frac{1}{2}(t^2 g^{ad})\left(\frac{g_{cd,t}}{t}\right)\tilde{v}^0 \frac{dx^c}{d\lambda} = 0 \end{aligned} \tag{4.26}$$

where ${}^{(2)}\Gamma^a_{bc}(g)$ are the Christoffel symbols of the metric $g$ defined (from the chosen CMC initial data) by Eqs. (2.12)–(2.18).

On an arbitrary $t$ = constant slice these equations are difficult to analyze but since we know that traces of resultant holonomy calculations will automatically be independent of the gnc slice chosen we can evaluate them in the limit as $t \to \infty$ by exploiting the facts (easily derived from Eqs. (2.12)–(2.18)) that

$$\begin{aligned} &\lim_{t\to\infty}\left(\frac{2}{t^2} g_{ab}\right) = (\rho_\mathcal{T})_{ab} \\ &\lim_{t\to\infty}\left(\frac{t^2}{2} g^{ab}\right) = (\rho_\mathcal{T})^{ab} \end{aligned} \tag{4.27}$$

$$\lim_{t\to\infty}\left(\frac{g_{ab,t}}{t}\right) = (\rho_\mathcal{T})_{ab}$$
$$\lim_{t\to\infty}\ {}^{(2)}\Gamma^a_{bc}(g) = \lim_{t\to\infty}\ {}^{(2)}\Gamma^a_{bc}\left(\frac{2}{t^2}g\right)$$
$$= {}^{(2)}\Gamma^a_{bc}(\rho_\mathcal{T})$$

and that the solutions of the linear equations (4.26) vary continuously with the coefficients. But the limiting equations so obtained,

$$\frac{d\tilde{v}^0}{d\lambda} + \frac{1}{2}(\rho_\mathcal{T})_{ab}\tilde{v}^a\frac{dx^b}{d\lambda} = 0,$$
$$\frac{d\tilde{v}^a}{d\lambda} +^{(2)}\Gamma^a_{bc}(\rho_\mathcal{T})\tilde{v}^b\frac{dx^c}{d\lambda} + \tilde{v}^0\frac{dx^a}{d\lambda} = 0, \tag{4.28}$$

are equivalent to the parallel propagation equations one would obtain (at an arbitrary instant of gnc time $t > 0$) for the "reference metric" ${}^{(3)}\eta_\mathcal{T}$ given above by Eq. (4.23).

But a flat metric of the form (4.23) defined on $\Sigma \times \mathbb{R}^+$ is isometric to a quotient of the interior of the future light cone of a point in 3-dimensional Minkowski space by a discrete subgroup of the (proper orthochronos) Lorentz group that fixes that point. The subgroup in question must be homomorphic to a representation of the fundamental group $\pi_1(\Sigma)$, of the higher genus surface $\Sigma$ and can be recovered from $(\Sigma \times \mathbb{R}^+,\ {}^{(3)}\eta_\mathcal{T})$ by the computation of a complete, independent set of $(6\,\mathrm{genus}\,(\Sigma) - 6)$ traces of holonomies. Conversely, a specification of these traces determines the $\rho_\mathcal{T}$ needed in the metric form (4.23) up to isometry (i.e., up to the pull back action by a diffeomorphism of $\Sigma$).

But this implies, as stated above, that the target metric $\rho_\mathcal{T}$ computed from an arbitrary CMC slice in the evolving spacetime $(\Sigma \times \mathbb{R}^+,\ {}^{(3)}g)$ must satisfy $\rho_\mathcal{T} = \varphi^*_\mathcal{T}\rho_{\overset{\circ}{\mathcal{T}}} = \varphi^*_\mathcal{T}\rho$ for some diffeomorphism $\varphi_\mathcal{T} : \Sigma \to \Sigma$ of $\Sigma$ which, by continuity of the evolution, is necessarily isotopic to the identity. The foregoing implies that the corresponding curve of (uniformized) metric $\gamma_\mathcal{T}$ satisfies both (4.21) and (4.22) with $\rho_\mathcal{T} = \varphi^*_\mathcal{T}\overset{\circ}{\rho}$. But, if $\varphi_\mathcal{T} \neq \mathrm{Id}$, the Eells, Earle global cross section of the bundle (4.19) is disjoint from that obtained upon replacing $\overset{\circ}{\rho}$ with $\rho_\mathcal{T} = \varphi^*_\mathcal{T}\overset{\circ}{\rho}$ since, in view of the covariance of the construction, the latter cross section is obtained from the former by pulling back each metric by the same diffeomorphism. It follows that since $\gamma_\mathcal{T}$ satisfies both (4.21) and (4.22), we must have $\varphi_\mathcal{T} = \mathrm{Id}$ throughout the evolution.

It is worth mentioning here that the metric ${}^{(3)}g$ takes the special form ${}^{(3)}\eta_\mathcal{T}$ (for which the gaussian time slices are also CMC) only if the initial data for ${}^{(3)}g$ satisfies $\overset{\circ}{k}{}^{TT}_{ab} = 0$ which, of course, is generically not the case. Roughly speaking, the "reference metric" ${}^{(3)}\eta_\mathcal{T}$ is constructed using only half of the data needed for the specification of the actual spacetime metric ${}^{(3)}g$. There are different ways of invariantly prescribing the "missing" data needed to fully characterize ${}^{(3)}g$. For example, in the Witten approach (cf. Ref [**1**]), one supplements the (linear) holonomies discussed above with certain translational holonomies defined in regarding $(\Sigma \times \mathbb{R}^+, {}^{(3)}g)$ as a quotient of Minkowski space by a suitably chosen discrete subgroup of the full inhomogeneous Lorentz (or Poincaré) group. We shall not pursue that approach here, but instead, following the proposal of Puzio [**7**], will characterize the remaining data in terms of the conserved quantities defined by a complete solution of the associated Hamilton-Jacobi equation.

## 5. Hamilton-Jacobi theory and the Dirichlet energy

In this section we shall establish a remarkable relationship between the Dirichlet energy for the Gauss map discussed in Sec. 3 and solutions to the Hamilton-Jacobi equation for the reduced Einstein equations discussed in the previous section. In fact, we shall derive a dynamically complete solution to the Hamilton-Jacobi equation by exploiting this relationship and thereby arrive at an implicit formula for the general solution to the reduced Einstein equations in CMCSH gauge.

In Sec. 3 we found that whenever any two metrics $\gamma$ and $\rho$ (satisfying ${}^{(2)}R(\gamma) = {}^{(2)}R(\rho) = -1$) are related by Eq. (2.28), for some choice of $\tau = \text{constant} < 0, k^{TT}$ a $TT$ tensor with respect to $\gamma$ and $\lambda$ determined uniquely by Eq. (2.23), then the identity map from $(\Sigma, \gamma)$ to $(\Sigma, \rho)$ is harmonic and its Dirichlet energy (which depends only upon the conformal class of $\gamma$) is expressible as

$$\mathcal{A}(\gamma, \rho, Id) = \frac{1}{2} \int_{\Sigma} d\mu_\gamma \gamma^{ab} \rho_{ab}. \tag{5.1}$$

Recalling the change of notation defined by Eq. (2.29), one can re-express the relationship between $\gamma$ and $\rho$ through equations (2.30) and (2.31) which are autonomous relative to $\tau$. In Ref. [**8**], Michael Wolf used these latter equations (with no apparent relativistic or associated Gauss map interpretation) to prove that, for any such fixed metric $\gamma$ (with ${}^{(2)}R(\gamma) = -1$), one could smoothly parametrize the space of metrics $\rho$ satisfying

(i) ${}^{(2)}R(\rho) = -1$, and
(ii) $\text{Id} : (\Sigma, \gamma) \to (\Sigma, \rho)$ is harmonic

by the space of $TT$ tensors relative to $\gamma$ (i.e., by the tensors $\tilde{k}^{TT}_{ab}$ appearing in these formulas, normally referred to as holomorphic quadratic differentials in the mathematics literature). More precisely, Wolf showed that the foregoing formulas define a global diffeomorphism between the space of $TT$ tensors defined relative to a fixed $\gamma$ and the space of uniformized metrics $\rho$ satisfying (i) and (ii) above. Thus for any fixed pair of metrics $\gamma$ and $\rho$ (each having scalar curvature $= -1$ and satisfying (ii) above) and any choice of $\tau = \text{constant} < 0$ there exists a unique $TT$ tensor $k^{TT}$ such that equations (2.28) and (2.23) hold.

We want to compute the variation of the energy defined by Eq. (5.1) above, holding $\rho$ fixed and allowing $\gamma$ to vary over that global cross section of $\mathcal{M}_{-1}(\Sigma)$ defined by the requirement that $\mathrm{Id} : (\Sigma, \gamma) \to (\Sigma, \rho)$ be harmonic. Writing $\gamma_{ab} = \gamma_{ab}(x^c, q^\alpha)$ as in the previous section we evaluate the energy $\mathcal{A}(\gamma(q), \rho, \mathrm{Id})$ and compute its partial derivatives with respect to the $\{q^\alpha\}$. The result is

$$\frac{\partial \mathcal{A}(\gamma(q), \rho, \mathrm{Id})}{\partial q^\alpha} = -\frac{1}{2} \int_\Sigma d\mu_\gamma \left(\gamma^{ac}\gamma^{bd} - \frac{1}{2}\gamma^{cd}\gamma^{ab}\right) \rho_{ab} \frac{\partial \gamma_{cd}}{\partial q^\alpha}. \tag{5.2}$$

But substituting the expression (2.28) for $\rho$ (as justified by Wolf's result) one arrives at

$$\begin{aligned} \frac{\partial \mathcal{A}}{\partial q^\alpha}(\gamma(q), \rho, \mathrm{Id}) &= -\int_\Sigma d\mu_\gamma \tau k^{TT}_{ab} \gamma^{ac}\gamma^{bd} \frac{\partial \gamma_{cd}}{\partial q^\alpha} \\ &= -\tau \int_\Sigma p^{TTcd} \frac{\partial \gamma_{cd}}{\partial q^\alpha} = \tau \int_\Sigma p_\beta m^{TTcd(\beta)} \frac{\partial \gamma_{cd}}{\partial q^\alpha} \\ &= \tau p_\alpha \end{aligned} \tag{5.3}$$

where we have used equations (2.4), (4.3), and (4.8) to simplify the result.

It thus follows that the gradient of the rescaled Dirichlet energy function, $\frac{1}{\tau}\mathcal{A}(\gamma(q), \rho, \mathrm{Id})$, with respect to the coordinates $\{q^\alpha\}$ (a global chart for $\mathcal{T}(\Sigma)$) yields precisely the canonical momentum components $\{p_\alpha\}$ such that the vacuum Einstein spacetime determined by the data $\{q^\alpha, p_\alpha, t = -\frac{1}{\tau}\}$ has asymptotic rescaled metric (c.f., Eq. (2.19)) given by $\rho$. As we saw in Sec. 3, this metric $\rho$ corresponds to the asymptotic conformal geometry invariantly defined by the linear holonomies of the chosen vacuum spacetime.

For later convenience, let us now write $T$ (instead of the more generic $t$), for the preferred time coordinate $-\frac{1}{\tau}$ and define

$$\mathcal{S}(q^\alpha, \rho, T) = -T \left[\mathcal{A}(\gamma(q), \rho, \mathrm{Id}) - \int_\Sigma d\mu_\gamma \, {}^{(2)}R(\gamma)\right] \tag{5.4}$$

wherein $\int_\Sigma d\mu_\gamma \, {}^{(2)}R(\gamma)$ of course is just the Gauss-Bonnet invariant of $\Sigma$. One now has

$$\begin{aligned} p_\alpha &= \frac{\partial \mathcal{S}}{\partial q^\alpha}(q^\alpha, \rho, T), \\ -\frac{\partial \mathcal{S}}{\partial T} &= \mathcal{A}(\gamma(q), \rho, \mathrm{Id}) - \int_\Sigma d\mu_\gamma^{(2)} R(\gamma) \end{aligned} \tag{5.5}$$

but it follows from Eqs. (3.6) and (5.3) that the right hand side of this last equation is equal to $H_{\mathrm{reduced}}(q^\alpha, p_\alpha, T) \Big|_{p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha}}$. In other words, that $\mathcal{S}(q^\alpha, \rho, T)$, for fixed $\rho$ satisfies the Hamilton-Jacobi equation for the reduced Einstein equations in CMCSH gauge

$$-\frac{\partial \mathcal{S}}{\partial T} = H_{\mathrm{reduced}}\left(q^\alpha, \frac{\partial \mathcal{S}}{\partial q^\alpha}, T\right). \tag{5.6}$$

This $\mathcal{S}$ is just a particular solution determined by the chosen target metric $\rho$ but the choice of $\rho$ was arbitrary. We are free to let $\rho$ range over $\mathcal{M}_{-1}(\Sigma)$ but, in view of the $\mathcal{D}_0(\Sigma)$ invariance of the Dirichlet energy functional, there is no essential loss of generality in restricting $\rho$ to lie in a global cross section for the bundle

$$\mathcal{M}_{-1}(\Sigma) \to \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \approx \mathcal{T}(\Sigma) \tag{5.7}$$

and hence in a model for Teichmüller space. Choosing global coordinates $\{Q^\alpha\}$ for this model one can now write, with a slight abuse of notation

$$\mathcal{S}(q^\alpha, Q^\alpha, T) = \mathcal{S}(q^\alpha, \rho(Q), T) \tag{5.8}$$

and regard this function $\mathcal{S}$ as globally defined on $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \times \mathbf{R}^+$.

To see that this $\mathcal{S}$ is dynamically complete, we note that the coordinates $\{q^\alpha\}$ label the arbitrary (at time $T \in \mathbf{R}^+$) initial conformal geometry which, upon making an arbitrary choice of global cross section, gets represented by the metric $\gamma_{ab}(x^c, q^\alpha)$ lying in that cross section. For fixed $\gamma$ the freedom to allow the target metric $\rho(Q)$ to vary over an independent global cross section of $\mathcal{M}_{-1}(\Sigma)$ (in particular over all those metrics for which $\mathrm{Id} : (\Sigma, \gamma) \to (\Sigma, \rho)$ is harmonic) provides precisely (via Wolf's diffeomorphism result) the freedom to complement $\gamma_{ab}$ with an arbitrary $TT$ tensor $k^{TT}_{ab}$. Thus we get fully general Cauchy data sets $\{\gamma_{ab}, k^{TT}_{ab}, T = -\frac{1}{\tau}\}$ by varying $\{q^\alpha, Q^\alpha\}$ freely.

A well known result in Hamilton-Jacobi theory is that one can derive a complementary set of constants of the motion to the $\{Q^\alpha\}$'s by differentiating the complete solution $\mathcal{S}(q^\alpha, Q^\alpha, T)$ with respect to these $Q^\alpha$'s. More precisely the quantities $P_\alpha$ defined by

$$\begin{aligned} P_\alpha &= -\frac{\partial \mathcal{S}}{\partial Q^\alpha}(q^\alpha, Q^\alpha, T) \\ &= +T\frac{\partial}{\partial Q^\alpha}(\mathcal{A}(\gamma(q), \rho(Q), \mathrm{Id})) \end{aligned} \tag{5.9}$$

are constants of the motion for the solutions of the reduced Hamilton equations and together with the $\{Q^\alpha\}$ form a complete set of canonically conjugate variables that are all constants of the notion. Since the solution curves are now implicitly determined by setting $\{Q^\alpha, P_\alpha\}$ equal to suitable values and solving equations (5.9) for $\{q^\alpha(T, Q, P)\}$ with the $p_\alpha$'s then given by $p_\alpha(T, Q, P) = \frac{\partial \mathcal{S}}{\partial q^\alpha}(q(T, Q, P), Q)$ it is of interest to express these equations more explicitly.

Writing $\rho_{ab}(x^c, Q^\alpha)$ for the $\rho_{ab}$ in Eq. (5.1) and differentiating this formula with respect to the $Q^\alpha$'s yields

$$\frac{P_\alpha}{T} = \frac{1}{2}\int_\Sigma d\mu_\gamma \gamma^{ab} \frac{\partial \rho_{ab}}{\partial Q^\alpha}(x^c, Q^\alpha) \tag{5.10}$$

with $\gamma_{ab} = \gamma_{ab}(x^c, q^\alpha)$ in the above. Since $\rho_{ab}$ is only varying over metrics satisfying ${}^{(2)}R(\rho) = -1$ and in fact only over a global cross section representing Teichmüller space the partial derivatives are always expressible in the form

$$\frac{\partial \rho_{ab}}{\partial Q^\alpha}(x^c, Q^\alpha) = \tilde{\ell}^{TT(\alpha)}_{ab}(x^c, Q^\alpha) + (\mathcal{L}_{{}^{(2)}X(x^c,Q^\alpha)}\rho(x^c, Q^\alpha))_{ab} \tag{5.11}$$

(c.f. Eq. (2.22) of Ref. (2) and associated footnotes) where here the $\tilde{\ell}^{TT(\alpha)}_{ab}$ $(x^c, Q^\alpha)$ provide, at each fixed $\{Q^\alpha\}$ a basis for the $TT$ symmetric tensors with respect to $\rho_{ab}(x^c, Q^\alpha)$ and where the vector fields ${}^{(2)}X(x^c, Q^\alpha)$ depend upon the chosen cross-section but are determined by that choice uniquely. We shall now show that the contributions from these Lie-derivative terms drop out of the formula for $P_\alpha$ and hence are ignorable in the following.

For any ${}^{(2)}X$ we can integrate by parts to get

$$\begin{aligned}
\int_\Sigma d\mu_\gamma \gamma^{ab}\,(\mathcal{L}_{{}^{(2)}X}\rho)_{ab} &= -\int_\Sigma (\mathcal{L}_{{}^{(2)}X}\mu_\gamma\gamma^{-1})^{ab}\rho_{ab} \\
&= -\int_\Sigma d\mu_\gamma \left\{\left(\frac{1}{2}\gamma^{cd}\gamma^{ab} - \gamma^{ac}\gamma^{bd}\right)(X_{c|d} + X_{d|c})\rho_{ab}\right\} \\
&= +\int_\Sigma d\mu_\gamma\{[X^{a|b} + X^{b|a} - \gamma^{ab}X^c_{|c}]\rho_{ab}\} \\
&= \int_\Sigma d\mu_\gamma\{[X^{a|b} + X^{b|a} - \gamma^{ab}X^c_{|c}]2\tau k^{TT}_{ab}\} \\
&= 0
\end{aligned} \tag{5.12}$$

where we have used Eq. (2.28) for $\rho_{ab}$ in the last step, exploited the tracelessness of $(\mathcal{L}_{{}^{(2)}X}\mu_\gamma\gamma^{-1})^{ab}$ and the transverse-tracelessness of $k^{TT}_{ab}$ with respect to $\gamma_{ab}$ to complete the reduction. Thus we get

$$\frac{P_\alpha}{T} = \frac{1}{2}\int_\Sigma d\mu_\gamma\gamma^{ab}\tilde{\ell}^{TT(\alpha)}_{ab}(x^c, Q^\alpha) = -\tau P_\alpha \tag{5.13}$$

where $\gamma_{ab} = \gamma_{ab}(x^c, q^\alpha)$ and the $\{\tilde{\ell}^{TT(\alpha)}_{ab}(x^c, Q^\alpha)\}$ yield a basis for the $TT$ tensors with respect to $\rho_{ab}(x^c, Q^\alpha)$.

We shall see below that every solution has the property that

$$\gamma_{ab}(x^c, q^\alpha(T, Q, P)) \longrightarrow \rho_{ab}(x^c, Q^\alpha) \text{ as } T \longrightarrow \infty \tag{5.14}$$

which is clearly compatible with Eq. (5.13) above. Indeed this equation contains in principle complete information about the solutions to the reduced Einstein equations but, lacking a more explicit representation for the family of metrics $\gamma_{ab}(x^c, q^\alpha)$ which fill out a cross section defined by the property that Id : $(\Sigma, \gamma) \to (\Sigma, \rho)$ be harmonic, we cannot convert this result into a very explicit formula for the solution curves. Wolf's diffeomorphism result provides a coordinate system for metrics satisfying this condition but under the restriction that $\gamma$ is held fixed and $\rho$ varies. We want the opposite. In the following sections we shall show how to derive a complementary result to that of Wolf in which the (Teichmüller) space of metrics $\gamma$ satisfying harmonicity of Id : $(\Sigma, \gamma) \to (\Sigma, \rho)$ for fixed $\rho$ is globally parametrized by the space of $TT$ tensors with respect to the fixed metric $\rho$. This will lead us to a more explicit representation of the solution curves but one which necessitates the solution of an associated Monge-Ampêre equation.

## 6. Analysis of the Gauss Map equation

As discussed in Sec. 2 the Gauss map equation takes the form

$$\rho_{ab} = (k_c^d k_d^c) g_{ab} + 2(tr_g k) \left( k_{ab} - \frac{1}{2} g_{ab} tr_g k \right) \tag{6.1}$$

where $g_{ab}$ and $k_{ab} = g_{bc} k_a^c$ are respectively the first and second fundamental forms induced on a CMC slice in a vacuum Einstein spacetime with Cauchy surfaces diffeomorphic to a higher genus surface $\Sigma$. The momentum constraint on the Cauchy data $(g_{ab}, k_{ab})$ is equivalent (when the mean curvature $tr_g k := \tau$ is constant as we have assumed) to harmonicity of the identity mapping from $(\Sigma, g)$ to $(\Sigma, \rho)$ and the Hamiltonian constraint is equivalent to the equation ${}^{(2)}R(\rho) = -1$ which we also assume imposed.

By the uniformization theorem we can always set $g_{ab} = e^{2\lambda} \gamma_{ab}$ for some uniquely determined metric $\gamma$ which satisfies $R(\gamma) = -1$ and uniquely determined smooth function $\lambda$. When $\tau = tr_g k = g^{ab} k_{ab}$ is constant the traceless part of $k_{ab}$ defined by

$$k_{ab}^{TT} = k_{ab} - \frac{1}{2} g_{ab} g^{cd} k_{cd} \tag{6.2}$$

is in fact transverse-traceless (by the momentum constraint) with respect to $g$ or, by the conformal invariance of this condition, with respect to any metric conformal to $g$ such as $\gamma$. Thus we can rewrite the Gauss map equation in the form

$$\rho_{ab} = \left( e^{-2\lambda} \gamma^{ce} \gamma^{df} k_{cd}^{TT} k_{ef}^{TT} + \frac{1}{2} \tau^2 e^{2\lambda} \right) \gamma_{ab} + 2\tau k_{ab}^{TT} \tag{6.3}$$

where

$$\begin{aligned} &{}^{(2)}R(\rho) = -1, {}^{(2)}R(\gamma) = -1 \\ &\tau = tr_g k = g^{ab} k_{ab} = \text{constant} \end{aligned} \tag{6.4}$$

and where

$$k_{ab}^{TT} = k_{ab} - \frac{1}{2} g_{ab} tr_g k \tag{6.5}$$

is transverse-traceless with respect to $\gamma_{ab}$. The Hamiltonian constraint corresponds to the satisfaction of Lichnerowicz's equation by the conformal factor $\lambda$.

In the gauge we have chosen, the metric $\rho$, which is determined up to isometry by the linear holonomies of the vacuum spacetime under study, remains fixed while the metric $\gamma$ evolves within a (global) cross-section of the space $\mathcal{M}_{-1}(\Sigma)$ which represents the Teichmüller space $\mathcal{T}(\Sigma) \approx \mathcal{M}_{-1}(\Sigma)$ $/\mathcal{D}_0(\Sigma)$. The cross-section in question is the smooth submanifold of $\mathcal{M}_{-1}(\Sigma)$ consisting of those metrics $\gamma'$ such that the identity map from $(\Sigma, \gamma')$ to $(\Sigma, \rho)$ is harmonic. In this same (CMCSH) gauge $\tau$ plays the role of time and labels the CMC slices of a global foliation of the spacetime. The lapse function $N$ and shift field $X$ are uniquely determined throughout the evolution by the elliptic equations (4.15) and (4.17) discussed in Sec. 4.

Since $\rho_{ab}$ is fixed during evolution in the chosen gauge and since $\lambda$ is uniquely determined in terms of $(\gamma_{ab}, k^{TT}_{ab}, \tau)$ from Lichnerowicz's equation it appears that the curve of $TT$ tensors $k^{TT}_{ab}$ should be determined, via Eq. (6.3), from the curve of metrics $\gamma_{ab}$ evolving within the chosen cross-section of $\mathcal{M}_{-1}(\Sigma)$. To see this more explicitly, compute

$$\begin{aligned} tr_\gamma \rho &:= \gamma^{ab}\rho_{ab} \\ &= 2\left(e^{-2\lambda}\left|k^{TT}\right|^2_\gamma + \frac{1}{2}\tau^2 e^{2\lambda}\right) \end{aligned} \tag{6.6}$$

where

$$\left|k^{TT}\right|^2_\gamma = \gamma^{cd}\gamma^{ef}k^{TT}_{ce}k^{TT}_{df} \tag{6.7}$$

and rewrite Eq. (6.3) in the form

$$\rho^{tr}_{ab} := \rho_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{ef}\rho_{ef} = 2\tau k^{TT}_{ab}. \tag{6.8}$$

Thus, up to a factor of $2\tau$, $k^{TT}_{ab}$ is simply the traceless part, $\rho^{tr}_{ab}$, of the fixed metric $\rho_{ab}$ computed with respect to the moving metric $\gamma_{ab}$. Rewriting Eq. (6.6) in this notation yields

$$tr_\gamma \rho := \gamma^{ab}\rho_{ab} = \tau^2 e^{2\lambda} + \frac{\left|\rho^{tr}\right|^2_\gamma e^{-2\lambda}}{2\tau^2} \tag{6.9}$$

where

$$\left|\rho^{tr}\right|^2_\gamma = \gamma^{df}\gamma^{ce}\rho^{tr}_{cf}\rho^{tr}_{de}. \tag{6.10}$$

Solving the associated quadratic equation for $e^{2\lambda}$ then gives

$$\tau^2 e^{2\lambda} = \frac{tr_\gamma\rho + \sqrt{(tr_\gamma\rho)^2 - 2|\rho^{tr}|^2_\gamma}}{2} \tag{6.11}$$

which shows how $\lambda$ is determined by $\rho$ and the moving metric $\gamma$. Recalling Eq. (3.5) we also have

$$\mu_\rho = \sqrt{\det \rho_{ab}} = -\mu_g {}^{(2)}R(g) \tag{6.12}$$

and hence

$$ {}^{(2)}R(g) = -\frac{\mu_\rho}{\mu_g} = -\frac{\mu_\rho e^{-2\lambda}}{\mu_\gamma}. \tag{6.13}$$

It was argued in Sec. 5 that the motion of $\gamma$ within the chosen cross section of $\mathcal{M}_{-1}(\Sigma)$ is determined implicitly by the equation

$$-\tau P_\alpha = \frac{1}{2}\int_\Sigma d\mu_\gamma\, \gamma^{ab}\, \tilde{\ell}^{TT}_{ab(\alpha)} \tag{6.14}$$

where $\{\tilde{\ell}^{TT}_{(\alpha)} \mid \alpha = 1, \ldots, 6\,\text{genus}(\Sigma) - 6\}$ is a fixed basis of $TT$ tensors with respect to $\rho_{ab}$ and $\{P_\alpha\}$ is a set of 6 genus $(\Sigma) - 6$ arbitrary constants which, together with the complimentary 6 genus $(\Sigma) - 6$ independent holonomies that determine $\rho_{ab}$, form a complete set of constants of the motion determining a vacuum spacetime. To convert this formula into a more explicit characterization of the motion of $\gamma$ it is essential to develop a more explicit characterization of the cross-section of metrics in which $\gamma$ is moving.

We begin by setting:

$$\zeta^b_a := \mu_g k^{TTb}_a = \mu_g g^{bc} k^{TT}_{ac} \tag{6.15}$$

and solving Eq. (6.3) algebraically for the (inverse-) metric $g^{ab}$. This solution can be expressed as

$$\begin{aligned} \mu_g g^{a\ell} &= 2\tau\rho^{ab}\zeta^\ell_b + \sqrt{1 + \frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}}\,\mu_\rho\rho^{a\ell} \\ \tau^2\mu_g &= \mu_\rho\left(1 + \sqrt{1 + \frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}}\right) \end{aligned} \tag{6.16}$$

where

$$|\zeta|^2 = \zeta^f_e \zeta^e_f. \tag{6.17}$$

Now, $\zeta^b_a$ is required to be transverse traceless (and symmetric) with respect to $g_{ab}$ (or any metric $\gamma_{ab}$ conformal to $g_{ab}$) but we should like to re-express these conditions on $\zeta^b_a$ relative to the fixed metric $\rho_{ab}$. First of all note that

$$\mu_g g^{a\ell}\zeta^m_a = 2\tau\rho^{ab}\zeta^\ell_b\zeta^m_a + \sqrt{1 + \frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}}\,\mu_\rho\rho^{a\ell}\zeta^m_a \tag{6.18}$$

so that, since the middle term is automatically symmetric, it follows that the contravariant form of $\zeta_a^m$ relative to the $g$ metric is symmetric if and only if its contravariant form relative to the $\rho$ metric is symmetric. The condition that $\zeta_a^m$ be transverse is equivalent (when $\tau = \text{constant}$) to the momentum constraint and this in turn, as we saw above, is equivalent to the requirement that the identity map from $(\Sigma, g)$ to $(\Sigma, \rho)$ be harmonic. One can express this latter formulation as the requirement that

$$\tilde{\nabla}_b(\rho)(\mu_g g^{ab}) = 0 \tag{6.19}$$

where $\tilde{\nabla}_b(\rho)$ signifies covariant differentiation with respect to $\rho_{ab}$. Computing the $\rho$-divergence of Eq. (6.16) and imposing the condition (6.19) leads immediately to

$$2\tau\tilde{\nabla}_\ell(\rho)\left(\frac{\zeta_b^\ell}{\mu_\rho}\right) + \partial_b\sqrt{1 + \frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}} = 0 \tag{6.20}$$

as a differential condition on $\zeta_a^m$ which only involves the metric $\rho_{ab}$. Thus $\zeta_a^m$ is required to be a traceless tensor density which is both symmetric with respect to $\rho$ and satisfies Eq. (6.20).

To solve this equation we first decompose $\zeta_b^\ell$ into a transverse-traceless summand (relative to $\rho$) and an $L^2$-orthogonal conformal Killing form

$$\zeta_b^\ell = \zeta_b^{TT\ell} + \mu_\rho\rho_{ab}(\tilde{\nabla}^a(\rho)Y^\ell + \tilde{\nabla}^\ell(\rho)Y^a - \rho^{a\ell}\tilde{\nabla}_m(\rho)Y^m) \tag{6.21}$$

where $\tilde{\nabla}^m(\rho) = \rho^{mr}\tilde{\nabla}_r(\rho)$.

Recalling that $\rho$ has constant curvature (since ${}^{(2)}R(\rho) = -1$) one finds readily that, in terms of the foregoing expansion of $\zeta_b^\ell$

$$\tilde{\nabla}_\ell(\rho)\left(\frac{\zeta_b^\ell}{\mu_\rho}\right) = \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)(\rho_{bc}Y^c) - \frac{1}{2}\rho_{bc}Y^c. \tag{6.22}$$

Now the vector field $Y^c$ has an $L^2$-orthogonal expansion of the form

$$Y^c = Y^{tr_c} + \rho^{cd}\Lambda_{,d} \tag{6.23}$$

where $\tilde{\nabla}_c(\rho)Y^{tr_c} = 0$ but it is straightforward to show that the second order elliptic operator on the right hand side of Eq. (6.22) maps divergence free vectors to the space orthogonal to gradients and furthermore, that this operator has trivial kernel within the space of divergence free vector fields.

Thus the only possible solutions of Eq. (6.20) must be expressible in the form of Eq. (6.21) where however, $Y^c = \rho^{cd}\Lambda_{,d}$ for some function $\Lambda$.

Substituting this result into Eq. (6.20) one gets the following equation for $\Lambda$

$$(6.24)\qquad \tilde{\nabla}_\ell(\rho)\left(\frac{\zeta_b^\ell}{\mu_\rho}\right) = \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)(\Lambda_{,b}) - \frac{1}{2}\Lambda_{,b} = -\frac{1}{2\tau}\partial_b\sqrt{1+\frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}}$$

where

$$(6.25)\qquad \frac{\zeta_b^\ell}{\mu_\rho} = \left(\frac{\zeta_b^{TT\ell}}{\mu_\rho}\right) + 2\tilde{\nabla}_b(\rho)\tilde{\nabla}^\ell(\rho)\Lambda - \delta_b^\ell(\tilde{\nabla}_r(\rho)\tilde{\nabla}^r(\rho)\Lambda).$$

Fortunately, one easily proves again using the constancy of curvature of $\rho$, that

$$(6.26)\qquad \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda_{,b} = \partial_b\left[\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda - \frac{1}{2}\Lambda\right]$$

so that Eq. (6.24) becomes

$$(6.27)\qquad \partial_b\left\{\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda - \Lambda + \frac{1}{2\tau}\sqrt{1+\frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}}\right\} = 0.$$

Thus we must have

$$(6.28)\qquad 2\tau(\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda - \Lambda) + \sqrt{1+\frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}} = C = \text{constant}$$

where $C$ could only depend upon $\tau$ at most. Note however that, if we define

$$(6.29)\qquad \Lambda^\dagger = 2\tau\Lambda + C$$

then $\Lambda^\dagger$ satisfies

$$(6.30)\qquad \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \Lambda^\dagger + \sqrt{1+\frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2} = 0$$

where

$$(6.31)\qquad 2\tau\frac{\zeta_b^\ell}{\mu_\rho} = \left(\frac{2\tau\zeta_b^{TT\ell}}{\mu_\rho}\right) + 2\tilde{\nabla}_b(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \delta_b^\ell\left(\tilde{\nabla}_m(\rho)\tilde{\nabla}^m(\rho)\Lambda^\dagger\right)$$

and wherein the undetermined $C$ plays no role.

Equation (6.30) is, in view of the Hessian of $\Lambda^\dagger$ which appears in the expression (6.31) for $\zeta_b^\ell$, a fully non-linear equation for $\Lambda^\dagger$. We shall show however that it always has a unique, smooth solution for an arbitrary choice of the TT tensor $\left(\frac{\zeta_b^{TT\ell}}{\mu_\rho}\right)$ which therefore parametrizes the space of solutions $\Lambda^\dagger$ and hence, through Eq. (6.31), $\left(\frac{\zeta_b^\ell}{\mu_\rho}\right)$.

Let us first, however, show that the density $\zeta_b^{TT\,\ell}$ is necessarily a constant of the motion (i.e., satisfies $\partial_\tau \zeta_b^{TT\,\ell} = 0$) for any solution of the reduced field equations. This follows from substituting the expression for $\mu_\gamma \gamma^{a\ell} = \mu_g g^{a\ell}$ given by Eq. (6.16) into Eq. (6.14), expanding out $\zeta_b^\ell$ through the use of Eq. (6.21) and then exploiting the transverse-traceless character of $\tilde{\ell}^{TT}_{a\ell(\alpha)}$ (with respect to the fixed metric $\rho_{ab}$) to finally derive that

$$-P_\alpha = \int_\Sigma \tilde{\ell}^{TT}_{a\ell(\alpha)} \rho^{ab} \zeta_b^{TT\,\ell}. \tag{6.32}$$

Thus the components of $\zeta_b^{TT\,\ell}$ in the basis defined by the time-independent family of (transverse-traceless) tensors $\{\tilde{\ell}^{TT}_{(\alpha)} \mid \alpha = 1, \dots 6\,\text{genus}\,(\Sigma) - 6\}$ are simply the constants of the motion $\{-P_\alpha\}$ identified previously.

## 7. Some basic properties of the $\Lambda^\dagger$ equation

At a point $x_{\min} \in \Sigma$ where a solution $\Lambda^\dagger$ to equation (6.30) achieves its minimum value $\Lambda^\dagger_{\min}$ one evidently has (since $\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger(x_{\min}) \geq 0$)

$$\Lambda^\dagger_{\min} = \Lambda^\dagger(x_{\min}) \geq \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2 (x_{\min})} \geq 1. \tag{7.1}$$

Thus $\Lambda^\dagger_{\min} \geq 1$ for any solution. On the other hand, if we square equation (6.30) and integrate over $\Sigma$, we obtain the integral formula

$$\int_\Sigma \left(\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \Lambda^\dagger\right)^2 d\mu_\rho = \int_\Sigma \left(1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2\right) d\mu_\rho \tag{7.2}$$

which is also satisfied by an arbitrary solution. Upon substituting the expression (6.31) for $\frac{2\tau\zeta}{\mu_\rho}$ and simplifying the result through the use of ${}^{(2)}R(\rho) = -1$ and the transverse-traceless character of $\zeta_b^{TT_\ell}$ one easily derives

(7.3)
$$\int_\Sigma \left\{[(\Lambda^\dagger)^2 - 1] + (\tilde{\nabla}_\ell(\rho)\Lambda^\dagger)(\tilde{\nabla}^\ell(\rho)\Lambda^\dagger)\right\} d\mu_\rho = 2\tau^2 \int_\Sigma \left(\frac{\zeta_b^{TT_\ell}}{\mu_\rho}\frac{\zeta_\ell^{TT_b}}{\mu_\rho}\right) d\mu_\rho$$

from which, rather surprisingly, all second derivatives have canceled.

An immediate consequence of Eq. (7.3) is that if either $\tau = 0$ or $\tau \neq 0$ but $\zeta_b^{TT_\ell} = 0$, so that the right hand side vanishes, then since $\Lambda^\dagger \geq 1$ a priori, we get that $\Lambda^\dagger = 1$ identically on $\Sigma$. We are interested in curves of solutions to Eq. (6.30) determined by data $\{\rho_{ab}, \zeta_b^{TT_\ell}\}$ specified on $\Sigma$ and parametrized by $\tau \in [0, -\infty)$. The above result has shown that any such curve will necessarily satisfy $\Lambda^\dagger|_{\tau=0} = 1$ on $\Sigma$.

To establish the uniqueness of solutions to the $\Lambda^\dagger$ equation (for arbitrary but fixed $\rho_{ab}$, $\zeta_a^{TT_\ell}$, and $\tau$) assume that $\Lambda^\dagger = \theta$ and $\Lambda^\dagger = \psi$ are any two such solutions (corresponding to the same given data) and consider the curve of functions $\chi_t = t\theta + (1-t)\psi$ for $0 \leq t \leq 1$. Clearly $\chi_0 = \psi, \chi_1 = \theta$ and $\frac{d\chi}{dt} = \theta - \psi$. Noting that

$$\begin{aligned}(7.4)\qquad &\int_0^1 dt \frac{d}{dt}\left\{\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\chi_t - \chi_t + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta_t}{\mu_\rho}\right|^2}\right\} \\ &= \left[\tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\chi_t - \chi_t + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta_t}{\mu_\rho}\right|^2}\right]\Bigg|_{t=0}^{t=1} = 0\end{aligned}$$

(where $\zeta_t$ is the expression for $\zeta$ with $\Lambda^\dagger$ replaced by $\chi_t$) we carry out the $t$-differentiation explicitly on the left hand side and express the result as

$$(7.5)\qquad \int_0^1 dt \left\{\rho^{\ell m} + \rho^{mb}\frac{\left(\frac{2\tau}{\mu_\rho}\zeta_t\right)^\ell_{\ b}}{\sqrt{1 + \frac{1}{2}\left(\frac{2\tau}{\mu_\rho}\zeta_t\right)^2}}\right\}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)(\theta - \psi) - (\theta - \psi) = 0.$$

Our aim is to show that this is an elliptic equation for $(\theta - \psi)$ of the form

$$(7.6)\qquad \bar{g}^{\ell m}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)(\theta - \psi) - (\theta - \psi) = 0$$

where $\bar{g}^{\ell m}$ is a positive definite (inverse) metric on $\Sigma$. It will then follow from an elementary maximum principle argument that both $\theta - \psi \geq 0$ on $\Sigma$ and $\psi - \theta \geq 0$ on $\Sigma$ from which one thus gets that $\theta = \psi$ on $\Sigma$ and hence that solutions are unique.

The set of (inverse) Riemannian metrics on $\Sigma$ is an open cone in the set of symmetric tensor fields so that to show that

$$(7.7)\qquad \bar{g}^{\ell m} := \int_0^1 dt \left\{\rho^{\ell m} + \rho^{mb}\frac{\left(\frac{2\tau}{\mu_\rho}\zeta_t\right)^\ell_{\ b}}{\sqrt{1 + \frac{1}{2}\left(\frac{2\tau}{\mu_\rho}\zeta_t\right)^2}}\right\}$$

is Riemannian it suffices to show that

$$\bar{g}_t^{\ell m} := \rho^{\ell m} + \frac{\frac{2\tau\zeta_t^{\ell m}}{\mu_\rho}}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta_t}{\mu_\rho}\right|^2}} \tag{7.8}$$

is positive definite for each $t \in [0,1]$ (where here we write $\zeta_t^{\ell m}$ for $\rho^{mb}(\zeta_t)_b^\ell$). At any point of $\Sigma$ let $v^a = \rho^{ab} v_b$ be an arbitrary unit vector with respect to $\rho$ (i.e., $\rho_{ab} v^a v^b = 1$) and compute $\bar{g}_t^{\ell m} v_\ell v_m$ using the formula above. For convenience choose coordinates so that $\rho^{\ell n} = \delta^{\ell n}$ at the chosen point and, if needed, make a further, orthogonal transformation to diagonalize the (real-symmetric-traceless) matrix $\frac{2\tau\zeta_t^{\ell m}}{\mu_\rho}$ at the chosen point, writing the result as

$$\left(\frac{2\tau\zeta_t^{\ell n}}{\mu_\rho}\right) = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \tag{7.9}$$

with $\rho^{ab} v_a v_b = v_1^2 + v_2^2 = 1$ at the chosen point. Evaluating $\bar{g}_t^{\ell n} v_\ell v_n$ at this point we get

$$\bar{g}_t^{\ell m} v_\ell v_m = 1 + \frac{u(v_1^2 - v_2^2)}{\sqrt{1+u^2}}. \tag{7.10}$$

Since $\left|v_1^2 - v_2^2\right| \leq 1$ and $|\frac{u}{\sqrt{1+u^2}}| < 1$ it follows that $\bar{g}_t^{\ell n}$ is positive definite at the chosen point which was an arbitrary point of $\Sigma$.

The foregoing calculation is also directly relevant to establishing the ellipticity of the equation for $\Lambda^\dagger$, a result we shall need for proving the existence of solutions via the method for continuity. To see this define, for fixed $\{\rho_{ab}, \tau, \zeta_b^{TT\ell}\}$, the non-linear operator

$$\mathcal{F}(\cdot, \Lambda^\dagger) = \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \Lambda^\dagger + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2} \tag{7.11}$$

where, as before,

$$\frac{2\tau\zeta_b^\ell}{\mu_\rho} = \frac{2\tau\zeta_b^{TT\ell}}{\mu_\rho} + 2\tilde{\nabla}_b(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \delta_b^\ell \tilde{\nabla}_m(\rho)\tilde{\nabla}^m(\rho)\Lambda^\dagger \tag{7.12}$$

and compute the first variation about an arbitrary $C^2$ configuration $\Lambda^\dagger$. Designating the variation of $\Lambda^\dagger$ by $\delta\Lambda^\dagger$ one gets

$$\begin{aligned} D\mathcal{F}(\cdot, \Lambda^\dagger) \cdot \delta\Lambda^\dagger &= \left\{ \rho^{\ell m} + \frac{\frac{2\tau\zeta^{\ell m}}{\mu_\rho}}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}} \right\} \tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)\delta\Lambda^\dagger - \delta\Lambda^\dagger \\ &= \tilde{g}^{\ell m}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)\delta\Lambda^\dagger - \delta\Lambda^\dagger \end{aligned} \tag{7.13}$$

from which the ellipticity (i.e., injectivity of the principle symbol) follows from the calculation done just above which showed that

$$\tilde{g}^{\ell m} = \rho^{\ell m} + \frac{\frac{2\tau\zeta^{\ell m}}{\mu_\rho}}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}} \tag{7.14}$$

is Riemannian.

Since the operator $\tilde{g}^{\ell m}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)$ clearly plays an important role in our analysis it is of some interest to explore its relationship to other natural elliptic operators arising in this context. Note that, for any arbitrary $C^2$-function $\lambda$ we can write, using Eq. (6.16)

$$\mu_g g^{a\ell}\lambda_{,a} = 2\tau\rho^{ab}\lambda_{,a}\zeta_b^\ell + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}\,\mu_\rho\rho^{a\ell}\lambda_{,a}. \tag{7.15}$$

Since $\mu_g g^{a\ell}\lambda_{,a}$ is a vector density its ordinary divergence $\partial_\ell(\mu_g g^{a\ell}\lambda_{,a})$ can be identified with its covariant divergence relative to the $\rho$ metric whence

$$\begin{aligned}\tilde{\nabla}_\ell(\rho)[\mu_g g^{a\ell}\lambda_{,a}] &= \partial_\ell[\mu_g g^{a\ell}\lambda_{,a}] = \mu_g(\tilde{\nabla}_\ell(g)\tilde{\nabla}^\ell(g)\lambda) = \mu_g\Delta_g\lambda \\ &= \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}\,\mu_\rho\tilde{\Delta}_\rho\lambda + 2\tau\rho^{ab}\zeta_b^\ell\left[\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_a(\rho)\lambda\right]\end{aligned} \tag{7.16}$$

where we have used Eq. (6.20) to simplify the $\rho$-divergence of the right hand side of Eq. (7.16) above. Rewriting this result as

$$\frac{\mu_g}{\mu_\rho\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}}\Delta_g\lambda = \left\{\rho^{\ell m} + \frac{\frac{2\tau\zeta^{\ell m}}{\mu_\rho}}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}}\right\}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)\lambda \tag{7.17}$$

which can be further re-expressed using the formula

$$\frac{\tau^2\mu_g}{\mu_\rho} = 1 + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}, \tag{7.18}$$

which follows from Eq. (6.16), one thus finally has

$$\left(\frac{1 + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}}\right)\left(\frac{1}{\tau^2}\Delta_g\lambda\right) = \tilde{g}^{\ell m}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)\lambda \tag{7.19}$$

as a formula relating these basic elliptic operators.

## 8. Existence of solutions to the $\Lambda^\dagger$ equation

To establish the existence of solutions to the $\Lambda^\dagger$ equation, we first need to specify more precisely the function spaces for which the equation is defined. We shall then apply the standard method of continuity to show that for any fixed metric $\rho$ and $TT$ tensor density $\zeta_a^{TTb}$ there exists a unique solution $\Lambda^\dagger$ for all $\tau$ in the interval $[0, -\infty)$.

Let $H^s(\Sigma)$ designate the Sobolev space of square integrable functions on $\Sigma$ having square integrable (distributional) derivatives up to order $s$ and let $\mathcal{M}^s(\Sigma)$ represent the corresponding space of $H^s$-Riemannian metrics on $\Sigma$. In the following, we shall assume that $\rho \in \mathcal{M}^s(\Sigma)$ for $s > 4$ and, as usual, also require that ${}^{(2)}R(\rho) = -1$ on $\Sigma$. For any such $\rho$ let $\mathcal{S}_d^{TT_{s-1}}(\Sigma)$ designate the (finite dimensional) space of $H^{s-1}$ tensor densities of type $(1,1)$ which, relative to the chosen $\rho$, are symmetric (i.e., satisfy $\rho^{ab}\zeta_b^{TT_c} = \rho^{cb}\zeta_b^{TT_a}$), transverse and traceless on $\Sigma$. In addition, consider for the same chosen $s > 4$, functions $\Lambda^\dagger \in H^{s+1}(\Sigma)$. The Schauder ring property of $H^s$ maps in 2 dimensions then guarantees that the map

$$\mathcal{F} : \mathcal{S}_d^{TT_{s-1}}(\Sigma) \times H^{s+1}(\Sigma) \to H^{s-1}(\Sigma) \tag{8.1}$$

$$\left(\zeta^{TT}, \Lambda^\dagger\right) \mapsto \tilde{\nabla}_\ell(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \Lambda^\dagger + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}$$

where, as above,

$$\frac{2\tau\zeta_b^\ell}{\mu_\rho} = \frac{2\tau\zeta_b^{TT_\ell}}{\mu_\rho} + 2\tilde{\nabla}_b(\rho)\tilde{\nabla}^\ell(\rho)\Lambda^\dagger - \delta_b^\ell\left(\tilde{\nabla}_m(\rho)\tilde{\nabla}^m(\rho)\Lambda^\dagger\right) \tag{8.2}$$

is a smooth (i.e., $C^\infty$) map between the indicated function spaces. Here, as before, $\tau$ is a real constant in the range $[0, -\infty)$.

In Sec. 7 we showed that the equation $\mathcal{F}(\zeta^{TT}, \Lambda^\dagger) = 0$ has the unique solution $\Lambda^\dagger = 1$ when $\tau\zeta^{TT} = 0$, on $\Sigma$ and that the equation was elliptic at an arbitrary configuration which is sufficiently smooth (the latter being here guaranteed by our Hilbert space assumptions for $\rho, \zeta^{TT}$ and $\Lambda^\dagger$). We want to appeal to the (Banach space version of the) implicit function theorem to show that a solution $\Lambda^\dagger$ to $\mathcal{F}(\zeta^{TT}, \Lambda^\dagger) = 0$ is implicitly determined in terms of $\tau\zeta^{TT}$ on some neighborhood of any given solution. We already know that any such solution must be unique so we need only check that the Frechet derivative of $\mathcal{F}$ with respect to its second argument $\Lambda^\dagger$ defines (at any "background" configuration $(\tau\zeta^{TT}, \Lambda^\dagger) \in \mathcal{S}_d^{TT_{s-1}}(\Sigma) \times H^{s+1}(\Sigma)$) an isomorphism of the function spaces $H^{s+1}(\Sigma)$ and $H^{s-1}(\Sigma)$, i.e., that

$$D_2\mathcal{F}(\zeta^{TT}, \Lambda^\dagger) \cdot \delta\Lambda^\dagger = \sigma \in H^{s-1}(\Sigma) \tag{8.3}$$

is uniquely solvable for $\delta\Lambda^\dagger \in H^{s+1}(\Sigma)$ for arbitrary $\sigma \in H^{s-1}(\Sigma)$. Here $D_2\mathcal{F}$ is given by the first variation formula (7.13). At a background for which $\tau\zeta^{TT} = 0$ (so that $\Lambda^\dagger = 1$) this equation reduces to

$$\rho^{\ell m}\tilde{\nabla}_\ell(\rho)\tilde{\nabla}_m(\rho)\delta\Lambda^\dagger - \delta\Lambda^\dagger = \sigma \tag{8.4}$$

but, for any $H^{s+1}$ metric $\rho$, the operator $\Delta_\rho - 1$ provides a well-known isomorphism of $H^{s+1}(\Sigma)$ and $H^{s-1}(\Sigma)$ via the Fredholm alternative.

When $\tau \neq 0$, we can combine Eqs. (7.13), (7.14) and (7.19) to re-express the linearized equation (8.3) as

$$\frac{\left(1 + \sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}\right)}{\sqrt{1 + \frac{1}{2}\left|\frac{2\tau\zeta}{\mu_\rho}\right|^2}} \frac{1}{\tau^2}\Delta_g\delta\Lambda^\dagger - \delta\Lambda^\dagger = \sigma \tag{8.5}$$

where $g$ is the $H^{s-1}$ metric defined (at the background solution) by Eqs. (6.16) and (8.2). The factor involving the square roots lies in $H^{s-1}(\Sigma)$ and so does not cause a serious problem. Multiplying the equation by its inverse merely converts the source $\sigma$ to another element of $H^{s-1}(\Sigma)$ and replaces the coefficient of $\delta\Lambda^\dagger$ by a strictly positive coefficient that is sufficiently smooth for the application of the standard elliptic theory argument.

A difficulty seems to arise however, though the fact that the $g$ metric only lies in $H^{s-1}$ and thus that its Christoffel symbols generically only lie in $H^{s-2}(\Sigma)$ and not in $H^{s-1}(\Sigma)$, as was true of the $\rho$ metric. This seems to interfere with the desired $H^{s+1}$-smoothness of $\delta\Lambda^\dagger$ (when the source function is taken to lie in $H^{s-1}$). Fortunately, however, our setup ensures that the identity map from $(g,\Sigma)$ to $(\rho,\Sigma)$ is harmonic and thus that

$$g^{ab}(\Gamma^c_{ab}(g) - \tilde{\Gamma}^c_{ab}(\rho)) = 0. \tag{8.6}$$

Using this equation to re-express the $g$-Laplacian on functions we get that

$$\Delta_g\delta\Lambda^\dagger = g^{ab}\tilde{\nabla}_a(\rho)\tilde{\nabla}_b(\rho)\delta\Lambda^\dagger \tag{8.7}$$

and hence that $\Delta_g$ maps $H^{s+1}$ to $H^{s-1}$ as desired with no fatal extra loss of derivatives.

It follows from the standard Fredholm argument again that $D_2\mathcal{F}(\zeta^{TT},\Lambda^\dagger)$ yields the needed isomorphism between $H^{s-1}$ and $H^{s+1}$ and hence that the equation $\mathcal{F}(\zeta^{TT},\Lambda^\dagger) = 0$ uniquely and implicitly determines $\Lambda^\dagger$ as a smooth functional of $\zeta^{TT}$ on some neighborhood of any particular solution. Since we know the unique solution when $\tau = 0$, we deduce that, for any chosen $\zeta^{TT} \in \mathcal{S}_d^{TT_{s-1}}(\Sigma)$, there exists a $\tau_0 \in (0,-\infty)$ such that a unique solution $\Lambda^\dagger$ exists for all $\tau \in [0,\tau_0)$.

To show directly that every solution extends to the full interval $\tau \in [0,-\infty)$ we need to establish that the $H^{s+1}(\Sigma)$ -norm of $\Lambda^\dagger$ cannot blow up until $\tau$ exhausts this interval. Ideally, this result should follow from estimates derived directly from the Monge-Ampêre equation itself and indeed the derivation of such estimates is part of the aim of an ongoing project with S.-T. Yau which seeks to sharply characterize the asymptotics of solutions in the limit as $\tau \searrow -\infty$ (i.e., at their big bang initial singularities). We shall describe this project more fully in the concluding section below, but to complete the present argument, shall here instead give an elementary proof (based upon a slight refinement of the ideas from Ref. [**6**]) that all the solution curves (of the reduced Hamiltonian system) do indeed exhaust the interval $\tau \in [0, -\infty)$.

The reduced Hamiltonian system under study is a globally smooth system of first order ordinary differential equations defined on $T^*\mathcal{T}(\Sigma) \times \mathbf{R}^+$ (when expressed in terms of the time variable $T = -\frac{1}{\tau} \in \mathbf{R}^+$). From Eq. (5.5) and the properties of the function $\mathcal{S}$ it follows that the momenta, $p_\alpha(T) = \frac{\partial \mathcal{S}}{\partial q^\alpha}(q^\alpha(T), \rho, T)$, remain well-defined (and hence, through Eq. (4.12) yield a correspondingly well-defined $p^{TTab}(x^c, T)$) so long as the base curve, expressed in coordinates through functions $\{q^\alpha(T)\}$, persists as a curve in $\mathcal{T}(\Sigma)$. From the smooth character of the associated differential equations (i.e., Hamilton's equations) the premature breakdown of such a solution could only occur if the base curve runs "off-the-edge" of Teichmüller space before $\tau \searrow -\infty$ (or, equivalently, before $T \searrow 0$). We shall exclude such a breakdown by exploiting known properties of the Dirichlet energy (with the target metric $\rho$ held fixed) as a proper function (i.e., an exhaustion function) on $\mathcal{T}(\Sigma)$ and by deriving estimates which show that this Dirichlet energy cannot blow up along a solution curve until $\tau \searrow -\infty$. These same estimates will also show that the Dirichlet energy tends to its (unique) infimum in the opposite limit (as $\tau \nearrow 0$ or $T \nearrow \infty$) and, again from known properties of this energy function, that this implies that the "moving," conformal metric $\gamma_T$ tends to the target metric $\rho$ as $T \to \infty$, a result which also follows from the method of continuity argument.

First of all note that, using Eqs. (2.4) and (2.5), Eq. (4.15) can be re-expressed as

$$\tau^2 = -\Delta_g N + N\left[\left|k^{TT}\right|_g^2 + \frac{1}{2}\tau^2\right]. \tag{8.8}$$

Using the inequality given by (2.44), that $\left|k^{TT}\right|_g^2 < \frac{\tau^2}{2}$, one easily derives the maximum principle bounds for the lapse function $N$:

$$1 < N \leq 2 \quad \text{on } \Sigma. \tag{8.9}$$

Furthermore, the integral of Eq. (8.8) over $\Sigma$ yields the formula

$$\tau^2 \int_\Sigma d\mu_g(2 - N) = 2\int_\Sigma d\mu_g N \left|k^{TT}\right|_g^2. \tag{8.10}$$

Now, from Eqs. (3.6) and (3.7), taking $T = -\frac{1}{\tau}$ as before, we find that

$$(8.11)\quad \begin{aligned} \frac{\partial}{\partial T}\int_\Sigma d\mu_g \left(k_c^{TT\,d} k_d^{TT\,c}\right) &= \frac{\partial}{\partial T}\left\{\frac{1}{2}\int_\Sigma \tau^2 d\mu_g + \int_\Sigma d\mu_g\, {}^{(2)}R_{(g)}\right\} \\ &= \tau^3 \int_\Sigma d\mu_g \left(1 - \frac{N}{2}\right) = \tau \int_\Sigma d\mu_g N \left|k^{TT}\right|_g^2 \\ &= -\frac{1}{T}\int_\Sigma d\mu_g N \left(k_c^{TT\,d}\, k_d^{TT\,c}\right) \end{aligned}$$

where, in intermediate steps, we have used the ADM field equation

$$(8.12)\quad \begin{aligned} g_{ab,T} &= \frac{2N}{\mu_g}(\pi_{ab} - g_{ab} tr_g \pi) + (\mathcal{L}_{(2)X}\, g)_{ab} \\ &= -2N k_{ab} + (\mathcal{L}_{(2)X}\, g)_{ab} \end{aligned}$$

to evaluate $\partial_T \mu_g = \frac{1}{2}\mu_g g^{ab}\partial_T g_{ab}$.

In view of the upper and lower bounds on $N$ given by (8.9), one easily derives from Eq. (8.11) the following bounds on

$$(8.13)\quad F := \int_\Sigma d\mu_g (k_c^{TT\,d}\, k_d^{TT\,c}).$$

Either

(i) $F(T) = 0\, \forall\, T \in [0,\infty)$,

or

(ii) $F(T_0)\left(\frac{T_0}{T_1}\right)^2 \leq F(T_1) < F(T_0)\frac{T_0}{T_1}$
$\forall\, T_0, T_1$ such that $T_0 < T_1$ with $T_0, T_1 \in (0,\infty)$.

Thus, unless $F(T) = 0$ identically, one has

$$\frac{\text{const}}{T^2} \leq F(T) < \frac{\text{const}}{T} \text{ as } T \to \infty$$

and

$$\frac{\text{const}}{T} < F(T) \leq \frac{\text{const}}{T^2} \text{ as } T \to 0$$

and thus that the Dirichlet energy function (c.f., Eqs. (3.6) and (3.7))

$$(8.14)\quad \begin{aligned} \mathcal{A}(\gamma_T, \rho, Id) &= 2\int_\Sigma d\mu_g k_c^{TT\,d}\, k_d^{TT\,c} - \int_\Sigma d\mu_g\, {}^{(2)}R(g) \\ &= 2F(T) - \int_\Sigma d\mu_g\, {}^{(2)}R(g) \end{aligned}$$

cannot blow up until $T \to 0$ but definitely does blow up in this limit unless $F(T) = 0$ identically (which corresponds to the trivial solution $\gamma_T = \rho\ \forall\ T \in [0,\infty)$). Furthermore, all solutions have the property that

$$(8.15)\quad \mathcal{A}(\gamma_T, \rho, Id) \underset{T \longrightarrow \infty}{\longrightarrow} -\int_\Sigma d\mu_g\, {}^{(2)}R(g)$$

so that this energy asymptotes to its global infimum in the indicated limit. From the aforementioned well-known properties of this function (c.f., the discussion in Ref. [**14**]), one sees that all nontrivial solution curves run "off-the-edge" of Teichmüller space precisely as $T \searrow 0$ and that every solution curve (including the trivial ones with $\gamma_T = \rho$) has the property that $\gamma_T \underset{T\to\infty}{\longrightarrow} \rho$ since the (two-point) Dirichlet energy achieves its global infimum precisely at coincident points.

This latter result can also be recovered from the method of continuity argument given above. We showed therein that, for fixed $\rho \in \mathcal{M}^s(\Sigma), s > 4$ and arbitrary $\zeta^{TT} \in \mathcal{S}_d^{TT_{s-1}}(\Sigma)$ the Monge-Ampêre equation yielded $\Lambda^\dagger \in H^{s+1}(\Sigma)$ as a smooth functional of $\tau\zeta^{TT}$ for $\tau$ in some interval of the form $[0, \tau_0), \tau_0 < 0$. Furthermore, the limiting value as $\tau \nearrow 0$ was always given by the unique solution, $\Lambda^\dagger = 1$, of this equation when $\tau\zeta^{TT} = 0$. Now, recalling Eqs, (6.16), (6.17) and (6.31) we see that the "rescaled" metric $g^*_{ab} := \tau^2 g_{ab}$ satisfies

$$g^{*a\ell} = \frac{\left[2\tau\frac{\rho^{ab}}{\mu_\rho}\zeta^\ell_b + \sqrt{1 + \frac{2\tau^2|\zeta|}{(\mu_\rho)^2}}\rho^{a\ell}\right]}{\left(1 + \sqrt{\left(1 + \frac{2\tau^2|\zeta|^2}{(\mu_\rho)^2}\right)}\right)} \tag{8.16}$$

and has a well-defined limit as $\tau \nearrow 0$ given by

$$g^*_{ab} \underset{\tau\nearrow 0}{\longrightarrow} 2\rho_{ab} \tag{8.17}$$

since $\tau\zeta^b_a \to 0$ in that limit. Since $\gamma_{ab}$ is obtained from $g_{ab}$ (or equivalently from $g^*_{ab}$) by uniformization, it follows that $\gamma_T \underset{T\to\infty}{\longrightarrow} \rho$ in this limit.

## 9. Einstein solution curves and ray structures on Teichmüller space

In the previous section we saw that, for any fixed metric $\rho$ having ${}^{(2)}R(\rho) = -1$, the $TT$ symmetric tensors relative to $\rho$ (i.e., the tensors $\frac{\rho_{ab}}{\mu_\rho}\zeta^{TT_b}_c$ formed from the mixed $TT$ densities $\zeta^{TT_b}_a$ used there) determine solution curves to the reduced vacuum Einstein equations in CMCSH (constant-mean-curvature-spatially-harmonic) gauge. These curves fall naturally into one parameter families generated by the scale invariance of Einstein's vacuum field equations – replacing $\zeta^{TT}$ by $\lambda\zeta^{TT}$, where $\lambda$ is a constant greater than zero, yields a rescaled solution for which the slice originally having mean curvature $\tau$ now has mean curvature $\tau/\lambda$. To obtain all possible vacuum solutions one must, in addition, allow $\rho$ to vary over a model of Teichmüller space for the given surface $\Sigma$, e.g., over a global cross-section for the trivial $\mathcal{D}_0(\Sigma)$-bundle $\mathcal{M}_{-1}(\Sigma) \to \mathcal{M}_{-1}(\Sigma)/\mathcal{D}_0(\Sigma) \approx \mathcal{T}(\Sigma)$ where $\mathcal{M}_{-1}(\Sigma)$ represents the space of metrics $\gamma$ with ${}^{(2)}R(\gamma) = -1$, $\mathcal{D}_0(\Sigma)$ designates the group of diffeomorphisms of $\Sigma$ isotopic to the identity and $\mathcal{T}(\Sigma)$ is the abstract

Teichmüller space of $\Sigma$. One can construct such cross sections in at least two distinct ways using harmonic maps as discussed for example by Tromba in [**14**, Sec. (3.4)]. In one such construction (due to Earle and Eells [**16**]) the section consists of all metrics $\rho \in \mathcal{M}_{-1}(\Sigma)$ such that the identity map from a fixed domain $(\gamma, \Sigma)$ to $(\rho, \Sigma)$ is harmonic. In a complementary section discussed by Tromba one holds the target $(\rho, \Sigma)$ fixed and considers all $\gamma \in \mathcal{M}_{-1}(\Sigma)$ such that again the identity from $(\gamma, \Sigma)$ to $(\rho, \Sigma)$ is harmonic. Our result shows that in effect $\rho$ and its conjugate variable $\zeta^{TT}$ represent asymptotic "Cauchy data" which, when prescribed in the limit $\tau \nearrow 0$, determine solution curves uniquely via the resolution of the $\Lambda^{\dagger}$ equation discussed above.

We know from the work in [**6**] that every such solution curve extends to the full interval $\tau \in [0, -\infty)$ which represents cosmological expansion from a big bang singularity at $\tau \to -\infty$ to the limit of infinite area as $\tau \nearrow 0$. Each corresponding curve of metrics $g_\tau$ can be smoothly and uniquely uniformized to yield a curve $\gamma_\tau$ which lies in a particular Tromba-section of $\mathcal{M}_{-1}(\Sigma)$, namely the submanifold consisting of all metrics $\gamma$ such that $Id : (\gamma, \Sigma) \to (\rho, \Sigma)$ is harmonic for the appropriately chosen asymptotic conformal metric $\rho$. The results in [**6**] show that, except for the trivial, fixed point solutions generated by $\zeta^{TT} = 0$, every solution curve runs "off-the-edge" of Teichmüller space as $\tau \to -\infty$. In this section, we want to consider the families of such uniformized solution curves which all have the same asymptotic limit $\rho$ (i.e., the curves determined by the asymptotic data $(\rho, \zeta^{TT})$ for arbitrary $\zeta^{TT}$). The aim is to show that such families (one for each choice of $\rho$) define so-called "ray structures" on Teichmüller space which are distinct from but somewhat complementary to the ray structures defined by Wolf [**8**]. In a similar spirit, we want to show how one can use the $TT$ tensors at $\rho$ to define a global coordinate system for Teichmüller space that is complementary to the one defined by Wolf (wherein one used $TT$ tensors relative to the domain metric of a harmonic mapping rather than at the target as we do).

Thus we fix $\rho \in \mathcal{M}_{-1}(\Sigma)$ and focus on that subset of solutions determined by data $\{\rho, \zeta^{TT}\}$ where $\zeta^{TT}$ is transverse-traceless and symmetric with respect to $\rho$. For the moment, let us exclude the trivial solution corresponding to $\zeta^{TT} = 0$ and look only at the non-fixed-points. We claim that no two distinct solutions, aside from those obtained by rescaling any given one, ever intersect (except asymptotically where they all tend to $\rho$). To see this, suppose an intersection did exist at some (uniformized) metric $\gamma$. By rescaling we can always arrange that the intersection point for each of the two curves corresponds to the same value of the mean curvature $\tau$. The complementary $TT$ tensors $\overset{(1)}{k}{}^{TT}_{ab}$ and $\overset{(2)}{k}{}^{TT}_{ab}$ which, together with the common $\gamma_{ab}$, make up the two sets of conformal Cauchy data at mean curvature time $\tau$ for these solutions must be distinct (i.e., have $\overset{(1)}{k}{}^{TT} \neq \overset{(2)}{k}{}^{TT}$) since otherwise, by uniqueness of solutions of the reduced Einstein equations in CMCSH gauge (an elementary ODE uniqueness result in $2+1$ dimensions) they would yield identical solutions contrary to assumption. However our setup would then show that Wolf's basic equation (c.f., Eq. (2.28) above)

$$\rho_{ab} = \left\{ \left( e^{-4\lambda} \gamma^{de} \gamma^{cf} k^{TT}_{ce} k^{TT}_{df} + \frac{1}{2}\tau^2 \right) e^{2\lambda} \gamma_{ab} + 2\tau k^{TT}_{ab} \right\} \tag{9.1}$$

(where $\lambda$ is uniquely determined by the solution of the Lichnerowicz equation (2.23)) is then satisfied for fixed $\{\rho, \gamma, \tau\}$ by two distinct choices for $k^{TT}$, namely $\overset{(1)}{k}{}^{TT}$ and $\overset{(2)}{k}{}^{TT}$. But this is impossible by Wolf's result that shows that the $TT$ tensors $k^{TT}$ relative to a fixed metric $\gamma$ define a global coordinate chart for the Earle-Eells section of $\mathcal{M}_{-1}(\Sigma)$ representing Teichmüller space $\mathcal{T}(\Sigma)$. In other words, a given target metric $\rho$ satisfying Eq. (9.1) corresponds to a unique $k^{TT}$ (note that Wolf, who was not considering Einstein's equations, has effectively taken $\tau = -1$ in our formulation).

By a similar argument we can show that every metric $\gamma$ in a Tromba-section based on $\rho$ is attained by a solution curve (unique up to scaling if $\gamma \neq \rho$) which is asymptotic to $\rho$. The result is trivial if $\gamma = \rho$ since one only need take the fixed point solution so assume that $\gamma \neq \rho$ but that $\gamma$ lies in the Tromba-section based on $\rho$ (i.e., all $\gamma \in \mathcal{M}_{-1}(\Sigma)$ such that $Id : (\gamma, \Sigma) \to (\rho, \Sigma)$ is harmonic). By Wolf's result (after choosing say $\tau = -1$ for convenience to eliminate the scaling freedom) there exists, for the chosen $\gamma$ and $\rho$, a unique $k^{TT}$ satisfying Eq. (9.1) (with $\lambda$ determined uniquely by the Lichnerowicz equation (2.23)). However, our arguments from Sec. 8 have shown that this conformal Cauchy data generates a solution whose asymptotic conformal metric is $\rho$. To compute the complementary asymptotic data $\zeta^{TT}$ for the corresponding solution curve we can appeal to Eqs. (6.15) and (6.21) which show that $\zeta^{TTb}_a$ is simply the transverse-traceless summand (relative to the $\rho$ metric decomposition here!) of the density

$$\zeta^b_a = \mu_g g^{bc} k^{TT}_{ac} = \mu_\gamma \gamma^{bc} k^{TT}_{ac} \tag{9.2}$$

and further (in view of Eqs. (6.14)–(6.16) and recalling that $\mu_g g^{ab} = \mu_\gamma \gamma^{ab}$) that this projection is independent of the value of $\tau$ at which $(\gamma_{ab}, k^{TT}_{ab})$ are evaluated (and hence of the arbitrary choice $\tau = -1$ made for convenience above).

## 10. Lagrangian foliations of $T^*\mathcal{T}(\Sigma)$

Various Lagrangian foliations of the cotangent bundle of Teichmüller space have been discussed in the mathematics literature [**17, 18**]. On the other hand, Hamilton-Jacobi theory is closely connected to the construction of Lagrangian submanifolds or Lagrangian foliations of the phase spaces of suitable Hamiltonian systems and we are here in the optimal circumstances of having a globally defined, complete solution to the Hamilton-Jacobi equation for the reduced Einstein equations in 2+1 dimensions. This allows us not only to define two distinct (one parameter families of) Lagrangian foliations of $T^*\mathcal{T}(\Sigma)$ but also to give a simple geometrical interpretation of the leaves of these foliations in terms of the ray structures discussed in the previous section.

As shown in Sec. 5 the Dirichlet energy function $\mathcal{A}$, after subtraction of the Gauss-Bonnet invariant and rescaling by the factor $\frac{1}{\tau}$, yields a global solution of the Hamilton-Jacobi equation for reduced $2+1$ gravity for any choice of the (uniformized) target metric $\rho$. By virtue of the $\mathcal{D}_0(\Sigma)$ covariance of the formalism, there is no essential loss of generality involved in constraining $\rho$ to lie in a model for Teichmüller space (e.g., in an Earle-Eells section of the bundle $\mathcal{M}_{-1}(\Sigma) \to \frac{\mathcal{M}_{-1}(\Sigma)}{\mathcal{D}_0(\Sigma)} \approx \mathcal{T}(\Sigma)$) which can be globally coordinatized (since $\mathcal{T}(\Sigma) \approx \mathbf{R}^{6\,\mathrm{genus}(\Sigma)-6}$) by a single chart $\{Q^\alpha \mid \alpha = 1, 2, \ldots, 6\,\mathrm{genus}(\Sigma) - 6\}$. Since the Dirichlet energy construction is conformally invariant with respect to the metric on the domain, we can also, without loss of generality, think of the domain metric $\gamma \in \mathcal{M}_{-1}(\Sigma)$ as constrained to lie in a model for $\mathcal{T}(\Sigma)$ which is globally coordinatized by another single chart $\{q^\alpha \mid \alpha = 1, 2, \ldots, 6\,\mathrm{genus}(\Sigma) - 6\}$. Our Hamilton-Jacobi function $\mathcal{S}$ can thus be regarded as expressible as a globally smooth real-valued function defined on $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \times \mathbf{R}^-$ and written in coordinates in the form $\mathcal{S}(q^\alpha, Q^\alpha, \tau)$, where

$$\mathcal{S}(q, Q, \tau) = \frac{\mathcal{A}(q, Q)}{\tau} - \frac{1}{\tau} \int_\Sigma d\mu_\gamma^{(2)} R(\gamma) \tag{10.1}$$

or, in terms of the "Newtonian" time $T = -\frac{1}{\tau}$ which ranges over $(0, +\infty)$, as $\mathcal{S}(q, Q, T) = -T\left[\mathcal{A}(q, Q) - \int_\Sigma d\mu_\gamma^{(2)} R(\gamma)\right]$. It satisfies the reduced Hamilton-Jacobi equation

$$-\frac{\partial \mathcal{S}}{\partial T} = H_{\mathrm{reduced}}\left(q^\alpha, \frac{\partial \mathcal{S}}{\partial q^\alpha}, T\right) \tag{10.2}$$

$\forall\ \{q^\alpha, Q^\alpha, T\}$ ranging over (the global charts for) $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma) \times \mathbf{R}^+$.

In the usual way, we can introduce a global chart for $T^*\mathcal{T}(\Sigma)$ with bundle coordinates $\{(q^\alpha, p_\alpha) \mid \alpha = 1, 2, \ldots 6\,\mathrm{genus}(\Sigma) - 6\}$ in terms of which the canonical symplectic form is expressible as $\omega = \sum_\alpha dq^\alpha \wedge dp_\alpha$. By graphing the gradient of $\mathcal{S}(q, Q, T)$, for fixed $\{Q^\alpha, T\}$ one gets a Lagrangian submanifold of $T^*\mathcal{T}(\Sigma)$ expressible in coordinates as $\{(q^\alpha, p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha})\}$. This clearly has the (maximal) dimension, namely $6\,\mathrm{genus}(\Sigma) - 6$, and is Lagrangian since the pullback of the symplectic form to this submanifold vanishes by virtue of the induced formula for the differentials $dp_\alpha$, i.e.,

$$dp_\alpha = \sum_\beta \frac{\partial^2 \mathcal{S}}{\partial q^\alpha \partial q^\beta} dq^\beta \tag{10.3}$$

which yields $\sum_{\alpha\beta} \frac{\partial^2 \mathcal{S}}{\partial q^\alpha \partial q^\beta} dq^\alpha \wedge dq^\beta \equiv 0$ by symmetry of $\frac{\partial^2 \mathcal{S}}{\partial q^\alpha \partial q^\beta}$. By allowing the $\{Q^\alpha\}$'s to vary, still holding $T$ fixed, one generates a family of such leaves which, taken all together, define a Lagrangian foliation of $T^*\mathcal{T}(\Sigma)$.

To see this, first note that the points on any given leaf $\{(q^\alpha, p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha}) \mid (Q^\beta, T) \text{fixed}\}$ represent the initial data sets (at time $T$) for all those solution curves which tend asymptotically to the conformal geometry represented by $\{Q\}$ as $T \to \infty$. Thus no two distinct leaves can intersect since a point of intersection would have to correspond to a solution curve having two distinct asymptotic conformal geometries. Furthermore, there is a (necessarily unique) leaf through any point of $T^*\mathcal{T}(\Sigma)$ since, thanks to Wolf's results, there is, for any base point $\{q^\alpha\}$ a unique covector $p_\alpha dq^\alpha$ (usually represented as a $TT$ tensor or quadratic differential at a representative domain metric $\gamma \in \mathcal{M}_{-1}(\Sigma)$) such that the map given by Eq. (9.1) yields any desired target $\{Q^\alpha\}$. In fact, by Wolf's diffeomorphism result, there is a bijective correspondence between the target points $\{Q^\alpha\}$ and the covectors $p_\alpha dq^\alpha$ at $\{q^\alpha\}$ and in our Hamilton-Jacobi formulation the latter are realized as gradients of $\mathcal{S}$ via $p_\alpha dq^\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha}(q, Q, T) dq^\alpha$. Thus every point $\{q^\alpha, p_\alpha\}$ in phase space $T^*\mathcal{T}(\Sigma)$ is realized as a pair $\{q^\alpha, \frac{\partial \mathcal{S}}{\partial q^\alpha}(q, Q, T)\}$ when, for any fixed $T \in (0, \infty)$, the pair $(q^\alpha, Q^\alpha)$ ranges over a global chart for $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$.

A complimentary Lagrangian foliation is defined by graphs of the form $\{(Q^\alpha, P_\alpha = \frac{\partial \mathcal{S}}{\partial Q^\alpha}(q, Q, T))\}$ where again $(q^\alpha, Q^\alpha)$ ranges over a (globally defined) chart for $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ at fixed $T > 0$. A particular leaf in this case (corresponding to fixed $\{q^\alpha\}$) consists of all asymptotic data points $\{Q^\alpha, P_\alpha\}$ belonging to solution curves which passed through $\{q^\alpha\}$ at time $T > 0$. Leaves cannot intersect by virtue of our bijectivity result from Sec. 9 which shows that for a fixed asymptotic geometry $\{Q^\alpha\}$ there is a bijective correspondence between the asymptotic covectors $P_\alpha dQ^\alpha$ (usually expressed in terms of $TT$ tensors, c.f., Eq. (5.9)–(5.13)) and the points $\{q^\alpha\}$ exhausting Teichmüller space. This correspondence insures that every point $\{(Q^\alpha, P_\alpha)\}$ in (a global chart for) $T^*\mathcal{T}(\Sigma)$ is achieved as, in addition, $\{Q^\alpha\}$ is allowed to range over its chart for $\mathcal{T}(\Sigma)$.

In the above, $T$ was held fixed. If it is allowed to vary, then the foliations "evolve" in a smooth way that is directly related to the scale invariance of Einstein's equations noted above. This follows from the observation that $p_\alpha = \frac{\partial \mathcal{S}}{\partial q^\alpha} = -T \frac{\partial \mathcal{A}}{\partial q^\alpha}$ and $P_\alpha = \frac{\partial \mathcal{S}}{\partial Q^\alpha} = -T \frac{\partial \mathcal{A}}{\partial Q^\alpha}$ wherein $T$ only appears as an overall multiplicative factor which rescales the momenta as it is varied.

Suppose that $\{Q^\alpha(p)\}$ and $\{q^\alpha(p)\}$ label the same point $p$ in abstract Teichmüller space. Then it is not difficult to see that $\frac{\partial \mathcal{S}}{\partial q^\alpha}(q(p), Q(p), T) = \frac{\partial \mathcal{S}}{\partial Q^\alpha}(q(p), Q(p), T) = 0$. This corresponds to the statement that the only solution curve passing through a given point in Teichmüller space at time $T \in (0, \infty)$ and also asymptoting to that point as $T \to \infty$ is in fact the trivial fixed point solution corresponding to the chosen point.

## Concluding remarks

Solutions to the vacuum field equations on $\Sigma \times \mathbf{R}$ can each be used, by simply taking a product with the circle, to generate corresponding vacuum solutions to the $3+1$ dimensional Einstein equations on $\Sigma \times S^1 \times \mathbf{R}$. Each of these $3+1$ dimensional Lorentz manifolds is still flat and spatially compact (with space sections diffeomorphic to $\Sigma \times S^1$) and has by construction a spacelike Killing field tangent to the circular factors. As such it provides a very specialized example of vacuum solutions definable on $S^1$-bundles over $\Sigma \times \mathbf{R}$ which have a Killing symmetry imposed along the circular fibers of the bundle in question. The most general such ($U(1)$-isometric) solution can be regarded (upon adopting a Kaluza-Klein viewpoint and projecting fields down to the base) as a non-vacuum solution to the $2+1$ dimensional field equations defined on $\Sigma \times \mathbf{R}$ with a certain very specific type of matter source. Perhaps the most elegant form of these reduced field equations is that wherein the $2+1$ dimensional metric is coupled to a wave map field with target space given by the Poincaré half-plane. If trivial bundles are considered this wave map can be polarized (i.,e. restricted so that its image lies along a geodesic in the half-plane) so that the wave map equations reduce to a wave equation but in the general case of non-trivial bundles over $\Sigma \times \mathbf{R}$ such a polarization is incompatible with the topology of the bundle [**19, 20, 21**].

The global study of such $U(1)$-invariant vacuum solutions to the $3+1$ dimensional field equations is an important and still largely open problem in general relativity. Some significant progress on it has been made by Y. Choquet-Bruhat and the present author by assuming Cauchy data which is sufficiently small (as a non-linear perturbation of the trivial, constant wave map solutions) and evolving only in the direction of cosmological expansion [**19, 20, 21**], a device which sidesteps complications produced by the big bang singularities. However, some independent work by these same authors together with J. Isenberg has exploited so-called Fuchsian methods to deal rigorously with the singularities themselves at least when the solutions are "half-polarized" in a well-defined sense [**22, 23**]. The generic non-polarized solution is however, anticipated not to be amenable to this kind of Fuchsian analysis and instead to exhibit a singularity of oscillatory type [**24**].

It seems plausible that the formation of black holes in these $U(1)$-isometric models is suppressed by the symmetry imposed and thus that there is no singular behavior expected for the cosmologically expanding direction at all. In other words, one should have large data global existence for all solutions to this problem. Of course, it is extremely unlikely that one could prove such a result without first learning how to treat $2+1$ dimensional wave maps globally on a background Lorentz manifold. In the present problem the Lorentzian metric is not a background but instead a functional of the evolving wave map and the Teichmüller parameters. Indeed even the (polarized) special case of a wave equation is highly non-linear because of this metrical dependence upon the evolving wave and, at present, only the small data results mentioned above are known.

Another problem related to that considered in the present article is that of constructing CMC foliations of *flat*, higher dimensional, spatially compact Lorentz manifolds. For any compact hyperbolic manifold $(\mathbf{H}^n/\Gamma, h)$, where $h$ has constant negative curvature as a Riemannian metric, one can construct the trivial, flat Lorentz cone spacetime on $\mathbf{H}^n/\Gamma \times \mathbf{R}$ for any $n \geq 2$ which is naturally foliated by CMC slices. For $n \geq 3$, Mostow rigidity forbids any "obvious" deformation of the metrics on $\mathbf{H}^n/\Gamma \times \mathbf{R}$ which preserves flatness since there is no corresponding Teichmüller space of hyperbolic metrics in these cases, but nevertheless, for some choices of $\mathbf{H}^n/\Gamma$ there can exist non-trivial moduli spaces of flat Lorentzian metrics on $\mathbf{H}^n/\Gamma \times \mathbf{R}$ and, for these, it would be of interest to construct CMC foliations. This problem has already been dealt with using indirect methods by L. Andersson who exploited earlier results by G. Mess and others on the properties of such spaces [**25**]. However, it might also be possible to attack this problem directly with methods that parallel, to some extent, those developed in the present paper. In particular, harmonicity of the Gauss map for a CMC slice in such a flat, higher dimensional spacetime will continue hold and the analogue of Eq. (6.3) above is easy to derive. Thus it is of interest to see whether one could derive an elliptic equation (or system) which plays the role for these higher dimensional problems that our Monge-Ampêre equation for $\Lambda^\dagger$ plays here. One would expect the role of holomorphic quadratic differentials to be played by the traceless Codazzi tensors of a fixed hyperbolic metric $h$ on $\mathbf{H}^n/\Gamma$.

Finally, let us return to the problem alluded to at the end of Sec. 8, namely the derivation of estimates for solutions of the Monge-Ampêre equation sufficiently sharp so as to characterize their asymptotics as $\tau \searrow -\infty$ (which limit corresponds to the big bang singularities in these models (c.f. Ref. [**6**])). This problem is currently under study with S.-T. Yau.

As in the case of Wolf's ray structures the natural conjecture here would seem to be that the ray structures introduced in Sec. 9, representing families of Einstein solution curves, have limits, as $\tau \searrow -\infty$, at Thurston boundary points of $\mathcal{T}(\Sigma)$. More precisely, the idea is that, for any fixed $\rho$, the collection of non-trivial solution curves emerging (at $\tau = 0$) from this (arbitrary) interior point of $\mathcal{T}(\Sigma)$ effectively attach the Thurston boundary to $\mathcal{T}(\Sigma)$ in the sense that each solution ray limits to a boundary point and every such boundary point is the limit of a unique ray. That this should be true is, to a large extent, already known from indirect, barrier-estimate arguments due to L. Andersson [**26**] which in turn are based upon the fundamental work of R. Benedetti and E. Guadagnini [**27**]. The latter authors use a "cosmic time" slicing (essentially a Gaussian normal slicing but having the big bang singularity itself as a $t = 0$ level "surface") and explicitly construct a (dense) subset of the full solution space by suitably "cutting and pasting" negatively curved Friedman-Roberton-Walker and flat Kazner metrics along certain leaves of geodesic laminations of $\Sigma$. This representation of the solutions induces different families of curves in $\mathcal{T}(\Sigma)$ from those we get (since generically their slicings are neither smooth nor CMC whereas ours are) but they show that their solution curves in $\mathcal{T}(\Sigma)$ do indeed limit to Thurston boundary points. Andersson's barrier arguments are designed to control the asymptotics of CMC slicings relative to the cosmic time ones near the big bang singularities and thereby to show that Thurston boundary points are attained, in the limit, by the former as well as the latter. Our Monge-Ampêre analysis yields, in principle, a direct characterization of all CMC sliced solutions (and not only the "simplicial" ones dealt with by Andersson) and thus affords the possibility of computing their (CMC-sliced) singularity structures more explicitly.

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## References

[1] E. Witten, 2+1 *Dimensional Gravity as an Exactly Soluble System*, Nucl. Phys. **B311** (1988), 46–78.

[2] V. Moncrief, *Reduction of the Einstein equations in* 2+1 *dimensions to a Hamiltonian system over Teichmüller space*, J. Math. Phys. **20** (1989), 2907–2914 .

[3] S. Carlip, *Quantum Gravity in* 2 + 1 *Dimensions (Cambridge Monographs on Mathematical Physics)*, Cambridge University Press (1998).

[4] L. Andersson and V. Moncrief, *Elliptic-Hyperbolic Systems and the Einstein Equations*, Ann. Henri Poincaré **4** (2003), 1–34 .

[5] A. Fischer and V. Moncrief, *Hamiltonian reduction, the Einstein flow, and collapse of* 3*-manifolds*, Nucl. Phys. B (Proc. Suppl.) **88** (2000), 83–102.

[6] L. Andersson, V. Moncrief and A. Tromba, *On the global evolution problem in* 2 + 1 *gravity*, J. Geom. Phys. **23** (1997), 191–205.

[7] R. Puzio, *The Gauss map and* 2 + 1 *gravity*, Class. Quantum Grav. **11** (1994), 2667–2675.

[8] M. Wolf, *The Teichmüller Theory of Harmonic Maps*, J. Diff, Geom. **29** (1989), 449–479.

[9] B. Tabak, *A Geometric Characterization of Harmonic Diffeomorphisms Between Surfaces*, Math. Ann. **270** (1985), 147–157.
[10] L. M. Sibner and R. J. Sibner, *A non-linear Hodge-deRham theorem*, Acta. Math. **125** (1970), 57–73.
[11] E. A. Ruh and J. Vilms, *The tension field of the Gauss map*, Trans. Am. Math. Soc. **149** (1970), 569–573.
[12] V. Moncrief, *Reduction of Einstein's Equations for Vacuum Space-Times with Spacelike U(1) Isometry Groups*, Ann. Phys. (NY) **167** (1986), 118–142. See appendix herein for an application of the sub and super solutions method.
[13] J. L. Kasdan, *Some applications of Partial Differential Equations to Problems in Geometry*, Surveys in Geometry (copyright by J. L. Kasdan 1983). Some of this material appears in the appendix of Einstein Manifolds by A. Besse (Springer, 2002)
[14] A. J. Tromba, *Teichmüller Theory in Riemannian Geometry (Lectures in Mathematics)* Birkhaüser 1992. See especially section 3.4 of this reference for a discussion of the relevant global cross-sections of $\mathcal{M}_{-1}(\Sigma)$.
[15] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer (1984). See section 3 of chapter 2 for a discussion of the Hopf (strong) maximum principle.
[16] C. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Diff. Geom. **3** (1969), 19–43.
[17] M. Wolf, *Measured Foliations and Harmonic Maps of Surfaces*, J. Diff. Geom. **49** (1998), 437–467.
[18] J. Hubbard and H. Masur, *Quadratic Differentials and Foliations*, Acta Math. **142** (1979), 221–274.
[19] Y. Choquet-Bruhat and V. Moncrief, *Future global in time Einsteinian spacetimes with $U(1)$ isometry group*, Ann. Henri Poincaré **2** (2001), 1007–1064.
[20] ———, *Non-linear stability of Einsteinian spacetimes with $U(1)$ isometry group* in "Partial Differential Equations and Mathematical Physics" in honor of J. Leray, Kajitani and Vaillant, eds. Birkhaüser (2003).
[21] Y. Choquet-Bruhat, *Future Complete $U(1)$ Symmetric Einsteinian Spacetimes, the Unpolarized Case* in "The Einstein Equations and the Large Scale Behavior of Gravitational Fields, P. T. Chrusciel and H. Friedrich, eds. Birkhaüser (2004).
[22] Y. Choquet-Bruhat, J. Isenberg and V. Moncrief, *Topologically general $U(1)$ symmetric Einstein spacetimes with AVTD behavior*, Nuovo Cim. **119B** (2004), 625–638.
[23] J. Isenberg and V. Moncrief, *Asymptotic Behavior of Polarized and Half-Polarized $U(1)$ Symmetric Vacuum Spacetimes*, Class. Quantum Grav. **19** (2002), 5361–5386.
[24] B. K. Berger and V. Moncrief, *Signature for local Mixmaster dynamics in $U(1)$ symmetric cosmologies*, Phys. Rev. **D 62** (2000), 123501. See also Phys. Rev. **D 62**, 023509 by the same authors.
[25] L. Andersson, *Constant mean curvature foliations of flat spacetimes*, Comm. Anal. Geom. **10** (2002), 1094-1115; arXiv: math. DG/0110245.
[26] ———, *Constant mean curvature folitaions of simplicial flat spacetimes*; arXiv: math. DG/0307338 v1 (2003).
[27] R. Benedetti and E. Guadagnini, *Cosmological Time in $(2+1)$-Gravity*, Nucl. Phys. **B613** (2001), 330–352 arXiv: gr-qc/0003055 v2 (2001).

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