A Survey on the Kähler-Ricci Flow and Yau's Uniformization Conjecture

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ABSTRACT. Yau's uniformization conjecture states: a complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$. The Kähler-Ricci flow has provided a powerful tool in understanding the conjecture, and has been used to verify the conjecture in several important cases. In this article we present a survey of the Kähler-Ricci flow with focus on its application to uniformization. Other interesting methods and results related to the study of Yau's conjecture are also discussed.

1. Introduction

A fundamental problem in complex geometry is to generalize the classical uniformization theorems on Riemann surfaces to higher dimensions. In Kähler geometry, the problem is to determine how curvature affects the underlying holomorphic structure of a Kähler manifold. In one complex dimension it is well known that a complete simply connected Riemann surface (M, g) is biholomorphic to either the Riemann sphere (when M is compact) or the complex plane (when M is noncompact) if the curvature is positive, and it is biholomorphic to the open unit disc if the curvature is negative and bounded from above away from zero.

In higher dimensions, for the compact case, the famous conjecture of Frankel says that a compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{CP}^n$. Frankel's conjecture was proved by Siu-Yau [52]. The stronger Hartshorne conjecture was proved by Mori [37]. In case of compact Kähler manifolds with nonnegative holomorphic bisectional curvature the uniformization of such manifolds was determined by Bando [2] for complex dimension three and by Mok [35] for all dimensions.

In the complete noncompact case there is a long standing conjecture due to Yau [**58**] in 1974 predicting analogous results for positively curved non-compact Kähler manifolds:

Yau's Conjecture [**58**] *A complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to $\mathbb{C}^n$.*

Or in Yau's words [**58**, p. 620]: *The question is to demonstrate that every noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex eulcidean space.*

The conjecture continues to generate much research activity, and although the full conjecture remains unproved, this research has produced many nice results and useful techniques and methods. In this survey we will discuss the method of using evolution equations, in particular the Kähler-Ricci flow, to study the conjecture.

In [**49**], Shi (under the supervision of Yau) began a program to use the Kähler-Ricci flow equation

$$(1.1) \qquad\qquad \frac{dg_{i\bar{j}}}{dt} = -R_{i\bar{j}}$$

to prove Yau's uniformization conjecture[1]. (1.1) is a geometric evolution equation which deforms an initial Kähler metric in the direction of its Ricci curvature (see §2). The equation is a strictly parabolic system for $g_{i\bar{j}}$, so one generally expects the geometry to improve under this evolution. The idea is to show that the geometry improves to the point that it determines the holomorphic structure of the manifold. This idea has been successfully applied in many cases now. In this article we survey the progress on Yau's conjecture and the Kähler-Ricci flow program. We will also survey some more general results relating to the conjecture. Since there are many works, the survey cannot be exhaustive.

An outline of the paper is as follows. In §2 we introduce the Kähler-Ricci flow and its basic theory on complete noncompact Kähler manifolds, and on nonnegatively curved Kähler manifolds in particular. In §3 we discuss the rate of curvature decay on a complete noncompact nonnegatively curved Kähler manifold, which is also related to volume growth. This turns out to be useful for latter application. The Steinness of non-negatively curved noncompact Kähler manifolds is discussed in §4, and this begins our actual survey of uniformization results. In §5 we introduce the Kähler-Ricci solitons and present our uniformization theorem for steady and expanding gradient Kähler-Ricci solitons. We place this section here because Kähler-Ricci solitons are canonical models for the cases in §6 and the analogy drawn there

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[1]The Kähler-Ricci flow had been used in Bando's work [**2**] (under the supervision of Yau) and Mok's work [**35**] on the uniformization of compact Kähler manifolds with nonnegative holomorphic bisectional curvature. The use of the Kähler-Ricci flow in these works however was not very extensive and involves only short-time behavior of the solutions. The study of long time behavior of the Kähler-Ricci flow in the compact case was initiated by Cao in [**6**].

between the Kähler-Ricci flow and complex dynamical systems. In §6 the connection between the Kähler-Ricci flow and complex dynamical systems is developed and used to prove a uniformization theorem for eternal solutions to the Kähler-Ricci flow and normalized Kähler-Ricci flow. As a corollary, we state a uniformization theorem for nonnegatively curved Kähler manifolds of average quadratic curvature decay, in particular for manifolds with maximum volume growth. Finally, in §7 we present the gap theorems for nonnegatively curved Kähler manifolds of faster than quadratic curvature decay. Whenever it is possible, we will sketch the main ideas of the proofs of the results.

2. Kähler-Ricci flow

The Ricci flow on a complete Riemannian manifold (M, g) is the following evolution equation for the metric g:

$$(2.1) \qquad \begin{cases} \dfrac{dg(t)}{dt} = -2Rc(t) \\[2mm] g(0) = g, \end{cases}$$

where $Rc(t)$ is the Ricci tensor of $g(t)$. The Ricci flow was introduced by Hamilton [**27**] to study the Poincaré conjecture. In this seminal work, Hamilton proved (2.1) has a short time solution on any smooth compact Riemannian manifold. Hamilton then showed that the solution exists for all time and converges after rescaling on any compact three manifold with positive Ricci curvature.

We are interested in complete noncompact Riemannian manifolds. The first fundamental result for the Ricci flow on noncompact manifolds is the following short time existence result of Shi [**48**]:

THEOREM 2.1. [**48**] *Let (M^n, g) be a complete noncompact Riemannian manifold with bounded sectional curvature. Then there exists $0 < T < \infty$, depending only on the initial curvature bound, such that (2.1) has a solution $g(t)$ on $M \times [0, T]$. Moreover, for all $t \in [0, T]$ we have*

(i) *$g(t)$ has bounded sectional curvature and is equivalent to g.*

(ii) *For any integer $m \geq 0$, there is a constant depending only on n, m and the bound of the curvature of the initial metric g such that*

$$\sup_{x \in M} |\nabla^m Rm|(x, t) \leq \frac{C}{t^m}$$

for all $0 \leq t \leq T$. Here ∇ is the covariant derivative with respect to $g(t)$.

The following theorem of Shi [**50**] is fundamental to the theory and application of the Ricci flow to complete noncompact Kähler manifolds.

THEOREM 2.2. [**50**] *Let* (M, g) *be as in Theorem 2.1. Suppose* (M, g) *is Kähler. Then*

   (i) *the solution* $g(t)$ *in the Theorem 2.1 is Kähler for* $t \in [0, T]$*; and*
   (ii) $g(t)$ *has non-negative holomorphic bisectional curvature if the same is true for* g*.*

Hence in the Kähler category, we refer to the Ricci flow as the Kähler-Ricci flow and the equation becomes:

$$(2.2) \qquad \frac{dg_{i\bar{j}}}{dt} = -R_{i\bar{j}}.$$

For compact Kähler manifolds, part (i) of Theorem 2.2 was proved by Bando [**2**] and part (ii) was proved by Bando [**2**] for complex three manifolds then by Mok [**35**] for all dimensions. Shi's proof of (i) for the noncompact case is different from that in [**2**]. Shi's proof of (ii) is similar to that in [**2**] and [**35**] which uses a so-called null-vector condition introduced in [**2**]. The proofs of both (i) and (ii) use the maximum principle for noncompact Riemannian manifolds which relies on the following result which was proved by Shi [**50**] using an idea of Greene-Wu [**24**]: Let (M, g) be a complete noncompact manifold with bounded curvature. Then one can construct a smooth function with bounded gradient and bounded Hessian, which is uniformly equivalent to the distance function from a point.

We now focus on complete non-compact Kähler manifolds with non-negative holomorphic bisectional curvature. The main long time existence result for the Kähler-Ricci flow in this setting is the following theorem of Shi [**50**]. We will denote the scalar curvature by $\mathcal{R}$ and define the average scalar curvature $k(x, r)$ as:

$$k(x, r) := \frac{1}{V_x(r)} \int_{B_x(r)} \mathcal{R} dV.$$

THEOREM 2.3. [**50**] *Let* (M^n, g) *be a complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature. Suppose*

$$(2.3) \qquad k(x, r) \le \frac{C}{(1 + r)^\epsilon}$$

for some constants C *and* $0 < \epsilon \le 2$*, and all* x *and* r*. Then (2.2) has a long time solution* $g(t)$*. Moreover,*

   (i)

$$\mathcal{R}(x, t) \le Ct^{\frac{2(1-\epsilon)}{\epsilon}},$$

   for some constant C *for all* x *and* t*.*

(ii) *For any integer $m \geq 0$, there is a constant C such that*

$$|\nabla^m Rm|(x,t) \leq C t^{\frac{2(1-\epsilon)}{\epsilon}(m+2)}$$

for all x and t.

For example, if $\epsilon = 2$, then $t\mathcal{R}$ will be uniformly bounded on spacetime. The proof of the theorem is based on estimating the volume element

$$F(x,t) = \log \frac{\det(g_{i\bar{j}}(x,t))}{\det(g_{i\bar{j}}(x,0))}.$$

Shi proved that F stays bounded from below on any finite time interval. Note that the bound $F \leq 0$ from above follows from Theorem 2.2. Using this and a parabolic version of the third order estimate for the complex Monge-Ampère equation, Shi then showed that the Kähler-Ricci flow cannot develop a singularity in finite time. In the case of $\epsilon > 1$, there is also a method by Ni-Tam [43] for proving the long time existence. Namely, by extending the method of Mok-Siu-Yau [36], Ni-Shi-Tam [41] were able to construct a potential u_0, for the Ricci form of the initial metric, having uniformly bounded gradient. It is readily seen that $u = u_0 - F$ is a potential of the Ricci form for $g(t)$. Using the maximum principle, one can show that u also has uniformly bounded gradient. One then shows that $|\nabla u|^2 + \mathcal{R}$ is a subsolution of the time dependent heat equation, and applying the maximum principle again, one concludes that $|\nabla u|^2 + \mathcal{R}$ is bounded by its maximum at $t = 0$. Since $g(t)$ has nonnegative holomorphic bisectional curvature, the Riemannian curvature can also be bounded from this.

On the other hand, the estimate of F depends on the fact that

$$\Delta_0 F \leq \mathcal{R}(0) + e^F \frac{\partial F}{\partial t}$$

and some mean value inequalities (see [43] for example). Here Δ_0 is the Laplacian of the initial metric. Once we estimate F we can then estimate $\mathcal{R}$ using the fact that

$$F(x,t) = - \int_0^t \mathcal{R}(x,\tau) d\tau$$

and the Li-Yau-Hamilton type Harnack inequality Theorem 6.7 of Cao [7] which implies that $t\mathcal{R}$ is nondecreasing in time. In any case, the estimates of the covariant derivatives of the curvature tensor follow from the general method developed by Shi [48].

Note that in the theorem, condition (2.3) is uniform in x. In many cases this condition is at least true at a point by [42]. Using this and the pseudolocality of Ricci flow by Perelman [44], which can be generalized to complete noncompact manifolds, Yu and the authors [16] obtained the following result on long time existence:

THEOREM 2.4. *Let (M^n, g) be a complete non-compact Kähler manifold with non-negative holomorphic bisectional curvature with injectivity radius bounded away from zero such that*

$$|Rm|(x) \to 0$$

as $x \to \infty$. Then the Kähler-Ricci flow with initial data g has a long time solution $g(t)$ on $M \times [0, \infty)$.

It would be nice if one could remove the condition on the injectivity radius.

3. Curvature decay rate

In order to apply Kähler-Ricci flow to Yau's conjecture, the following question becomes important in light of Theorem 2.3: What can we expect of the the curvature decay rate on complete noncompact Kähler manifolds with positive bisectional curvature? There are some results on the volume and curvature of manifolds with nonnegative holomorphic bisectional curvature in [**41**]. For example, it was proved that if the scalar curvature of a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature decays quadratically in the pointwise and average sense, and if the Ricci curvature is positive at some point, then the manifold must have maximum volume growth. On the other hand, the following was proved by Chen-Zhu [**21**]:

THEOREM 3.1. [**21**] *Let (M^n, g) be a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then for any $x \in M$ there is a constant C which may depend on x such that*

$$(3.1) \qquad \frac{1}{V_x(r)} \int_{B_x(r)} \mathcal{R} dV \leq \frac{C}{1+r}.$$

Later Ni and Tam [**42**] obtained the following, which generalizes the above results:

THEOREM 3.2. [**42**] *Let (M^n, g) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature.*
  (i) *Suppose M is simply connected, then $M = N \times M'$ holomorphically and isometrically, where N is a compact simply connected Kähler manifold, M' is a complete noncompact Kähler manifold and both N and M' have nonnegative holomorphic bisectional curvature. Moreover, the scalar curvature $\mathcal{R}'$ of M' satisfies the linear decay condition (3.1).*
  (ii) *If the holomorphic bisectional curvature of M is positive at some point, then M itself satisfies the linear decay condition (3.1).*

The idea of the proof is to construct a strictly plurisubharmonic function u of linear growth. If such a u exists, then one can use L^2 theory to construct nontrivial holomorphic section s of the canonical line bundle using u as a weight function. The growth rate of u will give an estimate of the growth rate of the L^2 norm of the length $||s||$ of s. Using the Bochner type differential inequality:

$$(3.2) \qquad \Delta \log ||s||^2 \geq \mathcal{R}$$

at the points where $s \neq 0$, one can show that $\log ||s||$ is at most linear growth by the mean value inequality of Li and Schoen [32]. Then the growth rate of $\log ||s||$ will give an estimate of the average of $\mathcal{R}$ over geodesic balls because of (3.2).

If the holomorphic bisectional curvature is positive, one just considers the Busemann function which is strictly plurisubharmonic at a point by a well known result of Wu [55]. In general, one may solve the heat equation with the Buesmann function as initial data. Then by a careful study of the solution for $t > 0$, one obtains a suitable strictly plurisubharmonic function in the case of Theorem 3.2 (ii).

Hence in some sense, linear decay is the slowest decay rate for the curvature of complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature.

In case that the manifold has maximum volume growth, one may expect the curvature will decay faster. In fact, it was conjectured by Yau [58]: If M has maximal volume growth in the sense that $V_p(r) \geq Cr^{2n}$ for some $C > 0$ for some $p \in M$ for all r, then the curvature must decay quadratically in the average sense. Assuming the curvature is bounded, this was confirmed by Chen-Tang-Zhu [20] for complex surfaces and Chen-Zhu [21] for higher dimension under a much stronger assumption that the manifold has nonnegative curvature operator. Finally, Ni [39] proved this conjecture of Yau in general. More precisely, Ni obtained the following:

THEOREM 3.3. [39] *Let (M^n, g) be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Suppose M has maximum volume growth. Then the scalar curvature $\mathcal{R}$ decay quadratically. That is to say, there exists $C > 0$ such that*

$$(3.3) \qquad \frac{1}{V_x(r)} \int_{B_x(r)} \mathcal{R} \, dV \leq \frac{C}{(1+r)^2}$$

for all $x \in M$ and for all $r > 0$.

The main steps of the proof are as follow: First, use the results in [42], in particular Theorem 3.2, to prove that a non-flat gradient shrinking Kähler-Ricci soliton with nonnegative bisectional curvature must have zero asymptotic volume ratio. A gradient shrinking Kähler-Ricci soliton is a Kähler

metric $g_{i\bar{j}}$ satisfying

$$R_{i\bar{j}} - \rho g_{i\bar{j}} = f_{i\bar{j}}, \quad f_{ij} = 0$$

for some smooth real-valued function f and for some $\rho > 0$. For an n dimensional Riemannian manifold with nonnegative Ricci curvature, the asymptotic volume ratio is defined as:

$$\mathcal{V}(g) = \lim_{r \to \infty} \frac{V_x(r)}{r^n}.$$

The limit exists and is independent of the base point x by the Bishop volume comparison theorem. In the case of maximal volume growth, this limit is non-zero.

The second step is to prove that any non-flat complete ancient solution of the Kähler-Ricci flow on a Kähler manifold with bounded and nonnegative holomorphic bisectional curvature must also have zero asymptotic volume ratio for all t. This is accomplished by showing that if this is not true, then by a blow down argument as in Perelman [44] for Riemannian manifolds with nonnegative curvature operator, one can construct a non-flat gradient shrinking Kähler-Ricci soliton with nonnegative bisectional curvature which has *nonzero* asymptotic volume ratio.

Now suppose (M, g) satisfies the conditions in Theorem 3.3, then one can solve the Kähler-Ricci flow equation with solution $g(t)$ with initial condition g, and can prove that $\mathcal{V}(g(t)) = \mathcal{V}(g)$ for all t. One then uses the second step above to prove that $g(t)$ must exist for all time and the scalar curvature must satisfy

$$\mathcal{R}(x, t) \leq \frac{C}{1 + t}$$

for some C for all x and t. This is proved by contradiction: if this were not so, one could construct an ancient solution as above having non-zero asymptotic volume ratio, which contradicts the previous assertion. Finally, one may use this asymptotic behavior of $\mathcal{R}$ as $t \to \infty$ to get information of $\mathcal{R}(x, 0)$ as $x \to \infty$ and obtain (3.3). Here an argument similar to that in Perelman [44] is also used.

In order to get more information on the question on curvature decay rate, it would be helpful to construct examples. This is considerably easier in the Riemannian setting: constructing complete noncompact Riemannian manidfolds having positive sectional curvature. It is not easy to construct complete Kähler metrics on $\mathbb{C}^n$ with positive holomorphic bisectional curvature. The first example is by Klembeck [31] which has positive holomorphic bisectional curvature. Klembeck's example has linear curvature decay with volume growth like $V(r) \sim r^n$. Later, Cao [8, 9] constructed examples of $U(n)$ invariant metrics on $\mathbb{C}^n$ having positive holomorphic bisectional curvature. The examples in [8] have linear curvature decay and volume growth like

$V(r) \sim r^n$ while those in [**9**] have quadratic curvature decay and maximum volume growth.

Complete $U(n)$ invariant Kähler metrics with positive holomorphic bisectional curvature on $\mathbb{C}^n$ have been classified and examples have been constructed by Wu and Zheng [**56**]. They have developed a systematic way to construct examples having positive holomorphic bisectional curvature such that the scalar curvature $\mathcal{R}$ satisfies various decay rates. In particular, for any $1 \leq \theta \leq 2$ there are examples such that

$$(3.4) \qquad \frac{1}{V_x(r)} \int_{B_x(r)} \mathcal{R} \, dV \leq \frac{C}{1 + r^\theta}$$

for some C for all r and for all x.

Finally, there are interesting results by Chen-Zhu [**21**] on volume growth of positively curved noncompact Kähler manifolds. For example, they proved that the volume of geodesic balls $B_p(r)$ in a complete noncompact Kähler manifold with positive holomorphic bisectional curvature must grow at least like r^n, where n is the complex dimension of the manifold. One may compare this with the well-known result of Calabi and Yau (see [**57**], for example) that in case of Riemannian manifold with nonnegative Ricci curvature, the growth rate of geodesic balls is at least linear.

4. Steinness of nonnegatively curved manifolds

In [**51**], Siu asked whether a complete noncompact Kähler manifold with positive holomorphic bisectional is Stein. This is a very interesting problem, and an affirmative answer to this should strongly support Yau's conjecture. In fact, Greene and Wu [**24**] first proved the following result on the complex structure of complete noncompact Kähler manifolds with positive curvature, which is part of the motivation behind Yau's conjecture:

THEOREM 4.1. [**24**] *Let (M, g) be a complete noncompact Kähler manifold with positive sectional curvature. Then M is Stein.*

The idea is to produce a smooth strictly plurisubharmonic exhaustion function. Then by a well-known result of Grauert, M will be Stein. As in the study of Riemannian manifolds with positive or nonnegative sectional curvature by Cheeger and Gromoll [**17**], the method of Greene-Wu uses the Busemann function $\mathcal{B}(x)$. Here the Busemann function is an exhaustion function because the sectional curvature is nonnegative [**17**]. It is also plurisubharmonic if the holomorphic bisectional curvature is nonnegative and strictly plurisubharmonic at the points where the holomorphic bisectional curvature is positive by Wu [**55**]. However, $\mathcal{B}(x)$ is only Lipschitz, and so one needs to approximate $\mathcal{B}(x)$ by a smooth strictly plurisubharmonic exhaustion function. This can be done using the fact that $\mathcal{B}(x)$ is strictly plurisubharmonic and the method developed by Greene-Wu [**24**].

Because of these, Theorem 4.1 can be improved. For example, the theorem is still true if we only assume that M has nonnegative sectional curvature and positive holomorphic bisectional curvature.

In the case that $\mathcal{B}$ is only plurisubharmonic, it is more efficient to use another method to approximate $\mathcal{B}(x)$. Namely, we solve the heat equation with initial data $\mathcal{B}(x)$. Ni and Tam [42] proved that if $u(x,t)$ is the solution to this, then $u(x,t)$ is still plurisubharmonic for $t > 0$. Moreover, they showed that the kernel $\mathcal{K}(x,t)$ of the complex Hessian of $u(x,t)$ is a parallel distribution. Using this one can obtain the following:

THEOREM 4.2. [42] *Let (M, g) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose either M has maximum volume growth or M has a pole. Then M is Stein.*

Here one uses the fact that $\mathcal{B}(x)$ is still an exhaustion function by [47] and so $u(x,t)$ is also an exhaustion function. As in the proof of Theorem 3.2, one can conclude that u is also strictly plurisubharmonic for $t > 0$. In the proof, one needs the following fact [42]: If (M, g) is a complete Kähler manifold with nonnegative holomorphic bisectional curvature which supports a nontrivial linear growth harmonic function, then the universal cover of M is a holomorphically isometric to the product of the complex $\mathbb{C}$ and another complete Kähler manifold with nonnegative holomorphic bisectional curvature.

Without the assumption that the manifold has maximum volume growth, if the Busemann function is an exhaustion function, then one can conclude that the universal cover is a product of a compact Hermitian symmetric manifold and a Stein manifold [42]. Using this, Fangyang Zheng (see [42]) obtained the following:

THEOREM 4.3. *Let (M, g) be a complete noncompact Kähler manifold with nonnegative sectional curvature. Then its universal cover is of the form $\widetilde{M} = \mathbb{C}^k \times \widetilde{N} \times \widetilde{L}$ where $\widetilde{N}$ is a compact Hermitian symmetric manifold, $\widetilde{L}$ is Stein and $\widetilde{L}$ contains no Euclidean factor. Moreover, M is a holomorphic and Riemannian fiber bundle with fiber $\widetilde{N} \times \widetilde{L}$ over a flat Kähler manifold $\mathbb{C}^k/\Gamma$. If in addition, the Ricci curvature is positive at some point, then M is simply connected and $M = \widetilde{M} = \widetilde{N} \times \widetilde{L}$ where $\widetilde{L}$ is diffeomorphic to the Euclidean space.*

So far the above results have made no assumptions on the boundedness of curvature. When the curvature is bounded, one can use the more powerful Kähler-Ricci flow (2.2), which can be solved for a short time by Theorem 2.1. The results in §2 suggests the study of $g(t)$ in connection to Yau's conjecture or more generally, the Steinness of M. When $g(t)$ exists for all time $0 \leq t < \infty$ we expect the asymptotics of $g(t)$ to be particularly useful. In [15] the authors proved the following:

THEOREM 4.4. [**15**] *Let (M^n, g_0) be a complete non-compact Kähler manifold with bounded non-negative holomorphic bisectional curvature. Suppose the scalar curvature of g_0 is such that $k(x, r) \leq k(r)$ for some function $k(r)$ satisfying*

(4.1)
$$k(r) \leq \frac{C}{r}$$

as $r \to \infty$ for some $C > 0$. Then M is holomorphically covered by a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$. Moreover, if M has positive bisectional curvature and is simply connected at infinity, then M is biholomorphic to a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$, and in particular, M is Stein.

When $k(r) = \frac{C}{1+r^{1+\epsilon}}$, for $\epsilon > 0$, the result that M is biholomorphic to a pseudoconvex domain was proved by Shi [**50**] under the additional assumption that (M, g) has positive sectional curvature. Under the same decay condition and assuming maximum volume growth, similar results were obtained by Chen-Zhu [**19**]. The condition of positive sectional curvature in [**50**] was used to produce a convex compact exhaustion of M. The maximum volume growth condition in [**19**] was used to control the injectivity radius for $g(t)$. Note that the decay condition in Theorem 4.4 is almost optimal because of Theorem 3.2.

The idea of proof of Theorem 4.4 is as follows. Let g_0 be as in the theorem. Then one can find $g(t)$ which solves the Kähler-Ricci flow equation (2.2) with initial data $g(0) = g_0$ satisfying the conclusion in Theorem 2.3. By taking the universal cover and by using the result of Cao [**10**], we may assume that $g(t)$ has positive Ricci curvature. Let $g_i = g(i)$. Then for any fixed $p \in M$,

- **(a1)** $cg_i \leq g_{i+1} \leq g_i$ for some $1 > c > 0$ for all i.
- **(a2)** $|\nabla Rm(g_i)| + |Rm(g_i)| \leq c'$ for some c' on $B_i(p, r_0)$, and for some $r_0 > 0$ for all i where $B_i(p, r_0)$ is the geodesic ball around p with respect to g_i.
- **(a3)** g_i is contracting in the following sense: For any ϵ, for any i, there $i' > i$ with
$$g_{i'} \leq \epsilon g_i$$
  in $B_i(p, r_0)$.

(a1)–**(a2)** follow from Theorem 2.3, and **(a3)** follows from Theorem 6.8 in §6.

Hence Theorem 4.4 is a consequence of the following:

THEOREM 4.5. [**15**] *Let M^n be a complex noncompact manifold. Suppose there exist a sequence of complete Kähler metrics g_i, $i \geq 1$ on M with properties (a1)–(a3). Then M is covered by a pseudoconvex domain in $\mathbb{C}^n$ which is homeomorphic to $\mathbb{R}^{2n}$.*

We now sketch some key ideas in the proof Theorem 4.5 . Fix some point $p \in M$. By **(a2)** we may lift the metric $g(t)$ to the tangent space via the exponential map at p and use L^2 theory to construct a local biholomorphism $\Phi_i : D(r) \to M$ for each i so that $\Phi_i^*(g_i)$ is equivalent to the Euclidean metric in $D(r)$ and equal to the Euclidean metric at 0, where $D(r)$ is the Euclidean ball of radius r with center at the origin (see Proposition 2.1 in [**14**]). Thus Φ_i provides a holomorphic normal coordinate at p, and one would like to consider an appropriate change of coordinate map from Φ_i and Φ_{i+1}. It is not clear one can do this however as Φ_i is generally not injective and may not even be a covering. Nevertheless, using **(a1)** and **(a2)**, we can find $F_{i+1} : D(r) \to \mathbb{C}^n$ such that $\Phi_i = \Phi_{i+1} \circ F_{i+1}$ for a smaller r if necessary yet independent of i. Note that this means that $F_{i+1}(D(r))$ is in the domain of Φ_{i+1}. Also, we may choose this r so that F_{i+1} is a biholomorphism onto its image. By **(a3)** one can show that these maps are essentially contracting in the sense that there are $n_i \uparrow \infty$ such that

$$(4.2) \qquad F_{n_{i+1}} \circ \cdots F_{n_i+2} \circ F_{n_i+1}(D(r)) \subset D\left(\frac{r}{2}\right)$$

for every i. Now let us suppose each F_i can be extended to a biholomorphism of $\mathbb{C}^n$, which we still denote as F_i, on $\mathbb{C}^n$. We may then let

$$S_i = (F_{n_i} \circ \cdots \circ F_2)^{-1}(D(r))$$

and $\Omega = \bigcup_i S_i$. Then Ω will be a pseudoconvex domain in $\mathbb{C}^n$ and it will be homeomorphic to $\mathbb{R}^{2n}$ because $S_i \subset S_{i+1}$. Let Ψ be defined as

$$\Psi(z) = \Phi_{n_i} \circ F_{n_i} \circ \cdots \circ F_2(z)$$

for $z \in S_i$. Note that

$$\Phi_{n_{i+1}} \circ F_{n_{i+1}} \circ \cdots \circ F_{n_i} \circ \cdots \circ F_2(z) = \Phi_{n_i} \circ F_{n_i} \circ \cdots \circ F_2(z)$$

if $z \in S_i$ and thus Ψ is a well-defined nondegenerate map from Ω to M. Moreover, using **(a3)** and that g_i is 'shrinking', one can prove that Ψ is surjective thus proving Theorem 4.5. In case M is simply connected, one can also prove that Ψ is injective.

Now in general, F_i can not be extended to a biholomorphism of $\mathbb{C}^n$. The key is to show that the maps $F_i : D(r) \to \mathbb{C}^n$ can be approximated well enough by biholomorphisms of $\mathbb{C}^n$. For this one uses a theorem of Anderson-Lempert [**1**] which states that if F is a biholomorphism from a star-shape domain in $\mathbb{C}^n$ onto a Runge domain in $\mathbb{C}^n$, then F can be uniformly approximated by biholomorphisms of $\mathbb{C}^n$ on compact subsets of the domain. To use the result it is thus sufficient to show that the image of $F_i : D(r) \to \mathbb{C}^n$ is Runge. Once it is established that F_i can be approximated in this way, one uses the approximations to construct a map Ψ as above having the desired properties.

From the above proof, one expects to obtain stronger results depending on how much more can be said of the maps F_i. We will see this to be the case in the following sections §5 and §6.

5. Gradient Kähler-Ricci solitons

In very special cases, the maps F_i from the previous section will all be equal to a single map F. This is the case when $g(t)$ is a gradient Kähler-Ricci soliton.

Let us recall that Kähler-Ricci solitons are solutions to the Kähler-Ricci flow for which the metric evolves only by dilation and pull back along a one parameter family of biholomorphisms. More specifically, $(M, g_{i\bar{j}})$ is said to be a Kähler-Ricci soliton if there is a family of biholomorphisms ϕ_t on M, given by a holomorphic vector field V, such that $g_{ij}(x, t) = \phi_t^*(g_{ij}(x))$ is a solution of the normalized Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g_{i\bar{j}}(x, t) = -R_{i\bar{j}}(x, t) - \kappa g_{i\bar{j}}(x, t)$$
(5.1)
$$g_{i\bar{j}}(x, 0) = g_{i\bar{j}}(x)$$

for some constant κ. The soliton is said to be *steady* if $\kappa = 0$, and *expanding* if $\kappa > 0$ (which will be normalized to be 1). In particular, note that $\phi_t : (M, g(t)) \to (M, g)$ is an isometry for every t, and thus Kähler-Ricci solitons can be viewed as generalized fixed points for the Kähler-Ricci flow. If in addition, the holomorphic vector field V is the gradient of a real valued function f, then the soliton is a gradient Kähler-Ricci soliton and the metric $g_{i\bar{j}}$ satisfies

$$f_{i\bar{j}} = R_{i\bar{j}} + \kappa g_{i\bar{j}},$$
(5.2)
$$f_{ij} = 0.$$

Conversely, if these equations are satisfied with bounded curvature, then the corresponding solution to (2.2) is a gradient Kähler-Ricci soliton by the uniqueness theorem of Chen-Zhu [**22**]. These equations are thus known as gradient Kähler-Ricci soliton equations.

Solitons are extremely important in studying the formation of singularities under the flow and hence the underlying structure of the manifolds. As described in [**29**] by Hamilton, solitons typically arise as limit solutions when one takes a dilation limit around a singularity forming under the flow (see [**28**] on limit solutions to the Ricci flow). In [**7**], Cao proved the following classification of limit solutions to the Kähler-Ricci flow.

THEOREM 5.1. [**7**] *Let $(M, g(t))$ be a family of complete Kähler metrics on a noncompact complex manifold M which form a solution to (5.1) for $t \in [0, \infty)$ if $\kappa > 0$ and for $t \in (-\infty, \infty)$ if $\kappa = 0$ such that the holomorphic bisectional curvature is nonnegative and the Ricci curvature is positive. Assume that the scalar curvature of $g(t)$ assumes its maximum in space time.*

Then $(M, g(t))$ is a gradient Kähler-Ricci soliton which is steady if $\kappa = 0$, and expanding if $\kappa = 1$.

The proof of Theorem 5.1 relies on the LYH type Harnack inequality Theorem 6.7 of Cao [**7**]. The theorem suggests that studying the uniformization of gradient Kähler-Ricci solitons should give insight into the uniformization of more general manifolds as in §6.

Let ϕ_t be the family of biholomorphisms defining a Kähler-Ricci soliton so that $g(t) = \phi_t^*(g(0))$. Assume the holomorphic bisectional curvature is bounded and nonnegative. Suppose p is a fixed point of the flow ϕ_t and let $F = \phi_1 = \phi_t|_{t=1}$. Then the injectivity radii of p with respect to $g(t)$ are constant. Hence one can construct biholomorphism Φ_i for $g(i)$ as in the proof of Theorem 4.5. In this case, we can actually take Φ_i such that $\Phi_{i+1}^{-1} = \Phi_1^{-1} \circ (\phi_1)^i$ and thus $\Phi_{i+1}^{-1} \circ \Phi_i = \Phi_1^{-1} \circ \phi \circ \Phi_1 = F_{i+1}$. In other words, the F_i's from §4 can in this case be taken as the single map. And studying the F_i's reduces here to studying the discrete complex dynamical system generated by F. Recall that p is a fixed point for F. In [**46**] Rosay-Rudin proved the following result for *attractive basins* of a fixed points for biholomorphisms of $\mathbb{C}^n$, which was pointed out to hold on general complex M by Varolin [**54**]:

THEOREM 5.2. [**46**] *Let F be a biholomorphism from a complex manifold M^n to itself and let $p \in M^n$ be a fixed point for F. Fix a complete Riemannian metric g on M and define the basin of attraction*

$$\Omega := \{x \in M : \lim_{k \to \infty} dist_g(F^k(x), p) = 0\}$$

where $F^k = F \circ F^{k-1}, F^1 = F$.

Then Ω is biholomorphic to $\mathbb{C}^n$ provided Ω contains an open neighborhood around p.

Now in many cases, the dynamical system generated by $F = \phi_1$ above can be shown to have a unique attractive fixed point with the whole manifold as a basin of attraction. By Theorem 5.2, such a soliton must then be biholomorphic to $\mathbb{C}^n$. This was observed by the authors in [**11**] where they proved:

THEOREM 5.3. [**11**] *If $(M, g_{i\bar{j}})$ is a complete non-compact gradient Kähler-Ricci soliton which is either steady with positive Ricci curvature so that the scalar curvature attains its maximum at some point, or expanding with non-negative Ricci curvature, then M is biholomorphic to $\mathbb{C}^n$.*

This result was obtained independently by Bryant [**5**] for the steady case.

To prove the theorem one only needs to check that there is a unique fixed point of the biholomorphisms ϕ_t and that ϕ_t is *contracting* on M. These are easily verified using the condition on the positivity or nonnegativity of the Ricci curvature.

The proof of Theorem 5.2 relies on the fact that F can be transformed to have a normal form around p. This fact is due to Sternberg [**53**] for real systems, and was later independently proved by Rosay-Rudin [**45**] for complex systems. We sketch the proof here for the case $M = \mathbb{C}^n$. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a biholomorphism such that $F(0) = 0$. One then modifies F by a biholomorphism T near the origin, so that (i) $T \circ F \circ T^{-1}$ is close to an *upper triangular* map G; and (ii) $T'(0) = I$, the identity map. Here $G = (g_1, \ldots, g_n)$ is an upper triangular map, if $g_1(z) = c_1 z_1$, $g_2(z) = c_2 z_2 + h(z_1), \ldots, g_n(z) = c_n z_n + h(z_1, \ldots, z_{n-1})$ for some constants $c_1, \ldots, c_n$. By (i) we mean that for any m, we can choose T and G so that $G^{-1} \circ T \circ F - T = O(|z|^m)$, with G being independent of m when m is large enough. It can then be shown that $\Psi = \lim_{k \to \infty} G^{-k} \circ T \circ F^k$ exists and is a biholomorphism from the basin onto $\mathbb{C}^n$. For any z in the basin, $F^k(z)$ is defined if k is large enough. On the other hand, F is shrinking so G is expanding. Hence one may expect the image will be the whole $\mathbb{C}^n$.

In special cases, the Kähler-Ricci flow actually performs this uniformization and the soliton metric converges in the re-scaled subsequence sense to a complete flat Kähler metric under the flow. The following result was obtained in [**12**].

THEOREM 5.4. [**12**] *Let $(M, g_{i\bar{j}})$ be a complete non-compact gradient Kähler-Ricci soliton as in Theorem 5.3 with smooth potential f and equilibrium point p. let $g_{i\bar{j}}(x, t)$ be the corresponding solution to (5.1) and let $\mathbf{v}_p \in T_p^{1,0}(M)$ be a fixed nonzero vector with $|\mathbf{v}_p|_0 = 1$. Then for any sequence of times $t_k \to \infty$, the sequence of complete Kähler metrics $\frac{1}{|\mathbf{v}_p|_{t_k}^2} g_{i\bar{j}}(x, t_k)$ subconverges on compact subsets of M to a complete flat Kähler metric $h_{i\bar{j}}$ on M if and only if $R_{i\bar{j}}(p) = \beta g_{i\bar{j}}(p)$ at $t = 0$ for some constant β. In particular, if this condition is satisfied then M is biholomorphic to $\mathbb{C}^n$.*

Note that the Theorem suggests in general, we do not expect to prove uniformization by rescaling a solution $g(t)$ to the Kähler-Ricci flow to obtain a complete Kähler flat metric as a limit.

6. Eternal solutions to the normalized Kähler-Ricci flow

In this section we generalize Theorem 5.3 for Kähler-Ricci solitons to *eternal* solutions to the normalized Kähler-Ricci flow. As a corollary of this we will present a uniformization theorem for the case of average quadratic curvature decay. This is a critical case for uniformization in light of the gap phenomenon for manifolds with faster than quadratic curvature decay discussed in §7. The first major result in this case was the following theorem of Mok [**34**].

THEOREM 6.1. [**34**] *Let (M^n, g) be a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Suppose that the*

following conditions are satisfied for some $p \in M^n$,

> (i) $Vol(B_p(r)) \geq C_1 r^{2n}$, *for all* $r \geq 0$
> (ii) $\mathcal{R}(x) \leq \frac{C_2}{(d(p,x)+1)^2}$. *for some* $C_1, C_2, \epsilon > 0$.

Then M^n is biholomorphic to an affine algebraic variety.

It is well known that a complete noncompact Riemannian manifold with positive sectional curvature is diffeomorphic to the Euclidean space [**26**]. Using a result of Ramanujam [**45**] which states that an algebraic surface which is homeomorphic to $\mathbb{R}^4$ must be biholomorphic to $\mathbb{C}^2$, Mok [**34**] concluded:

COROLLARY 6.2. *Let (M^2, g) be a complete noncompact Kähler surface with positive sectional curvature satisfying conditions (i) and (ii) in the above theorem. Then M is biholomorphic to $\mathbb{C}^2$.*

The method of Mok is to construct enough polynomial growth holomorphic functions to embed M into some $\mathbb{C}^N$ so that the image will be an affine algebraic variety. The proof uses algebraic geometric methods. Later, in separate works, Chen-Tang-Zhu, Chen-Zhu and Ni used the Kähler-Ricci flow to improve Mok's result in Theorem 6.1. In particular in the case of $n = 2$, Chen-Zhu [**20, 21**] obtained the same result as in the Corollary 6.2 by only assuming positive and bounded bisectional curvature and maximal volume growth. Ni [**39**] further weakened the condition of positive holomorphic bisectional curvature to nonnegative holomorphic bisectional curvature. The main idea is to show that maximum volume growth still implies quadratic curvature decay condition, as mentioned in Theorem 3.3. Then one can still prove that M is homeomorphic to $\mathbb{R}^4$ and produce enough polynomial growth holomorphic functions to carry over Mok's method and to use the result of Ramanujam. Chen-Zhu [**18**] also proved that if a Kähler surface with bounded and positive sectional curvature is such that the integral of $(Ric)^2$ is finite, then the surface is biholomorphic to $\mathbb{C}^2$, using Ramanujam's result again. Ramanujam's theorem however is only true for complex surfaces, and for higher dimensions we need other methods.

We would like to use the results on Kähler-Ricci flow by Shi in §2 to generalize Theorem 5.3 for Kähler-Ricci solitons to general solutions $g(t)$ to the Kähler-Ricci flow. It is natural here to consider *eternal* solutions to (5.1), in other words solutions defined for $t \in (-\infty, \infty)$. It is readily seen that a steady or expanding gradient Kähler-Ricci soliton is indeed an eternal solution to (5.1). In light of this, one may expect that Theorem 5.3 is still true when $g(t)$ is an eternal solution to (5.1) with nonnegative uniformly bounded holomorphic bisectional curvature. This expectation is confirmed in Theorem 6.3 and was proved by the authors in [**14**]. Before stating the theorem, we first discuss the case of quadratic curvature decay in relation to eternal solutions.

Suppose (M,g) is complete noncompact with bounded and nonnegative holomorphic bisectional curvature so that its scalar curvature satisfies the quadratic decay condition (3.3). Then by the results in §2, (6.1) has a long time solution $\tilde{g}_{i\bar{j}}$ with initial data g, and the scalar curvature $\widetilde{\mathcal{R}}$ will satisfy $t\widetilde{R} \leq C$ for some constant C uniform on spacetime. Now if we let $g(t) = e^{-t}\tilde{g}_{i\bar{j}}(e^t)$, then $g(t)$ will be an eternal solution to the normalized Kähler-Ricci flow (5.1) for $\kappa = 1$. Moreover, it is easy to see that $g(t)$ has uniformly bounded nonnegative holomorphic bisectional curvature. Conversely, given such an eternal solution $g(t)$ one sees that $\widetilde{g}(t) = tg(\log t)$ solves the unnormalized Kähler-Ricci flow

$$(6.1) \qquad \frac{\partial}{\partial t}\widetilde{g}_{i\bar{j}} = -\widetilde{R}_{i\bar{j}}.$$

for $t \geq 1$, and that $t\widetilde{R} \leq C$ for some uniform constant C. This in turns implies that the scalar curvature of the initial metric $g(0)$ satisfies the quadratic decay condition (3.3) see [**43, 39**].

Now we state our theorem on eternal solution:

THEOREM 6.3. [**14**] *Let M^n be a noncompact complex manifold. Suppose there is a smooth family of complete Kähler metrics $g(t)$ on M such that for $\kappa = 0$ or 1, $g(t)$ satisfies*

$$(6.2) \qquad \frac{\partial}{\partial t}g_{i\bar{j}}(x,t) = -R_{i\bar{j}}(x,t) - \kappa g_{i\bar{j}}(x,t)$$

for all $t \in (-\infty, \infty)$ such that for every t, $g(t)$ has uniformly bounded nonnegative holomorphic bisectional curvature on M independent of t. Then M is holomorphically covered by $\mathbb{C}^n$.

By the remarks preceding the theorem, we have the following result by the authors:

THEOREM 6.4. [**14**] *Suppose (M^n, g) has holomorphic bisectional curvature which is bounded, non-negative and has average quadratic curvature decay. Then M is holomorphically covered by $\mathbb{C}^n$.*

Combining this with Theorem 3.3, we conclude:

COROLLARY 6.5. *Let (M^n, g) be a complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature such that M has maximum volume growth then M is biholomorphic to $\mathbb{C}^n$.*

REMARK 1. Corollary 6.5 was proved before Theorem 6.4 by the authors in [**13**].

As noted earlier, if we assume the holomorphic bisectional curvature is bounded and nonnegative, then Theorem 6.3 is basically a direct generalization of Theorem 5.3 for gradient Kähler-Ricci solitons. However, the proof

of Theorem 6.3 is much more complicated. Beginning with a solution $g(t)$ to the Kähler-Ricci flow as in Theorem 6.3, fix some point $p \in M$ and construct maps Φ_i as in the proof of Theorem 4.5 in §4. For simplicity, we will assume that Φ_t is injective for every t (thus M is simply connected). In other words, we assume the injectivity radius of $g(t)$ at p is bounded from below independently of t. Such a bound exists in the case of [13], where maximum volume growth is assumed and removing the dependence on this bound is the essential generalization made in [14]. We may also assume that $g(t)$ has positive Ricci curvature because of a dimension reduction result of Cao [10].

Now for $N > 0$ sufficiently large, as in §4, we can find a sequences of biholomorphisms F_i from $D(r)$ onto its image which is inside $D(r)$:

$$(6.3) \qquad F_{i+1} = \Phi_{(i+1)N}^{-1} \circ \Phi_{iN} : D(r) \to D(r)$$

for $i \geq 1$. These F_i's are basically the same as those in the proof of Theorem 4.5, which as noted in §5, can be chosen to be a single map F when $g(t)$ is gradient Kähler-Ricci soliton. One would now like to imitate the proof of Rosay-Rudin's Theorem 5.2. A key step in their proof was to transform F into a particularly nice form. Now the main difficulty here is simultaneously transforming the sequence $\{F_i\}$ into a likewise nice form. In [30], Johnsson-Varolin showed that this can be done provided that asymptotically they behave close enough to a single map F. This closeness is essentially in terms of the Lyapunov regularity of the F_i's (see [3] for the terminology). In terms of the Kähler-Ricci flow, the authors proved [14] that this transformability is possible due to the Lyapunov regularity of $g(t)$ as described in the following:

THEOREM 6.6. [14] *Let $M^n, g(t)$ be as in Theorem 6.3 such that the Ricci curvature of $g(t)$ is positive. Let $p \in M$ be fixed and let $\lambda_1(t) \geq \ldots \geq \lambda_n(t) > 0$ be the eigenvalues of $R_{i\bar{j}}(p,t)$ relative to $g_{i\bar{j}}(p,t)$. Then $\lim_{t\to\infty} \lambda_i(t)$ exists for $1 \leq i \leq n$ and there is a constant $C > 0$ such that $\lambda_n(t) \geq C$ for all $t \geq 0$. Moreover, if we let $\mu_1 > \ldots > \mu_l > 0$ be the distinct limits with multiplicities $d_1, \ldots, d_l$, then $V = T_p^{(1,0)}(M)$ can be decomposed orthogonally with respect to $g(0)$ as $V_1 \oplus \cdots \oplus V_l$ so that the following are true:*

(i) *If v is a nonzero vector in V_i for some $1 \leq i \leq l$, then $\lim_{t\to\infty} |v_i(t)| = 1$ and thus $\lim_{t\to\infty} Rc(v(t), \bar{v}(t)) = \mu_i$ and*

$$\lim_{t\to\infty} \frac{1}{t} \log \frac{|v|_t^2}{|v|_0^2} = -\mu_i - 1.$$

Moreover, the convergences are uniform over all $v \in V_i \setminus \{0\}$.

(ii) *For $1 \leq i, j \leq l$ and for nonzero vectors $v \in V_i$ and $w \in V_j$ where $i \neq j$, $\lim_{t\to\infty} \langle v(t), w(t)\rangle_t = 0$ and the convergence is uniform over all such nonzero vectors v, w.*

(iii) $\dim_{\mathbb{C}}(V_i) = d_i$ *for each* i.
(iv)

$$\sum_{i=1}^{l}(-\mu_i - 1)\dim_{\mathbb{C}} V_i = \lim_{t\to\infty}\frac{1}{t}\log\frac{\det(g_{i\bar{j}}(t))}{\det(g_{i\bar{j}}(0))}.$$

The theorem basically says that the eigenvalues of the Ricci tensor have limits and the eigenspaces are almost the same.

The proof of the theorem relies on an important differential Li-Yau-Hamilton(LYH) inequality for the Kähler-Ricci flow by Cao [7]:

THEOREM 6.7. [7] *Let* $g(t)$ *be a complete solution to the Kähler-Ricci flow (2.2) on* $M \times [0, T)$ *with bounded and nonnegative holomorphic bisectional curvature. For any* $p \in M$ *and holomorphic vector* V *at* p, *let*

$$Z_{i\bar{j}} = \frac{\partial}{\partial t}R_{i\bar{j}} + R_{i\bar{k}}Z_{k\bar{j}} + R_{i\bar{j},k}V_{\bar{k}} + R_{i\bar{j},\bar{k}}V_k + R_{i\bar{j}k\bar{l}}V_{\bar{k}}V_l + \frac{1}{t}R_{i\bar{j}}.$$

Then
$$Z_{i\bar{j}}W^i W^{\bar{j}} \geq 0$$
for all holomorphic vectors W *at* p.

Using this differential inequality the authors proved that

THEOREM 6.8. [15] *Let* $g(t)$ *be a complete solution to (2.2) with nonnegative holomorphic bisectional curvature such that for any* $T > 0$, $g(t)$ *has bounded curvature for all* $t \in [0, T]$. *Fix some* $p \in M$ *and let* $\lambda_i(t)$ *be the eigenvalues of* $Rc(p, t)$ *arranged in increasing order. Then*

$$t\lambda_k(t)$$

is nondecreasing in t *for all* $1 \leq k \leq n$.

Now under the condition of Theorem 6.6, the proof of Theorem 6.8 directly implies that $\lambda_i(t)$ is nondecreasing in t for every $1 \leq i \leq n$. This will imply that $\lim_{t\to\infty}\lambda_i(t)$ exists for all i. From this, one argues as in the proof of Theorem 5.1 in [9] to prove that $g(t)$ behaves like gradient Kähler-Ricci soliton with fixed point at p as $t \to \infty$ in the following sense: For any $t_k \to \infty$, there is a subsequence of $g(t + t_k)$ such that $(M, g(t + t_k))$ converge to a gradient Kähler-Ricci soliton. To prove this one actually only needs the convergence of the scalar curvature $R(p, t)$. In case the manifold has maximum volume growth, a more general result similar to this was obtained by Ni [40] independently. Now it is easy to see that if $g(t)$ is a gradient Kähler-Ricci soliton with fixed point p then Theorem 6.6 is true, and that in this case we do not even have to take limits. Observing this, one then argues carefully to obtain the results in Theorem 6.6.

We now return to our sketch of proof of Theorem 6.3. The $F_i's$ define a randomly iterated complex dynamical system on $D(r)$ with fixed point

at the origin. Moreover, using that the Ricci curvature is bounded away
from zero at p for all t by Theorem 6.6, one can show that the maps F_i are
uniformly contracting at the origin and that $\lim_{i \to \infty} F_i \circ \cdots \circ F_1(D(r)) = 0$.
This is one of the biggest differences between the F_i's here and those in §4.
Although the maps there were eventually contracting, they are in general
not uniformly contracting.

Theorem 6.6 can now be translated into Lyapunov regularity of the system $\{F_i\}$ which can roughly be described as follows. Let $A_i = F_i'(0)$. Then
the system is Lyapunov regular at 0 if one can decomposed $\mathbb{C}^n$ orthogonally with respect the Euclidean metric as $E_1, \ldots, E_l$ such that if $E_k^{(i+1)} = A_{i+1}(E_k^{(i)})$, then $E_k^{(i)}$ are asymptotically orthogonal and for each k, A_i is
asymptotically contracting at a constant rate on $E_k^{(i)}$.

Once this is established, one follows the construction in [**30**] to construct
a sequence of biholomorphisms $G_i : \mathbb{C}^n \to \mathbb{C}^n$, and $T_i : D(r) \to D(r)$ (it
may be necessary to take r smaller here, but independently of i). Here, the
G_i's represent approximations of the F_i's in $\mathrm{Aut}(\mathbb{C}^n)$ which are lower triangular in a certain sense while the T_i's represent a sequence of holomorphic
coordinate changes of $D(r)$. The following Lemma describes the extent of
this approximation ([**13**] Lemma 5.7).

LEMMA 6.9. *Let $k \geq 0$ be an integer. Then*

$$\Psi_k = \lim_{l \to \infty} G_{k+1}^{-1} \circ G_{k+2}^{-1} \circ \cdots \circ G_{k+l}^{-1} \circ T_{k+l} \circ F_{k+l} \circ \cdots \circ F_{k+2} \circ F_{k+1}$$

*exists and is a nondegenerate holomorphic map from $D(r)$ into $\mathbb{C}^n$. Moreover,
there is a constant $\gamma > 0$ which is independent of k such that*

$$\gamma^{-1} D(r) \subset \Psi_k(D(r)) \subset \gamma D(r).$$

Since F_i is uniformly contracting, one can show that the sets
$\Omega_i = \Phi_{iT}(D(r))$ exhaust M as $i \to \infty$. This together with Lemma 6.9 and
the definition of the $F_i's$ tell us that the sequence of maps

$$S_i = G_1^{-1} \circ \cdots G_i^{-1} \circ T_i \circ \Phi_{iN}^{-1} : \Omega_i \to \mathbb{C}^n$$

converges to a biholomorphic map Ψ from M into $\mathbb{C}^n$. It is shown in ([**13**],
§5) that Ψ is onto, and thus M is biholomorphic to $\mathbb{C}^n$.

Now in case that the injectivity radius of p with respect to $g(t)$ is not
bounded away from zero, one works on the pullback metrics $\hat{g}(t)$ of $g(t)$ under
the exponential maps. In this setting the injectivity radius will be bounded
away from zero and one can show that $\hat{g}(t)$ still behaves like gradient Kähler-
Ricci soliton locally near 0 as $t \to \infty$. One then constructs maps F_i as in
the proof of Theorem 4.5, which one then shows to be Lyapunov regular,
and proceeds as above to obtain Theorem 6.3.

The fact that $\hat{g}$ behave like gradient Kähler-Ricci solitons locally near
0 as $t \to \infty$ can be used to prove the following corollary to the Theorem 6.3.

COROLLARY 6.10. *Let $M, g(t)$ as in Theorem 6.3. Suppose the Ricci curvature is positive with respect to $g(0)$ at some point p. Then M is simply connected and is biholomorphic to $\mathbb{C}^n$. In particular, if (M, g) is a complete noncompact Kähler manifold with bounded positive holomorphic bisectional curvature which satisfies the quadratic decay condition (3.3), then M is biholomorphic to $\mathbb{C}^n$.*

One might want to compare the last assertion of the corollary with the statement of Yau's conjecture and the result of Zheng's Theorem 4.3. The main point of the proof of the corollary is to show that the first fundamental group of M is finite. Since M is covered by $\mathbb{C}^n$, M must then be simply connected by a well-known result, see [**4**].

7. A Theorem of Mok-Siu-Yau and its generalizations

One may expect stronger results when the curvature decays faster than quadratic. In fact, there are gap theorems which tell us that curvature of a nonflat complete noncompact Kähler manifold (M, g) with nonnegative holomorphic bisectional curvature cannot decay too fast. These can be viewed as converses to the curvature decay Theorem 3.1. The following classic gap theorem of Mok-Siu-Yau [**36**] in 1981 was the first result supporting Yau's conjecture.

THEOREM 7.1. [**36**] *Let M be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose that the following conditions are satisfied for some $p \in M$ and some $\epsilon > 0$:*

  (i) *$Vol(B_p(r)) \geq C_1 r^{2n}$, for all $r \geq 0$.*
  (ii) *$\mathcal{R}(x) \leq \frac{C_2}{(d(p,x)+1)^{2+\epsilon}}$. for some $C_1, C_2, \epsilon > 0$.*
 (iii) *Either M is Stein or M has nonnegative sectional curvature.*

Then M is isometrically biholomorphic to $\mathbb{C}^n$.

By Theorem 4.2, condition (iii) is superfluous because of (i). The condition of maximum volume growth (i) however seems rather strong. In [**19**], Chen-Zhu proved the following:

THEOREM 7.2. [**19**] *Let M be a complete noncompact Kähler manifold with nonnegative and bounded holomorphic bisectional curvature. Suppose that for some positive real function $\epsilon(r)$ satisfying $\lim_{r \to \infty} \epsilon(r) = 0$ we have*

$$(7.1) \qquad \frac{1}{Vol(B_x(r))} \int_{B_x(r)} \mathcal{R} dV \leq \frac{\epsilon(r)}{(1+r)^2}$$

for all $x \in M$ and for all $r > 0$. Then the universal cover of M is isometrically biholomorphic to $\mathbb{C}^n$.

The theorem says that if M has nonnegative holomorphic bisectional curvature such that the curvature is bounded and decays faster than quadratic on average *uniformly* at all points in M, then M is flat. It is not hard to see these hypothesis are implied by (i) and (ii) in Theorem 7.1. The proof of Theorem 7.2 uses the Kähler-Ricci flow. In fact, one can see from the proof of Theorem 2.3 that if we consider the normalized flow (5.1) with $\kappa = 1$, then the scalar curvature will tend to zero at infinity. Hence it must be zero initially by the monotonicity derived from Theorem 6.7. Note that in order to use the Kähler-Ricci flow, one needs to assume that the curvature is bounded. On the other hand, there is a way to prove a stronger result in this case without using the Kähler-Ricci flow by modifying the method of [36].

Let us go back to the proof of Mok-Siu-Yau [36]. Their method is as follows: First one solves the Poisson equation $\frac{1}{2}\Delta u = \mathcal{R}$ with good estimates. This can be done because assumptions (i) and (ii) in Theorem 7.1 give a good estimate of the Green's function. Secondly, one can show that $||\sqrt{-1}\partial\bar{\partial}u - \text{Ric}||$ is subharmonic using the fact that M has nonnegative holomorphic bisectional curvature. Using the estimate of u, one may get an integral estimate of $||\partial\bar{\partial}u||^2$ on geodesic balls. Then by a mean value inequality, we conclude that $\sqrt{-1}\partial\bar{\partial}u = \text{Ric}$. In particular, u is plurisubharmonic. Finally, one proves that u is constant implying that M is flat. One proves this by contradiction: assuming u is not constant, one produces a function v which at a point is strictly plurisubharmonic and satisfies $(\sqrt{-1}\partial\bar{\partial}v)^n \equiv 0$. If the manifold is Stein one embeds M in $\mathbb{C}^N$ for some N and proceeds to use the coordinate functions of $\mathbb{C}^N$ to construct such a v. If M has nonnegative sectional curvature one uses the Busemann function, together with u, to construct v.

Following this line of argument, Ni-Shi-Tam in [41] obtained a general result on the existence of a solution of the Poisson equation on complete noncompact Riemannian manifold with nonnegative Ricci curvature without any volume growth condition. In particular, it was shown there that if (7.1) holds at some point p, then one can still solve:

$$\frac{1}{2}\Delta u = \mathcal{R}$$

with

(7.2) $$\limsup_{r\to\infty} \frac{u(x)}{\log d(p, x)} \leq 0.$$

These follow from the classic results of Li-Yau [33] on the heat kernel estimates on manifolds with nonnegative Ricci curvature, which provide an estimate for the Green's function without assuming maximum volume growth.

In order to prove that $\sqrt{-1}\partial\bar{\partial}u = \text{Ric}$, one needs the condition on the L^2 norm of $\mathcal{R}$:

$$(7.3) \qquad \liminf_{r\to\infty}\left[\exp(-ar^2)\int_{B_p(r)}\mathcal{R}^2 dV\right] < \infty$$

for some $a > 0$. Note that $\mathcal{R}$ may be allowed to grow like $\exp(a'r^2)$ for some $a' > 0$. From this, one may then solve the heat equation with initial data u and use the maximum principle and the mean value inequality as developed by Li-Schoen [**32**] to conclude that u is indeed a potential of the Ric form. Here (7.3) is used to apply the maximum principle.

Finally, one can prove that u is constant by (7.2) and the following Liouville property for pluirsubharmonic functions of Ni-Tam [**42**]:

THEOREM 7.3. [**42**] *Let (M, g) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose u is a continuous plurisubharmonic function satisfying (7.2), then u is constant.*

We may assume here that u is bounded from below. The idea of proof of Theorem 7.3 uses methods as in the proof of Theorem 3.2 and uses the following result of Ni [**38**] to conclude that u is actually harmonic:

THEOREM 7.4. [**38**] *Let (M^n, g) be a complete noncompact Kähler manifold with nonnegative Ricci curvature. Suppose u is a plurisubharmonic function on M satisfying (7.2). Then $(\partial\bar{\partial}u)^n \equiv 0$.*

Hence u must be harmonic and therefore constant by a classical result of Cheng-Yau [**23**] on harmonic function on complete manifolds with nonnegative Ricci curvature. From these results Ni-Tam [**42**] proved the following, which is the best result up to now in the generalization of Theorem 7.1:

THEOREM 7.5. [**42**] *Let (M, g) be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Suppose the scalar curvature $\mathcal{R}$ satisfies (7.1) and (7.3) for some p. Then the universal cover of M is isometrically biholomorphic to $\mathbb{C}^n$.*

It is still unknown whether the condition (7.3) can be removed.

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