Collapsed Manifolds with Bounded Sectional Curvature and Applications

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Abstract. This is a survey article on collapsed Riemannian manifolds with bounded sectional curvature. Instead of attempting to cover many results in related topics, we will concentrate on one path that includes most of the main ideas and techniques developed in the last two decades.

1. Introduction

Consider a complete n-manifold M of sectional curvature normalized to be bounded in absolute value, $|\sec_M| \leq 1$. Given $\epsilon > 0$, there is an ϵ *thick-thin* decomposition of M: the thick part consists of points whose injectivity radius is $\geq \epsilon$ and the complement is the thin part. According to [**12**], the local topology of the thick part is under control: after a small perturbation of the boundary, any ball of radius one in the thick part has only finitely many possible topological types. On the other hand, when $\epsilon < \epsilon(n)$ (a constant depending only on n), there exists a special geometric/topological structure on any unit ball in the thin part [**14**], consisting of a sort of generalized foliation with orbits consisting of nilmanifolds.

Unless otherwise specified, a *collapsed manifold* means a complete Riemannian manifold M with $|\sec_M| \leq 1$, whose injectivity radii are less than $\epsilon(n)$ everywhere, i.e., M is thin. Since the 1980s, Riemannian geometry has experienced an explosive development, and one of the most important achievements is the theory of collapsed manifolds.

Before discussing collapsing in detail, we recall the Cheeger-Gromov compactness theorem [**11, 12, 36**], which, in its pointed version, controls the thick part. A sequence of Riemannian n-manifolds, (M_i, g_i), is said to converge in the $C^{1,\alpha}$-topology to a $C^{1,\alpha}$ manifold (M_∞, g_∞) if there are diffeomorphisms $f_i : M_\infty \to M_i$ such that the pullback metrics

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converge to g_∞ in the $C^{1,\alpha}$ sense, where g_∞ is a $C^{1,\alpha}$ metric. Specifically, there is an atlas on M_∞, with $C^{2,\alpha}$ transition functions, such that in local coordinates corresponding to each chart, the convergence of the $g_{i,j}$ is in the $C^{1,\alpha}$-topology.

THEOREM 1.1. *Given positive constants, n, d, v, and a sequence of closed n-manifolds M_i satisfying*

$$|\sec_{M_i}| \leq 1, \qquad \operatorname{diam}(M_i) \leq d, \qquad \operatorname{vol}(M_i) \geq v > 0,$$

there is a subsequence converging in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$-manifold.

An important consequence of Theorem 1.1 is that for any n, d, v, there are only finitely many closed manifolds in the given class up to diffeomorphism. Essentially, this is obtained in [**11, 12**] by estimating a uniform lower bound on the injectivity radius and by constructing an atlas whose charts are normal coordinate systems defined on balls of a definite size, for which the transition functions are controlled; compare [**45**]. It is also observed in [**11**] that assuming additional bounds on higher covariant derivatives of curvature gives correspondingly better control of the transition functions. In an unpublished work of Cheeger (part of which was the subject of a lecture at the Summer Institute on Global Analysis held at Stanford in 1973), under the assumptions of Theorem 1.1, Lipschitz control of the metric was obtained via regularization arguments. One should point out, however, that from the standpoint of regularity normal coordinates systems are far from optimal.

In [**36**], Gromov noted that employing distance function coordinates gives control of one more derivative of the transition functions and of metric, i.e., C^2 and C^1 control, respectively. He also made the powerful observation that Toponogov's comparison theorem for geodesic triangles has a formulation which passes to limits under such convergence, or even under (the weaker) Gromov-Hausdorff convergence.

Given the assumptions of Theorem 1.1, harmonic coordinate systems on balls of a definite size, in which the metric has definite $C^{1,\alpha}$-bounds, were constructed in [**40**]. Harmonic coordinates were used in [**30**] to obtain the optimal regularity in Theorem 1.1.

A natural question is: What can be said if the assumption of positive lower bound on volume in Theorem 1.1 is removed? In general, one asks the same question when removing bounds on diameter and volume but assuming that the local volume is arbitrarily small (equivalently, the injectivity radii are everywhere uniformly small).

The first non-trivial example of collapsing was observed by M. Berger. It is obtained from the standard metric on S^3 by multiplying the component tangent to the Hopf fibration on S^3 by ϵ^2 while keeping the metric in the orthogonal complement. Then as $\epsilon \to 0$ the sectional curvature lies in $[\epsilon^2, 4 - 3\epsilon^2]$ while the injectivity radii converge to zero everywhere. The first theorem on collapsing is Gromov's characterization of "almost flat" manifolds [**32**], which became a cornerstone of the subsequent more general collapsing theory (see Section 3). Gromov classified the maximally collapsed situation, i.e., when the diameter of M is very small. He showed that a finite normal covering of M is diffeomorphic to a nilpotent manifold [**32**].

A simple but powerful idea used in [**32**] is the notion of Gromov-Hausdorff distance, which measures the closeness of metric spaces, and a compactness theorem for this distance (see Section 2), whose importance, as further developments showed, cannot be overstated.

The more general collapsing theory was established in the 1980s in the works of Cheeger-Gromov, [**15, 16**], Fukaya [**24, 28**] and Cheeger-Fukaya-Gromov (see Sections 4 and 5). Since the early 1990s, several interesting applications of collapsing theory have been obtained (see Section 6).

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2. Gromov-Hausdorff distances and compactness

Let (Z, d) be a metric space. The collection of all compact subsets of Z forms a metric space under the *Hausdorff distance*, $d_H(A,B) = \max\{d(x,A), d(y,B),\ x \in A,\ y \in B\}$. Comparing this to the distance, $d(A,B) = \min \{d(x,y),\ x \in A,\ y \in B\}$, notice that $d(A,B) = 0$ if and only if $A \cap B \neq \emptyset$ while $d_H(A,B) = 0$ if and only if $A = B$. Hence, $d_H(A,B)$ measures the "uniform closeness" of A and B.

Gromov introduced an abstract version of the Hausdorff distance between any two compact metric spaces X and Y. A metric on the disjoint union, $X \coprod Y$, is called *admissible* if it extends the metrics on X and on Y. For example, any disjoint isometric embedding of X and Y into the metric product, $X \times Y \times [0,1]$, induces an admissible metric on $X \coprod Y$.

Definition 2.1 (Gromov-Hausdorff distance). For any two compact metric spaces X and Y, we call

$$d_{GH}(X,Y) = \inf\left\{d_H(X,Y),\ \text{all admissible metrics on } X \coprod Y\right\},$$

the Gromov-Hausdorff distance (simply, the GH-distance).

It is easy to check that d_{GH} satisfies the triangle inequality, and $d_{GH}(X,Y) = 0$ if and only if X is *isometric* to Y. Let $\mathcal{M}et_c$ denote the set of isometric classes of all compact metric spaces. Then $(\mathcal{M}et_c, d_{GH})$ is a metric space. Observe that for $X \in \mathcal{M}et_c$ and given any $\epsilon > 0$, $d_{GH}(X,A) < \epsilon$ for any finite ϵ-dense subset $A \subset X$. This shows that d_{GH} may not measure differences in local geometry. The power and usefulness of d_{GH} lie in its pre-compact criterion below, which applies to many interesting geometric situations in Riemannian geometry (see Lemma 2.1).

Let's first observe two obvious properties of a Cauchy sequence, $\{X_i\} \subset \mathcal{M}et_c$:

(2.1) There is a uniform upper bound on the diameter of X_i.

(2.2) Given $\epsilon > 0$, each X_i has an ϵ-dense subset $A_i(\epsilon)$ of size $|A_i(\epsilon)| \le \ell(\epsilon)$, a constant depending only on ϵ.

We now verify that (2.1) and (2.2) are also sufficient conditions for any sequence in $\mathcal{M}et_c$ to contain a convergent subsequence (so $(\mathcal{M}et_c, d_{GH})$ is a complete metric space): To construct a limit for X_i, we may assume, passing to a subsequence if necessary, that for all i, an admissible metric $d_{i,i+1}$ on $X_i \coprod X_{i+1}$ such that $d_{i,i+1,H}(X_i, X_{i+1}) < 2^{-i}$. We then define an admissible metric d on $Y = \coprod X_i$ by assigning an admissible metric $d_{i,i+j}$ on each $X_i \coprod X_{i+j}$:

$$d(x_i, x_{i+j}) = \inf_{x_{i+k} \in X_{i+k}} \left\{ \sum_{k=0}^{j-1} d_{i+k,i+k+1}(x_{i+k}, x_{i+k+1}) \right\}.$$

It is easy to see that X_i is a Cauchy sequence in (Y, d) with respect to d_H. Let $X = \{\{x_i\}$: equivalent Cauchy sequences in $Y, x_i \in X_i\}$ with $\{x_i\}$ equivalent to $\{y_i\}$ if $d(x_i, y_i) \to 0$ as $i \to \infty$. Using (2.1) and (2.2), one verifies that X is a compact metric space with the metric $d_X(\{x_i\}, \{y_i\}) = \lim\limits_{i\to\infty} d(x_i, y_i)$. Finally, we define an admissible metric on $Y \coprod X$ by $d(y, \{x_i\}) = \lim\limits_{i\to\infty} d(y, x_i)$, and check that $d_{GH}(X_i, X) \le d_H(X_i, X) \to 0$. As a by-product, we see that $X_i \xrightarrow{d_{GH}} X$ can be understood as $X_i \xrightarrow{d_H} X$ in the compact metric space $Y \coprod X$. In particular, it makes sense to say that $x_i \in X_i$ and $x_i \to x \in X$.

The Bishop-Gromov volume comparison theorem asserts that if M is a complete n-manifold of Ricci curvature $\ge k(n-1)$, then for $p \in M$, the ratio of volumes of r-balls, $\mathrm{vol}(B_r(p))/\mathrm{vol}(B_r^k)$, is not increasing in r, where B_r^k is an r-ball in a simply connected n-space form of curvature k. As an application, Gromov observed that (2.2) is satisfied under the following geometric conditions:

LEMMA 2.1 (Precompactness). *Any sequence of closed n-manifolds, M_i, with Ricci curvature $\mathrm{Ric}_{M_i} \ge -k$ and diameter $\mathrm{diam}(M_i) \le d$, contains a d_{GH}-convergent subsequence.*

In the rest of this section, we will discuss the equivariant and pointed Gromov-Hausdorff convergence (a motivation will be given at the end of this subsection).

Consider $X_i \xrightarrow{d_{GH}} X$ such that each X_i also admits an effective and isometric action by a compact group G_i. It is natural to ask if there is a symmetry structure on X related to these G_i-actions. To give a positive answer, we need the following 'equivalent' definition of d_{GH}: a map $f: X \to Y$ is called an ϵ-GH approximation if $|d(x_1,x_2)-d(f(x_1),f(x_2))| < \epsilon$ and if $f(X)$ is ϵ-dense in Y. Define

$$\hat{d}_{GH}(X,Y) = \inf\big\{\epsilon,\ \ \exists\ \epsilon\text{-Hausdorff approximation from } X \text{ to } Y \\ \text{and vice versa}\big\}.$$

It turns out that $\frac{2}{3}d_{GH} \le \hat{d}_{GH} \le 2d_{GH}$, and thus we may view "$d_{GH} = \hat{d}_{GH}$" as far as the convergence is concerned (but $\hat{d}_{GH}$ may not satisfy the triangle inequality.).

By the above, $X_i \xrightarrow{d_{GH}} X$ is equivalent to the condition that given (decreasing) $\epsilon_i \to 0$, there are ϵ_i-GH approximations, $f_i : X_i \to X$ and $h_i : X \to X_i$. We now construct a limit group G of G_i as follows: take a sequence of finite ϵ_i-dense subsets, $A(\epsilon_i) \subset X$, such that $A(\epsilon_i) \subset A(\epsilon_j)$ for all $i<j$, and define, for each i, a sequence of maps: $\phi_j : G_j \to C(A(\epsilon_i);X)$, $\phi_j(g)(x) = f_j g h_j(x), g \in G_j, x \in A(\epsilon_i)$, where $C(A(\epsilon_i);X)$ denotes the compact metric space consisting of maps $\alpha : A(\epsilon_i) \to X$, with $d_i(\alpha,\alpha') = \max\{d_X(\alpha(x), \alpha'(x)), x \in A(\epsilon_i)\}$. Passing to a subsequence if necessary, we may assume that $\phi_j(G_j) \xrightarrow{d_{i,H}} G_i' \subset C(A(\epsilon_i);X)$. Clearly, $g \in G_i' : A(\epsilon_i) \hookrightarrow X$ is an isometric embedding. Because $A(\epsilon_i) \subset A(\epsilon_j)$ $(i<j)$, $G_j'|A(\epsilon_i) = G_i'$, we take the direct limit of $\{G_i'\}$, G, whose elements are isometric embeddings $\cup_i A(\epsilon_i) \hookrightarrow X$ and thus extend to isometries of X. Clearly, G is a closed subgroup of the isometry group of X and their quotient spaces satisfy $X_i/G_i \xrightarrow{d_{GH}} X/G$. Hence, we say that (X_i,G_i) *equivariantly GH-converges to* (X,G), *denoted by* $(X_i,G_i) \xrightarrow{d_{eqGH}} (X,G)$.

Summarizing the above discussion, we give

LEMMA 2.2 (Equivariant convergence). *Let X_i be a sequence of compact metric spaces such that X_i admits an isometric action by a closed group G_i. If $X_i \xrightarrow{d_{GH}} X$, then there is a closed group of isometries on X such that $(X_i,G_i) \xrightarrow{d_{eqGH}} (X,G)$ and $X_i/G_i \xrightarrow{d_{GH}} X/G$.*

Note that the above assertion on convergence actually means a subsequence converging. Let's now make it a convention for the rest of the paper that a 'convergence' means up to a subsequence.

The GH-convergence and the equivariant GH-convergence, and Lemmas 2.1 and 2.2, can be extended to the (pointed) non-compact metric spaces whose bounded subsets are precompact. We say that a sequence of such pointed metric spaces converges, $(X_i,x_i) \xrightarrow{d_{GH}} (X,x)$, if for all $r>0$, the sequence of closed r-balls, $B_r(x_i) \subset X_i$, d_{GH}-converges to $B_r(x) \subset X$ such that $x_i \to x$. For instance, if a closed group G_i acts isometrically on X_i, then $(X_i,x_i) \xrightarrow{d_{GH}} (X,x)$ implies that $(X_i,x_i,G_i) \xrightarrow{d_{eqGH}} (X,x,G)$ for some closed group of isometries of X and that $(X_i/G_i,\bar{x}_i) \xrightarrow{d_{GH}} (X/G,\bar{x})$.

However, a significant difference for pointed GH-convergence is that for a sequence, different choice of base points may yield different limits.

We conclude this section by applying the above results to two convergent sequences associated to a given sequence of Riemannian manifolds $M_i \xrightarrow{d_{GH}} X$:
(2.3)

$$\begin{array}{ccccccc}
(\tilde{M}_i, \tilde{x}_i, \Gamma_i) & \xrightarrow{d_{eqGH}} & (\tilde{X}, \tilde{x}, \Gamma) & & (F(M_i), O(n)) & \xrightarrow{d_{eqGH}} & (Y, O(n)) \\
\downarrow \pi_i & & \downarrow \bar{\pi} & \text{and} & \downarrow p_i & & \downarrow \bar{p} \\
M_i & \xrightarrow{d_{GH}} & X = \tilde{X}/\Gamma & & M_i & \xrightarrow{d_{GH}} & X = Y/O(n)
\end{array}$$

where $\pi_i : \tilde{M}_i \to M_i$ is the Riemannian universal covering, $\Gamma_i = \pi_1(M_i)$ is the group of deck transformations and $F(M_i)$ is the orthogonal frame bundle equipped with a canonical metric: the parallel transport on M_i defines 'horizontal subspaces' on $T(F(M))$, and thus introduces a canonical metric on $F(M)$ (up to a choice of a bi-invariant metric on $O(n)$) such that $p : F(M) \to M$ is a Riemannian submersion.

A reason for studying the above associated sequences is that more information on $M_i \xrightarrow{d_{GH}} X$ may be seen from $(\tilde{M}_i, \tilde{x}_i, \Gamma_i) \xrightarrow{d_{eqGH}} (\tilde{X}, \tilde{x}, \Gamma)$ (especially when $\dim(X) < \dim(\tilde{X})$) and from $(F(M_i), O(n)) \xrightarrow{d_{eqGH}} (Y, O(n))$ (see the next two sections).

3. Almost flat manifolds

Gromov's theorem on almost flat manifolds is the first result on collapsing, and it has become a cornerstone of the collapsing theory ([**32**], cf. [**4, 14, 28, 53**]).

A closed manifold M is called almost flat if the scaling invariant, $\max|\sec| \cdot \operatorname{diam}^2(M)$, is very small. A flat manifold is almost flat, but an almost flat manifold may admit no flat metric (Example 3.1). If we scale the metric so that $|\sec_M| \leq 1$, then M is almost flat if and only if M is maximally collapsed, i.e., M is close to a point with respect to d_{GH}.

THEOREM 3.1 (Almost flat manifolds). *There exist positive constants $\epsilon(n)$ and $w(n)$ such that if a closed n-manifold satisfies $\max|\sec|\cdot\operatorname{diam}^2(M) < \epsilon(n)$, then M is finitely covered by a nilpotent manifold (the quotient of a simply connected nilpotent group $\tilde{N}$ by a cocompact discrete subgroup Γ) with order $\leq w(n)$.*

Ruh [**53**] improved Theorem 3.1, showing that M itself is diffeomorphic to the quotient, $\Gamma \backslash \tilde{N}$, where $\Gamma \subset \tilde{N} \ltimes \operatorname{Aut}(\tilde{N})$ (the group of automorphisms on $\tilde{N}$) and $[\Gamma : \Gamma \cap \tilde{N}] \leq w(n)$. Such a manifold is called an infra-nilmanifold.

The key ingredient in Theorem 3.1 is the following Margulis lemma [**32**].

LEMMA 3.1. *If G is a connected Lie group, then its identity has a neighborhood U_e such that if Γ is a discrete subgroup, then $\Gamma \cap U_e$ generates a nilpotent subgroup.*

Lemma 3.1 follows from the property $d(e, g_1 g_2 g_1^{-1} g_2^{-1}) \leq C d(e, g_1) d(e, g_2)$ for any g_1, g_2 close to the identity $e \in G$ (equipped with a left invariant metric), which is seen by (twice) applying the mean value theorem to $f(t) = d(e, g_1(t) g_2(t) g_1(t)^{-1} g_2(t)^{-1})$, where $g_i(t)$ is a geodesic from g_i to e and C is a constant.

Lemma 3.1 easily implies Theorem 3.1 in the following special situation: let Γ be a cocompact discrete subgroup of a simply connected Lie group $\tilde{N}$. Assume that $\tilde{N}$ admits a left invariant metric such that $|\sec| \leq 1$ and $\operatorname{diam}(\Gamma \setminus \tilde{N}) < \epsilon$. Then $\Gamma \cap U_e$ generates Γ and $\exp_e^{-1}(\Gamma \cap U_e)$ spans the Lie algebra $\hbar$ of $\tilde{N}$. Thus Γ is nilpotent (Lemma 3.1), which implies that $\tilde{N}$ is nilpotent.

On the other hand, given a simply connected nilpotent group, one can construct a family of left invariant metrics via *inhomogeneous* rescaling so that the diameter of any compact subset goes to zero (Example 3.1). This implies that for any discrete cocompact subgroup, the quotient is almost flat.

EXAMPLE 3.1. A Lie group G is nilpotent if $[G, G_k] = 1$ for some natural number k, where $G_{i+1} = [G, G_i]$ denotes the commutator of $G_0 = G$ and G_i. Then G_{i+1} is a normal subgroup of G_i such that G_i/G_{i+1} is abelian. If $\hbar_i$ denotes the Lie algebra of G_i, then $[\hbar_i, \hbar] \subset \hbar_{i+1}$, and thus one can choose a basis for $\hbar$, $\{e_{ik}\}$, such that $\{e_{jl}, i \leq j\}$ spans $\hbar_i$, and

$$[e_{ik}, e_{jl}] = \sum_{s \geq i} \sum_{p} C^p_{ijkl} e_{sp}, \quad \sum_{j \leq s} \sum_{p} |C^p_{ijkl}| \leq C \quad (C^p_{ijkl}, C \text{ are constants}).$$

Given any left invariant metric g, one can estimate the curvature tensor, $|R(U,V)W| \leq 6\|ad\|_g \cdot |U| \cdot |V| \cdot |W|$, where $\|ad\|_g = \max\{|[U,V]|, |U| = |V| = 1, U, V \in \hbar\}$. We now define a one-parameter family of left invariant metrics by assigning $\{e_{ik}\}$, an orthogonal basis, with norm $g_\epsilon(e_{ik}, e_{ik}) = \epsilon^{2^i}$ (*inhomogeneous scaling*). It is easy to check that $|ad|_{g_\epsilon} \leq C$ for all ϵ, and thus g_ϵ has the desired property.

One can easily extend the above construction in a fibration setting: let $M \to N$ be a fibration with fiber a nilpotent manifold with a flat connection, and let M have a metric such that when restricting to a fiber, parallel fields are Killing fields. Thus the structural group is a subgroup of the affine automorphisms of a fiber. By collapsing a fiber to a point as in the above, one obtains a sequence of metrics, g_ϵ, on the total space of the fibration such that $(M, g_\epsilon) \xrightarrow{d_{GH}} N$ with $|\sec_{g_\epsilon}| \leq 1$ [**27**].

SKETCH OF A PROOF OF THEOREM 3.1. Recall that a Lie group has the unique canonical flat connection, i.e., left invariant fields are parallel, and thus the torsion is parallel. Conversely, if a simply connected manifold

M admits a flat connection with a parallel torsion, then parallel fields form a Lie algebra which then determines a Lie group structure on M. The goal of the proof is to construct a flat Riemannian connection with a parallel torsion on the Riemannian universal covering space $\tilde{M}$ such that the deck transformations preserve the flat connection. It then follows that $\tilde{M}$ is a Lie group and $\pi_1(M) \subset \tilde{M} \ltimes \mathrm{Aut}(\tilde{M})$. By the discussion following Lemma 3.1, we can conclude the desired result.

By an obvious contradiction argument, it suffices to prove Theorem 3.1 for a sequence (see Lemma 2.1),

$$M_i \xrightarrow{d_{GH}} \mathrm{pt}, \qquad |\sec_{M_i}| \le i^{-1}. \tag{3.1}$$

Let $F(M_i)$ denote the orthogonal frame bundle. Since we will work on $F(M_i)$ with a canonical metric where a bound on curvature is required, we will need a bound on the covariant derivative of the curvature tensor. Deforming the metric g_i on M_i a short (but definite) time along the Ricci flows, one gets another almost flat metric with the required regularity [**39, 50**]. Hence, without loss of generality, we may assume that g_i satisfies this extra regularity.

For the sake of exposition, let's assume that (M, g_i) is obtained by slightly perturbing a 'left invariant' almost flat metric g_i' on a nilpotent manifold $\Lambda \setminus N$, as in Example 3.1. Because the injectivity radius of $\tilde{g}_i'$ is infinite, it is expected that

(3.2) the injectivity radius of $(\tilde{M}_i, \tilde{g}_i)$ is bounded below by a constant $\rho(n) > 0$.

Since $|\sec_{\tilde{M}_i}| \le 1$, (3.2) implies a positive lower bound for the convexity radius of $\tilde{M}_i$, say $\rho_c(n) > 0$. Given a finite number points $\{\tilde{y}_j\}$ in a ball $B_i \subset \tilde{M}_i$ of radius $\rho_c(n)$, the function $\tilde{h}(\tilde{x}) = \frac{1}{2}\sum_j d^2(\tilde{y}_j, \tilde{x}) : B_i \to \mathbb{R}$ is strictly convex and thus achieves the minimum at a unique point, call the *center of mass* for $\{\tilde{y}_j\}$.

Assuming (3.2) (whose proof will be delayed until the next section), we will first construct a cross section for $F(M) \to M$ via the technique of 'the center of mass': fixing $\tilde{x} \in \tilde{M}$, $\alpha(\tilde{x}) \in F(\tilde{M})$, by parallel translation of $\alpha(\tilde{x})$ along radial geodesics in $B_\rho(\tilde{x})$, one obtains a cross section, $\alpha : B_\rho(\tilde{x}) \to F(B_\rho(\tilde{x}))$. Of course, α may not be '$\pi_1(M)$-invariant,' i.e., $\alpha(\gamma(\tilde{y})) \ne \gamma_*(\alpha(\tilde{y}))$, where $\tilde{y} \in B_\rho(\tilde{x}), \gamma \in \pi_1(M)$ such that $\gamma(\tilde{y}) \in B_\rho(\tilde{x})$. However, for any $\tilde{z} \in B_\rho(\tilde{x})$, if the following inclusion holds for the finite set $A(\tilde{z})$,

$$\begin{aligned} A(\tilde{z}) &= \{\gamma_*(\alpha(\tilde{y})),\ \tilde{y} \in B_\rho(\tilde{z}), \gamma \in \pi_1(M) \text{ such that } \gamma(\tilde{y}) = \tilde{z}\} \\ &\subset B_{\rho'}(\alpha(\tilde{z})), \quad (\rho' > 0 \text{ is the convex radius of } F(\tilde{M})), \end{aligned} \tag{3.3}$$

then the map $\tilde{z} \to$ the center of mass of $A(\tilde{z})$, is well-defined and defines a $\pi_1(M)$-invariant cross section, and thus a cross section on $F(M) \to M$ (note that $B_\rho(\tilde{x}) \to M$ is onto, because $\operatorname{diam}(M) \ll \rho$)).

We now verify (3.3). Consider an equivariant sequence, $(\tilde{M}_i, \tilde{x}_i, \Gamma_i) \xrightarrow{d_{eqGH}} (\tilde{X}, \tilde{x}, \Gamma)$, associated to (3.1) as in (2.3). By (3.1) and (3.2), we may identify $(\tilde{X}, \tilde{x}) = (\mathbb{R}^n, 0)$, and thus G is a closed subgroup of $\operatorname{Isom}(\mathbb{R}^n) = \mathbb{R}^n \ltimes O(n)$. Because $\mathbb{R}^n/G$ is a point, $G = \mathbb{R}^n \ltimes H$, where H is a subgroup of $O(n)$. By an argument similar to the proof of (3.2), one concludes that H is *finite*. This implies, from the equivariant convergence, that a short geodesic loop, γ_i (representing a nontrivial element in Γ_i), has either a non-small holonomy or has a very small holonomy compared to its length. This implies a homomorphism, $\phi_i : \Gamma_i \to H$, whose kernel, $\Lambda_i = \ker(\phi_i)$, has a very small holonomy. In other words, if $\hat{M}_i = \tilde{M}_i/\Lambda_i$, then (3.3) is satisfied for i large, and therefore we obtain a cross section, $\hat{M}_i \to F(\hat{M}_i)$.

The flat connection may be viewed as a small perturbation of the Levi-Civita connection on M_i, and thus its torsion should be very small. Using some PDE techniques, one may deform the flat connection by a gauge transformation so as to obtain a flat connection with a parallel torsion [**53, 29**]. By now we can conclude, following the discussion at the beginning of the proof, that $\hat{M}_i$ is a nilpotent manifold and $[\Gamma_i : \Lambda_i] = |H|$. Therefore, $M_i = \hat{M}_i/H$ is an infra-nilpotent manifold. □

Note that the flat connection constructed in the above proof may depend on the choice of $\alpha(\tilde{x})$. Removing this dependence is necessary for reducing the structural group of the fibration to an affine automorphism group, as seen in Example 3.1 [**14**].

4. Collapsed manifolds with bounded diameter

After Theorems 1.1 and 3.1, we consider a sequence of closed n-manifolds, $M_i \xrightarrow{d_{GH}} X$, with $|\sec_{M_i}| \le 1$, $\operatorname{diam}(M_i) \le d$, and $0 < \dim(X) < n$. Without loss of generality, we may assume the sequence of orthogonal frame bundles, $p_i : F(M_i) \to M_i$, equipped with a canonical metric (see Section 2), satisfy (2.3):

$$\begin{array}{ccc} (F(M_i), O(n)) & \xrightarrow{d_{eqGH}} & (Y, O(n)) \\ \Big\downarrow p_i & & \Big\downarrow \bar{p} \\ M_i & \xrightarrow{d_{GH}} & X = Y/O(n) \end{array} \tag{4.1}$$

The main issue is to investigate

(4.2) links between geometrical and topological structures of M_i and X.

A significant consequence of the two-sided bound on curvature is that Y is a manifold [**25**]. This essentially reduces (4.1) to the following special situation [**14, 24, 25, 9**]:

THEOREM 4.1 (Fibration). *Let a compact Lie group G act isometrically on manifolds M, N, which satisfy*

$$\sec_M \geq -1, \qquad |\sec_N| \leq 1, \qquad \text{injrad }(N) \geq i_0 > 0.$$

There is a constant $\epsilon(m, i_0) > 0$ ($m = \dim(M)$) such that if $d_{eqGH}((M,G), (N,G)) < \epsilon \leq \epsilon(n, i_0)$, then there is a G-invariant fibration map, $f : (M,G) \to (N,G)$, with a connected fiber F such that:

- (4.1.1) *$d(x, f(x)) < \tau(\epsilon)$, where d is an admissible metric on $M \coprod N$, $\tau(\epsilon) \xrightarrow{\epsilon \to 0} 0$.*
- (4.1.2) *$f : M \to N$ is an almost Riemannian submersion: any vector ξ orthogonal to F satisfies that $e^{-\tau(\epsilon)} \leq |df(\xi)|/|\xi| \leq e^{\tau(\epsilon)}$.*
- (4.1.3) *If $sec_M \leq 1$, then the second fundamental form of fibers $|II_F| \leq c(m)$.*

SKETCH OF PROOF. We will present a proof with $G = \{e\}$, and the general case can be obtained with suitable 'equivariant' modification.

Given an admissible metric d on $M \coprod N$ such that $d_H(M, N) < \epsilon$, there is a natural projection that maps $x \in M$ to $y \in N$ which is closest to x, but this projection may not even be continuous if y is not unique. Using the geometric bounds on M and N, one overcomes this ambiguity by constructing a smooth embedding, $\Phi : N \hookrightarrow \mathbb{R}^s$, and a C^1-map, $\Psi : M \to \mathbb{R}^s$, such that $\Psi(M)$ is contained in a tube U of $\Phi(N)$ where the projection $P : U \to \Phi(N)$, to the nearest point in $\Phi(N)$, is smooth, and then defining $f = \Phi^{-1} \circ P \circ \Psi : M \to N$. Furthermore, f will satisfy (4.1.2) if Φ and Ψ are also 'C^1-close' in 'horizontal directions' as follows: let $u_i \in N, v_i \in M$ such that $d(u_1, u_i) = i_0/10$ ($i = 2, 3$) and $d(u_i, v_i) < \epsilon$, and let ξ and η be tangent vectors of the minimal geodesics from u_1 to u_2 and from v_1 to v_2 respectively. Then there are constants $C, \tau(\epsilon)$ such that

$$|d\Phi(\xi) - d\Psi(\eta)|_{\mathbb{R}^s} \leq C \cdot \tau(\epsilon) \cdot |\xi|, \qquad \lim_{\epsilon \to 0} \tau(\epsilon) = 0. \tag{4.3}$$

To construct Φ and Ψ, we first choose a pair of closed 'ϵ-nets' in M and in N (i.e., $\{x_i\} \subset N$ and $\{y_i\} \subset M$ are ϵ-dense in N and Y respectively such that $d(x_i, x_j) \geq \epsilon$ and $d(x_i, y_i) < \epsilon$), and a smooth 'cut-off' function, $\rho(t) \geq 0$, with $h((-\infty, 0]) = \rho(0) = 1$, $\text{supp}(\rho) = (-\infty, i_0/100]$, $\rho'(t) \leq 0$ and $\rho'(t) \sim -t^{-1}$ near 0. Then, define $\Phi(x) = (\rho(d(x, x_i))) \in \mathbb{R}^s$ and $\Psi(y) = (\rho(\psi_i(y))) \in \mathbb{R}^s$, where $s = |\text{net}_\epsilon(N)|$ and $\psi_i(y)$ is the average distance from y to $z \in B_\epsilon(y_i)$ (this guarantees that $\rho(d(y, y_i))$ is C^1 smooth). The bounds on curvature and injectivity radius of N guarantee (4.1.1) and that Φ is an embedding (the verification is somewhat tedious). The 'C^1-close' in

(4.3) can be verified from the 'angle-close' from $d_{GH}(M,N) < \epsilon$: Let α be the angle between segments $\overline{u_1u_i}$, β the angle between $\overline{v_1v_i}$, $i = 2, 3$. Then using the Toponogov comparison theorem, one can show that $|\alpha - \beta| < \tau(\epsilon)$ [**13**].

One may prove (4.1.3) by contradiction, and with a suitable rescaling and taking pointed convergence for a sequence of counterexamples, one ends up with a Riemannian submersion of a flat manifold to $\mathbb{R}^m$ whose fiber is not totally geodesic, a contradiction. □

REMARK 4.1. Note that the image, $f(x)$, depends only on the local geometry around $x \in M \coprod N$ (because of a cut-off function). Thus there is a local version of Theorem 4.1 (and 5.1), see Corollary 5.1.

Using Theorem 4.1, we can give a proof of (3.2) and thus complete the proof of Theorem 3.1. This, in turn, implies that in (4.1.3), a fiber is almost flat.

SKETCH OF PROOF OF (3.2).

We argue by contradiction: let $(M_i, g_i) \xrightarrow{d_{GH}} pt$ be as in (3.1) and γ_i be a non-trivial geodesic loop at $x_i \in M_i$ such that γ_i is homotopically trivial and $\text{length}(\gamma_i) \le 2\text{diam}(M_i) = 2\ell_i \to 0$. By scaling, we may assume that $(M_i, \ell_i^{-2} g_i) \xrightarrow{d_{GH}} X$ with $\text{diam}(X) = 1$ (Lemma 2.1). We claim that X is a flat manifold, and this implies that $\text{length}_{\ell_i^{-2} g_i}(\gamma_i,) \to 0$; otherwise, we may assume that γ_i converges to a non-trivial geodesic loop in X which is homotopically trivial, a contradiction.

Let's first assume the claim and derive a contradiction. By Theorem 4.1, we obtain a fibration, $f_i : M_i \to X$, with fiber F_i an almost flat manifold. We may assume that γ_i is homotopically equivalent, through curves of length $\le 100 \cdot \text{length}(\gamma_i)$, to a geodesic loop $\hat{\gamma}_i$ in a fiber F_i ($\hat{\gamma}_i$ may not be a geodesic in M_i). Note that $\hat{\gamma}_i$ is not trivial because $\sec_{M_i} \le i^{-1}$ implies that γ_i is not homotopically trivial through short curves. We now proceed by induction on n, and we will show that a short geodesic loop in M_i cannot be homotopically trivial. By the inductive assumption, we conclude that $\hat{\gamma}_i$ is not homotopically trivial in F_i. On the other hand, from the homotopy exact sequence of $M_i \to X$ and $\pi_2(X) = 0$, we conclude that $\pi_1(F_i) \to \pi_1(M_i)$ is an injection; a contradiction.

Finally, we verify the claim. Let $B_i(0_i) \subset T_{x_i}M_i$ denote the ball of radius $i\pi/2$. Then $\exp_{x_i} : B_i(0_i) \to M_i$ is non-singular and thus there is a pull-back metric $\tilde{g}_i$. Furthermore, short geodesic loops at x_i generate a pseudo-group that acts isometrically on $B_i(0_i)$ (e.g., the Γ_i-orbit at $X \in B_i(0_i)$ is $\exp_{x_i}^{-1}(\exp_{x_i}(X)) \cap B_i(0_i)$) [**28**]. We may assume that $(B_i(0_i), \Gamma_i, 0_i) \xrightarrow{d_{eqGH}} (\mathbb{R}^n, G, 0)$, and thus $M_i = B_i(0_i)/\Gamma_i \xrightarrow{d_{GH}} X_1 = \mathbb{R}^n/G$ (see the discussion at the end of Section 1). It suffices to show that G acts freely on $\mathbb{R}^n$.

If G_0 denotes the identity component of G, G_0 is normal and thus every G_0-orbit is isometric to $G_0(0) = \mathbb{R}^k$. This implies that G_0 acts freely on $\mathbb{R}^n$. If $1 \neq t \in G$ such that $t(0) = 0$, then $t^m = 1$. Let $t_i \in \Gamma_i$ such that $t_i \to t$ (see Section 1). By the discreteness of G/G_0 and the compactness of $X = (\mathbb{R}^{n-k}/G_0)/(G/G_0) = \mathbb{R}^n/G$, we can see that $t_i^m = 1$, a contradiction because t_i fixes the center of mass of $\{t_i(x_i), \dots, t^m(x_i)\}$ for i large. □

A natural question is if the converse of Theorem 4.1 holds. Motivated by Example 3.1, a positive answer requires that the structural group of the fibration in Theorem 4.1 reduce to a subgroup of affine automorphisms. This issue will be resolved below.

We first return to (4.1): $M_i \xrightarrow{d_{GH}} X$ and X is not a manifold. As mentioned there, (4.1) can be answered through studying the convergent sequence of the frame bundles, $F(M_i) \xrightarrow{d_{GH}} Y$ and Y is always a manifold.

A *pure nilpotent Killing structure* on $F(M)$ (with a canonical metric) is a fibration, $N \to F(M) \xrightarrow{f} Y$, with fiber N a nilpotent manifold (equipped with a flat connection) on which parallel fields are Killing fields and the $O(n)$-action preserves the affine fibration. The underlying $O(n)$-invariant affine bundle structure is called a *pure N-structure* and a metric for which the N-structure becomes a nilpotent Killing structure is called *invariant.* Let Y be equipped with a metric such that f is a Riemannian submersion. By the $O(n)$-invariance, the $O(n)$-action on $F(M)$ descends to an isometric $O(n)$-action on Y so that f is an $O(n)$-map. Because a general N-fiber meets an $O(n)$-orbit transversally, the $O(n)$-action on Y is effective. Furthermore, the N-fibration descends to a possible singular fibration on M such that the following diagram commutes:

$$\begin{array}{ccc} (F(M), O(n)) & \xrightarrow{f} & (Y, O(n)) \\ \Big\downarrow p & & \Big\downarrow \bar{p} \\ M & \xrightarrow{\bar{f}} & X = Y/(n) \end{array} \tag{4.4}$$

We call a torus bundle, $T^k \to F(M) \xrightarrow{h} Z$, a sub-bundle of $N \to F(M) \xrightarrow{f} Y$, if each T^k-fiber is contained in a fiber N. A pure N-structure has a natural T^k-sub-bundle determined by the center of the nilpotent group, called a canonical *pure F-structure.* Observe that if $\pi_1(M)$ is finite, so is $\pi_1(F(M))$, and the homotopy exact sequence of $N \to F(M) \to Y$ yields that $\pi_1(N)$ is abelian. This implies that $N = T^k$, i.e., the pure N-structure coincides with a pure F-structure [**47**]. Observe that on a simply connected manifold, a pure F-structure is equivalent to a torus action.

Combining Theorems 3.1 and 4.1, we obtain the following result [**14, 24, 25**].

THEOREM 4.2 (Singular fibration). *Let a sequence of closed n-manifolds $M_i \xrightarrow{d_{GH}} X$ with $|\sec_{M_i}| \le 1$ and X be a compact metric space. Then:*

(4.2.1) *The frame bundles $F(M_i)$, equipped with a canonical metric, converge to Y, which is homeomorphic to a manifold and on which $O(n)$ acts isometrically.*

(4.2.2) *There is an $O(n)$-invariant fibration, $\tilde f : F(M_i) \to Y$, satisfying the conditions in Theorem 3.3, which becomes, for $\epsilon > 0$, a nilpotent Killing structure with respect to an ϵ C^1-closed metric.*

Note that Theorem 4.2 provides a satisfactory answer to (3.1).

SKETCH OF PROOF OF THEOREM 4.2.

(4.2.1) Let $(F(M_i), O(n)) \xrightarrow{d_{eqGH}} (Y, O(n))$ be the associated sequence in (2.3). We first show that any $y \in Y$ has a manifold neighborhood. Let $(x_i, \alpha_i) \in F(M_i)$ such that $(x_i, \alpha_i) \to y$, where α_i is an orthogonal basis at $x_i \in M_i$. Let B_i denote the unit ball at x_i, and let $\tilde B_i$ denote the unit ball in the tangent space $T_{x_i}M_i$. The short geodesic loops at y generate a pseudogroup, Γ_i, that 'acts' isometrically on $\tilde B_i$ (equipped with the pullback metric by the exponential map) so that $\tilde B_i/\Gamma_i = B_i$ [**26**]. Via the differentials, Γ_i acts isometrically on $F(\tilde B_i)$ such that $F(\tilde B_i)/\Gamma_i = F(B_i)$. Because the injectivity radius at the center of $\tilde B_i$ is at least $1/3$ (because $\sec_{M_i} \le 1$), the limit, $(\tilde B_i, \Gamma_i) \xrightarrow{d_{eqGH}} (Z, \Gamma)$, is a $C^{1,\alpha}$-manifold (a local version of Theorem 1.1) and thus the limit, $(F(\tilde B_i), \Gamma_i) \xrightarrow{d_{eqGH}} (F(Z), \Gamma)$, is the frame bundle of Z. Hence, the Γ-action on $F(Z)$ is *free* because it is induced from the Γ-action on Z (any nontrivial isometry acts freely on the frame bundle via its differential). Consequently, $F(B_i) = F(\tilde B_i)/\Gamma_i \xrightarrow{d_{GH}} F(Z)/\Gamma$ (see the end of Section 1) is a manifold neighborhood of y.

(4.2.2) For each i, let $g_{i,\epsilon}$ be the solution of the Ricci equation as in the proof of Theorem 3.1. From the above, it is clear that the extra regularity implies that the limit Y_ϵ of $(F(M_i), g_{i,\epsilon})$ is a smooth Riemannian manifold, and thus we can apply Theorem 4.1 to conclude that for all $\epsilon \le \epsilon_0$ (small), there are $O(n)$-invariant fibrations,

$$\begin{array}{ccc} (F(M_i)_\epsilon, O(n)) & \xrightarrow{\tilde f_\epsilon} & (Y_\epsilon, O(n)) \\ \Big\downarrow{\scriptstyle p_i} & & \Big\downarrow{\scriptstyle p} \\ M_{i,\epsilon} & \xrightarrow{d_{GH}} & X_\epsilon = Y_\epsilon/O(n) \end{array}$$

By the continuity, it is clear that $(F(M_i)_\epsilon,\ O(n))$ is conjugate to $(F(M_i)_{\epsilon_0}, O(n))$, and thus $(Y_\epsilon, O(n))$ is conjugate to $(Y_{\epsilon_0}, O(n))$. This implies that $Y_\epsilon \xrightarrow{d_{GH}} Y$ is equivalent to a convergent sequence of metrics on Y_{ϵ_0}, and thus $(Y, O(n))$ is conjugate to $(Y_{\epsilon_0}, O(n))$. Consequently, the composition of maps,

$$(F(M_i), O(n)) \simeq (F(M_i)_{\epsilon_0}, O(n)) \xrightarrow{f_{\epsilon_0}} (Y_{\epsilon_0}, O(n)) \simeq (Y, O(n)),$$

has the desired property. □

Let's look at a simple example of a singular fibration in Theorem 4.2: consider an isometric T^2-action on the unit S^3. Let $\mathbb{R}^1 \subset T^2$ be a dense subgroup. Then $\mathbb{R}^1$ acts isometrically on S^3 such that every orbit is one-dimensional. Write $g = g_1 + g_1^\perp$, and define, for $\epsilon > 0$, $g_\epsilon = \epsilon^2 g_1 + g_1^\perp$, where g_1 is the restriction of g on the tangent space of an $\mathbb{R}^1$-orbit, and $g_1^\perp$ is the orthogonal complement. Then $(S^3, g_\epsilon) \xrightarrow{d_{GH}} S^3/T^2 = [0, \pi/2]$ as $\epsilon \to 0$ such that $|\sec_{g_\epsilon}| \le C$. The $O(3)$-invariant fibration on $F(S^3) = O(4)$ is a principal T^2-bundle, $T^2 \to O(4) \to O(4)/T^2 = Y$ (defined by $dt : F(S^3) \to F(S^3)$, $t \in T^2$), and the induced singular fibration on S^3 coincides with the orbits of the T^2-action.

A natural question is whether M, carrying a pure nilpotent Killing structure with all orbits of positive dimension, admits a sequence of metrics with bounded curvature collapsing to the orbit space. In general, the answer is negative (there are such manifolds of non-vanishing signature, [**15**]). This clearly suggests a possible constraint on the pure nilpotent Killing structure arising in Theorem 4.2 (cf. [**18**]).

5. Collapsed manifolds (without a bound on diameters)

Consider a collapsed complete n-manifold, that is, M satisfies $|\sec_M| \le 1$ and $\mathrm{vol}(B_1(x)) < \epsilon$ for all $x \in M$. By a simple limiting argument, using Theorem 4.2 one can see that $B_1(x)$ is contained in some open set which admits a pure nilpotent Killing structure of some nearby metric. The main issue is how these 'charts' of local pure nilpotent Killing structures can be patched together.

To be precise, let's consider a collapsed metric on $(-R, R) \times T^1 \times T^1$:

$$g = dr^2 + e^{-(R+r)} d\theta_1^2 + e^{-(R-r)} d\theta_2^2.$$

Clearly, one gets (from the above) a pure T^2-structure around the point $(0, \theta_1, \theta_2)$ and pure T^1-structures near $(-R, \theta_1, \theta_2)$ and (R, θ_1, θ_2). This illustrates that the local pure structure cannot be made completely canonical, because a T^1-action cannot be continuously deformed to a T^2-action. However, in the region where a T^2- and a T^1-action meet, the latter is conjugate to a T^1-subgroup of the former (easily seen for an obvious topological reason). In general, the tool for this kind of compatibility is the rigidity of any two C^1-closed compact Lie group actions [**37**].

We now define a (*mixed*) *nilpotent Killing structure*. Consider a pure nilpotent Killing structure, $\mathcal{N}$, on an open subset $U \subset M$, $N \to F(U) \to Y$.

Another pure nilpotent Killing structure on U, $N_1 \to F(U) \to Y_1$, is called a *sub-nilpotent Killing structure* if every N_1-fiber is an affine submanifold of some N-fiber. If a nilpotent Killing structure does not coincide with its center, then its canonical F-structure is a proper substructure.

A (*mixed*) *nilpotent Killing structure* $\mathcal{N}$ on a complete manifold M consists of $\{(U_i, \mathcal{N}_i)\}$, where $\{U_i\}$ is a locally finite open cover for M, $\mathcal{N}_i$ is a pure nilpotent Killing structure on U_i such that if $U_i \cap U_j \neq \emptyset$, then $U_i \cap U_j$ is an invariant subsets of both $\mathcal{N}_i$ and $\mathcal{N}_j$, and $\mathcal{N}_i$ is a substructure of $\mathcal{N}_j$ or vice versa. By the compatibility, M decomposes into $\mathcal{N}$-*orbits*; an $\mathcal{N}$-orbit at x is the minimal invariant subset of all $(U_i, \mathcal{N}_i)$ that contains x.

Theorem 5.1 (Mixed nilpotent Killing structure, [**14**]). *There exists a constant $\epsilon(n) > 0$ such that if a complete n-manifold M satisfies*

$$|\sec_M| \leq 1, \qquad \mathrm{vol}(B_1(x)) < \epsilon(n) \qquad \forall x \in M,$$

then M admits a Killing nilpotent structure of some nearby metric (*with a higher regularity*) *whose orbits have positive dimension and diameter $<\epsilon(n)$.*

A consequence of the existence of such a nilpotent Killing structure is the vanishing of the Euler characteristic of M [**16**]. Another consequence (with Theorem 1.1) is the so-called *thick-thin* decomposition on any complete manifold of bounded sectional curvature (the thin part consists of points satisfying the conditions of Theorem 5.1). Its local structure is as follows:

Corollary 5.1 (Local structure). *There exists $\epsilon(n) > 0$ such that if M^n is a complete manifold with sectional curvature $|\sec_{M^n}| \leq 1$ and $x \in M$, then either $B_\epsilon(x)$ is diffeomorphic to a Euclidean ball or there is an open subset $U \supset B_\epsilon(x)$ such that the frame bundle $F(U)$ admits an $O(n)$-invariant fibration as in Theorem 4.2.*

Sketch of a Proof of Theorem 5.1.

By a simple limiting argument, one easily sees that for any $x \in M$, $B_1(x)$ is contained in an open subset $U \subset M$ such that U admits a pure nilpotent Killing structure $\mathcal{N}$ with respect to a nearby metric, i.e., $F(U)$ admits an $O(n)$-invariant fibration as in Theorem 4.2 (see Remark 4.1). Thus one obtains a locally finite open cover, $\{(U_i, \mathcal{N}_i)\}$, for M. Note that on $U_i \cap U_j \neq \emptyset$, $\mathcal{N}_i$ may not be a substructure of $\mathcal{N}_j$ or vice versa (because the construction of $\mathcal{N}_i$ cannot be made completely canonical). However, on $U_i \cap U_j$, the two pure nilpotent Killing structures should be close in a suitable sense, because both are constructed from the same geometry data.

The goal of the proof is to systematically modify $(U_i, \mathcal{N}_i)$ and $(U_j, \mathcal{N}_j)$ wherever $U_i \cap U_j \neq \emptyset$ so as to form new charts, still denoted by $\{(U_i, \mathcal{N}_i)\}$, with a compatibility condition: on $U_i \cap U_j \neq \emptyset$, $\mathcal{N}_i \subseteq \mathcal{N}_j$ or vice versa. For the sake of simple exposition, let's first consider the case where $\mathcal{N}_i = \mathcal{F}_i$, i.e., $\mathcal{N}_i$ coincides with its canonical F-structure $\mathcal{F}_i$. Note that $\mathcal{F}_i$ is also defined by some torus T^{k_i}-action on a finite normal covering of U_i, and thus the problem essentially reduces to showing that the two T^{k_i}- and T^{k_j}-actions are C^1-close [**37**]. A technical issue arises around a multiple intersection: in performing consecutively ordered modifications, it is necessary that each modification preserves the C^1-closeness up to a controlled factor (indeed, such a construction is not easy and quite technical).

In the general case, one performs the modification on the Riemannian universal covering space of $U_i \cap U_j$ by applying the technique of center of mass. □

Similar to the discussion at the end of last section, it is natural to ask if a (mixed) N-structure implies a collapsing (roughly, if the converse of Theorem 4.1 holds). The following implies a positive partial answer:

THEOREM 5.2. [**15**] *Suppose a manifold M admits an F-structure of positive rank. Then M admits a one-parameter family of invariant metrics g_ϵ such that $|\sec_{g_\epsilon}| \le 1$ and the diameters of all orbits uniformly converge to zero (consequently, $\mathrm{vol}_{g_\epsilon}(B_1(x)) \to 0$ $(\epsilon \to 0)$ for all $x \in M$). Moreover, if all local pure F-structures have orbits of constant dimension, then $\mathrm{vol}_{g_\epsilon}(M) \to 0$.*

Combining Theorems 5.1 and 5.2, one easily concludes a classification of collapsed 3-manifolds: a closed 3-manifold M admits a collapsed metric with bounded sectional curvature if and only if M is diffeomorphic to a graph manifold [**47**]. This also confirms the Gap conjecture of Gromov [**34**] for $n = 3$: there is $v_n > 0$ such that if a complete n-manifold M with $|\sec| \le 1$ has volume $< v_n$, then M admits a sequence of volume collapsed metrics (cf. [**19, 48**]).

An interesting problem is to prove Theorem 5.2 for a mixed N-structure with orbits of positive dimension [**14**] and [**43**].

6. Applications

In Sections 4 and 5, we construct a pure (resp. mixed) nilpotent Killing structure on a collapsed manifold with (resp. without) a bound on the diameter. In this section, we will present some applications based on the existence of such a structure; most are in various especially interesting geometric/topological situations. It is a special geometric/topological condition that puts additional constraints on a nilpotent Killing structure, which in turn implies additional topological constraints on the underlying manifold.

It turns out that in every collapsed situation discussed in this section, the pure/mixed nilpotent Killing structure arising from a collapsed metric actually coincides with its canonical F-structure. Recall that if a manifold of finite fundamental group admits a pure nilpotent Killing structure

$\mathcal{N}$, then $\mathcal{N} = \mathcal{F}$, i.e., it coincides with its canonical F-structure (see the discussion prior to Theorem 4.2). This implies that on a collapsed manifold with pinched positive sectional curvature, a pure nilpotent Killing structure coincides with the canonical F-structure. If M is a collapsed manifold with bounded non-positive sectional curvature, then a (mixed) nilpotent Killing structure also coincides with its canonical F-structure. Basically, this is due to the fact that a solvable subgroup of $\pi_1(M)$ is actually Bieberbach [**55, 31, 41**].

a. Finiteness results. A typical diffeomorphism finiteness theorem, such as Theorem 1.1, concerns non-collapsed manifolds, or equivalently, manifolds where there is a positive lower bound on (local) volume. An interesting application of Theorem 4.2 is the following finiteness result that includes collapsed manifolds:

THEOREM 6.1. [**20, 23, 44**] *For $n, d > 0$, there is a constant $c(n, \delta)$ such that the class of closed 2-connected n-manifolds satisfying*

$$|\sec| \leq 1, \qquad \operatorname{diam} \leq d,$$

contains at most $c(n, d)$ many diffeomorphic types.

Note that the conclusion of Theorem 6.1 remains true for $n \leq 6$ without assuming vanishing π_2 [**21, 56**], and is false if we remove either $\pi_2 = 0$ [**1**] or an upper bound on curvature without imposing further restrictions [**38**].

SKETCH OF A PROOF OF THEOREM 6.1. By Theorem 1.1, Theorem 6.1 is true if it holds for any collapsing sequence, $M_i \xrightarrow{d_{GH}} X$, as in (4.1), and $T^k \to SF(M_i) \xrightarrow{f_i} Y$ is a principal T^k-bundle as in (4.4) (Theorem 4.2), where $SF(M_i)$ denotes the $SO(n)$-frame bundle. Because the induced $SO(n)$-actions on Y are C^1-close, without loss of generality we may choose f_i so that the induced $SO(n)$-action is independent of i [**37**]. We will construct an $SO(n)$-conjugate map between $SF(M_i)$ and $SF(M_j)$. First, by the standard bundle theory, a principal T^k-bundle whose total space is 2-connected is unique up to a fiber automorphism, and thus we may assume a T^k-conjugate diffeomorphism, $f : SF(M_i) \to SF(M_j)$, such that f induces an identity map on Y. We will use the center of mass technique to modify f into an $SO(n)$-conjugate map. Because the $SO(n)$-action commutes with the T^k-action, one sees that f maps the T^k-fiber at $s(x)$ to that at $f(s(x))$, $s \in SO(n)$. Consequently, $s^{-1}f(s(x)) \subseteq T^k(f(x))$ for all s. Because the diameter of $T^k(f(x))$ is very small, we consider the lifting map, $Sp(n) \to \mathbb{R}^k(f(x))$, where $SP(n) \to SO(n)$ and $\mathbb{R}^k(f(x)) \to T^k(f(x))$ are the Riemannian universal coverings. By now we can take the center of mass of $Sp(n) \to \mathbb{R}^k(f(x))$, which is clearly invariant under the deck transformations, and thus obtain the desired $SO(n)$ conjugate map. $\square$

Using Theorem 4.2, one can prove the following finiteness result without assuming a lower volume bound:

THEOREM 6.2. [**22**] *For $n, d > 0$, there is a constant $v(n, d) > 0$ such that if a closed symplectic n-manifold of finite fundamental group satisfies*

$$|\sec| \leq 1, \qquad \operatorname{diam} \leq d,$$

then vol$(M) \geq v(n, d)$. In particular, M has finitely many possible diffeomorphic types depending on n and d.

Note that Theorem 6.2 is false if we remove the restriction on the fundamental group without imposing further restrictions (e.g., a flat torus).

SKETCH OF A PROOF OF THEOREM 6.2. Arguing by contradiction, we may assume a sequence satisfying Theorem 6.2, $M_i \xrightarrow{d_{GH}} X$ with $\dim(X) < n$. Then M_i admits a pure F-structure with orbits of positive dimensions (Theorem 4.2). Equivalently, the universal covering space $\tilde{M}_i$ admits a torus action without fixed points, and thus there is a circle subgroup without fixed points, a contradiction to a topological result obtained in [**22**]: any effective circle action on a closed symplectic manifold has a non-empty fixed point set. □

Using Theorem 4.2, one can also prove an isomorphism finiteness result for the q-th homotopy groups of closed n-manifolds in terms of n, q and bounds on curvature and diameter [**22, 50**]. Note that the homotopy group finiteness does not hold if we remove the upper curvature bound (compare to [**33**]).

b. Manifolds with pinched positive sectional curvature. Let M be a closed n-manifold of positive sectional curvature. Recall that the fundamental group $\pi_1(M)$ is finite, and if n is even and M is orientable, then $\pi_1(M) = 1$ (cf. [**13**]). However, in odd dimensions no general constraint on $\pi_1(M)$ is known that could distinguish positive curvature from non-negative curvature. A conjectured obstruction is that there is $\gamma \in \pi_1(M)$ such that the ratio $|\pi_1(M)|/|\gamma| \leq w(n)$, a constant depending only on n, where $|\gamma|$ is the order of γ [**50**].

Based on Theorem 4.2, one can partially verify the above conjecture.

THEOREM 6.3. [**50, 51**] *Let M be a closed n-manifold of δ-pinched curvature. If $|\pi_1(M)| \geq w(n, \delta)$, then $\pi_1(M)$ has a non-trivial normal cyclic subgroup of index at most $w(n)$.*

SKETCH OF A PROOF OF THEOREM 6.3. We may assume $\dim(M)$ is odd. The condition '$\sec_M \geq \delta$' implies that the diameter $\leq \pi/\sqrt{\delta}$ (Bonnet theorem) and the volume $\leq \frac{\operatorname{vol}(S^n_\delta)}{w(n,\delta)}$ (volume comparison) which is small when $w(n, \delta)$ is large, where S^n_δ is the n-sphere of constant curvature δ. Hence,

M admits a pure F-structure of rank $k \geq 1$ and a nearby invariant metric (Theorem 4.1). As it turns out, it is crucial to have an invariant metric of positive curvature. Based on the regularity of the Ricci flows, one can get a nearby invariant metric with $\delta/2$-pinched curvature [**50**].

The above symmetry structure of a positively curved metric is all one needs to prove the desired property of $\pi_1(M)$. For instance, if $k = 1$ and circle orbits on M form a fibration, then the homotopy class of a circle orbit generates a normal cyclic subgroup $< \sigma > \subseteq \pi_1(M)$ such that $\pi_1(M)/ < \sigma > \cong \pi_1(M^*)$, where M^* denotes the orbit space. Note that $\pi_1(M^*) = 1$ or $\mathbb{Z}_2$ because $\dim(M^*)$ is even and M^* has a unique metric so that $M \to M^*$ is a Riemannian submersion (thus $\sec_{M^*} \geq \delta/2$, Gray-O'Neill submersion equations).

The proof in general is quite involved, and the constant $w(n)$ is related to Gromov's Betti number estimate [**33**]. □

Recall that the injectivity radius of a closed even-dimensional manifold of $0 < \sec_M \leq 1$ is at least $\pi/2$ while there is no positive lower bound in odd dimensions (e.g., Berger sphere). The Klingenberg-Sakai conjecture says once δ is fixed, there is a positive lower bound on the injectivity radius of a δ-pinched metric depending on δ. However, there are infinitely many simply connected 7-manifolds of uniformly pinched positive sectional curvature [**1**]. Hence, to have a possible universal lower bound, i.e., one depending only on n and δ, additional restrictions are required.

THEOREM 6.4. [**46**] *Let M be a closed n-manifold of δ-pinched curvature. If M is 2-connected, then the injectivity radius of M is at least $\epsilon(n, \delta) > 0$, a constant depending only on n and δ.*

Note that in the above-mentioned 7-manifolds, each second homotopy group has rank one.

SKETCH OF A PROOF OF THEOREM 6.4. We argue by contradiction, assuming a sequence M_i satisfying Theorem 6.4 such that $M_i \xrightarrow{d_{GH}} X$, where $\dim(X) < n$. From the proof of Theorem 6.1, we may assume a manifold $M \simeq M_i$ admitting a T^k-action without fixed points and a sequence of invariant metrics g_i such that g_i collapses along $\mathcal{F}$, i.e., the diameters of all $\mathcal{F}$ converge uniformly to zero and the induced metrics d_i on M/T^k converge to d pointwise. As seen in the proof of Theorem 6.3, we may assume the invariant metrics are $\delta/2$-pinched. By now, we are in a situation similar to the collapsing of Berger's sphere (where, however, the minimal curvature converges to zero).

Indeed, given any sequence of metrics g_i on a manifold, $\lambda \leq \sec_{g_i} \leq 1$, collapsing along a (fixed) F-structure, one can construct a complete non-compact length space with curvature $\geq\lambda$, and the non-compactness forces

$\lambda \leq 0$ (a generalized Bonnet theorem, [**45**]). By now, one sees a contradiction to the above.

For simplicity, we will explain the idea with the special case $k = 1$. To get a contradiction, we take a finite open cover $\{U_\alpha\}$ for M such that each U_α is a tube of radius $\rho > 0$ with respect to d. Clearly, (U_α, g_i) converges to a ρ-ball in X. For $U_\alpha \cap U_\beta \neq \emptyset$, let $\phi_{\alpha\beta} : U_\alpha \supset U_\alpha \cap U_\beta \to U_\alpha \cap U_\beta \subset U_\beta$ denote a gluing map. On $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, these maps satisfy $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} \circ \phi_{\gamma\alpha} =$ id. Let $\tilde U \to U_\alpha$ denote the Riemannian universal covering space. Then there is a lifting map, $\tilde\phi_{\alpha\beta} : \tilde U_\alpha \supset \tilde U_\alpha \cap \tilde U_\beta \to \tilde U_\alpha \cap \tilde U_\beta \subset \tilde U_\beta$. However, these lifting maps do not satisfy the compatibility condition: $\tilde\phi_{\alpha\beta} \circ \tilde\phi_{\beta\gamma} \circ \tilde\phi_{\gamma\alpha} = \xi_{\alpha\beta\gamma} \neq \text{id}$. We may view $\{\xi_{\alpha\beta\gamma}\}$ as an obstruction to gluing $\{\tilde U_\alpha\}$ together. The key observation is that when taking limits, $(\tilde U_\alpha, x_\alpha, g_i) \xrightarrow{d_{GH}} (\tilde U_{\alpha,\infty}, x_\alpha)$ and $\tilde\phi_{\alpha\beta} \to \tilde\phi_{\alpha\beta,\infty}$ simultaneously, the collapsing condition implies that $\xi_{\alpha\beta\gamma} \to$ id as $i \to \infty$. Consequently, using $\{\tilde\phi_{\alpha\beta,\infty}\}$, one can glue $\{\tilde U_{\alpha,\infty}\}$ together to form a complete non-compact manifold (because $\tilde U_{\alpha,\infty} \simeq D^2 \times \mathbb{R}$) with curvature $\geq \lambda$ (e.g., for Berger's sphere, $N = S^2_{\frac{1}{2}} \times \mathbb{R}$). □

c. Collapsed manifolds with non-positive sectional curvature. A flat manifold (of small volume) is a trivial example of a collapsed manifold with (bounded) non-positive curvature, and a nontrivial example is that any graph 3-manifold whose fundamental group contains no cyclic subgroup of finite index admits a collapsed metric with (bounded) non-positive curvature [**34**]. S. Buyalo studied a collapsed 3-manifold M with $-1 \leq \sec_M \leq 0$ and found that there are a finite number of totally geodesic flat tori, $T^2_i \subset M$, such that each component U_j of $M - \bigcup_i T^2_i$ is a metric product, $U_j = \Sigma^2_j \times S^1$ [**5–7**]. By definition, M is a graph manifold with a graph system $\{T^2_i\}$. Note that $\{(U_i, T^1)\}$ actually defines an F-structure, called a *Cr-structure*, with the additional properties that U_i is a product (in general, up to a finite covering space) and the fundamental group of an orbit injects into $\pi_1(M)$.

One may describe the local metric product structure in terms of the subgroups of $\pi_1(M)$ generated by loops in orbits (up to a finite covering). Recall that for each non-trivial abelian subgroup $A \subset \pi_1(M)$, there is an isometric immersion of a flat torus, $i : T^{\text{rank}(A)} \to M$, such that the induced maps on the fundamental groups satisfy $i_*\pi_1(T^{\text{rank}(A)}) = A$ [**31, 41**]. More generally, the minimal set splits, $\text{Min}(A) = D \times \mathbb{R}^{\text{rank}(A)}$, where $\text{Min}(A)$ denotes the set of points in the Riemannian universal covering of M at which the displacement of any element in A achieves the minimum, and whose projection contains the immersed flat torus.

Let $\mathcal{A} = \{A_\alpha\}$ denote a collection of abelian subgroups $A_\alpha \subset \pi_1(M)$ which are preserved by conjugation. We say that $\mathcal{A}$ determines an *abelian structure* (resp. *a local splitting structure*) on M if the following conditions (resp. (6.5.1)) hold:

(6.1) The Riemannian universal covering space $\tilde{M} = \bigcup_{A_\alpha \in \mathcal{A}} \operatorname{Min}(A_\alpha)$.

(6.2) $\operatorname{Min}(A_\alpha) \cap \operatorname{Min}(A_\beta) \neq \emptyset$ if and only if A_α and A_β commute.

If $\pi_1(M)$ has a nontrivial normal abelian subgroup A, then $\mathcal{A} = \{A\}$ determines an abelian structure for any non-positively curved metric on M [**31, 41**].

Theorem 6.5. [**55**] *Let M be a closed n-manifold with $-1 \leq \sec_M \leq 0$. If $\operatorname{vol}(B_1(x)) < \epsilon(n)$ for all $x \in M$, then there is $\mathcal{A} = \{A_\alpha\} \subset \pi_1(M)$ that determines an abelian structure. Moreover, any abelian structure determines a canonical Cr-structure (i.e., one whose orbits are totally geodesic flat submanifolds).*

Theorem 6.5 was essentially conjectured by Buyalo, who also verified it for $n = 3, 4$. Indeed, one easily concludes that if a graph manifold admits a Cr-structure compatible with one metric of non-positive curvature, then it is compatible with every non-positively curved metric.

Recall that geometrical rigidity results often assert that a class of certain metrics are unique up to a scaling (e.g., the higher rank rigidity, [**2**]). In this spirit, one may view the above as a weak rigidity: the underlying Cr-structure captures the local splitting structure of every metric of non-positive curvature. (In this sense, all these metrics are alike.) It is conjectured that such a weak rigidity should hold in all dimensions.

The following result partially supports the conjecture.

Theorem 6.6. [**10**] *Let M be a closed manifold which admits a metric of non-positive sectional curvature. If M admits an F-structure, then every metric of non-positive sectional curvature has a splitting structure.*

Here we omit the outline of proofs. A remaining problem is to show that the local splitting structure in Theorem 6.6 satisfies (6.2).

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