Perspectives on geometric analysis

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This essay grew from a talk I gave on the occasion of the seventieth anniversary of the Chinese Mathematical Society. I dedicate the lecture to the memory of my teacher S.S. Chern who had passed away half a year before (December 2004).

During my graduate studies, I was rather free in picking research topics. I [**731**] worked on fundamental groups of manifolds with non-positive curvature. But in the second year of my studies, I started to look into differential equations on manifolds. However, at that time, Chern was very much interested in the work of Bott on holomorphic vector fields. Also he told me that I should work on Riemann hypothesis. (Weil had told him that it was time for the hypothesis to be settled.) While Chern did not express his opinions about my research on geometric analysis, he started to appreciate it a few years later. In fact, after Chern gave a course on Calabi's works on affine geometry in 1972 at Berkeley, S.Y. Cheng told me about these inspiring lectures. By 1973, Cheng and I started to work on some problems mentioned in Chern's lectures. We did not realize that the great geometers Pogorelov, Calabi and Nirenberg were also working on them. We were excited that we solved some of the conjectures of Calabi on improper affine spheres. But soon after we found out that Pogorelov [**563**] published his results right before us by different arguments. Nevertheless our ideas are useful in handling other problems in affine geometry, and my knowledge about Monge-Ampère equations started to broaden in these years. Chern was very pleased by my work, especially after I [**736**] solved the problem of Calabi on Kähler Einstein metric in 1976. I had been at Stanford, and Chern proposed to the Berkeley Math Department that they hire me. I visited Berkeley in 1977 for a year and gave a course on geometric analysis with emphasis on isometric embedding.

Chern nominated me to give a plenary talk at the International Congress in Helsinki. The talk [**737**] went well, but my decision not to stay at Berkeley

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did not quite please him. Nevertheless he recommended me for a position on the faculty at the Institute for Advanced Study. Before I accepted a faculty position at the Institute, I organized a special year on geometry in 1979 at the Institute at the invitation of Borel. That was an exciting year because most people in geometric analysis came.

In 1979, I visited China at the invitation of Professor L.K. Hua. I gave a series of talks on the bubbling process of Sacks-Uhlenbeck [**581**]. I suggested to the Chinese mathematicians that they apply similar arguments for a Jordan curve bounding two surfaces with the same constant mean curvature. I thought it would be a good exercise for getting into this exciting field of geometric analysis. The problem was indeed picked up by a group of students of Professor G.Y. Wang [**362**]. But unfortunately it also initiated some ugly fights during the meeting of the sixtieth anniversary of the Chinese Mathematical Society. Professor Wang was forced to resign, and this event hampered development of this beautiful subject in China in the past ten years.

In 1980, Chern decided to develop geometric analysis on a large scale. He initiated a series of international conferences on differential geometry and differential equations to be held each year in China. For the first year, a large group of the most distinguished mathematicians was gathered in Beijing to give lectures (see [**148**]). I lectured on open problems in geometry [**739**]. It took a much longer time than I expected for Chinese mathematicians to pick up some of these problems.

To his disappointment, Chern's enthusiasm about developing differential equations and differential geometry in China did not stimulate as much activity as he had hoped. Most Chinese mathematicians were trained in analysis but were rather weak in geometry. The goal of geometric analysis for understanding geometry was not appreciated. The major research center on differential geometry came from students of Chern, Hua and B.C. Su. The works of J.Q. Zhong (see, e.g., [**755, 527, 528**]) were remarkable. Unfortunately he died about twenty years ago. Q.K. Lu studied the Bergman metric extensively. C.H. Gu [**296**] studied gauge theory and considered harmonic map where the domain is $R^{1,1}$. J.X. Hong (see, e.g., [**345, 318**]) did some interesting work on isometric embedding of surfaces into $\mathbb{R}^3$. In the past five years, the research center at the Chinese University of Hong Kong, led by L.F. Tam and X.P. Zhu, has produced first class work related to Hamilton's Ricci flow (see, e.g., [**125, 126, 129, 130, 113**]).

In the hope that it will advance Chern's ambition to build up geometric analysis, I will explain my personal view to my Chinese colleagues. I will consider this article to be successful if it conveys to my readers the excitement of developments in differential geometry which have been taking place during the period when it has been my good fortune to contribute. I do not claim this article covers all aspects of the subject. In fact, I have given priority to those works closest to my personal experience, and, consequently, I have given insufficient space to aspects of differential geometry in which I have not participated. In spite of these shortcomings, I hope that my readers, particularly those too young to know the origins of geometric analysis, will be interested to learn how the field looks to someone who was there. I would like to thank comments given by R. Bryant, H.D. Cao, J. Jost, H. Lawson, N.C. Leung, T.J. Li, Peter Li, J. Li, K.F. Liu, D. Phong, D. Stroock, X.W. Wang, S. Scott, S. Wolpert, and S.W. Zhang. I am also grateful to J.X. Fu, especially for his help of tracking down references for the major part of this survey. When Fu went back to China, this task was taken up by P. Peng and X.F. Sun to whom I am grateful also.

In this whole survey, I follow the following:

Basic Philosophy:

Functions, tensors and subvarieties governed by natural differential equations provide deep insight into geometric structures. Information about these objects will give a way to construct a geometric structure. They also provide important information for physics, algebraic geometry and topology. Conversely it is vital to learn ideas from these fields.

Behind such basic philosophy, there are basic invariants to understand how space is twisted. This is provided by Chern classes [149], which appear in every branch of mathematics and theoretical physics. So far we barely understand the analytic meaning of the first Chern class. It will take much more time for geometers to understand the analytic meaning of the higher Chern forms. The analytic expression of Chern classes by differential forms has opened up a new horizon for global geometry. Professor Chern's influence on mathematics is forever.

An old Chinese poem says:

The reeds and rushes are abundant,

and the white dew has yet to dry.

The man whom I admire is on the bank of the river.

I go against the stream in quest of him,

But the way is difficult and turns to the right.

I go down the stream in quest of him,

and Lo! He is on the island in the midst of the water.

May the charm and beauty be always the guiding principle of geometry!

Contents

1. History and contributors of the subject

1.1. Founding fathers of the subject. Since the whole development of geometry depends heavily on the past, we start out with historical developments. The following are samples of work before 1970 which provided fruitful ideas and methods.

- **Fermat's principle of calculus of variation** (Shortest path in various media).
- **Calculus (Newton and Leibnitz)**: Path of bodies governed by law of nature.
- **Euler, Lagrange**: Foundation for the variational principle and the study of partial differential equations. Derivations of equations for fluids and for minimal surfaces.
- **Fourier, Hilbert**: Decomposition of functions into eigenfunctions, spectral analysis.
- **Gauss, Riemann**: Concept of intrinsic geometry.
- **Riemann, Dirichlet, Hilbert**: Solving Dirichlet boundary value problem for harmonic function using variational method.
- **Maxwell**: Electromagnetism, gauge fields, unification of forces.
- **Christoffel, Levi-Civita, Bianchi, Ricci**: Calculus on manifolds.
- **Riemann, Poincaré, Koebe, Teichmüller**: Riemann surface uniformization theory, conformal deformation.
- **Frobenius, Cartan, Poincaré**: Exterior differentiation and Poincaré lemma.
- **Cartan**: Exterior differential system, connections on fiber bundle.
- **Einstein, Hilbert**: Einstein equation and Hilbert action.
- **Dirac**: Spinors, Dirac equation, quantum field theory.
- **Riemann, Hilbert, Poincaré, Klein, Picard, Ahlfors, Beurling, Carlsson**: Application of complex analysis to geometry.
- **Kähler, Hodge**: Kähler metric and Hodge theory.
- **Hilbert, Cohn-Vossen, Lewy, Weyl, Hopf, Pogorelov, Efimov, Nirenberg**: Global surface theory in three space based on analysis.
- **Weierstrass, Riemann, Lebesgue, Courant, Douglas, Radó, Morrey**: Minimal surface theory.
- **Gauss, Green, Poincaré, Schauder, Morrey**: Potential theory, regularity theory for elliptic equations.
- **Weyl, Hodge, Kodaira, de Rham, Milgram-Rosenbloom, Atiyah-Singer**: de Rham-Hodge theory, integral operators, heat equation, spectral theory of elliptic self-adjoint operators.
- **Riemann, Roch, Hirzebruch, Atiyah-Singer**: Riemann-Roch formula and index theory.
- **Pontrjagin, Chern, Allendoerfer-Weil**: Global topological invariants defined by curvature forms.

- **Todd, Pontrjagin, Chern, Hirzebruch, Grothendieck, Atiyah**: Characteristic classes and K-theory in topology and algebraic geometry.
- **Leray, Serre**: Sheaf theory.
- **Bochner-Kodaira**: Vanishing of cohomology groups based on the curvature consideration.
- **Birkhoff, Morse, Bott, Smale**: Critical point theory, global topology, homotopy groups of classical groups.
- **De Giorgi-Nash-Moser**: Regularity theory for the higher dimensional elliptic equation and the parabolic equation of divergence type.
- **Kodaira, Morrey, Grauert, Hua, Hörmander, Bergman, Kohn, Andreotti-Vesentini**: Embedding of complex manifolds, $\bar{\partial}$-Neumann problem, L^2 method, kernel functions.
- **Kodaira-Spencer, Newlander-Nirenberg**: Deformation of geometric structures.
- **Federer-Fleming, Almgren, Allard, Bombieri, De Giorgi, Giusti**: Varifolds and minimal varieties in higher dimensions.
- **Eells-Sampson, Al'ber**: Existence of harmonic maps into manifolds with non-positive curvature.
- **Calabi**: Affine geometry and conjectures on Kähler Einstein metric.

1.2. Modern Contributors. The major contributors can be roughly mentioned in the following periods:

I. 1972 to 1982: M. Atiyah, R. Bott, I. Singer, E. Calabi, L. Nirenberg, A. Pogorelov, R. Schoen, L. Simon, K. Uhlenbeck, S. Donaldson, R. Hamilton, C. Taubes, W. Thurston, E. Stein, C. Fefferman, Y.T. Siu, L. Caffarelli, J. Kohn, S.Y. Cheng, M. Kuranishi, J. Cheeger, D. Gromoll, R. Harvey, H. Lawson, M. Gromov, T. Aubin, V. Patodi, N. Hitchin, V. Guillemin, R. Melrose, Colin de Verdière, M. Taylor, R. Bryant, H. Wu, R. Greene, Peter Li, D. Phong, S. Wolpert, J. Pitts, N. Trudinger, T. Hildebrandt, S. Kobayashi, R. Hardt, J. Spruck, C. Gerhardt, B. White, R. Gulliver, F. Warner, J. Kazdan.

Highlights of the works in this period include a deep understanding of the spectrum of elliptic operators, introduction of self-dual connections for four manifolds, introduction of a geometrization program for three manifolds, an understanding of minimal surface theory, Monge-Ampére equations and the application of the theory to algebraic geometry and general relativity.

II. 1983 to 1992: In 1983, Schoen and I started to give lectures on geometric analysis at the Institute for Advanced Study. J.Q. Zhong took notes on the majority of our lectures. The lectures were continued in 1985 in San Diego. During the period of 1985 and 1986, K.C. Chang and W.Y. Ding came to take notes of some part of our lectures. The book *Lectures on*

Differential Geometry was published in Chinese around 1989 [**606**]. It did have great influence for a generation of Chinese mathematicians to become interested in this subject. At the same time, a large group of my students made contributions to the subject. This includes A. Treibergs, T. Parker, R. Bartnik, S. Bando, L. Saper, M. Stern, H.D. Cao, B. Chow, W.X. Shi, F.Y. Zheng and G. Tian.

At the same time, D. Christodoulou, C.S. Lin, N. Mok, J.Q. Zhong, J. Jost, G. Huisken, D. Jerison, P. Sarnak, T. Ilmanen, C. Croke, D. Stroock, J. Bismut, Price, F. H. Lin, S. Zelditch, S. Klainerman, V. Moncrief, C.L. Terng, Michael Wolf, M. Anderson, C. LeBrun, M. Micallef, J. Moore, K. Fukaya, T. Mabuchi, John Lee, A. Chang, N. Korevaar were making contributions in various directions. One should also mention that in this period important work was done by the authors in the first group. For example, Donaldson, Taubes [**655**] and Uhlenbeck [**688, 689**] did spectacular work on Yang-Mills theory of general manifolds which led Donaldson [**195**] to solve the outstanding question on four manifold topology. Donaldson [**196**], Uhlenbeck-Yau [**691**] proved the existence of Hermitian Yang-Mills connection on stable bundles. Schoen [**590**] solved the Yamabe problem.

III. 1993 to now: Many mathematicians joined the subject. This includes P. Kronheimer, B. Mrowka, J. Demailly, T. Colding, W. Minicozzi, T. Tao, R. Thomas, Zworski, Y. Eliashberg, Toth, Andrews, L.F. Tam, N.C. Leung, Y.B. Ruan, W.D. Ruan, R. Wentworth, A. Grigor'yan, L. Saloff-Coste, J.X. Hong, X.P. Zhu, M. T. Wang, A.K. Liu, K.F. Liu, X.F. Sun, T.J. Li, X.J. Wang, J. Loftin, H. Bray, J.P. Wang, L. Ni, P.F. Guan, N. Kapouleas, P. Ozsváth, Z. Szabó and Y.I. Li. The most important event is of course the major breakthrough of Hamilton [**315**] in 1995 on the Ricci flow. I did propose to him in 1982 to use his flow to solve Thurston's conjecture. Only after this paper by Hamilton, it is finally realized that it is feasible to solve the full geometrization program by geometric analysis. (A key step was the estimates on parabolic equations initiated by Li-Yau [**445**] and accomplished by Hamilton for Ricci flow [**312, 313**].) In 2002, Perelman [**551, 552**] brought in fresh new ideas to solve important steps that remained in the program. Many contributors, including Colding-Minicozzi [**173**], Shioya-Yamaguchi [**616**] and Chen-Zhu [**129**], [**130**] have helped in filling gaps in the arguments of Hamilton-Perelman. Cao-Zhu has just finished a long manuscript which gives the first complete detailed account of the program. The paper appeared in Asian J. Math., **10**(**2**) (2006), 165–492 while the monograph will be published by International Press. In the other direction, we see the important development of Seiberg-Witten theory [**721**]. Taubes [**661, 662, 663, 664**] was able to prove the remarkable theorem for counting pseudo-holomorphic curves in terms of his invariants. Kronheimer-Mrowka [**402**] were able to solve the Thom conjecture that holomorphic curves provide the lowest genus surfaces in representing homology in algebraic surfaces. (Ozsváth-Szabó had a symplectic version [**548**].)

2. Construction of functions in geometry

The following is the basic principle [**737**]:

> **Linear or non-linear analysis is developed to understand the underlying geometric or combinatorial structure. In the process, geometry will provide deeper insight of analysis. An important guideline is that space of special functions defined by the structure of the space can be used to define the structure of this space itself.**

Algebraic geometers have defined the Zariski topology of an algebraic variety using ring of rational functions. In differential geometry, one should extract information about the metric and topology of the manifolds from functions defined over it. Naturally, these functions should be defined either by geometric construction or by differential equations given by the underlying structure of the geometry. (Integral equations have not been used extensively as the idea of linking local geometry to global geometry is more compatible with the ideas of differential equations.) A natural generalization of functions consists of the following: differential forms, spinors, and sections of vector bundles.

The dual concepts of differential forms or sections of vector bundles are submanifolds or foliations. From the differential equations that arise from the variational principle, we have minimal submanifolds or holomorphic cycles. Naturally the properties of such objects or the moduli space of such objects govern the geometry of the underlying manifold. A very good example is Morse theory on the space of loops on a manifold (see [**518**]).

I shall now discuss various methods for constructing functions or tensors of geometric interest.

2.1. Polynomials from ambient space.

If the manifold is isometrically embedded into Euclidean space, a natural class of functions are the restrictions of polynomials from Euclidean space. However, isometric embedding in general is not rigid, and so functions constructed in such a way are usually not too useful.

On the other hand, if a manifold is embedded into Euclidean space in a canonical manner and the geometry of this submanifold is defined by some group of linear transformations of the Euclidean space, the polynomials restricted to the submanifold do play important roles.

2.1.1. *Linear functions being the harmonic function or eigenfunction of the submanifold.* For minimal submanifolds in Euclidean space, the restrictions of linear functions are harmonic functions. Since the sum of the norm square of the gradient of the coordinate functions is equal to one, it is fruitful to construct classical potentials using coordinate functions. This principle was used by Cheng-Li-Yau [**140**] in 1982 to give a comparison theorem for the heat kernel of minimal submanifolds in Euclidean space, sphere and hyperbolic space. Li-Tian [**439**] also considered a similar estimate for complex submanifolds of $\mathbb{C}P^n$. But this follows from [**140**] as such submanifolds can be lifted to a minimal submanifold in S^{2n+1}.

Another very important property of a linear function is that when it is restricted to a minimal hypersurface in a sphere S^{n+1}, it is automatically an eigenfunction. When the hypersurface is embedded, I conjectured that the first eigenfunction is linear and the first eigenvalue of the hypersurface is equal to n (see [**739**]). While this conjecture is not completely solved, the work of Choi-Wang [**155**] gives strong support. They proved that the first eigenvalue has a lower bound depending only on n. Such a result was good enough for Choi-Schoen [**153**] to prove a compactness result for embedded minimal surfaces in S^3.

2.1.2. *Support functions.* An important class of functions that are constructed from the ambient space are the support functions of a hypersurface. These are functions defined on the sphere and are related to the Gauss map of the hypersurface. The famous Minkowski problem reduces to solving some Monge-Ampère equation for such support functions. This was done by Nirenberg [**540**], Pogorelov [**560**], Cheng-Yau [**144**]. The question of prescribed symmetric functions of principal curvatures has been studied by many people: Pogorelov [**564**], Caffarelli-Nirenberg-Spruck [**92**], P.F. Guan and his coauthors (see [**298, 297**]), Gerhardt [**249**], etc. It is not clear whether one can formulate a useful Minkowski problem for higher codimensional submanifolds.

The question of isometric embedding of surfaces into three space can also be written in terms of the Darboux equation for the support function. The major global result is the Weyl embedding theorem for convex surfaces, which was proved by Pogorelov [**561, 562**] and Nirenberg [**540**]. The rigidity part was due to Cohn-Vossen and an important estimate was due to Weyl himself. For local isometric embeddings, there is work by C.S. Lin [**455, 456**], which are followed by Han-Hong-Lin [**318**]. The global problem for surfaces with negative curvature was studied by Hong [**345**]. In all these problems, infinitesimal rigidity plays an important role. Unfortunately they are only well understood for a convex hypersurface. It is intuitively clear that generically, every closed surface is infinitesimally rigid. However, significant works only appeared for very special surfaces. Rado studied the set of surfaces that are obtained by rotating a curve around an axis. The surfaces constructed depend on the height of the curve. It turns out that such surfaces are infinitesimally rigid except on a set of heights which form part of a spectrum of some Sturm-Liouville operator.

2.1.3. *Gradient estimates of natural functions induced from ambient space.* A priori estimates are the basic tools for nonlinear analysis. In general the first step is to control the ellipticity of the problem. In the case of the Minkowski problem, we need to control the Hessian of the support function. For minimal submanifolds and other submanifold problem, we need gradient estimates which we shall discuss in Chapter 4. In 1974 and 1975, S.Y. Cheng and I [**143, 147**] developed several gradient estimates for linear or quadratic polynomials in order to control metrics of submanifolds in Minkowski spacetime or affine space. This kind of idea can be used to deal with many different metric problems in geometry.

The first theorem concerns a spacelike hypersurface M in the Minkowski space $\mathbb{R}^{n,1}$. The following important question arose: Since the metric on $\mathbb{R}^{n,1}$ is $\sum(dx^i)^2 - dt^2$, the restriction of this metric on M need not be complete even though it may be true for the induced Euclidean metric. In order to prove the equivalence of these two concepts for hypersurfaces whose mean curvatures are controlled, Cheng and I proved the gradient estimate of the function

$$\langle X, X\rangle = \sum_i (x^i)^2 - t^2$$

restricted on the hypersurface.

By choosing a coordinate system, the function $\langle X, X\rangle$ can be assumed to be positive and proper on M. For any positive proper function f defined on M, if we prove the following gradient estimate

$$\frac{|\nabla f|}{f} \leq C$$

where C is independent of f, then we can prove the induced metric on M is complete. This is obtained by integrating the inequality to get

$$|\log f(x) - \log f(y)| \leq Cd(x, y)$$

so that when $f(y) \to \infty$, $d(x, y) \to \infty$. Once we knew the metric was complete, we proved the Bernstein theorem which says that maximal spacelike hypersurface must be linear. Such work was then generalized by Treibergs [**685**], C. Gerhardt [**248**] and R. Bartnik [**40**] for hypersurfaces in more general spacetime. (It is still an important problem to understand the behavior of a maximal spacelike hypersurface foliation for general spacetime when we assume the spacetime is evolved by Einstein equation from a nonsingular data set.)

Another important example is the study of affine hypersurfaces M^n in an affine space A^{n+1}. These are the improper affine spheres

$$\det(u_{ij}) = 1$$

where u is a convex function or the hyperbolic affine spheres

$$\det(u_{ij}) = \left(-\frac{1}{u}\right)^{n+2}$$

where u is convex and zero on $\partial\Omega$ and Ω is a convex domain. Note that these equations describe hypersurfaces where the affine normals are either parallel or converge to a point.

For affine geometry, there is an affine invariant metric defined on M which is

$$(\det h_{ij})^{-\frac{1}{n+2}} \sum h_{ij} dx^i dx^j$$

where h_{ij} is the second fundamental form of M. A fundamental question is whether this metric is complete or not.

A coordinate system in A^{n+1} is chosen so that the height function is a proper positive function defined on M. The gradient estimate of the height function gives a way to prove completeness of the affine metric. Cheng and I [**147**] did find such an estimate which is similar to the one given above.

Once completeness of the affine metric is known, it is straight forward to obtain important properties of the affine spheres, some of which were conjectured by Calabi. For example we proved that an improper affine sphere is a paraboloid and that every proper convex cone admits a foliation of hyperbolic affine spheres. The statement about improper affine sphere was first proved by Jörgens [**364**], Calabi [**94**] and Pogorelov [**563**]. Conversely, we also proved that every hyperbolic affine sphere is asymptotic to a convex cone. (The estimate of Cheng-Yau was reproduced again by a Chinese mathematician who claimed to prove the result ten years afterwards.) Much more recently, Trudinger and X. J. Wang [**687**] solved the Bernstein problem for an affine minimal surface, thereby settling a conjecture by Chern. They found a counterexample for dim≥ 10. These results are solid contributions to fourth order elliptic equations.

The argument of using gradient estimates for some naturally defined function was also used by me to prove that the Kähler Einstein metric constructed by Cheng and myself is complete for any bounded pseudo-convex domain [**145**]. (It appeared in my paper with Mok [**526**] who proved the converse statement which says that if the Kähler Einstein metric is complete, the domain is pseudo-convex.)

It should be noted that in most cases, gradient estimates amount to control of ellipticity of the nonlinear elliptic equation.

> **Comment**: To control a metric, find functions that are capable of describing the geometry and give gradient or higher order estimates for these functions.

2.2. Geometric construction of functions.

2.2.1. *Distance function and Busemann function.* When manifolds cannot be embedded into the linear space, we construct functions adapted to the metric structure. Obviously the distance function is the first major function to be constructed. A very important property of the distance function is that when the Ricci curvature of the manifold is greater than the Ricci curvature of a model manifold which is spherical symmetric at one point, the Laplacian of the distance function is not greater than the Laplacian of the distance function of the model manifold in the sense of distribution. This fact was used by Cheeger-Yau [**124**] to give a sharp lower estimate of the heat kernel of such manifolds. An argument of this type was also used by Perelman in his recent work.

Gromov [**284**] developed a remarkable Morse theory for the distance function (a preliminary version was developed by K. Grove and K. Shiohama [**295**]) to compare the topology of a geodesic ball to that of a large ball, thereby obtaining a bound on the Betti numbers of compact manifolds with nonnegative sectional curvature. (He can also allow the manifolds to have negative curvature. But in this case the diameter and the lower bound of the curvature will enter into the estimate.)

We can also take the distance function from a hypersurface and compute the Hessian of the distance function. In general, one can prove comparison theorems, and the principle curvatures of the hypersurface will come into the estimates. However, the upper bound of the Laplacian of the function depends only on the Ricci curvature of the ambient manifold and the mean curvature of the hypersurface. This kind of calculation was used in the sixties by Penrose and Hawking to study the focal locus of a closed surface under the assumption that the surface is "trapped," which means the mean curvatures are negative in both the ingoing and the outgoing null directions. This information allowed them to prove the first singularity theorem in general relativity (which demonstrates that the black hole singularity is stable under perturbation). The distance to hypersurfaces can be used as barrier functions to prove the existence of a minimal surface as was shown by Meeks-Yau [**507**], [**508**]. T. Frankel used the idea of minimizing the distance between two submanifolds to detect the topology of minimal surfaces. In particular, two maximal spacelike hypersurfaces in spacetime which satisfy the energy condition must be disjoint if they are parallel at infinity.

Out of the distance function, we can construct the Busemann function in the following way:

Given a geodesic ray $\gamma : [0,\infty) \to M$ *so that*

$$\textit{distance}(\gamma(t_1),\gamma(t_2)) = t_2 - t_1,$$

where $\| \frac{d\gamma}{dt} \| = 1$, *one defines*

$$B_\gamma(x) = \lim_{t\to\infty}(d(x,\gamma(t)) - t).$$

This function generalizes the notion of a linear function. For a hyperbolic space form, its level set defines horospheres. For manifolds with positive curvature, it is concave. Cohn-Vossen (for surface) and Gromoll-Meyer [**279**] used it to prove that a complete noncompact manifold with positive curvature is diffeomorphic to $\mathbb{R}^n$.

A very important property of the Busemann function is that it is superharmonic on complete manifolds with nonnegative Ricci curvature in the sense of distribution. This is the key to prove the splitting principle of Cheeger-Gromoll [**119**]. Various versions of this splitting principle have been important for applications to the structure of manifolds. When I [**736**] proved the Calabi conjecture, the splitting principle was used by me and others to prove the structure theorem for Kähler manifolds with a nonnegative first Chern Class. (The argument for the structure theorem is due to Calabi [**93**] who knew how to handle the first Betti number. Kobayashi [**387**] and Michelsohn [**516**] wrote up the formal argument and Beauville [**45**] had a survey article on this development.)

In 1974, I was able to use the Busemann function to estimate the volume of complete manifolds with nonnegative Ricci curvature [**734**]. After long discussions with me, Gromov [**285**] realized that my argument of Busemann function amounts to compare volumes of geodesic balls. The comparison theorem of Bishop-Gromov had been used extensively in metric geometry.

If we consider $\inf_\gamma B_\gamma$, where γ ranges from all geodesic rays from a point on the manifold, we may be able to obtain a proper exhaustion of the manifold. When M is a complete manifold with finite volume and its curvature is pinched by two negative constants, Siu and I [**634**] did prove that such a function gives a concave exhaustion of the manifold. If the manifold is also Kähler, we were able to prove that one can compactify the manifold by adding a point to each end to form a compact complex variety. In the other direction, Schoen-Yau [**603**] was able to use the Busemann function to construct a barrier for the existence of minimal surfaces to prove that any complete three dimensional manifold with positive Ricci curvature is diffeomorphic to Euclidean space.

The Busemann function also gives a way to detect the angular structure at infinity of the manifold. It can be used to construct the Poisson kernel of hyperbolic space form. For a simply connected complete manifold with bounded and strongly negative curvature, it is used as a barrier to solve the Dirichlet problem for bounded harmonic functions, after they are mollified at infinity. This was achieved by Sullivan [**647**] and Anderson [**8**]. Schoen and Anderson [**9**] obtained the Harnack inequality for a bounded harmonic function and identified the Martin boundary of such manifolds. W. Ballmann [**27**] then studied the Dirichlet problem for manifolds of non-positive curvature. Schoen and I [**606**] conjectured that nontrivial bounded harmonic function exists if the manifold has bounded geometry and a positive first eigenvalue. Many important cases were settled in [**606**]. Lyons-Sullivan [**487**] proved the existence of nontrivial bounded harmonic functions using the non-amenability of groups acting on the manifold.

The abundance of bounded harmonic functions on the universal cover of a compact manifold should mean that the manifold is "hyperbolic". Hence if the Dirichlet problem is solvable on the universal cover, one expects the Gromov volume of the manifold to be greater than zero.

The Martin boundary was studied by L. Ji and MacPherson (see [**303, 361**]) for the compactification of various symmetric spaces. For product of manifolds with negative curvature, it was determined by A. Freire [**232**]. For rank one complete manifolds with non-positive curvature, work has been done by Ballman-Ledrappier [**28**] and Cao-Fan-Ledrappier [**105**]. It should be nice to generalize the work of L.K. Hua on symmetric spaces with higher rank to general manifolds with non-positive curvature. Hua found that bounded harmonic functions satisfy extra equations (see [**348**]).

2.2.2. *Length function defined on loop space.* If we look at the space of loops in a manifold, we can take the length of each loop and thereby define a natural function on the space of loops. This is a function for which Morse theory found rich application. Bott [**71**] made use of it to prove his periodicity theorem. Bott [**68, 70**] and Morse also developed a formula for computing index of a geodesic. Bott showed that the index of a closed geodesic and its linearized Poincaré map determine the indexes of iterates of this geodesic. Starting from the famous works of Poincaré, Birkhoff, Morse and Ljusternik-Shnirel'man, there has been extensive work on proving the existence of a closed geodesic using Morse theory on the space of loops. Klingenberg and his students developed powerful tools (see [**386**]). Gromoll-Meyer [**280**] did important work in which they proved the existence of infinitely many closed geodesics assuming the Betti number of the free loop space of the manifold grows unboundedly. They used the results of Bott [**70**], Serre and some version of degenerate Morse theory. There was also later work by Ballmann, Ziller, G. Thorbergsson, Hingston and Kramer (see, e.g., [**30, 328, 401**]), who improved the Gromoll-Meyer theorem to give a low estimate of the growth of the number of geometrically distinct closed geodesics of length $\leq t$. In most cases, they grow at least as fast as the prime numbers. The classical important question that every metric on S^2 supports an infinite number of closed geodesics was also solved affirmatively by Franks [**228**], Bangert [**35**] and Hingston [**329**]. An important achievement was made by Vigué-Poirrier and D. Sullivan [**695**] who proved that the Gromoll-Meyer condition for the existence of infinite numbers of closed geodesics is satisfied if and only if the rational cohomology algebra of the manifold has at least two generators. They made use of Sullivan's theory of the rational homotopic type. When the metric is Finsler, the most recent work of Victor Bangert and Yiming Long [**36**] showed the existence of two closed geodesics on the two dimensional sphere. (Katok [**377**] produced an example which shows that two is optimal.) Length function is a natural concept in Finsler geometry. In the last fifty years, Finsler geometry has not been popular in western world. But under the leadership of Chern, David Bao, Z. Shen, X. H. Mo and M. Ji did develop Finsler geometry much further (see, e.g., [**37**]).

A special class of manifolds, all of whose geodesics are closed, has occupied quite a lot of interest of distinguished geometers. It started from the work of Zoll (1903) for surfaces where Guillemin did important contributions. Bott [**69**] has determined the cohomology ring of these manifolds. The well known Blaschke conjecture was proved by L. Green [**270**] for two dimension and by M. Berger and J. Kazdan (see [**51**]) for higher dimensional spheres. Weinstein [**713**] and C.T. Yang [**727, 728, 729**] made important contributions to the conjecture for other homotopic types.

2.2.3. *Displacement functions.* When the manifold has negative curvature, the length function of curves is related to the displacement function defined in the following way:

If γ is an element of the fundamental group acting on the universal cover of a complete manifold with non-positive curvature, we consider the function $d(x, \gamma(x))$: The study of such a function gives rise to properties of compact manifolds with non-positive curvature. For example, in my thesis, I generalized the Preissmann theorem to the effect that every solvable subgroup of the fundamental group must be a finite extension of an Abelian group which is the fundamental group of a totally geodesic flat sub-torus [**731**]. Gromoll-Wolf [**281**] and Lawson-Yau [**412**] also proved that if the fundamental group of such a manifold has no center and splits as a product, then the manifold splits as a metric product. Strong rigidity result for a discrete group acting on product of manifolds irreducibly was obtained by Jost-Yau [**372**] where they proved that these manifolds are homogeneous if the discrete group also appears as fundamental group of compact manifolds with nonpositive curvature.

When the manifold has bounded curvature, Margulis studied those points where $d(x, \gamma(x))$ is small and proved the famous Margulis lemma which was used extensively by Gromov [**282**] to study the structure of manifolds with non-positive curvature.

> **Comment**: The lower bound of sectional curvature (or Ricci curvature) of a manifold gives upper estimate of the Hessian (or the Laplacian) of the distance functions. Since most functions constructed in geometry come from distance functions, we have partial control of the Hessian of these functions. The information provides us with basic tools to construct barrier functions for harmonic analysis or to produce convex functions. The Hessian of distance functions come from computations of second variation of geodesics. If we consider the second variation of closed geodesic loops, we get information about the Morse index of the loop, which enable us to link global topology to the existence of many closed geodesics or curvatures of the manifold.
>
> We always look for canonical objects through geometric constructions and deform them to find their global properties.

2.3. Functions and tensors defined by linear differential equations. Direct construction of functions or tensors based on geometric intuitions alone is not rich enough to handle the very complicated geometric world. One should produce global geometric objects based on global differential equations. Often the construction depends on the maximal principle, integration by part, or the method of contradictions, and they are not necessarily geometric intuitive. On the other hand, the basic principle of global differential equations does fit well with modern geometry in relating local data to global behavior. In order for the theory to be effective, the global differential operator has to be constructed from a geometric structure naturally.

The key to understanding any self-adjoint linear elliptic differential operator is to understand its spectral resolution and the detail of the structure of objects in the process of the resolution: eigenvalues or eigenfunctions are particularly important for their relation to geometry. Low eigenvalues and low eigenfunctions give deep information about global geometry such as topology or isoperimetric inequalities. High eigenvalues and high eigenfunctions are related to local geometry such as curvature forms or characteristic forms. Semiclassical analysis in quantum physics give a way to relate these two ends. This results in using either the heat equation or the hyperbolic equation.

There are many important first order differential operators: d, δ, $\bar{\partial}$, Dirac operator. All these operators have contributed to a deeper understanding of geometry. They form systems of equations. Our understanding of them is not as deep as our understanding of the Laplacian acting on functions. The future of geometry will rest on an understanding of global systems of equations and their relation to global topology. The index theorem has given many important contributions as it provides significant information about the dimension of the kernel (or cokernel). However, a deeper understanding of the spectrum of these operators is still needed.

2.3.1. *Laplacian.*

(a)*. Harmonic functions.* The spectral resolution of the Laplacian gives rise to eigenfunctions. Harmonic functions are therefore the simplest functions that play important roles in geometry.

If the manifold is compact, the maximum principle shows that harmonic functions are constant. However, when we try to understand the singularities of compact manifolds, we may create noncompact manifolds by scaling and blowing up processes, at which point harmonic functions can play an important role.

The first important question about harmonic functions on a complete manifold is the Liouville theorem. I started my research on analysis by understanding the right formulation of the Liouville theorem. In 1971, I thought that it is natural to prove that for complete manifolds with a nonnegative Ricci curvature, there is no nontrivial harmonic function [**732**]. I also thought that in the opposite case, when a complete manifold has strongly negative curvature and is simply connected, one should be able to solve Dirichlet problem for bounded harmonic functions.

The gradient estimates [**732**] that I derived for a positive harmonic function come from a suitable interpretation of the Schwarz lemma in complex analysis. In fact, I generalized the Ahlfors Schwarz lemma before I understood how to work out the gradient estimates for harmonic functions. The generalized Schwarz lemma [**738**] says that holomorphic maps, from a complete Kähler manifold with Ricci curvature bounded from below to a Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant, are distance decreasing with constants depending only on the bound on the curvature. This generalization has since found many applications such as the study of the geometry of moduli spaces by Liu-Sun-Yau [**471, 472**]. They used it to prove the equivalence of the Bergman metric with the Kähler-Einstein metric on the moduli space. They also proved that these metrics are equivalent to the Teichmüller metric and the McMullen metric.

The classical Liouville theorem has a natural generalization: Polynomial growth harmonic functions are in fact polynomials. Motivated by this fact and several complex variables, I asked whether the space of polynomial growth harmonic functions with a fixed growth rate is finite dimension with the upper bound depending only on the growth rate [**741**]. This was proved by Colding-Minicozzi [**168**] and generalized by Peter Li [**437**]. (Functions can be replaced by sections of bundles). In a beautiful series of papers (see, e.g., [**440, 441**]), P. Li and J.P. Wang studied the space of harmonic functions in relation to the geometry of manifolds. In the case when harmonic functions are holomorphic, they form a ring. I am curious about the structure of this ring. In particular, is it finitely generated when the manifold is complete and has a nonnegative Ricci curvature? A natural generalization of such a question is to consider holomorphic sections of line bundles, especially powers of canonical line bundles. This is part of Mori's minimal model program.

(b). *Eigenvalues and eigenfunctions.* Eigenvalues reflect the geometry of manifolds very precisely. For domains, estimates of them date back to Lord Rayleigh. Hermann Weyl [**711**] solved a problem of Lorentz's on the asymptotic behavior of eigenvalues in relation to the volume of the domain and hence initiated a new subject of spectral geometry. Pólya-Szegö, Faber, Krahn and Levy gave estimates of eigenvalues of various geometric problems. On a general manifold, Cheeger [**114**] was the first person to relate a lower estimate of the first eigenvalue with the isoperimetric constant (now called the Cheeger constant). One may note that many questions on the eigenvalue for domains are still unsolved. The most noted one is the Pólya conjecture which gave a sharp lower estimate of the Dirichlet problem in terms of volume. Li-Yau [**444**] did settle the average version of the Pólya conjecture.

The gradient estimate that I found for harmonic functions can be generalized to cover eigenfunctions and Peter Li [**436**] was the first one to apply it to finding estimates for eigenvalues for manifolds with positive Ricci curvature. (If the Ricci curvature has a positive lower bound, this is due to Lichnerowicz.) Li-Yau [**442**] then solved the well-known problem of estimating eigenvalues of manifolds in terms of their diameter and the lower bound on their Ricci curvature. Li-Yau conjectured the sharp constant for their estimates, and Zhong-Yang [**755**] were able to prove this conjecture by sharpening Li and Yau's arguments. A probabilistic argument was later developed by Chen and Wang [**133**] to derive these inequalities. The precise upper bound for the eigenvalue was first obtained by S. Y. Cheng [**137**] also in terms of diameter and lower bound of the Ricci curvature. Cheng's theorem gives a very good demonstration of how the analysis of functions provides information about geometry. As a corollary of his theorem, he proved that if a compact manifold M^n has a Ricci curvature $\geq n-1$ and the diameter is equal to π, then the manifold is isometric to the sphere. He used a lower estimate for eigenvalues based on the work of Lichnerowicz and Obata. Colding [**167**] was able to use functions with properties close to those of the first eigenfunction to prove a pinching theorem which states that: When the Ricci curvature is bounded below by $n-1$ and the volume is close to that of the unit sphere, the manifold is diffeomorphic to the sphere. There is extensive work by Colding-Cheeger [**116, 117, 118**] and Perelman (see, e.g., [**88**]) devoted to the understanding of Gromov's theory of Hausdorff convergence for manifolds. The tools they used include the comparison theorem, the splitting theorem of Cheeger and Gromoll, and the ideas introduced earlier by Colding.

A very precise estimate of eigenvalues of the Laplacian has been important in many areas of mathematics. For example, the idea of Szegö [**651**]-Hersch [**327**] on the upper bound of the first eigenvalue in terms of the area alone was generalized by me to the higher genus in joint works with P. Yang [**730**] and P. Li [**443**]. For genus one, this was Berger's conjecture, as I was informed by Cheng. After Cheng showed me the paper of Hersch, I realized how to create trial functions by taking the branched conformal cover of S^2. While the constant in the paper of Yang-Yau [**730**] for torus is not the best possible, the recent work of Jakobson, Levitin, Nadirashvili, Nigam and Polterovich [**358**] demonstrated that the constant for a genus two surface may be the best possible and may be achieved by Bolza's surface.

Shortly afterwards, I applied the argument of [**730**] to prove that a Riemann surface defined by an arithmetic group must have a relative high degree when it is branched over the sphere. This observation of using Selberg's estimate coupled with Li-Yau [**443**] was made in 1985 when I was in San Diego, where I also used similar idea to estimate genus of mini-max surface in three dimensional manifolds and also to prove positivity of Hawking mass. After I arrived in Harvard, I discussed the idea with my colleague N. Elkies and B. Mazur. The paper was finally written up and published in 1995 [**745**]. In the meanwhile, ideas of using my work on eigenvalue coupled with Selberg's work to study congruence subgroup was generalized by D. Abramovich [**1**] (my idea was conveyed by Elkies to him) and by P. Zograf [**758**] to the case where the curve has cusps. Most recently Ian Agol [**2**] also used a similar idea to study arithmetic Kleinian reflection groups.

In a beautiful article, N. Korevaar [**397**] gave an upper bound, depending only on genus and n, for the n-th eigenvalue λ_n of a Riemann surface. His result answered a challenge of mine (see [**739**]) when I met him in Utah in 1989. Grigor'yan, Netrusov and I [**278**] were able to give a simplified proof and apply the estimate to bound the index of minimal surfaces. There are also works by P. Sarnak (see, e.g., [**586, 357**]) on understanding eigenfunctions for such Riemann surfaces. Iwaniec-Sarnack [**357**] showed that the estimate of the maximum norm of the n-th eigenfunction on an arithmetic surface has significant interest in number theory. Wolpert [**725**] analyzes perturbation stability of embedded eigenvalues and applies asymptotic perturbation theory and harmonic map theory to show that stability is equivalent to the non-vanishing of certain standard quantities in number theory. There was also the work of Schoen-Wolpert-Yau [**595**] on the behavior of eigenvalues $\lambda_1, \cdots, \lambda_{2g-3}$ for a compact Riemann surface of genus g. These are eigenvalues that may tend to zero for metrics with curvature -1. However, λ_{2g-2}, λ_{2g-1}, $\cdots$, λ_{4g-1} always appear in $[c_g, \frac{1}{4}]$ where $c_g > 0$ depends only on g. It will be nice to find the optimal c_g.

In this regard, one may mention the very deep problem of Selberg on lower estimate of λ_1 for surfaces defined by an arithmetic group. Selberg proved that it is greater than $\frac{3}{16}$ and it was later improved by Luo-Rudnick-Sarnak [**482**]. For a higher dimensional locally symmetric space, there is a similar question of Selberg and results similar to Selberg's were found by J.S. Li [**425**] and Cogdell-Li-Piatetski-Shapiro-Sarnak [**166**]. Many researchers attempt to use Kazdhan's property T for discrete groups to study Selberg's problem.

There are many important properties of eigenfunctions that were studied in the seventies. For example, Cheng [**138**] found a beautiful estimate of multiplicities of eigenvalues of Riemann surfaces based only on genus. The idea was used by Colin de Verdière [**175**] to embedded graphes into $\mathbb{R}^3$ when they satisfy nice combinatorial properties. The connectivity and the topology of nodal domains are very interesting questions. Melas [**510**] did prove that for a convex planar domain, the nodal line of second eigenfunctions must intersect the boundary in exactly two points. Very little is known about the number of nodal domains except the famous theorem of Courant that the number of nodal domains of the m-th eigenfunction is less than m.

There are several important questions related to the size of nodal sets and the number of critical points of eigenfunctions. I made a conjecture (see [**739**]) about the area of nodal sets, and significant progress toward its resolution was made by Donnelly-Fefferman [**207**], Dong [**206**] and F.H. Lin [**458**]. The number of critical points of an eigenfunction is difficult to determine. I [**746**] managed to prove the existence of a critical point near the nodal set. Jakobson and Nadirashvili [**359**] gave a counterexample to my conjecture that the number of critical points of the n-th eigenfunction is unbounded when n tends to infinity. I believe the conjecture is true for generic metrics and deserves to be studied extensively. Nadirashvili and his coauthors [**344, 319**] were also the first to show that the critical sets of eigenfunctions in n-dimensional manifold have a finite H^{n-2}-Hausdorff measure. Afterwards, Han-Hardt-Lin [**317**] gave an explicit estimate.

When there is potential V, the eigenvalues of $-\triangle+V$ are also important. When V is convex, with Singer, Wong and Stephen Yau, I applied the argument that I had with Peter Li to estimate the gap $\lambda_2 - \lambda_1$ [**624**]. When V is arbitrary, I [**747**] observed how this gap depends on the lower eigenvalue of the Hessian of $-\log\psi$, where ψ is the ground state. The method of continuity was used by me in 1980 to reprove the work of Brascamp-Lieb [**79**] on the convexity of $-\log\psi$ when V is convex. (This work appeared in the appendix of [**624**].) When V is the scalar curvature, this was studied by Schoen and myself extensively. In fact, in [**604**], we found an upper estimate of the first Dirichlet eigenvalue of the operator $-\triangle + \frac{1}{2}R$ in terms of $\frac{3\pi^2}{2r^2}$ where r is a certain concept of radius related to loops in a three dimensional manifold. (If we replace loops with higher dimensional spheres, one can define a similar concept of radius. It will be nice if such a concept can be related to eigenvalues of differential forms.) This operator is naturally related to conformal deformation, stability of minimal surfaces, etc. (The works of D. Fischer-Colbrie and Schoen [**223**], Micallef [**512**], Schoen-Yau [**597, 603**] on stable minimal surfaces all depend on an understanding of spectrum of this operator.) The parabolic version appears in the recent work of Perelman.

If there is a closed non-degenerate elliptic geodesic in the manifold, Babič [**25**], Guillemin and Weinstein [**302**] found a sequence of eigenvalues of the Laplacian which can be expressed in terms of the length, the rotation angles and the Morse index of the geodesic.

> **Comment**: It is important to understand how harmonic functions or eigenfunctions oscillate. Gradient estimate is a good tool to achieve this. Gradient estimate for the log of the eigenfunction can be used to prove the Louville theorem or give a good estimate of eigenvalues. For higher eigenfunctions, it is important to understand its zero set and its growth. By controlling this information, one can estimate the dimension of these functions. A good upper estimate for eigenvalues depends on geometric intuition which may lead to construction of trial functions that are more adaptive to geometry. It should be emphasized that a clean and sharp estimate for the linear operator is key to obtaining good estimates for the nonlinear operator.

(c). *Heat kernel.* Most of the work on the heat kernel over Euclidean space can be generalized to those manifolds where Sobolev and Poincaré inequalities hold. (It should be noted that Aubin [**22, 24**] and Talenti [**652**] did find the best constant for various Sobolev inequalities on Euclidean space.) These inequalities are all related to isoperimetric inequalities. C. Croke [**178**] was able to follow my work [**733**] on Poincaré inequalities to prove the Sobolev inequality depending only on volume, diameter and the lower bound of Ricci curvature. Arguments of John Nash were then used by Cheng-Li-Yau [**139**] to give estimates of the heat kernel and its higher derivatives. In this paper, an estimate of the injectivity radius was derived and this estimate turns out to play a role in Hamilton's theory of Ricci flow. A year later, Cheeger-Gromov-Taylor [**122**] made use of the wave kernel to reprove this estimate. In other direction, D. Stroock (see [**538**]) used Malliavin's calculus to give remarkable estimates for the heat kernel at a pair of points where one point is at the cut locus of another point.

The estimate of the heat kernel was later generalized by Davies [**185, 186**], Saloff-Coste [**582**] and Grigor'yan [**276, 277**] to complete manifolds with polynomial volume growth and volume doubling property. Since these are quasi-isometric invariants, their analysis can be applied to graphs or discrete groups. See Grigor'yan's survey [**277**] and Saloff-Coste's survey [**583**].

On the other hand, the original gradient estimate that I derived is a pointwise inequality that is much more adaptable to nonlinear theory. Peter Li and I [**445**] were able to find a parabolic version of it in 1984. We observed its significance for estimates on the heat kernel and its relation to the variational principle for paths in spacetime. Coupled with the work of Cheeger-Yau [**124**], this gives a very precise estimate of the heat kernel. Such ideas turn out to provide fundamental estimates which are crucial for the analysis of Hamilton's Ricci flow [**312, 313**].

Not much is known about the heat kernel on differential forms or differential forms with twisted coefficients. The fundamental idea of using the heat equation to prove the Hodge theory came from Milgram-Rosenbloom. The heat kernel for differential forms with twisted coefficients does play an important role in the analytic proof of the index theorem, as was demonstrated by Atiyah-Bott-Patodi [**13**]. It is the alternating sum that exhibits cancellations and gives rise to index of elliptic operators. When t is small, the alternating sum reduces to a calculation of curvature forms. When t is large, it gives global information on harmonic forms. Since the index of the operator is independent of t, we can relate the index to characteristic forms.

If a compact manifold is the quotient of a non-compact manifold by a discrete group and if the heat kernel of the non-compact manifold can be computed explicitly, it can be averaged to give the heat kernel of the quotient manifold. Since the integral of the later kernel on the diagonal can be computed by the spectrum to be $\sum e^{-t\lambda_i}$, one can relate the displacement function of the discrete group to the spectrum. This is the Selberg trace formula relating length of closed geodesics to the spectrum of the Laplacian.

Heat kernel converges to delta function when t approaches zero. This property was used by Kefeng Liu [**469, 470**] in an elegant way to obtain various localization formulas on the moduli space of bundles. Liu's idea was used later by Bismut to treat the formula of E. Verlinde [**54**].

> **Comment**: Understanding the heat kernel is almost the same as understanding the heat equation. However, heat kernel satisfies semi-group properties, which enables one to give a good estimate of the maximum norm or higher order derivative norms as long as the Sobolev inequality can be proved. It is useful to look at the heat equation in spacetime where the Li-Yau gradient estimate is naturally defined. The estimate provides special pathes in spacetime for the estimate of the kernel. However, the effects of closed geodesics have not been found in the heat equation approach. A sharp improvement of the Li-Yau estimate may lead to such information.

(d). *Isoperimetric inequalities.* Isoperimetric inequality is a beautiful subject. It has a long history. Besides its relation to eigenvalues, it reviews the deep structure of manifolds. The best constant for the inequality is important. Pólya-Szegö [**565**], G. Faber (1923), E. Krahn (1925) and P. Lévy (1951) made fundamental contributions. Gromov generalized the idea of Lévy to obtain a good estimate of Cheeger's constant (see [**289**]). C. Croke [**179**] and Cao-Escobar [**104**] have worked on domains in a simply connected manifold with non-positive curvature. It is assumed that the inequality holds for any minimal subvariety in Euclidean space. But it is not known to be true for the best constant. Li-Schoen-Yau [**438**] did prove it in the case of a minimal surface with a connected boundary, and E. Lutwak, Deane Yang and G.Y. Zhang did some beautiful work in the affine geometry case (see, e.g., [**484, 485**]). In Hamilton's proof of Ricci flow convergent to the round metric on S^2, he demonstrated that the isoperimetric constant of the metric is improving and geometry of the manifold is controlled.

> **Comment**: The variational principle has been the most important method in geometry since the Greek mathematicians. Fixing the area of the domain and minimizing the length of the boundary is the most classical form of isoperimetric inequality. This principle has been generalized to much more general situations in geometry and mathematical physics. In most cases, one tries to prove existence of the extremal object and establish isoperimetric inequalities by calculating corresponding quantities for the extremal object. There is also the idea of rearrangement or symmetrization to prove isoperimetric inequalities. In the other direction, there is the duality

> principle in the calculus of variation: instead of minimizing the length of the boundary, one can fix it and maximize the area it encloses. The principle can be effective in complicated variational problems.

(e). *Harmonic analysis on discrete geometry.* There are many other ideas in geometric analysis that can be discretized and applied to graph theory. This is especially true for the theory of spectrum of graphs. Some of these were carried out by F.Chung, Grigor'yan and myself (see the reference of Chung's survey [**164**]). But the results in [**164**] are far away from being exhaustive. On the other hand, Margulis [**491**] and Lubotzky-Phillips-Sarnak [**480**] were able to make use of discrete group and number theory to construct expanding graphs. Methods to construct and classify these expanding graphs are important for application in computer science. It should be noted that Kazhdan's property (T) [**380**] did play an important role in such discussions. It is also important to see how to give a good decomposition of any graph using the spectral method.

The most important work for the geometry of a finitely presented group was done by Gromov [**284**]. He proved the fundamental structure theorem of groups where volume grows at most polynomially. These groups must be virtually nilpotent. Geometric ideas were developed by Varopoulos and his coauthors [**693, 38**] on the precise behaviors of the heat kernel in terms of volume growth. As an application of the theory of amenable groups, R. Brooks [**81**] was able to prove that if a manifold covers a compact set by a discrete group Γ, then it has positive eigenvalue if and only if Γ is non-amenable.

Gromov [**283**] also developed a rich theory of hyperbolic groups using concepts of isoperimetric type inequalities. It would be nice to characterize these groups that are fundamental groups of compact manifolds with non-positive curvature or locally symmetric spaces.

> **Comment**: The geometry of a graph or complex can be used as a good testing ground for geometric ideas. They can be important in understanding smooth geometric structures. Many rough geometric concepts, such as isoperimetric inequalities, can be found on graphs, and in fact they play some roles in computer network theory. On the other hand, many natural geometric concepts should be generalized to graphs: for example, the concept of the fiber bundle, bundle theory over graphs and harmonic forms. It is likely that one needs to have a good way to define the concept of equivalence between such objects. When we approximate a smooth manifold by a graph or complex, we only care about the limiting object and therefore some equivalence relations should be allowed. In the case of Cayley graph of a finitely generated group, it depends on the choice of the generating set, and

> properties independent of this generating set are preferable if we are only interested in the group itself. In the other direction, computer networks and other practical subjects have independent interest in graph theory. A close collaboration between geometer and computer scientists would be fruitful.

(f). *Harmonic analysis via hyperbolic operators.* There are important works of Fefferman, Phong, Lieb, Duistermaat, Guillemin, Melrose, Colin de Verdier, Taylor, Toth, Zelditch and Sarnak on understanding the spectrum of the Laplacian from the point of view of semi-classical analysis (see, e.g. [**221, 210, 325, 587**]). Some of their ideas can be traced back to the geometric optics analysis of J. Keller. The fundamental work of Duistermaat and Hörmander [**209**] on propagation of singularities was also used extensively. There has been a lot of progress on the very difficult question of determining when one "Can hear the shape of a drum" by, among others, Melrose (see [**511**]), Guillemin [**299**] and Zelditch [**753**]. (Priori to this, Guillemin and Kazhdan [**300**] proved that no negatively curved closed surface can be isospectrally deformed.) The first counterexample for closed manifolds was given by J. Milnor [**519**] on a 16 dimensional torus. The idea was generalized by Sunada [**649**], Gordon-Wilson [**263**]. For domains in Euclidean spaces, there were examples by Urakawa in three dimensions. Two dimensional counterexamples were given by Gordon-Webb-Wolpert [**262**], Wilson and Szabó [**650**]. Most of the ideas for counterexamples are related to the Selberg trace formula discussed in the section of heat kernel. The semi-classical analysis based on the hyperbolic operator also gives a very precise estimate or relation between the geodesic and the spectrum. The support of the singularities of the trace of the wave kernel $\sum e^{\sqrt{-1}t\sqrt{\lambda_i}}$ is a subset of the set of the lengths of closed geodesics. It is difficult to achieve such results by elliptic theory. However, most results are asymptotic in nature. It will be remarkable if both methods can be combined.

> **Comment**: Fourier expansion has been a very powerful tool in analysis and geometry. Practically any general theorem in classical Fourier analysis should have a counterpart in analysis of the spectrum of the Laplacian. The theory of geometric optics and the propagation of a singularity gives deep understanding of the singularity of a wave kernel. Geodesic and closed geodesic becomes an important means to understand eigenvalues. However, the theory has not been fruitful for the Laplacian acting on differential forms. Should areas of minimal submanifolds play a role? In the case of Kähler manifolds, holomorphic cycles or the volume of special Lagrangian cycles should be important, as the length of close geodesics appear in the exponential decay term of the heat kernel. It would be useful to sharpen the heat equation method to capture this lower order information.

(g). *Harmonic forms.* Natural generalizations of harmonic or holomorphic functions are harmonic or holomorphic sections of bundles with connections. The most important bundles are the exterior power of cotangent bundles. Using the Levi-Civita connection, harmonic sections are harmonic forms which, by the theory of de Rham and Hodge, give canonical representation of cohomology classes. The major research on harmonic forms comes from Bochner's vanishing theorem [**59**]. But our understanding is still poor except for 1-forms or when the curvature operator is positive, in which case the Bochner argument proved the manifold to be a homology sphere. If there is any nontrivial operator which commutes with the Laplacian, the eigenforms split accordingly. Making use of special structures of such splitting, the Bochner method can be more effective. For example, when the manifold is Kähler, differential forms can be decomposed further to (p, q)-forms and the Kodaira vanishing theorem [**390**] yields much more powerful information, when the (p, q) forms are twisted with a line bundle or vector bundles. Similar arguments can be applied to manifolds with a special holonomy group depending on the representation theory of the holonomy group. When the complex structure moves holomorphically, the subbundles of (p, q) forms in the bundle of $(p + q)$ forms do not necessarily deform holomorphically. The concept of Hodge filtration is therefore introduced. When we deform the complex structure around a point where the complex structure degenerates, there is a monodromy group acting on the Hodge filtration. The works of Griffiths-Schmid [**275**] and Schmid's $SL_2(\mathbb{R})$ theorem [**588**] give powerful control on the degeneration of the Hodge structure. Deligne's theory of mixed Hodge structure [**187**] plays a fundamental role for studying singular algebraic varieties. The theory of variation of Hodge structures is closely related to the study of period of the differential forms. This theory also appears in the subject of mirror symmetry. It is desirable to give a precise generalization of these works to higher dimensional moduli spaces where Kaplan-Cattani-Schmid made important contributions.

Harmonic forms give canonical representation to de Rham cohomology. However, the wedge product of harmonic forms need not be harmonic. The obstruction comes from secondary cohomology cooperation. K.T. Chen [**132**] studied the case of 1-forms and Sullivan [**646**] studied the general case and gave a minimal model theory for a rational homotopic type of a manifold. Using $\partial\bar{\partial}$-lemma of Kähler manifolds, Deligne-Griffiths-Morgan-Sullivan [**188**] showed that the rational homotopic type is formal for Kähler manifolds.

The importance of harmonic forms is that they give canonical representation to the de Rham cohomology which is isomorphic to singular cohomology over real numbers. It gives a powerful tool to relate local geometry to global topology. In fact the vanishing theorem of Bochner-Kodaira-Lichnerowicz allows one to deduce from sign of curvature to vanishing of cohomology.

This has been one of the most powerful tools in geometry in the past fifty years.

The idea of harmonic forms came from fluid dynamics and Maxwell equations. The non-Abelian version is the Yang-Mills theory. Most of the works on Yang-Mills theory have been focused on these gauge fields where the absolute minimum is achieved by some (topological) characteristic number. (These are called BPS state in physics literature.) When the dimension of the manifold is four, the star operator maps two form to two form and it makes sense to require the curvature form to be self-dual or anti-self-dual depending on whether the curvature form is invariant or anti-invariant under the star operator. These curvature forms can be interpreted as non-Abelian harmonic forms. The remarkable fact is that when the metric is Kähler, the anti-self-dual connections give rise to holomorphic bundles. The moduli space of such bundles can often be computed using tools from algebraic geometry.

On the product space $M \times \mathcal{M}$ where M is the four dimensional manifold and $\mathcal{M}$ is the moduli space of anti-self-dual connections, there is a universal bundle V over $M \times \mathcal{M}$. By studying the slant product and the Chern classes of V, we can construct polynomials on the cohomology of M that are invariants of the differentiable structure of M. These are Donaldson polynomials (see [**204**]). In general $\mathcal{M}$ is not compact and Donaldson has to construct cycles in $\mathcal{M}$ for such operations. Donaldson invariants are believed to be equivalent to Seiberg-Witten invariants, where the vanishing theorem can apply and powerful geometric consequences can be found. Kronheimer and Mrowka [**402**] built an important concept of simple type for Donaldson invariants. It is believed that Donaldson invariants of algebraic surfaces of general type are of simple type.

If the manifold is symplectic, we can look at the moduli space of pseudo-holomorphic curves. (These are J-invariant maps from Riemann surfaces to the manifold. J is an almost complex structure that is tame to the symplectic form.) Symplectic invariants can be created and they are called Gromov-Witten invariants. Y. Ruan [**579**] has observed that they need not be diffeomorphic invariants. It may still be interesting to know whether Gromov-Witten invariants are invariants of differentiable structures for Calabi-Yau manifolds.

De Rham cohomology can only capture the non-torsion part of the singular cohomology. Weil [**710**] and Allendoerfer-Eells [**5**] attempted to use differential forms with poles to compute cohomology with integer coefficients. Perhaps one should study Chern forms of a complex bundle with a connection that satisfies the Yang-Mills equation and whose curvature is square integrable. The singular set of the connection may be allowed to be minimal submanifolds. The moduli space of such objects may give information about integral cohomology. It should be noted that Cheeger-Simons [**123**] did develop a rich theory of differential character with values in $\mathbb{R}/\mathbb{Z}$.

It depends on the connections of the bundle. Witten managed to integrate the Chern-Simons forms [**152**] on the space of connections to obtain the knot invariants of Jones [**363**].

When we look for different operators acting on different forms, we may have to look into different kinds of harmonic forms. For example, if we are interesting in $\partial\bar{\partial}$ cohomology, we may look for the operator $\left(\partial\bar{\partial}\right)^* \partial\bar{\partial} + \partial\partial^* + \bar{\partial}\bar{\partial}^*$. It would be interesting to see how super-symmetry may be used to generalize the concept of harmonic forms.

> **Comment**: The theory of harmonic form is tremendously powerful because it provides a natural link between global topology, analysis, geometry, algebraic geometry and arithmetic geometry. However, our analytic understanding of high degree forms is poor. For one forms, we can integrate along paths. For two forms, we can take an interior product with a vector field to create a moment map. For closed $(1,1)$-forms in a Kähler manifold, we can express them locally as $\partial\bar{\partial}f$. However, we do not have good ways to reduce a high degree form to functions which are easier to understand. Good estimates of higher degree forms will be very important.

2.3.2. *$\bar{\partial}$-operator.* Construction of holomorphic functions or holomorphic sections of vector bundles and holomorphic curves are keys to understanding complex manifolds.

In order to demonstrate the idea behind the philosophy of determining the structure of manifolds by function theory, I was motivated to generalize the uniformization theory of a Riemann surface to higher dimensions when I was a graduate student. During this period, I was influenced by the works of Greene-Wu [**273**] in formulating these conjectures. Greene and Wu were interested in knowing whether the manifolds are Stein or not.

When the complete Kähler manifold is compact with positive bisectional curvature, this is the Frankel conjecture, as was proved independently by Mori [**530**] and Siu-Yau [**633**]. Both arguments depend on the construction of rational curves of low degree. Mori's argument is stronger, and it will be good to capture his result by the analytic method. When the manifold has nonnegative bisectional curvature and positive Ricci curvature, Mok-Zhong [**527**] and Mok [**523**], using ideas of Bando [**31**] in his thesis on Hamilton's Ricci flow, proved that the manifold is Hermitian symmetric unless it is biholomorphic to projective space.

When the complete Kähler manifold is noncompact with positive bisectional curvature, I conjectured that it must be biholomorphic to $\mathbb{C}^n$ (see [**739**]). Siu-Yau [**632**] made the first attempt to prove such a conjecture by using the L^2-method of Hörmander [**347**] to construct holomorphic functions with slow growth. (Note that Hörmander's method goes back to Kodaira, which was also generalized by Calabi-Vesentini [**95**].) Singular weight functions were used in this paper and later much more refined arguments were developed by Nadel [**535**] and Siu [**630**] using what is called the multiplier ideal sheaf method. Siu found important applications of this method in algebraic geometry and also related the idea to the powerful work of J. Kohn on weakly pseudo-convex domain.

This work of Siu-Yau was followed by Siu-Mok-Yau [**524**] and Mok [**521, 522**] under assumptions about the decay of curvature and volume growth. Shi [**611, 612, 613**] introduced Hamilton's Ricci flow to study my conjecture, and his work is fundamental. This was followed by beautiful works of Cao [**101, 102**], Chen-Zhu [**127, 128**] and Chen-Tang-Zhu [**125**]. Assuming the manifold has maximal Euclidean volume growth and bounded curvature, Chen-Tang-Zhu [**125**] (for complex dimension two) and then Ni [**539**] (for all higher dimension) were able to prove the manifold can be compactified as a complex variety. Last year, Albert Chau and Tam [**113**] were finally able to settle the conjecture assuming maximal Euclidean volume growth and bounded curvature. An important lemma of L. Ni [**539**] was used, where a conjecture of mine (see [**742**] or the introduction of [**539**]) was proved. The conjecture says that maximal volume growth implies scalar curvature decays quadratically in the average sense.

While we see great accomplishments for Kähler manifolds with positive curvature, very little is known for Kähler manifolds, which are complete simply connected with strongly negative curvature. It is conjectured to be a bounded domain in $\mathbb{C}^n$. (Some people told me that Kodaira considered a similar problem. But I cannot find the appropriate reference.) The major problem is to construct bounded holomorphic functions.

The difficulty of construction of bounded holomorphic functions is that the basic principle of the L^2-method of Hörmander comes from Kodaira's vanishing theorem. It is difficult to obtain elegant results by going from weighted L^2 space to bounded functions. In this connection, I was able to show that non-trivial bounded holomorphic functions do not exist on a complete manifold with non-negative Ricci curvature [**738**].

If the manifold is the universal cover of a compact Kähler manifold M which has a homotopically nontrivial map to a compact Riemann surface with genus > 1, then one can construct a bounded holomorphic function, using arguments of Jost-Yau [**370**]. In particular, if M has a map to a product of Riemann surfaces with genus > 1 with nontrivial topological degree, the universal cover should have a good chance to be a bounded domain.

Of course, this kind of construction is based on the fact that holomorphic functions are harmonic. Certain rigidity based on curvature forced the converse to be true. For functions, the target space has no topology and rigidity is not expected. Bounded holomorphic functions can not be constructed by solving the Dirichlet problem unless some boundary condition is assumed. This would make good sense if the boundary has a nice CR structure. Indeed, for odd dimensional real submanifold in $\mathbb{C}^n$ which has maximal complex linear subspace on each tangent plane, Harvey-Lawson [**321, 322**] proved the remarkable theorem that they bound complex submanifolds. Unfortunately the boundary of a complete simply connected manifold with bounded negative curvature does not have a smooth boundary. It will be nice to define a CR structure on such a singular boundary. One may mention the remarkable work of Kuranishi [**405, 406, 407**] on embedding of an abstract CR structure.

Historically a motivation for the development of the $\bar{\partial}$ operator came from the Levi problem, which was solved by Morrey, Grauert and greatly improved by Kohn and Hörmander. Their methods are powerful in studying pseudoconvex manifolds.

In this regard, one may mention the conjecture of Shafarevich that the universal cover of an algebraic manifold is pseudoconvex. Many years ago, I conjectured that if the second homotopy group of the manifold is trivial, its universal cover can be embedded into a domain of some algebraic manifold where the covering transformations act on the domain by birational transformations. One may also mention the work of S. Frankel [**227**] on proving that an algebraic manifold is Hermitian symmetric if the universal cover is a convex domain in complex Euclidean space.

> **Comment**: The $\bar{\partial}$ operator is the fundamental operator in complex geometry. Classically it was used to solve the uniformization theorem, the Levi problem and the Corona problems. We have seen much progress on the higher dimensional generalizations of the first two problems. However, due to poor understanding of the construction of bounded holomorphic functions, we are far away from understanding the Corona problem in higher dimensional manifolds and many related geometric questions.

2.3.3. *Dirac operator.* A very important bundle is the bundle of spinors. The Dirac operator acting on spinors is the most mysterious but major geometric operator. Atiyah-Singer were the first mathematicians to study it in detail in geometry and by thoroughly understanding the Dirac operator, they were able to prove their celebrated index theorem [**20**]. On a Kähler manifold, the Dirac operator can be considered as a $\bar{\partial} + \bar{\partial}^*$ operator acting on differential forms with coefficients on the square root of the canonical line bundle. Atiyah-Singer's original proof can be traced back to the celebrated Riemann-Roch-Hirzebruch formula and the Hirzebruch index formula. The formulas of Gauss-Bonnet-Chern and Atiyah-Singer-Hirzebruch should certainly be considered as the most fundamental identities in geometry. The vanishing theorem of Lichnerowicz [**453**] on harmonic spinors over spin manifolds with positive scalar curvature gives strong information. Through the Atiyah-Singer index theorem, it gives the vanishing theorem for the $\hat{A}$-genus and the α invariants for spin manifolds with positive scalar curvature. The method was later sharpened by Hitchin [**331**] to prove that every Einstein metric over $K3-$surfaces must be Kähler and Ricci flat. An effective use of Lichnerowicz formula for a spin$_{\mathbb{C}}$ structure for a four dimensional manifold is important for Seiberg-Witten theory, which couples the Dirac operator with a complex line bundle. Lawson-Yau [**413**] were able to use Lichnerowicz's work coupled with Hitchin's work to prove a large class of smooth manifolds have no smooth non-Abelian group action and, by using modular forms, K.F. Liu proved a loop space analogue of the Lawson-Yau's theorem for the vanishing of the Witten genus in [**467**].

On the basis of the surgery result of Schoen-Yau [**597, 600**] and Gromov-Lawson [**290, 291**], one expects that a suitable converse to Lichnerowicz's theorem exists. The chief result is that surgery on spheres with codimension ≥ 3 preserves a class of metrics with positive scalar curvature. Once geometric surgery is proved, standard works on cobordism theory allow one to deduce existence results for simply connected manifolds with positive scalar curvature. The best work in this direction is due to Stolz [**641**] who gave a complete answer in the case of simply connected manifolds with dimension greater then 4. I also suggested the possibility of performing surgery on an asymptotic hyperbolic manifold with conformal boundary whose scalar curvature is positive. This is related to the recent work of Witten-Yau [**722**] on the connectedness of the conformal boundary.

The study of metrics with positive scalar curvature is the first important step in understanding the positive mass conjecture in general relativity. Schoen-Yau [**598, 602**] gave the first proof using ideas of minimal surfaces. Three years later, Witten [**716**] gave a proof using harmonic spinors. Both approaches have been fundamental to questions related to mass and other conserved quantities in general relativity. In the other direction, Schoen-Yau [**600**] generalized their argument in 1979 to find topological obstructions for higher dimensional manifolds with positive scalar curvature. Subsequently Gromov-Lawson [**290, 291**] observed that the Lichnerowicz theorem can be coupled with a fundamental group and give topological obstructions for a metric with positive scalar curvature. This work was related to the Novikov conjecture where many authors, including Lusztig [**483**], Rosenberg [**571**], Weinberger [**712**] and G.L. Yu [**751**] made contributions.

Besides its importance in demonstrating the stability of Minkowski spacetime, the positive mass conjecture was used by Schoen [**590**] in a remarkable manner to finish the proof of the Yamabe problem where Trudinger [**686**] and Aubin [**21**] made substantial contributions.

> **Comment**: The Dirac operator is perhaps one of the most mysterious operators in geometry. When it is twisted with other bundles, it gives the symbol of all first order elliptic operators. When it couples with a complex line bundle it gives the Seiberg-Witten theory which provides powerful information for four manifolds. On the other hand, there were two different methods to study metrics with positive scalar

> curvature. It should be fruitful to combine both methods: the method of Dirac operator and the method of minimal submanifolds.

2.3.4. *First order operator twisted by vector fields or endomorphisms of bundles.* Given a vector field X on a manifold, we can consider the complex of differential forms ω so that $L_X\omega = 0$. On such a complex, $d + \iota_X$ defines a differential and the resulting cohomology is called equivariant cohomology.

During the seventies, Bott [**72**] and Atiyah-Bott [**12**] developed the localization formula for equivariant cohomology. Both the concepts of a moment map and equivariant cohomology have become very important tools for computations of various geometric quantities, especially Chern numbers of natural bundles. The famous work of Atiyah, Guillemin-Sternberg on the convexity of the image of the moment map gives a strong application of equivariant cohomology to toric geometry. The formula of Duistermaat-Heckman [**208**] played an important role in motivation for evaluation of path integrals. These works have been used by Jeffrey and Kirwan [**360**] and by K.F. Liu and his coauthors on several topics: the mirror principle (Lian-Liu-Yau [**449, 450, 451, 452**]), topological vertex (Li-Liu-Liu-Zhou [**430**]), etc. The idea of applying localization to enumerative geometry was initiated by Kontsevich [**393**] and later by Givental [**257**] and Lian-Liu-Yau [**449**] independently. (Lian-Liu-Yau [**449**] formulated a functorial localization formula which has been fundamental for various calculations in mirror geometry.) These works solve the identities conjectured by Candelas et al [**99**] based on mirror symmetry, and provide good examples of the ways in which conformal field theory can be a source of inspiration when looking at classical problems in mathematics.

If we twist the $\bar{\partial}$ operator with an endomorphism valued holomorphic one form s so that $s \circ s = 0$, it gives rise to a complex $\bar{\partial} + s$. This was the Higgs theory initiated by Hitchin [**332**] and studied extensively by Simpson [**621**]. There is extensive work of Zuo Kang and Jost-Zuo (see [**759**]) on Higgs theory and representation of fundamental groups of algebraic manifold.

In string theory, there is a three form H and the cohomology of $d^c + H$ has not been well understood. It would be interesting to develop a deeper understanding of such twisted cohomology and its localization.

> **Comment**: The idea of deforming a de Rham operator by twisting with some other zero order operators has given powerful information to geometry. Witten's idea of the analytic proof of Morse theory is an example. Equivariant cohomology is another example. We expect to see more works in such directions.

2.3.5. *Spectrum and global geometry.* Weyl made a famous address in the early fifties. The title of his talk was *The Eigenvalue Problem Old and New*. He was excited by the work of Minakshisundaram and Pleijel which asserts that the zeta function $\zeta(s) = \sum_\lambda \lambda^{-s}$, where λ are eigenvalues of the

Laplacian, not only makes sense for Re's large, but also has meromorphic extension to the whole complex s-plane, the position of whose poles could be described explicitly. In particular, it is analytic near $s = 0$. Formally $\frac{d\zeta(s)}{ds}\mid_{s=0}$ can be viewed as $-\log\det(\Delta)$. This gives a definition of determinant of Laplacian which entered into the fundamental work of Ray-Singer relating Reidemeister's combinational invariant of a manifold with analytic torsion defined by determinants of the Laplacians acting on differential forms of various degrees. Other application of zeta function expressed in terms of kernel is the calculation of the asymptotic growth of eigenvalues in terms of volume of the manifold. Tauberian type theorem is needed.

This initiated the subject of finding a formula to relate spectrum of manifolds with their global geometry. Atiyah and Singer [**20**] were the most important contributors to this beautiful subject. Atiyah-Bott-Patodi [**13**] applied the heat kernel expansion to a proof of the local index theorem where Gilkey [**256**] also made an important contributions. Atiyah-Patodi-Singer [**17, 18, 19**] initiated the study of spectrum flow and gave important global spectral invariants on odd dimensional manifolds. These global invariants become boundary terms for the L^2-index theorem developed by Atiyah-Donnelly-Singer [**14**] and Mark Stern [**640**]. (A method of Callias [**96**] has been used for such calculations.) Witten [**717, 718**] has introduced supersymmetry and analytic deformation of the de Rham complex to Morse theory, and thereby revealed a new aspect of the connection between global geometry and theoretical physics. Witten's work has been generalized by Demailly [**190**] and Bismut-Zhang [**56, 57**] to study the holomorphic Morse inequality and analytic torsion. Novikov [**541**] also studied Morse theory for one forms. Witten's work on Morse theory inspired the work of Floer (see, e.g., [**224, 225, 226**]) who used his ideas in Floer cohomology to prove Arnold's conjecture in the case where the manifold has vanishing higher homotopic group. Floer's theory is related to knot theory (through Chern-Simon's theory [**152**]) on three manifolds. Atiyah, Donaldson, Taubes, Dan Freed, P. Braam, and others (see, e.g., [**10, 658, 77, 229**]) all contributed to this subject. Fukaya-Ono [**241**], Oh [**544**], Kontsevich [**394**], Hofer-Wysocki-Zehnder [**341**], G. Liu-Tian [**466**], all studied such a theory in symplectic geometry. Some part of Arnold's conjecture on fixed points of groups acting on symplectic manifolds was claimed to be proven. But a satisfactory proof has not been forthcoming.

One should also mention here the very important work of Cheeger [**115**] and Müller [**533**] in which they verify the conjecture of Ray-Singer equating analytic torsion with the combinational torsion of the manifold. The fundamental idea of Ray-Singer [**567**] on holomorphic torsion is still being vigorously developed. It appeared in the beautiful work of Vafa et al [**50**]. Many more works on analytic torsion were advanced by Quillen, Todorov, Kontsevich, Borcherds, Bismut, Lott, Zhang, and Z.Q. Lu (see [**55**] and its reference, [**365**], [**64, 65**]). The local version of the index theorem by

Atiyah-Bott-Patodi [**13**] was later extended by Bismut [**53**] to an index theorem for a family of elliptic operators. However, the pushed forward Chern forms have not been calculated and the formula has not been used effectively. (The local index argument dates back to the foundational work of McKean-Singer [**499**] where methods were developed to calculate coefficients of heat kernel expansion.) The study of elliptic genus by Witten [**719**], Bott-Taubes [**73**], Taubes [**657**], K.F. Liu [**468**] and M. Hopkins [**346**] has built a bridge between topology and modular form.

> **Comment**: The subject of relating the spectrum to global topology is extremely rich. It is likely that we have only touched part of this rich subject. The deformation of spectrum associated with the deformation of geometric structure is always a fascinating subject. Global invariants are created by spectral flows. Determinants of elliptic operators are introduced to understand measures of infinite dimensional space. Geometric invariants that are created by asymptotic expansion of heat or wave kernels are in general not well understood. It will be a long time before we have a much better understanding of the global behavior of spectrum.

3. Mappings between manifolds and rigidity of geometric structures

There is a need to exhibit a geometric structure in a simpler space: hence we embed algebraic manifolds into complex projective space, we isometrically embed a Riemannian manifold into Euclidean space and we classify structures such as bundles by studying maps into Grassmannian.

We are also interested in probing the structure of a manifold by mapping Riemann surfaces inside the manifold, an important example being holomorphic curves in algebraic manifolds. Of course, we are also interested in maps that can be used to compare the geometric structures of different manifolds.

3.1. Embedding.

3.1.1. *Embedding theorems.* Holomorphic sections of holomorphic line bundles have always been important in algebraic geometry. The Riemann-Roch formula coupled with vanishing theorems gave very powerful existence results for sections of line bundles. The Kodaira embedding theorem [**391**] which said that every Hodge manifold is projective has initiated the theory of holomorphic embedding of Kähler manifolds. For example, Hirzebruch-Kodaira [**330**] proved that every odd (complex) dimensional Kähler manifold diffeomorphic to projective space is biholomorphic to projective space. (I proved the same statement for even dimensional Kähler manifolds based on Kähler Einstein metric.)

Given an orthonormal basis of holomorphic sections of a very ample line bundle, we can embed the manifold into projective space. The induced metric is the Bergman metric associated with the line bundle. Note that the original definition of the Bergman metric used the canonical line bundle and L^2-holomorphic sections.

In the process of understanding the relation between stability of a manifold and the existence of the Kähler Einstein metric, I [**741**] proposed that every Hodge metric can be approximated by the Bergman metric as long as we allow the power of the line bundle to be large. Following the ideas of the paper of Siu-Yau [**632**], Tian [**676**] proved the C^2 convergence in his thesis under my guidance. My other student W. D. Ruan [**575**] then proved C^∞ convergence in his thesis. This work was followed by Lu [**479**], Zelditch [**752**] and Catlin [**110**] who observed that the asymptotic expansion of the kernel function follows from some rather standard expressions of the Szegö kernel, going back to Fefferman [**220**] and Boutet de Monvel-Sjöstrand [**75**] on the circle bundle associated with the holomorphic line bundle over the Kähler manifold. Recently, Dai, Liu, Ma and Marinescu [**182**] [**488**] obtained the asymptotic expansion of the kernel function by using the heat kernel method, and gave a general way to compute the coefficients, thus also extended it to symplectic and orbifold cases.

Kodaira's proof of embedding Hodge manifolds by the sufficiently high power of a positive line bundle is not effective. Matsusaka [**493**] and later Kollár [**392**], Siu [**628**] were able to provide effective estimate of the power. Demailly [**192, 193**] and Siu [**628, 630**] made a significant contribution toward the solution of the famous Fujita conjecture [**237**] (see also Ein and Lazarsfeld [**213**]). Siu's powerful method also leads to a proof of the deformation invariance of plurigenera of algebraic manifolds [**629**]. It should be noticed that the extension theorem of Ohsawa-Takegoshi played an important role in this last work of Siu.

> **Comment**: The idea of embedding a geometric structure is clearly important because once they are put in the same space, we can compare them and study the moduli space of the geometric structure. For example, one can define Chow coordinate of a projective manifold and we can study various concepts of geometric stability of these structures. However, there is no natural universal space of Kähler manifolds or complex manifolds as we may not have a positive holomorphic line bundle over such manifolds to embed into complex projective space. In a similar vein, it will be nice to find a universal space for symplectic manifolds.

3.1.2. *Compactification.* The problem of compactification of the manifold dates back to Siegel, Satake, Baily-Borel [**26**] and Borel-Serre [**67**]. They are important for representation theory, for algebraic geometry and for number theory.

For geometry of non-compact manifolds, we like to control behavior of differentiable forms at infinity. A good exhaustion function is needed.

Construction of a proper exhaustion function with a bounded Hessian on a complete manifold with bounded curvature was achieved by Schoen-Yau [**606**] in 1983 in our lectures in Princeton. Based on this exhaustion function, M. Dafermos [**180**] was able to give a transparent proof of the theorem of Cheeger-Gromov [**121**] that such manifolds admit an exhaustion by compact hypersurfaces with bounded second fundamental form. Such exhaustions are useful to understand characteristic forms on noncompact manifolds as the boundary term can be controlled by the second fundamental form of the hypersurfaces.

After my work with Siu [**634**] on compactification of a strongly, negatively curved Kähler manifold with finite volume, I proposed that every complete Kähler manifold with bounded curvature, finite volume and finite topology should be compactifiable to be a compact complex variety. I suggested this problem to Mok and Zhong in 1982 who did significant work [**528**] in this direction. (The compactification by Mok-Zhong is not canonical and it is desirable to find an algebraic geometric analogue of Borel-Baily compactification [**26**] so that we can study the L^2-cohomology in terms of the intersection cohomology of the compactification.) Recall that the important conjecture of Zucker on identifying L^2-cohomology with the intersection cohomology of the Borel-Baily compactification was settled by Saper-Stern [**585**] and Looijenga [**476**]. (Intersection cohomology was introduced by Goresky-MacPherson [**265, 266**]. It is a topological concept and hence the Zucker conjecture gives a topological meaning of the L^2-cohomology.) It would be nice to find compactification for algebraic varieties so that a suitable form of intersection cohomology can be used to understand L^2 cohomology. Goresky-Harder-MacPherson [**264**] and Saper [**584**] have contributed a lot toward this kind of question. For moduli space of bundles, or polarized projective structures, compactification means studying of degeneration of these structures in a suitable canonical manner. For algebraic curves, there is Deligne-Mumford compactification [**189**] which has played a fundamental role in understanding algebraic curves. Geometric invariant theory (see [**534**]) gives a powerful method to introduce the concept of stable structures. Semi-stable structures can give points at infinity. The compactification based on the geometric invariant theory for moduli space of surfaces of the general type was done by Gieseker [**254**]. For a higher dimension, this was done by Viehweg [**694**]. Detailed analysis of the divisors at infinity is still missing.

> **Comment**: Compactification of a manifold is very much related to the embedding problem. One needs to construct functions or sections of bundles near infinity. For the moduli space of geometric structures, it amounts to study of degeneration of the structures canonically, e.g., the degeneration of Hermitian Yang-Mills connections and Kähler Einstein metrics.

3.1.3. *Isometric embedding.* Given a metric tensor on a manifold, the problem of isometric embedding is equivalent to find enough functions $f_1, ...,$ f_N so that the metric can be written as $\sum (df_i)^2$. Much work was accomplished for two dimensional surfaces as was mentioned in section 2.1.2. Isometric embedding for the general dimension was solved by the famous work of J. Nash [**536, 537**]. Nash used his famous implicit function theorem which depends on various smoothing operators to gain derivatives. In a remarkable work, Günther [**307**] was able to avoid the Nash procedure. He used only the standard Hölder regularity estimate for the Laplacian to reproduce the Nash isometric embedding with the same regularity result. In his book [**287**], Gromov was able to lower the codimension of the work of Nash. He called his method the h-principle.

When the dimension of the manifold is n, the expected dimension of the Euclidean space for the manifold to be isometrically embedded is $\frac{n(n+1)}{2}$. It is important to understand manifolds isometrically embedded into Euclidean space with this optimal dimension. Only in such a dimension does it make sense to talk about rigidity questions. It remains a major open problem whether one can find a nontrivial smooth family of isometric embeddings of a closed manifold into Euclidean space with an optimal dimension. Such a nontrivial family was found for a polyhedron in Euclidean three space by Connelly [**176**]. For a general manifold, it is desirable to find a canonical isometric embedding into a given Euclidean space by minimizing the L^2 norm of its mean curvature within the space of isometric embeddings.

Chern told me that he and H. Lewy studied local isometric embedding of a three manifold into six dimensional Euclidean space. But they didn't have any publication on it. The major work was done by E. Berger, Bryant, Griffiths and Yang [**85**], [**47**]. They showed that a generic three dimensional embedding system is strictly hyperbolic, and the generic four dimensional system is a real principal type. Local existence is true for a generic metric using a hyperbolic operator and the Nash-Moser implicit function theorem.

If the target space of isometric embedding is a linear space with indefinite metric, it is possible that the problem is easier. For example, by a theorem of Pogorelov [**561, 562**], any metric on the two dimensional sphere can be isometrically embedded into a three dimensional hyperbolic space-form (where the sectional curvature may be a large negative constant). Hence it can always be embedded into the hyperboloid of the Minkowski spacetime. This statement may also be true for surfaces with higher genus. The fundamental group may cause obstruction, hence the first step should be an attempt to canonically embed any complete metric (with bounded curvature) on a simply connected surface into a three dimensional hyperbolic space form. It should be also very interesting to study the rigidity problem of a space-like surface in Minkowski spacetime. Besides requesting the metric to be the induced metric, we shall need one more equation. Such an equation should be related to the second fundamental form. A candidate appeared in the work of M. Liu-Yau [**464, 465**] on the quasi-local mass in general relativity.

In the other direction, Calabi found the condition for a Kähler metric to be isometrically and holomorphically embedded into Hilbert space with an indefinite signature. In the course of his investigation, he introduced some kind of distance function that can be defined by the Kähler potential and enjoys many interesting properties. Calabi's work in this direction which should be relevant to the flat coordinate appeared in the recent works of Vafa et al [**50**].

> **Comment**: The theory of isometric embedding is a classical subject. But our knowledge is still rather limited, especially in dimension greater than three. Many difficult problems are related to nonlinear mixed type equation or hyperbolic differential systems over a closed manifold.

3.2. Rigidity of harmonic maps with negative curvature. One can define the energy of maps between manifolds and the critical maps are called harmonic maps. In 1964, Eells-Sampson [**212**] and Al'ber [**3**] independently proved the existence of such maps in their homotopy class if the image manifold has a non-positive curvature.

When I was working on manifolds with non-positive curvature, I realized that it is possible to use harmonic map to reprove some of the theorems in my thesis. I was convinced that it is possible to use harmonic maps to study rigidity questions in geometry such as Mostow's theorem [**531**]. In 1976, I proved the Calabi conjecture and applied the newly proved existence of the Kähler Einstein metric and the Mostow rigidity theorem to prove uniqueness of a complex structure on the quotient of the ball [**735**]. Motivated by this theorem, I proposed to use the harmonic map to prove the rigidity of a complex structure for Kähler manifolds with strongly negative sectional curvature. I proposed this to Siu who carried out the idea when the image manifold satisfies a stronger negative curvature condition [**625**]. Jost-Yau [**368**] proved that for harmonic maps into manifolds with non-positive curvature, the fibers give rise to holomorphic foliations even when the map is not holomorphic. Such a work was found to be useful in the work of Corlette, Simpson et al.

A further result was obtained by Jost-Yau [**371**] and Mok-Siu-Yeung [**525**] on the proof of the superrigidity theorem of Margulis [**490**], improving an earlier result of Corlette [**177**] who proved superrigidity for a certain rank one locally symmetric space. Complete understanding of superrigidity for the quotient of a complex ball is not yet available. One needs to find more structures for harmonic maps which reflect the underlying structure of the manifold. The analytic proof of super-rigidity was based on an argument of Matsushima [**495**] as was suggested by Calabi. (This was a topic discussed by Calabi in the special year on geometry in the Institute for Advanced Study.)

The discrete analogue of harmonic maps is also important. When the image manifold is a metric space, there are works by Gromov-Schoen [**292**], Korevaar-Schoen [**399**] and Jost [**367**]. Margulis knew that the super-rigidity for both the continuous and the discrete case is enough to prove Selberg's conjecture for the arithmeticity of lattices in groups with rank ≥ 2. Unfortunately, the analytic argument mentioned above only works if the lattices are cocompact as it is difficult to find a degree one smooth map with finite energy for non-cocompact lattices. Harmonic maps into a tree have given interesting applications to group theory. When the domain manifold is a simplicial complex, there are articles by Ballmann-Śwwiątkowski [**29**] and M.T. Wang [**703, 704**], where they introduce maps from complices which are generalizations of buildings. They also generalized the work of H. Garland [**245**] on the vanishing of the cohomology group for p-adic buildings.

Using the concept of the center of gravity, Besson-Courtois-Gallot [**52**] give a metric rigidity theorem for rank one locally symmetric space. They also proved a rigidity theorem for manifolds with negative curvature: if the fundamental group can be split as a nontrivial free product over some other group C, the manifold can be split along a totally geodesic submanifold with the fundamental group C.

> **Comment**: The harmonic map gives the first step in matching geometric structures of different manifolds. Eells-Sampson derived it from the variational principle. One can also use different elliptic operators to define maps which satisfy elliptic equations. Higher dimensional applications are mostly based on the assumption that the image manifold has a metric with non-positive curvature. In such a case, existence is easier and uniqueness (as shown by Hartman) is also true. Up to now, significant results on higher dimensional harmonic maps are based on such assumptions. Generalization to Kähler manifold should be reasonable. The second homotopic group should play a role as one may look at it as a generalization of the work of Sacks-Uhlenbeck. It may be possible to use harmonic maps to study the moduli of geometric structure on a fixed manifold as was done by Michael Wolf for Riemann surfaces. It will also be nice to see how a harmonic map can be used to compare graphs.

3.3. Holomorphic maps. The works of Liouville, Picard, Schwarz-Pick and Ahlfors show the importance of hyperbolic complex analysis. Grauèrt-Reckziegel [**268**] generalized this kind of analysis to higher dimensional complex manifolds. Kobayashi [**388**] and H. Wu [**726**] put this theory in an elegant setting. Kobayashi introduced the concept of hyperbolic complex manifolds. Its elegant formulation has been influential. An important application of the negative curvature metric is the extension theorem for holomorphic maps, as was achieved by the work of Griffiths-Schmid [**275**] on maps to a period domain and by the extension theorem of Borel [**66**] on compactification of Hermitian symmetric space. A major question was Lang's conjecture: on an algebraic manifold of a general type, there exists a proper subvariety such that the image of any holomorphic map from $\mathbb{C}$ must be a subset of this subvariety. It has deep arithmetic geometric meaning. In terms of the Kobayashi metric, it says that the Kobayashi metric is nonzero on a Zariski open set. Many works were done towards subvarieties of Abelian variety by Bloch, Green-Griffiths, Kobayashi-Ochiai, Voitag and Faltings. For generic hypersurfaces in $\mathbb{C}P^n$, there is work by Siu [**631**]. They developed the tool of jet differentials and meromorphic connections. For algebraic surfaces with $C_1^2 > 2C_2$, Lu-Yau [**477**] proved Lang's conjecture, based on the ideas of Bogomolov.

> **Comment**: Holomorphic maps have been studied for a long time. There is no general method to construct such maps based on the knowledge of topology alone, except the harmonic map approach proposed by me and carried out by Siu, Jost-Yau and others. But the approach is effective only for manifolds with negative curvature. For rigidity questions, the most interesting manifolds are Kähler manifolds with non-positive Ricci curvature, which give the major chunk of algebraic manifolds of a general type. The Kähler-Einstein metric should provide tools to study such problems. Is there any intrinsic way, based on the metric, to find the largest subvariety where the image of all holomorphic maps from the complex line lie? Deformation theory of such a subvariety should be interesting. There is also the question of when the holomorphic image of the complex line will intersect a divisor. Cheng and I did find good conditions for the complement of a divisor to admit the complete Kähler-Einstein metric. For such a geometry, the holomorphic line should either intersect the divisor or a subset of some subvarieties. These kinds of questions are very much related to arithmetic questions if the manifolds are defined over number fields.

3.4. Harmonic maps from two dimensional surfaces and pseudo-holomorphic curves. Harmonic maps behave especially well for Riemann surface. Morrey was the first one who solved the Dirichlet problem for energy minimizing harmonic map into any Riemannian manifold.

Another major breakthrough was made by Sacks-Uhlenbeck [**581**] in 1978 where they constructed minimal spheres in Riemannian manifolds representing elements in the second homotopy group using a beautiful extension theorem of a harmonic map at an isolated point. By pushing their method further, Siu-Yau [**633**] studied the bubbling process for the harmonic map and made use of it to prove a stable harmonic map must be holomorphic under curvature assumptions. As a consequence, they proved the famous conjecture of Frankel that a Kähler manifold with positive bisectional curvature is $\mathbb{C}P^n$, as was discussed in Section [2.3.2].

Gromov [**286**] then realized that a pseudoholomorphic curve for an almost complex structure can be used in a similar way to prove rigidity of a symplectic structure on $\mathbb{C}P^n$. The bubbling process mentioned above was sharpened further to give compactification of the moduli space of pseudoholomorphic maps by Ye [**749**] and Parker-Wolfson [**549**]. Based on these ideas, Kontsevich [**393**] introduced the concept of stable maps and the compactification of their moduli spaces.

The formal definitions of Gromov-Witten invariants and quantum cohomology were based on these developments and the ideas of physicists. For example, quantum cohomology was initiated by Vafa (see, e.g., [**692**]) and his coauthors (the name was suggested by Greene and me). Associativity in quantum cohomology was due to four physicists WDVV [**720, 194**]. The mathematical treatment (done by Ruan [**579**] and subsequently by Ruan-Tian [**580**]) followed the gluing ideas of the physicists. Ruan-Tian made use of the ideas of Taubes [**656**]. But important points were overlooked. A. Zinger [**756, 757**] has recently completed these arguments.

In close analogy with Donaldson's theory, one needs to introduce the concept of virtual cycle in the moduli space of stable maps. The algebraic setting of such a concept is deeper than the symplectic case and is more relevant to the development for algebraic geometry. The major idea was conceived by Jun Li who also did the algebraic geometric counterpart of Donaldson's theory (see [**426, 431**]). (The same comment applies to the concept of the relative Gromov-Witten invariant, where Jun Li made the vital contribution in the algebraic setting [**428, 429**].) The symplectic version of Li-Tian [**432**] ignores difficulties, many of which were completed recently by A. Zinger [**756, 757**].

Sacks-Uhlenbeck studied harmonic maps from higher genus Riemann surfaces. Independently, Schoen-Yau [**601**] studied the concept of the action of an L_1^2 map on the fundamental group of a manifold. It was used to prove the existence of a harmonic map with prescribed action on the fundamental group. Jost-Yau [**369**] generalized such action on fundamental group to a more general setting which allows the domain manifold to be higher dimensional. Recently F. H. Lin developed this idea further [**460**]. He studied extensively geometric measure theory on the space of maps (see, e.g., [**457, 459**]). The action on the second homotopy group is much more difficult to understand. I think there should exist a harmonic map with nontrivial action on the second homotopic group if such a continuous map exists. Such an existence theorem will give interesting applications to Kähler geometry.

There is a supersymmetric version of harmonic maps studied by string theorists. This is obtained by coupling the map with Dirac spinors in different ways (which corresponds to different string theories). While this kind of world sheet theory is fundamental for the development of string theory, geometers have not paid much attention to the supersymmetric harmonic map. Interesting applications may be found. The most recent paper of Chen, Jost, Li and Wang [**134**] does address to a related problem where they studied the regularity and energy identities for Dirac-Harmonic maps.

> **Comment**: Maps from circle or Riemann surfaces into a Riemannian manifold give a good deal of information about the manifold. The capability to construct holomorphic or pseudo-holomorphic maps from spheres with low degree was the major reason that Mori, Siu-Yau and Taubes were able to prove the rigidity of algebraic or symplectic structures on the complex projective space. It will be desirable to find more ways to construct such maps from low genus curves to manifolds that are not of a rational type. Their moduli space can be used to produce various invariants. An outstanding problem is to understand the invariants on counting curves of a higher genus which appeared in the fundamental paper of Vafa et al [**50**].

3.5. Morse theory for maps and topological applications. The energy functional for maps from S^2 into a manifold does not quite give rise to Morse theory. But the perturbation method of Sacks-Uhlenbeck did provide enough information for Micallef-Moore [**513**] to prove some structure theorem for manifolds with positive isotropic curvature. (Micallef and Wang [**514**] then proved the vanishing of second Betti number in the even dimensional case. If the manifold is irreducible, has non-negative isotropic curvature and non-vanishing second Betti number, then they proved that its second Betti number equals to one and it is Kähler with positive first Chern class.)

If the image manifold has negative curvature, the theorem of Eells-Sampson [**212**] says that any map can be canonically deformed by the heat flow to a unique harmonic map. Hence the topology of the space of maps is given by the space of homomorphism between the fundamental groups of the manifolds. This gives some information of the topology of manifolds with negative curvature. Farrell and Jones [**218**] have done much deeper analysis on the differentiable structure of manifolds with negative curvature.

Schoen-Yau [**601**] exploited the uniqueness theorem for harmonic maps to demonstrate that only finite groups can act smoothly on a manifold which admits a non-zero degree map onto a compact manifold with negative curvature. The size of the finite group can also be controlled. If the image manifold has non-positive curvature, then the only compact continuous group actions are given by the torus.

The topology of the space of maps into Calabi-Yau manifolds should be very interesting for string theory. Sullivan [**648**] has developed an equivariant homology theory for loop space. It will be interesting to link such a theory with quantum cohomology when the manifold has a symplectic structure.

> **Comment**: Morse theory has been one of the most powerful tools in geometry and topology as it connects local to global information. One does not expect full Morse theory for harmonic maps as we have difficulty even proving their existence. However, if their existence can be proven, the perturbation technique may be used and powerful conclusions may be drawn.

3.6. Wave maps. In the early eighties, C.H. Gu [**296**] studied harmonic maps when the domain manifold is the two dimensional Minkowski spacetime. They are called wave maps. Unfortunately, good global theory took much longer to develop as there were not many good a priori estimates. This subject was studied extensively by Christodoulou, Klainerman, Tao, Tataru and M. Struwe (see, e.g., [**161, 385, 653, 654, 610**]). It is hoped that such theory may shed some light on Einstein equations.

> **Comment**: The geometric or physical meaning of wave maps should be studied. The problem of vibrating membrane gives a good motivation to study time-like minimal hypersurface in a Minkowski spacetime. One can study the vibration of a submanifold by looking into the minimal time-like hypersurface with the boundary given by the submanifold. It is a mystery how such vibrations can be related to the eigenvalues of the submanifold.

3.7. Integrable system. Classically, Bäcklund (1875) was able to find a nonlinear transformation to create a surface with constant curvature in $\mathbb{R}^3$ from another one. The nonlinear equation behind it is the Sine-Gordon equation. Then in 1965, Kruskal and Zabusky (see [**403**]) discovered solitons and subsequently in 1967, Gardner, Greene, Kruskal and Miura [**244**] discovered the inverse scatting method to solve the KdV equations. The subject of a completely integrable system became popular.

Uhlenbeck [**690**] used techniques from integrable systems to construct harmonic maps from S^2 to $U(n)$, Bryant [**83**] and Hitchin [**334**] also contributed to related constructions using twistor theory and spectral curves. These inspired Burstall, Ferus, Pedit and Pinkall [**89**] to construct harmonic maps from a torus to any compact symmetric space. In a series of papers, Terng and Uhlenbeck [**668, 669**] used loop group factorizations to solve the inverse scattering problem and to construct Bäcklund transformations for soliton equations, including Schrödinger maps from $\mathbb{R}^{1,1}$ to a Hermitian symmetric space. There have been recent attempts by Martin Schmidt [**589**] to use an integrable system to study the Willmore surface.

The integrable system also appeared naturally in several geometric questions such as the Schottky problem (see Mulase [**532**]) and the Witten conjecture on Chern numbers of bundles over moduli space of curves.

Geroch found the Backlund transformation for axially symmetric stationary solutions of Einstein equations. It will be nice to find such nonlinear transformations for more general geometric structures.

> **Comment**: It is always important to find an explicit solution to a nonlinear problem. Hopefully an integrable system can help us to understand general structures of geometry.

3.8. Regularity theory. The major work on regularity theory of harmonic maps in higher dimensions was done by Schoen-Uhlenbeck [**592, 593**]. (There is a weaker version due to Giaquinta-Giusti [**252**] and also the earlier work of Ladyzhenskaya-Ural'ceva and Hildebrandt-Kaul-Widman where the image manifolds for the maps are more restrictive.) Leon Simon (see [**620**]) made a deep contribution to the structure of harmonic maps or minimal subvarieties near their singularity. This was followed by F.H. Lin [**459**]. The following is still a fundamental problem: Are singularities of harmonic maps or minimal submanifolds stable when we perturb the metric of the manifolds? Presumably some of them are. Can we characterize them? How big is the codimension of generic singularities?

In the other direction Schoen-Yau [**596**] also proved that degree one harmonic maps are one to one if the image surface has a non-positive curvature. Results of this type work only for two dimensional surfaces. It will be nice to study the set where the Jacobian vanishes.

> **Comment**: There is a very rich theory of stable singularity for smooth maps. However, in most problems, we can only afford to deform certain background geometric structures, while the extremal objects are still constrained by the elliptic variational problem. Understanding this kind of stable singularity should play fundamental roles in geometry.

4. Submanifolds defined by variational principles

4.1. Teichmüller space. The totality of the pair of polarized Kähler manifolds with a homotopic equivalence to a fixed manifold gives rise to the Teichmüller space. For an algebraic curve, this is the classical Teichmüller space. This space is important for the construction of the mapping problem for minimal surfaces of a higher genus.

In fact, given a conformal structure on a Riemann surface Σ, a harmonic map from Σ to a fixed Riemannian manifold may minimize energy within a certain homotopy class. However, it may not be conformal and may not be a minimal surface. In order to obtain a minimal surface, we need to vary the conformal structure on Σ also. Since the space of conformal structures on a surface is not compact, one needs to make sure the minimum can be achieved.

If the map f induces an injection on the fundamental group of the domain surface, Schoen-Yau [**597**] proved the energy of the harmonic map is proper on the moduli space of conformal structure on this surface by making use of a theorem of Linda Keen [**381**]. Based on a theory of topology of the L_1^2 map, they proved the existence of incompressible minimal surfaces. As a product of this argument, it is possible to find a nice exhaustion function for the Teichmüller space. Michael Wolf [**723**] was able to use harmonic maps to give a compactification of Teichmüller space which he proved to be equivalent to the Thurston compactification. S. Wolpert studied extensively the behavior of the Weil-Petersson metric (see Wolpert's survey [**724**]). A remarkable theorem of Royden [**574**] says that the Teichmüller metric is the same as the Kobayashi metric. C. McMullen [**502**] introduced a new Kähler metric on the moduli space which can be used to demonstrate that the moduli space is hyperbolic in the sense of Gromov [**288**]. The great detail of comparison of various intrinsic metrics on the Teichmüller space had been a major problem [**741**]. It was accomplished recently in the works of Liu-Sun-Yau [**471, 472**]. Actually Liu-Sun-Yau introduced new metrics with bounded negative curvature and geometry and found the stability of the logarithmic cotangent bundle of the moduli spaces. Recently L. Habermann and J. Jost [**308, 309**] also studied the geometry of the Weil-Petersson metric associated to the Bergmann metric on the Riemann surface instead of the Poincaré metric.

> **Comment**: For a conformally invariant variational problem, Teichmüller space plays a fundamental role. It covers the moduli space of curves and in many ways behaves like a Hermitian symmetric space of noncompact type. Unfortunately, there is no good canonical realization of it as a pseudo-convex domain in Euclidean space. For example, we do not know whether it can be realized as a smooth domain or not.
>
> There is also Teichmüller space for other algebraic manifolds, such as Calabi-Yau manifolds. It is an important question in understanding their global behavior.

4.2. Classical minimal surfaces in Euclidean space. There is a long and rich history of minimal surfaces in Euclidean space. Recent contributions include works by Meeks, Osserman, Lawson, Gulliver, White, Hildebrandt, Rosenberg, Collin, Hoffman, Karcher, Ros, Colding, Minicozzi, Rodríguez, Nadirashvili and others (see the reference in Colding and Minicozzi's survey [**174**]) on embedded minimal surfaces in Euclidean space. They come close to classifying complete embedded minimal surfaces and give a good understanding of complete minimal surface in a bounded domain. For example, Meeks-Rosenberg [**503**] proved that the plane and helicoid are the only properly embedded simply connected minimal surfaces in $\mathbb{R}^3$.

Calabi also initiated the study of isometric embedding of Riemann surfaces into S^N as minimal surfaces. The geometry of minimal spheres and minimal torus was then pursued by many geometers [**107**], [**150**], [**83**], [**334**], [**410**].

> **Comment**: This is one of the most beautiful subjects in geometry where Riemann made important contributions. Classification of complete minimal surface is nearly accomplished. However, a similar problem for compact minimal surfaces in S^3 is far from being solved. It is also difficult to detect which set of disjoint Jordan curves can bound a connected minimal surface. The classification of moduli space of complete minimal surfaces with finite total curvature should be studied in detail.

4.3. Douglas-Morrey solution, embeddedness and application to topology of three manifolds. In a series of papers started in 1978, Meeks-Yau [**505, 506, 507, 508**] settled a classical conjecture that the Douglas solution for the Plateau problem is embedded if the boundary curve is a subset of a mean convex boundary. (One should note that Osserman [**546**] had already settled the old problem of non-existence of branched points for the Douglas solution while Gulliver [**306**] proved non-existence of false branched points.) We made use of the area minimizing property of minimal surfaces to prove these surfaces are equivariant with respect to the group action. Embedded surfaces which are equivariant play important roles for finite group actions on manifolds. Coupling with a theorem of Thurston, we can then prove the Smith conjecture [**748**] for cyclic groups acting on the spheres: that the set of fixed points is not a knotted curve.

The Douglas-Morrey solution of the Plateau problem is obtained by fixing the genus of the surfaces. However, it is difficult to minimize the area when the genus is allowed to be arbitrary large. This was settled by Hardt-Simon [**320**] by proving the boundary regularity of the varifold solution of the Plateau problem. In the other direction, Almgren-Simon [**7**] succeeded in minimizing the area among embedded disks with a given boundary in Euclidean space. The technique was used by Meeks-Simon-Yau [**504**] to prove the existence of embedded minimal spheres enclosing a fake ball. This theorem has been important to prove that the universal covering of an irreducible three manifold is irreducible. They also gave conditions for the existence of embedding minimal surfaces of a higher genus. This work was followed by topologists Freedman-Hass-Scott [**231**]. Pitts [**557**] used the mini-max argument for varifolds to prove the existence of an embedded minimal surfaces. Simon-Smith (unpublished) managed to prove the existence of an embedded minimax sphere for any metric on the three sphere. J. Jost [**366**] then extended it to find four mini-max spheres. Pitts-Rubinstein (see, e.g., [**558**]) continued to study such mini-max surfaces. Since such mini-max surfaces have Morse index one, I was interested in representing such a minimal surface as a Heegard splitting of the three manifolds. I estimated its genus based on the fact that the second eigenvalue of the stability operator is nonnegative. This argument (dates back to Szego-Hersch) is to map the surface conformally to S^2. Hence we can use three coordinate functions, orthogonal to the first eigenfunction, to be trial functions. The estimate gave an upper bound of the genus for mini-max surfaces in compact manifolds with positive scalar curvature. About twenty years ago, I was hoping to use such an estimate to control a Heegard genus as a way to prove Poincaré conjecture. While the program has not materialized, three manifold topologists did adapt the ideas of Meeks-Yau to handle combinational type minimal surfaces and gave applications in three manifold topology.

The most recent works of Colding and Minicozzi [**169, 170, 171, 172**] on lamination by minimal surfaces and estimates of minimal surfaces without the area bound are quite remarkable. They [**173**] made contributions to Hamilton's Ricci flow by bounding the total time for evolution. Part of the idea came from the above mentioned inequality.

> **Comment**: The application of minimal surface theory to three manifold topology is a very rich subject. However, one needs to have a deep understanding of the construction of minimal surfaces. For example, if minimal surfaces are constructed by the method of mini-max, one needs to know the relation of their Morse index to the dimension of the family of surfaces that we use to perform the procedure of mini-max. A detailed understanding may lead to a new proof of the Smale conjecture, as we may construct a minimal surface by a homotopic group of embeddings of surfaces. Conversely, topological methods should help us to classify closed minimal surfaces.

4.4. Surfaces related to classical relativity. Besides minimal surfaces, another important class of surfaces are surfaces with constant mean curvature and also surfaces that minimize the L^2-norm of the mean curvature. It is important to know the existence of such surfaces in a three dimensional manifold with nonnegative scalar curvature, as they are relevant to the questions in general relativity.

The existence of minimal spheres is related to the existence of black holes. The most effective method was developed by Schoen-Yau [**604**] where they [**599**] proved the existence theorem for the equation of Jang. It should be nice to find new methods to prove existence of stable minimal spheres. The extremum of the Hawking mass is related to minimization of the L^2 norm of mean curvature. Their existence and behavior have not been understood.

For surfaces with constant mean curvature, we have the concept of stability. (Fixing the volume it encloses, the second variation of area is nonnegative.) Making use of my work on eigenvalues with Peter Li, I proved with Christodoulou [**162**] that the Hawking mass of such a surface is positive. (This was part of my contribution to the proposed joint project with

Christodoulou-Klainerman which did not materialize.) This fact was used by Huisken and me [**354**] to prove uniqueness and the existence of foliation by constant mean curvature spheres for a three dimensional asymptotically flat manifold with positive mass. (We initiated this research in 1986. Ye studied our work and proved existence of similar foliations under various conditions, see [**750**].)

This foliation was used by Huisken and Yau [**354**] to give a canonical coordinate system at infinity. It defines the concept of center of gravity where important properties for general relativity are found. The most notable is that total linear momentum is equal to the total mass multiple with the velocity of the center of the mass. One expects to find good asymptotic properties of the tensors in general relativity along these canonical surfaces. We hope to find a good definition of angular momentum based on this concept of center of gravity so that global inequality like total mass can dominate the square norm of angular momentum.

The idea of using the foliation of surfaces satisfying various properties (constant Gauss curvature, for example) to study three manifolds in general relativity was first developed by R. Bartnik [**41**]. His idea of quasi-spherical foliation gives a good parametrization of a large class of metrics with positive scalar curvature.

Some of these ideas were used by Shi-Tam [**614**] to study quantities associated to spheres which bound three manifolds with positive scalar curvature. Such a quantity is realized to be the quasi-local mass of Brown-York [**82**]. At the same time, Melissa Liu and Yau [**464, 465**] were able to define a new quasi-local mass for general spacetimes in general relativity, where some of the ideas of Shi-Tam were used. Further works by M. T. Wang and myself generalized Liu-Yau's work by studying surfaces in hyperbolic space-form.

My interest in quasi-local mass dates back to the paper that I wrote with Schoen [**604**] on the existence of a black hole due to the condensation of matter. It is desirable to find a quasi-local mass which includes the effect of matter and the nonlinear effect of gravity. Hopefully one can prove that when such a mass is larger than a constant multiple of the square root of the area, a black hole forms. This has not been achieved.

> **Comment**: When surfaces theory appears in general relativity, we gain intuitions from both geometry and physics together. This is a fascinating subject.

4.5. Higher dimensional minimal subvarieties. Higher dimensional minimal subvarieties are very important for geometry. There are works by Federer-Fleming [**219**], Almgren [**6**] and Allard [**4**]. The attempt to prove the Bernstein conjecture, that minimal graphs are linear, was a strong drive for its development. Bombieri, De Giorgi and Giusti [**63**] found the famous counterexample to the Bernstein problem. It initiated a great deal of interest in the area minimizing cone (as a graph must be area minimizing).

Schoen-Simon-Yau [**591**] found a completely different approach to the proof of Bernstein problem in low dimensions. This paper on stable minimal hypersurfaces initiated many developments on curvature estimates for the codimension one stable hypersurfaces in higher dimension. There are also works by L. Simon with Caffarelli and Hardt [**91**] on constructing minimal hypersurfaces by deforming stable minimal cones. Recently N. Wickramasekera [**714, 715**] did some deep work on stable minimal (branched) hypersurfaces which generalizes Schoen-Simon-Yau.

Michael-Simon [**515**] proved the Sobolev inequality and mean valued inequalities for such manifolds. This enables one to apply the classical argument of harmonic analysis to minimal submanifolds. For a minimal graph, Bombieri-Giusti [**62**] used ideas of De Giorgi-Nash to prove gradient estimates of the graph. N. Korevaar [**396**] was able to reprove this gradient estimate based on the maximal principle.

The best regularity result for higher codimension was done by F. Almgren [**6**] when he proved that for any area minimizing variety, the singular set has the codimension of at least two. How such a result can be used for geometry remains to be seen.

It was observed by Schoen-Yau [**597**] that for a closed stable minimal hypersurface in a manifold with positive scalar curvature, the first eigenfunction of the second variational operator can be used to conformally deform the metric so that the scalar curvature is positive. This provides an induction process to study manifolds with a positive scalar curvature. For example, if the manifold admits a nonzero degree map to the torus, one can then construct stable minimal hypersurfaces inductively until we find a two dimensional surface with higher genus which cannot support a metric with positive scalar curvature. At this moment, the argument encounters difficulty for dimensions greater than seven as we may have problems of singularity. In any case, we did apply the argument to prove the positive action conjecture in general relativity. The question of which type of singularities for minimal subvariety are generic under metric perturbation remains a major one for the theory of minimal submanifolds.

Perhaps the most important possible application of the theory of minimal submanifolds is the Hodge conjecture: whether a multiple of a (p,p) type integral cohomology class in a projective manifold can be represented by an algebraic cycle. Lawson made an attempt by combining a result of Lawson-Simons [**411**] and work of J. King [**383**] and Harvey-Shiffman [**324**]. (Lawson-Simons proved that currents in $\mathbb{C}P^n$ which are minimizing with respect to the projective group action are complex subvarieties.) The problem of how to use the hypothesis of (p,p) type has been difficult. In general, the algebraic cycles are not effective. This creates difficulties for analytic methods. The work of King [**383**] and Shiffman [**615**] on complex currents may be relevant.

Perhaps one should generalize the Hodge conjecture to include general (p, q) classes, as it is possible that every integral cycle in $\bigoplus_{i=-k}^{k} H^{p-i,p+i}$ is rationally homologous to an algebraic sum of minimal varieties such that there is a $p - k$ dimensional complex space in the tangent space for almost every point of the variety: it may be important to assume the metric to be canonical, e.g. the Kähler Einstein metric.

A dual question is how to represent a homology class by Lagrangian cycles which are minimal submanifolds also. When the manifold is Calabi-Yau, these are special Lagrangian cycles. Since they are supposed to be dual to holomorphic cycles, there should be an analogue of the Hodge conjecture. For example, if $\dim_{\mathbb{C}} M = n$ is odd, any integral element in $\bigoplus_{i+j=n} H^{i,j}$ should be representable by special Lagrangian cycles up to a rational multiple provided the cup product of it with the Kähler class is zero.

A very much related question is: if the Chern classes of a complex vector bundle are of (p, p) type, does the vector bundle, after adding a holomorphic vector bundle, admit a holomorphic structure? If the above generalization of the Hodge conjecture holds, there should be a similar generalization for the vector bundle. It should also be noted that Voisin [**696**] observed that Chern classes of all holomorphic bundles do not necessarily generate all rational (p, p) classes. On the other hand, the Kähler manifold that she constructed is not projective.

These questions had a lot more success for four dimensional symplectic manifolds by the work of Taubes both on the existence of pseudoholomorphic curves [**665**] and on the existence of anti-self-dual connections [**655, 656**]. On a Kähler surface, anti-self-dual connections are Hermitian connections for a holomorphic vector bundle. In particular, Taubes gave a method to construct holomorphic vector bundles over Kähler surfaces. Unfortunately this theorem does not provide much information on the Hodge conjecture as it follows from Lefschetz theorem in this dimension.

Another important class of minimal varieties is the class of special Lagrangian cycles in Calabi-Yau manifolds. Such cycles were first developed by Harvey-Lawson [**323**] in connection to calibrated geometry. Major works were done by Schoen-Wolfson [**594**], Yng-Ing Lee [**417**] and Butscher [**90**]. One expects Lagrangian cycles to be mirror to holomorphic bundles and special Lagrangian cycles to be mirror to Hermitian-Yang-Mills connections. Hence by the Donaldson-Uhlenbeck-Yau theorem, it is related to stability. The concept of stability for Lagrangian cycles was discussed by Joyce and Thomas. Since the Yang-Mills flow for Hermitian connection exists for all time, Thomas-Yau [**671**] suggested an analogy with the mean curvature flow for Lagrangian cycles. For stable Lagrangian cycles, mean curvature flow should converge to special Lagrangian cycles. See M.T. Wang [**705, 706**], Smoczyk [**636**] and Smoczyk-Wang [**637**]. The geometry of mirror symmetry was explained by Strominger-Yau-Zaslow in [**643**] using a family of special Lagrangian tori. There are other manifolds with special holonomy group. They have similar calibrated submanifolds. Conan Leung has contributed to studies of such manifolds and their mirrors (see, e.g., [**420, 421**]).

Submanifolds of space forms are called isoparametric if the normal bundle is flat and the principal curvatures are constants along parallel normal fields. These were studied by E. Cartan [**108**]. Minimal submanifolds with constant scalar curvature are believed to be isoparametric surfaces. There is work done by Lawson [**409**], Chern-de Carmo-Kobayashi [**151**] and Peng-Terng [**550**]. Recently there have been works by Terng and Thorbergsson (see Terng's survey [**667**] and Thorbergsson [**672**]). Terng [**666**] related isometric embedded hyperbolic spaces in Euclidean space to soliton theory. A theory of Lax pair and loop groups related to geometry has been developed.

> **Comment**: The theory of higher dimensional minimal submanifolds is one of the deepest subjects in geometry. Unfortunately our knowledge of the subject is not mature enough to give applications to solve outstanding problems in geometry, such as the Hodge conjecture. But the future is bright.

4.6. Geometric flows. The major geometric flows are flows of submanifold driven by mean curvature, gauss curvature, inverse mean curvature. Flows that change geometric structures are Ricci flows and Einstein flow.

Mean curvature flow for varifolds was initiated by Brakke [**78**]. The level set approach was studied by many people: S. Osher, L. Evans, Giga, etc (see [**545, 217, 136**]). Huisken [**349, 350**] did the first important work when the initial surface is convex. His recent work with Sinestrari [**352, 353**] on mean convex surfaces is remarkable and gives a good understanding of the structure of singularities of mean curvature flow. Mean curvature flow has many geometric applications. For example, the work of Huisken-Yau mentioned in 4.4 was achieved by mean curvature flow. Mean curvature flow for spacelike hypersurfaces in Lorentzian manifolds should be very interesting. Ecker [**211**] did interesting work in this direction. It will be nice to find the Li-Yau type estimate for such flows.

The inverse mean curvature was proposed by Geroch [**250**] to understand the Penrose conjecture relating the mass with the area of the black hole. Such a procedure was finally carried out by Huisken-Ilmanen [**351**] when the scalar curvature is non-negative. There was a different proof by H. Bray [**80**] subsequently.

Ricci flow has had spectacular successes in recent years. However, not much progress has been made on the Calabi flow (see Chang's survey [**111**]) for Kähler metrics. They are higher order problems where the maximal principle has not been effective. An important contribution was made by Chrúsciel [**163**] for Riemann surface. Inspired by the concept of the Bondi mass in general relativity, Chruściel was able to give a new estimate for the

Calabi flow. Unfortunately, a higher dimensional analogue had not been found.

Natural higher order elliptic problems are difficult to handle. Affine minimal surfaces and Willmore surfaces are such examples. L. Simon [**619**] made an important contribution to the regularity of the Willmore surfaces. The corresponding flow problem should be interesting.

The dynamics of Einstein equations for general relativity is a very difficult subject. The Cauchy problem was considered by many people: A. Lichnerowicz, Y. Choquet-Bruhat, J. York, V. Moncrief, H. Friedrich, D. Christodoulou, S. Klainerman, H. Lindblad, M. Dafermos (see, e.g., [**454**], [**158**], [**157**], [**234**], [**159**], [**384**], [**461**], [**181**]). But the global behavior is still far from being understood. The major unsolved problem is to formulate and prove the fundamental question of Penrose on Cosmic censorship. I suggested to Klainerman and Christodoulou to consider small initial data for the Einstein system. The treatment of stability of Minkowski spacetime was accomplished by Christodoulou-Klainerman [**160**] under small perturbation of flat spacetime and fast fall off conditions. Recently Lindblad and Rodnianski [**461**] gave a simpler proof. A few years ago, N. Zipser (Harvard thesis) added Maxwell equation to gravity and still proved stability of Minkowski spacetime. There is remarkable progress on the problem of Cosmic censorship by M. Dafermos [**181**]. He made an important contribution for the spherical case. Stability for Schwarzschild or Kerr solutions is far from being known. Finster-Kamran-Smoller-Yau [**222**] had studied decay properties of Dirac particles with such background. The work does indicate the stability of these classical spacetimes.

The no hair theorem for stationary black holes is a major theorem in general relativity. It was proved by W. Israël [**355**], B. Carter [**109**], D. Robinson [**570**] and S. Hawking [**326**]. But the proof is not completely rigorous for the Kerr metric. In any case, the existing uniqueness theorem does assume regularity of the horizon of the black hole. It is not clear to me whether a nontrivial asymptotically flat solitary solution of a vacuum Einstein equation has to be the Schwarzschild solution. There is a possibility that the Killing field is spacelike. In that case, there may be a new interesting vacuum solution.

There is extensive literature on spacelike hypersurfaces with constant mean curvature. The foliation defined by them gives interesting dynamics of Einstein equation. These surfaces are interesting even for $\mathbb{R}^{n,1}$. A. Treiberges studied it extensively [**685**]. Li, Choi-Treibergs [**154**] and T. Wan [**699**] observed that the Gauss maps of such surfaces give very nice examples of harmonic maps mapping into the disk. Recently Fisher and Moncrief used them to study the evolution equation of Einstein in $2+1$ dimension.

> **Comment**: The dynamics of submanifolds and geometric structures reveal the true nature of these geometric objects deeply. In the process of arriving at a stationary object or a

solitary solution, it encounters singularities. Understanding the structures of such singularity will solve many outstanding conjectures in topology such as Shoenflies conjecture.

5. Construction of geometric structures on bundles and manifolds

A fundamental question is how to build geometric structures over a given manifold. In general, the group of topological equivalences that leaves this geometric structure invariant should be a special group. With the exception of symplectic structures, these groups are usually finite dimensional. When the geometric structure is unique (up to equivalence), it can be used to produce key information about the topological structure.

The study of special geometric structures dates back to Sophus Lie, Klein and Cartan. In most cases, we like to be able to parallel transport vectors along paths so that we can define the concept of holonomy group.

5.1. Geometric structures with noncompact holonomy group. When the holonomy group is not compact, there are examples of projective flat structure, affine flat structure and conformally flat structure. It is not a trivial matter to determine which topological manifolds admit such structures. Since the structure is flat, there is a unique continuation property and hence one can construct a developing map from a suitable cover of the manifold to the real projective space, the affine space and the sphere respectively. The map gives rise to a representation of the fundamental group of the manifold to the real projective group, the special linear group and the Möbius group respectively. This holonomy representation gives a great deal of information for the geometric structure. Unfortunately, the map is not injective in general. In the case where it is injective, the manifold can be obtained as a quotient of a domain by a discrete subgroup of the corresponding Lie group. In this case, a lot more can be said about the manifold as the theories of partial differential equations and discrete groups can play important roles.

5.1.1. *Projective flat structure.* If a projective flat manifold can be projectively embedded as a bounded domain, Cheng-Yau [**145**] were able to construct a canonical metric from the real Monge-Ampère equation which generalizes the Hilbert metric. When the manifold is two dimensional, there are works of C.P. Wang [**700**] and J. Loftin [**473**] on how to associate such metrics to a conformal structure and a holomorphic section of the cubic power of a canonical bundle. This is a beautiful theory related to the hyperbolic affine sphere mentioned in chapter one.

There are fundamental works by S.Y. Choi, W. Goldman (see the reference of Choi-Goldman [**156**]), N. Hitchin [**335**] and others on the geometric decomposition and the moduli of flat projective structures on Riemann surfaces. It should be interesting to extend them to three or four dimensional manifolds.

5.1.2. *Affine flat structure.* It is a difficult question to determine which manifolds admit flat affine structures. For example, it is still open whether the Euler number of such spaces is zero, although great progress was made by D. Sullivan [**644**]. W. Goldman [**259**] has also found topological constraints on three manifolds in terms of fundamental groups. The difficulty arises as there is no useful metric that is compatible with the underlining affine structure. This motivated Cheng-Yau [**146**] to define the concept of affine Kähler metric.

When Cheng and I considered the concept of affine Kähler metric, we thought that it was a natural analogue of Kähler metrics. However, compact nonsingular examples are not bountiful. Strominger-Yau-Zaslow [**643**] proposed the construction of mirror manifolds by constructing the quotient space of a Calabi-Yau manifold by a special Lagrangian torus. At the limit of the large Kähler class, it was pointed out by Hitchin [**336**] that the quotient space admits a natural affine structure with a compatible affine Kähler structure. But in general, we do expect singularities of such a structure. It now becomes a deep question to understand what kind of singularity is allowed and how we build the Calabi-Yau manifold from such structures. Loftin-Yau-Zaslow [**474**] have initiated the study of the structure of a "Y" type singularity. Hopefully one can find an existence theorem for affine structures over compact manifolds with prescribed singularities along codimension two stratified submanifolds.

5.1.3. *Conformally flat structure.* Construction of conformally flat manifolds is also a very interesting topic. Similar to projective flat or affine flat manifolds, there are simple constraints from curvature representation for the Pontrjagin classes. The deeper problem is to understand the fundamental group and the developing map. When the structure admits a conformal metric with positive scalar curvature, Schoen-Yau [**605**] proved the rather remarkable theorem that the developing map is injective. Hence such a manifold must be the quotient of a domain in S^n by a discrete subgroup of Möbius transformations. It would be interesting to classify such manifolds. In this regard, the Yamabe problem as was solved by Schoen [**590**] did provide a conformal metric with constant scalar curvature. One hopes to be able to use such metrics to control the conformal structure. Unfortunately the metric is not unique and a deep understanding of the moduli space of conformal metrics with constant scalar curvature is needed.

Kazdan-Warner [**379**] and Korevaar-Mazzeo-Pacard-Schoen [**398**] developed a conformal method to understand Nirenberg's problem on prescribed scalar curvature. It was followed by Chen-Lin [**131**], Chang-Gursky-Yang [**112**]. Chen-Lin have related this problem to mean field theory. Their computations in the relevant degree theory involve deep analysis. One should be able to generalize their works to functions which are sections of a flat line bundle because it is related to the previously mentioned work of Loftin-Yau-Zaslow [**474**]. In any case, the integrability condition of Kazdan and Warner is still not fully understood.

It is curious that while bundle theory was used extensively in Riemannian geometry, it has not been used in the study of these geometries. One can construct real projective space bundles, affine bundles or sphere bundles by mapping coordinate charts projectively, affinely or conformally to the corresponding model spaces (possible with dimensions different from the original manifold) and gluing the target model spaces together to form natural bundles. Perhaps one may study their associated Chern-Simons forms [**152**].

Many years ago, H.C. Wang [**702**] proved the theorem that if a compact complex manifold has trivial holomorphic tangent bundle, it is covered by a complex Lie group. It will be nice to generalize and interpret such a theorem in terms of Hermitian connections on the manifold with a special holonomy group and torsion.

This program was discussed in my paper [**744**] on algebraic characterization of locally Hermitian symmetric spaces. For a holomorphic stable vector bundle V, we can form a stable vector bundles from V by taking irreducible representation of $GL(n,\mathbb{C})$ from decomposition of the tensor product representation $\bigotimes_p V \bigotimes_q V^*$. By twisting with powers of canonical line bundle, we can form irreducible stable bundles with trivial determinant line bundle. In general, such bundles may not have holomorphic sections. If they do, the section must be parallel with respect to the Hermitian-Yang-Mills connection on the bundle, and the structure group of V can be reduced to a smaller group. Hermitian-Yang-Mills connections with reduced holonomy group have good geometric properties. We may formulate a principle: For stable holomorphic bundles, existence of nontrivial holomorphic invariants implies the existence of parallel tensors and therefore the reduction of structure group. If the holonomy group is reduced to a discrete group, the bundle will provide representations of the fundamental group into unitary group. This should compare with Wang's theorem when the bundle is the tangent bundle.

> **Comment**: Geometric structures with a noncompact holonomy group are less intuitive than Riemannian geometry. Perhaps we need to deepen our intuitions by relating them to other geometric structures, especially those structures that may carry physical meaning.

5.2. Uniformization for three manifolds. An important goal of geometry is to build a canonical metric associated to a given topology. Besides the uniformization theorem in two dimensions, the only (spectacular) work in higher dimensions is the geometrization program of Thurston (see [**674**]).

W. Thurston made use of ideas from Riemann surface theory, W. Haken's work [**310**] on three-manifolds, G. Mostow's rigidity [**531**] to build his manifolds. Many mathematicians have contributed to the understanding of this program of Thurston's (e.g., J. Morgan [**529**], C. McMullen [**500, 501**], J. Otal [**547**], J. Porti [**566**]). Thurston's orbifold program was finally settled by M. Boileau, B. Leeb and J. Porti [**61**]. However, one needs to assume that an incompressible surface (or corresponding surface in case of orbifold) exists. When R. Hamilton [**311**] had his initial success on his Ricci flow, I suggested (around 1981) to him to use his flow to break up the manifold and prove Thurston's conjecture. His generalization of the theory of Li-Yau [**445**] to Ricci flow [**312, 313**] and his seminal paper in 1996 [**315**] on breaking up the manifold mark a cornerstone of the remarkable program. Perelman's recent idea [**551, 552**] built on these two works and has gone deeply into the problem. Detailed discussions have been pursued by Hamilton, Zhu, Cao, Colding-Minicozzi, Shioya-Yamaguchi, and Huisken in the past two years. Hopefully it may lead to the final settlement of the geometrization program. This theory of Hamilton and Perelman should be considered as a crowning achievement of geometric analysis in the past thirty years. Most ideas developed in this period by geometric analysts are used.

Let me now explain briefly the work of Hamilton and Perelman.

In the early 90's, Hamilton [**313, 314, 315**] developed methods and theorems to understand the structure of singularities of the Ricci flow. Taking up my suggestions, he proved a fundamental Li-Yau type differential inequality (now called the Li-Yau-Hamilton estimate) for the Ricci flow with non-negative curvature in all dimensions. He gave a beautiful interpretation of the work of Li-Yau and observed the associated inequalities should be equalities for solitary solutions. He then established a compactness theorem for (smooth) solutions to the Ricci flow, and observed (also independently by T. Ivey[**356**]) a pinching estimate for the curvature for three-manifolds. By imposing an injectivity radius condition, he rescaled the metric to show that each singularity is asymptotic to one of the three singularity models. For type I singularities in dimension three, Hamilton established an isoperimetric ratio estimate to verify the injectivity radius condition and obtained spherical or neck-like structures. Based on the Li-Yau-Hamilton estimate, Hamilton showed that any type II model is either a Ricci soliton with a neck-like structure or the product of the cigar soliton with the real line. Similar characterization for type III model was obtained by Chen-Zhu [**126**]. Hence Hamilton had already obtained the canonical neighborhood structures (consisting of spherical, neck-like and cap-like regions) for the singularities of three-dimensional Ricci flow.

But two obstacles remained: one is the injectivity radius condition and the other is the possibility of forming a singularity modelled on the product of the cigar soliton with a real line which could not be removed by surgery.

Recently, Perelman [**551**] removed these two stumbling blocks in Hamilton's program by establishing a local injectivity radius estimate (also called "Little Loop Lemma" by Hamilton in [**314**]). Perelman proved the Little Loop Lemma in two ways, one with an entropy functional he introduced in [**551**], the other with a reduced distance function based on the same idea as Li-Yau's path integral in obtaining their inequality [**551**]. This reduced distance question gives rise to a Gaussian type integral which he called reduced volume. The reduced volume satisfies monotonicity property. Furthermore, Perelman [**552**] developed a refined rescaling argument (by considering local limits and weak limits in Alexandrov spaces) for singularities of the Ricci flow on three-manifolds to obtain a uniform and global version of the canonical neighborhood structure theorem.

After obtaining the canonical neighborhoods for the singularities, one performs geometric surgery by cutting off the singularities and continuing the Ricci flow. In [**315**], Hamilton initiated such a surgery procedure for four-manifolds with a positive isotropic curvature. Perelman [**552**] adapted Hamilton's geometric surgery procedure to three-manifolds. The most important question is how to prevent the surgery times from accumulations and make sure there are only a finite number of surgeries on each finite time interval. When one performs the surgeries with a given accuracy at each surgery time, it is possible that the errors may add up, which causes the surgery times to accumulate. Hence at each step of surgery one is required to perform the surgery more accurately than the former one. In [**553**], Perelman presented a clever idea on how to find "fine" necks, how to glue "fine" caps and how to use rescaling arguments to justify the discreteness of the surgery times. In the process of rescaling for surgically modified solutions, one encounters the difficulty of how to use Hamilton's compactness theorem, which works only for smooth solutions. The idea to overcome such difficulty consists of two parts. The first part, due to Perelman [**552**], is to choose cutoff radius (in neck-like regions) small enough to push the surgical regions far away in space. The second part, due to Chen-Zhu [**130**] and Cao-Zhu [**103**], is to show that the solutions are smooth on some uniform small time intervals (on compact subsets) so that Hamilton's compactness theorem can be used.

Once surgeries are known to be discrete in time, one can complete Schoen-Yau's classification [**603**] for three-manifolds with positive scalar curvature. For simply connected three manifolds, if one can show solution to the Ricci flow with surgery extincts in finite time, Poincaré conjecture will be proved. Recently, such a finite extinction time result was proposed by Perelman [**553**] and a proof appeared in Colding-Minicozzi [**173**].

For the full geometrization program, one still needs to find the long time behavior of surgically modified solutions. In [**316**], Hamilton studied the long time behavior of the Ricci flow on a compact three-manifold for a special class of (smooth) solutions called "nonsingular solutions". Hamilton proved that any (three-dimensional) nonsingular solution either collapses or subsequently converges to a metric with constant curvature on the compact manifold, or at large time it admits a thick-thin decomposition where the thick part consists of a finite number of hyperbolic pieces and the thin part collapses. Moreover, by adapting Schoen-Yau's minimal surface arguments to a parabolic version, Hamilton showed that the boundary of hyperbolic pieces are incompressible tori. Then by combining with the collapsing results of Cheeger-Gromov [**120**], any nonsingular solution to the Ricci flow is geometrizable.

In [**551, 552**], Perelman modified Hamilton's arguments to analyze the long-time behavior of arbitrary solutions to the Ricci flow and solutions with surgery in dimension three. Perelman also argued by showing a thick-thin decomposition, except that he can only show the thin part has (local) lower bound on sectional curvature. For the thick part, based on Li-Yau-Hamilton estimate, Perelman established a crucial elliptic type Harnack estimate to conclude that the thick part consists of hyperbolic pieces. For the thin part, he announced a new collapsing result which states that if a three-manifold collapses with a (local) lower bound on the sectional curvature, then it is a graph manifold. However, the proof of the new collapsing result has not been published. Shioya and Yamaguchi [**616, 617**] offered a proof for compact manifolds. Very recently, Cao-Zhu claimed to have a complete proof for compact manifolds based only on the Shioya-Yamaguchi's collapsing result.

Hopefully all these arguments can be checked thoroughly in the near future. It should also be interesting to see whether other famous problems in three manifold can be settled by analysis: Does every three dimensional hyperbolic manifold admit a finite cover with nontrivial first Betti number?

Hyperbolic metrics have been used by topologists to give invariants for three dimensional manifolds. Thurston [**673**] observed that the volume of a hyperbolic metric is an important topological invariant. The associated Chern-Simons [**152**] invariant, which is defined by mod integers, can be looked upon as a phase for such manifolds. These invariants appeared later in Witten's theory of $2 + 1$ dimensional gravity [**719**] and S. Gukov [**304**] was able to relate them to fundamental questions in knot theory.

> **Comment**: This is the most spectacular development in the last thirty years. Once the three manifold is hyperbolic, Ricci flow does not give much more information. Perhaps, one may obtain further information by performing reduction from four dimension Ricci flow to three dimension by circle action. Is there any effective way to understand the totality of all hyperbolic manifolds with finite volume by constructing flows that may break up topology?

5.3. Four manifolds. The major accomplishment of Thurston, Hamilton, Perelman et al is the ability to create a canonical structure on three manifolds. Such a structure has not even been conjectured for four manifolds despite the great success of Donaldson invariants and Seiberg-Witten invariants. Taubes [**659**] did prove a remarkable existence theorem for self-dual metrics on a rather general class of four dimensional manifolds. Unfortunately their moduli space is not understood and their topological implication is not clear at this moment. Since the twistor space of Taubes metric admits integrable complex structure, ideas from complex geometry may be helpful. Prior to the construction of Taubes, Donaldson-Friedman [**205**] and LeBrun [**414**] have used ideas from twistor theory to construct self-dual metrics on the connected sum of $\mathbb{C}P^2$.

The problem of the four manifold is the lack of good diffeomorphic invariants. Donaldson or Seiberg-Witten provide such invariants. But they are not powerful enough to control the full structure of the manifold. A true understanding of four manifolds probably should come from understanding the question of existence of the integrable complex structures. The Riemann-Roch-Hirzebruch formula has been the basic tool to find the integrability condition. In the last twenty-five years, there are nonlinear methods from Käher-Einstein metrics, harmonic maps, anti-self-dual connections and Seiberg-Witten invariants. However, one needs an existence theorem to find a canonical way to deform an almost complex structure to an integrable complex structure. What kind of obstructions do we expect? The work of Donaldson [**199, 201**] and Gompf [**261**] gave a good characterization of symplectic manifolds in terms of Lefschetz fibration. It may be useful to know under which condition such fibration will give rise to complex structures. I did ask several of my students to work on it. But no definite answer is known. J Jost and I [**370**] studied the rigidity part: if a Kähler surface has a topological map to a Riemann surface with higher genus, it can be deformed to be a holomorphic map by changing the complex structure of the Riemann surface. One can derive from the work of Griffiths [**274**] that every algebraic surface has a Zariski open set which admits a complete Kähler-Einstein metric with finite volume and is covered by a contractible pseudo-convex domain. Perhaps one can classify these manifolds by topological means.

While the Donaldson invariant gave the first counterexample to the h-cobordism theorem and irreducibility (nontrivial connected sum with manifolds not homotopic to $\mathbb{C}P^2$) of four manifolds, the Seiberg-Witten invariant gave the remarkable result that an algebraic surface of general type can not be diffeomorphic to rational or elliptic surfaces. It also solves the famous Thom conjecture that holomorphic curves realize the lowest genus for embedded surface in a Kähler surface (Kronheimer-Mrowka [**402**] and Ozsváth-Szabó [**548**]). One wonders whether one can construct a diffeomorphic invariant based on metrics which are a generalization of Kähler-Einstein metrics.

> **Comment**: A good conjectural statement needs to be made on the topology of four manifolds that may admit an integrable complex structure. Pseudo-holomorphic curve and fibration by Riemann surfaces should provide important information. Geometric flows may still be the major tool.

5.4. Special connections on bundles. In the seventies, theoretic physicists were very much interested in the theory of instantons: self-dual connections on four manifolds. Singer was able to communicate the flavor of this excitement to the mathematical community which soon led to his paper with Atiyah and Hitchin [**16**] and also the complete solution of the problem over the four sphere by Atiyah-Hitchin-Drinfel'd-Manin [**15**] using twistor technique of Penrose.

While the paper of Atiyah-Hitchin-Singer [**16**] laid the algebraic and geometric foundation for self-dual connections, the analytic foundation was laid by Uhlenbeck [**688, 689**] where she established the removable singularity theorem and compactness theorem for Yang-Mills connections. This eventually led to the fundamental works of Taubes [**656**] and Donaldson [**195**] which revolutionized four manifold topology.

In the other direction, Atiyah-Bott [**11**] applied Morse theory to the space of connections over Riemann surface. They solved important questions on the moduli space of holomorphic bundles which was studied by Narasimhan, Seshadri, Ramanathan, Newstead and Harder. In the paper of Atiyah-Bott, Morse theory, moment map and localization of equivariant cohomology were introduced on the subject of vector bundle. It laid the foundation of works in the last twenty years.

The analogue of anti-self dual connections over Kähler manifolds are Hermitian Yang-Mills connections, which was shown by Donaldson [**196**] for Kähler surfaces and Uhlenbeck-Yau [**691**] for general Kähler manifolds to be equivalent to the polystability of bundles. (That polystability of bundle is a consequence of the existence of Hermitian Yang-Mills connection was first observed by Lübke [**486**]. Donaldson [**197**] was able to make use of the theorem of Mehta-Ramanathan [**509**] and ideas of the above two papers to prove the theorem for projective manifold). It was generalized by C. Simpson [**621**], using ideas of Hitchin [**333**], to bundles with Higgs fields. It has important applications to the theory of variation of a Hodge structure [**622, 623**]. G. Daskalopoulos and R. Wentworth [**184**] studied such a theory for moduli space of vector bundles over curves. Li-Yau [**433**] generalized the existence of Hermitian Yang-Mills connections to non-Kähler manifolds. (Buchdahl [**87**] subsequently did the same for complex surfaces.) Li-Yau-Zheng [**435**] then used the result to give a complete proof of Bogomolov's theorem for class VII_0 surfaces. The only missing parts of the classification of non-Kähler surfaces are those complex surfaces with a finite number of holomorphic curves. It is possible that the argument of Li-Yau-Zheng can be used. One may want to use Hermitian Yang-Mills connections with poles along such curves. I expect more applications of Donaldson-Uhlenbeck-Yau theory to algebraic geometry.

It should be noted that the construction of Taubes [**659**] on the anti-self-dual connection is achieved by singular perturbation after gluing instantons from S^4. The method is rather different from Donaldson-Uhlenbeck-Yau. While it applies to arbitrary four manifolds, it does require some careful choice of Chern classes for the bundle. It will be nice to find a concept of stability for a general complex bundle so that a similar procedure of Donaldson-Uhlenbeck-Yau can be applied. The method of singular perturbation has an algebraic geometric counterpart as was found by Gieseker-Li [**255**] and O'Grady [**542**], who proved that moduli spaces of algebraic bundles with fixed Chern classes over algebraic surfaces are irreducible. Li [**427**] also obtained information about Betti number of such moduli space. Not many general theorems are known for bundles over algebraic manifolds of a higher dimension. It will be especially useful for bundles over Calabi-Yau manifolds.

D. Gieseker [**253**] developed the geometric invariant theory for the moduli space of bundles and introduced the Gieseker stability of bundles. Conan Leung [**419**] introduced the analytic counterpart of such bundles in his thesis under my guidance. While it is a natural concept, there is still an analytic problem to be resolved. (He assumed the curvature of the bundles to be uniformly bounded.)

There were attempts by de Bartolomeis-Tian [**43**] to generalize Yang-Mills theory to symplectic manifolds and also by Tian [**679**] to manifolds with a special holonomy group, as was initiated by the work of Donaldson and Thomas. However, the arguments for both papers are not complete and still need to be finished.

For a given natural structure on a manifold, we can often fix a structure and linearize the equation to obtain a natural connection on the tangent bundle. Usually we obtain Yang-Mills connections with the extra structure given by the holonomy group of the original structure. It is interesting to speculate whether an iterated procedure can be constructed to find an interesting metric or not. In any case, we can draw analogous properties between bundle theory and metric theory. The concept of stability for bundles is reasonably well understood for the holomorphic category. I believed that for each natural geometric structure, there should be a concept of stability. Donaldson [**197**] was able to explain stability in terms of moment map, generalizing the work of Atiyah-Bott [**11**] for bundles over Riemann surfaces. It will be nice to find moment maps for other geometric structures.

> **Comment**: Bundles with anti-self-dual connections or Hermitian Yang-Mills connections have been important for geometry. However, we do not have good estimates of the curvature of such connections. Such an estimate would be useful

to handle important problems such as the Hartshorne question (see, e.g., [**39**]) on the splitting of rank two bundle over high dimension complex projective space.

5.5. Symplectic structures. Symplectic geometry had many important breakthroughs in the past twenty years. A moment map was developed by Atiyah-Bott [**12**], Guillemin-Sternberg [**301**] who proved the image of the map is a convex polytope. Kirwan and Donaldson had developed such a theory to be a powerful tool. The Marsden-Weinstein [**492**] reduction has become a useful method in many branches of geometry. At around the same time, other parts of symplectic topology were developed by Donaldson [**200**], Taubes [**660**], Gompf [**260**], Kronheimer-Mrowka [**402**] and others.

The phenomenon of symplectic rigidity is manifested by the existence of symplectic invariants measuring the 2–dimensional size of a symplectic manifold. The first such invariant was discovered by Gromov [**286**] via pseudo-holomorphic curves. Hofer [**340**] then developed several symplectic invariants based on variational methods and successfully applied them to Weinstein conjectures. Ekeland-Hofer [**214**] introduced a concept of symplectic capacity and used it to provide a characterization of a symplectomorphism not involving any derivatives. The C^0-closed property of the symplectomorphism group as a subgroup of the diffeomorphism group then follows, which was independently established by Y. Eliashberg [**215**] via wave front methods. Hofer-Zehnder [**343**] introduced another capacity and discovered the displacement-energy on $\mathbb{R}^{2n}$. By relating the two invariants with the energy-capacity inequality, Hofer [**339**] found a bi-invariant norm on the infinite dimensional group of Hamiltonian symplectomorphisms of $\mathbb{R}^{2n}$. The existence of such a norm has now been established for general symplectic manifolds by Lalonde-McDuff [**408**] via pseudo-holomorphic curves and symplectic embedding techniques. The generalized Weinstein conjecture on the existence of periodic orbits of Reeb flows for many 3-manifolds including the 3-sphere was also established in Hofer [**340**] by studying the finite energy pseudo-holomorphic plane in the symplectization of contact 3-manifolds.

Eliashberg-Givental-Hofer [**216**] recently introduced the concept of symplectic field theory, which is about invariants of punctured pseudo-holomorphic curves in a symplectic manifold with cylindrical ends. Though it has not been rigorously established, some applications in contact and symplectic topology have been found.

By analyzing the singularities of pseudo-holomorphic curves in a symplectic 4–manifold, D. McDuff [**498**] established rigorously the positivity of intersections of two distinct curves and the adjunction formula of an irreducible curve. Applying these basic properties to symplectic 4-manifolds containing embedded pseudo-holomorphic spheres with self-intersections at least -1, she was able to construct minimal models of general symplectic 4–manifolds, and classify those containing embedded symplectic spheres with non-negative self-intersections.

A fundamental question in symplectic geometry is to decide which topological manifold admits a symplectic structure and how, as was pointed out by Smith-Thomas-Yau [**635**], mirrors of certain non-Kähler complex manifolds should be symplectic manifolds. Based on this point of view, they construct a large class of symplectic manifolds with trivial first Chern class by reversing the procedure of Clemens-Friedman on non-Kähler Calabi-Yau manifolds [**165, 233**]. In dimension four, the Betti numbers of such manifolds are determined by T. J. Li [**446**]. In the last ten years, there has been extensive work on symplectic manifolds, initiated by Gromov [**286**], Taubes [**661, 662, 663, 664**], Donaldson [**198, 199, 201**] and Gompf [**261**]. These works are based on the understanding of pseudo-holomorphic curves and Lefschetz fibrations. They are most successful for four dimensional manifolds. The major tools are Seiberg-Witten theory [**607, 608, 721**] and analysis. The work of Taubes on the existence of pseudo-holomorphic curves and the topological meaning of its counting is one of the deepest works in geometry. Based on this work, Taubes [**661**] was able to prove the old conjecture that there is only one symplectic structure on the standard $\mathbb{C}P^2$. However, the following question of mine is still unanswered: If M is a symplectic 4-manifold homotopic to $\mathbb{C}P^2$, is M symplectomorphic to the standard $\mathbb{C}P^2$? (The corresponding question for complex geometry was solved by me in [**735**].) On the other hand, based on the work of Taubes [**660**], T.J. Li and A.K. Liu [**447**] did find a wall crossing formula for four dimensional manifolds that admit metrics with a positive scalar curvature. Subsequently A. Liu [**462**] gave the classification of such manifolds. (The surgery result by Stolz [**641**] based on Schoen-Yau-Gromov-Lawson for manifolds with positive scalar curvature is not effective for four dimensional manifolds.) As another application of the general wall crossing formula in [**447**], it was proved by T.J. Li and A. Liu in [**448**] that there is a unique symplectic structure on S^2-bundles over any Riemann surface. A main result of D. McDuff in [**497**] is used here.

McDuff [**496**] also used a refined bordism type Gromov-Witten invariant to distinguish two cohomologous and deformation equivalent symplectic forms on $S^2 \times S^2 \times T^2$, showing that they are not isotopic. Notice that there are also cohomologous but non-deformation equivalent symplectic forms on $K3 \times S^2$ as shown by Y. Ruan [**578**]. In contrast, it is not known whether examples of this kind exist in dimension 4 or not. This phenomenon might be related to the special features of pseudo-holomorphic curves in a 4-manifold.

Fukaya and Oh [**239**] have developed an elaborate theory for symplectic manifolds with Lagrangian cycles. Pseudo-holomorphic disks appeared as trace of motions of curves according to Floer theory. Due to boundary bubbles, the Lagrangian Floer homology is not always defined. Oh [**543**] developed some works on pseudo-holomorphic curves with Lagrangian boundary conditions and extended the Lagrangian Floer homology to all monotone symplectic manifolds. In order to understand open string theory, Katz-Liu [**378**] and Melissa Liu [**463**] developed the theory in analogue of the Gromov-Witten invariant for a holomorphic curve with boundaries on a given Lagrangian submanifold. Fukaya [**238**] discovered the underlying A^∞ structure of the Lagrangian Floer homology on the chain level, leading to the Fukaya category. By carefully analyzing this A^∞ structure, Fukaya, Oh, Ohta and Ono in [**240**] have constructed a sequence of obstruction classes which elucidate the rather difficult Lagrangian Floer homology theory to a great extent. Seidel-Thomas [**609**] and W.D. Ruan [**577**] discussed Fukaya's category in relation to Kontsevich's homological mirror conjecture [**395**]. One wonders whether Fukaya's theory can help to construct canonical metrics for symplectic structures. For symplectic manifolds that admit an almost complex structure with zero first Chern class, it would be nice to construct Hermitian metrics with torsion that admit parallel spinor. Such structures may be considered as a mirror to the system constructed by Strominger on non-Kähler complex manifolds. Perhaps one can also gain some knowledge by reduction of G_2 or Spin(7) structures to six dimensions.

> **Comment**: Geometry from the symplectic point of view has seen powerful development in the past twenty years. Its relation to Seiberg-Witten theory and mirror geometry is fruitful. More interesting development is expected.

5.6. Kähler structure. The most interesting geometric structure is the Kähler structure. There are two interesting pre-Kähler structures. One is the complex structure and the other is the symplectic structure. The complex structure is rather rigid for complex two dimensional manifolds. However, it is much more flexible in dimension greater than two. For example, the twistor space of anti-self-dual four manifolds admits complex structures. Taubes [**659**] constructed a large class of such manifolds and hence a large class of complex three manifolds. There is also the construction of Clemens-Friedman for non-Kähler Calabi-Yau manifolds which will be explained later.

For quite a long time, it was believed that every compact Kähler manifold can be deformed to a projective manifold until C. Voisin [**697, 698**] found many counterexamples. We still need to digest the distinction between these two categories.

Besides some obvious topological obstruction from Hodge theory and the rational homotopic type theory of Deligne-Griffiths-Morgan-Sullivan [**188**], it has been difficult to decide which complex manifolds admit Kähler structure. The harmonic map argument does give some information. But it requires the fundamental group to be large.

Many years ago, Sullivan [**645**] proposed to use the Hahn-Banach theorem to construct Kähler metrics. This involves the concept of duality and hence closed currents. P. Gauduchon [**246**] has proposed those Hermitian metrics ω which is $\partial\bar{\partial}\omega^{n-1} = 0$. Siu [**627**] was able to use these ideas to prove that every $K3$ surface is Kähler. Demailly [**191**] did some remarkable work on regularization of closed positive currents. Singular Kähler metrics have been studied and used by many researchers. In fact, in my paper on proving the Calabi conjecture, I proved the existence of the Kähler metrics singular along subvarieties with control on volume element. They can be used to handle problems in algebraic geometry, including Chern number inequalities, and possibly problems arising in the minimal model program.

> **Comment**: The Kähler structure is one of the richest structures in geometry. Deeper understanding may require some more generalized structure such as a singular Kähler metric or balanced metrics.

5.6.1. *Calabi-Yau manifolds.* The construction of Calabi-Yau manifolds was based on the existence of a complex structure which can support a Kähler structure and a pluriharmonic volume form.

A fundamental question is whether an almost complex manifold admits an integrable complex structure when complex dimension is greater than two. The condition that the first Chern class is zero is equivalent to the existence of pluriharmonic volume for Kähler manifolds. Such a condition is no more true for non-Kähler manifolds. It would be nice to find a topological method to construct an integrable complex structure with pluriharmonic volume form.

Once we have an integrable complex structure, we can start to search for Hermitian metrics with special properties. As was mentioned earlier, if we would like the geometry to have supersymmetries, then a Kähler metric is the only choice if the connection is torsion free. Further supersymmetry would then imply the manifold to be Calabi-Yau. However, if we do not require the connection to be torsion free, Strominger [**642**] did derive a set of equations that exhibit supersymmetries without requiring the manifold to be Kähler. It is a coupled system of Hermitian Yang-Mills connections with Hermitian metrics. Twenty years ago, I tried to develop such a coupled system. The attempt was unsuccessful as I restricted myself to Kähler geometry. My student Bartnik with Mckinnon [**42**] did succeed in doing so in the Lorentzian case. They found non-singular solutions for such a coupled system. (The mathematically rigorous proof was provided by Smoller-Wasserman-Yau-Mcleod [**639**] and [**638**]).

The Strominger's system was shown to be solvable in a neighborhood of a Calabi-Yau structure by Jun Li and myself [**434**]. Fu and I [**235**] were also able to solve it for many complex manifolds which admit no Kähler structure. These manifolds are balanced manifolds and were studied by M. Michelsohn [**517**]. These manifolds can be used to explain some questions of flux in string theory (see, e.g., [**46, 106**]). Since Strominger has shown such manifolds admits parallel spinors, I have directed my student C.C. Wu to decompose cohomology group of such manifolds correspondingly. It is expected that many theorems in Kähler geometry may have counterparts in such geometry.

Such a structure may help to understand a proposal of Reid [**568**] in connecting Calabi-Yau manifolds with a different topology. This was initiated by a construction of Clemens [**165**] who proposed to perform complex surgery by blowing down rational curves with negative normal bundles in a Calabi-Yau manifold to rational double points. Friedman [**233**] found the condition to smooth out such singularities. Based on this Clemens-Friedman procedure, one can construct a complex structure on connected sums of $S^3 \times S^3$. It would be nice to construct Strominger's system on these manifolds.

The Calabi-Yau structure was used by me and others to solve important problems in algebraic geometry before it appeared in string theory. For example, the proof of the Torelli theorem (by Piatetskii-Shapiro and Shafarevich [**559**]) for a $K3$ surface by Todorov [**683**]-Siu [**627**] and the surjectivity of the period map of a $K3$ surface (by Kulikov [**404**]) by Siu [**626**]-Todorov [**683**] are important works for algebraic surfaces. The proof of the Bogomolov [**60**]-Tian [**675**]-Todorov [**684**] theorem also requires the metric. (One needs to use the statement that the holomorphic n-form is parallel. This was overlooked in [**675**].) The last theorem helps us to understand the moduli space of Calabi-Yau manifolds. It is important to understand the global behavior of the Weil-Petersson geometry for Calabi-Yau manifolds. C.L. Wang [**701**] was able to characterize these points which have finite distance in terms of the degeneration of the Hodge structure.

In my talk [**737**] in the Congress in 1978, I outlined the program and the results of classifying noncompact Calabi-Yau manifolds. Some of this work was written up in Tian-Yau [**681, 682**] and Bando-Kobayashi [**32, 33**]. During the period of 1984, there was an urgent request by string theorists to construct Calabi-Yau threefolds with a Euler number equal to ± 6. During the Argonne Lab conference, I [**740**] constructed such a manifold with a $\mathbb{Z}_3$ fundamental group by taking the quotient of a bi-degree $(1,1)$ hypersurface in the product of two cubics. Soon afterwards, more examples were constructed by Tian and myself [**680**]. However, it was pointed out by Brian Greene that all the manifolds constructed by Tian-Yau can be deformed to my original manifold. The idea of producing Calabi-Yau manifolds by the complete intersection of hypersurfaces in products of weighted projective space was soon picked up by Candelas et al [**97**]. By now, on the order of ten thousand examples of different homotopic types had been constructed. The idea of using toric geometry for construction was first performed by S. Roan and myself [**569**]. A few years later, the systematic study by Batyrev [**44**] on toric geometry allowed one to construct mirror pairs for a large class of Calabi-Yau manifolds, generalizing the construction of Greene-Plesser [**271**] and Candelas et al [**97**]. Tian and I [**680**] were also the first one to apply flop construction to change topology of Calabi-Yau manifolds. Greene-Morrison-Plesser [**272**] then made the remarkable discovery of isomorphic quantum field theory on two topological distinct Calabi-Yau manifolds. Most Calabi-Yau threefolds are a complete intersection of some toric varieties and they admit a large set of rational curves. It will be important to understand the reason behind it. Up to now all the Calabi-Yau manifolds that have a Euler number ± 6 and a nontrivial fundamental group can be deformed from the birational model of the manifold (or their mirrors) that I constructed. It would be important if one could give a proof of this statement.

The most spectacular advancement on Calabi-Yau manifolds come from the work of Greene-Plesser, Candelas et al on construction of pairs of mirror manifolds with isomorphic conformal field theories attached to them. It allows one to calculate Gromov-Witten invariants. Existence of such mirror pairs was conjectured by Lerche-Vafa-Warner [**418**] and rigorous proof of mirror conjecture was due to Givental [**258**] and Lian-Liu-Yau [**449**] independently. The deep meaning of the symmetry is still being pursued.

In [**643**], Strominger, Yau and Zaslow proposed a mathematical explanation for the mirror symmetry conjecture for Calabi-Yau manifolds. Roughly speaking, mirror Calabi-Yau manifolds should admit special Lagrangian tori fibrations and the mirror transformation is a nonlinear analog of the Fourier transformation along these tori.

This proposal has opened up several new directions in geometric analysis. The first direction is the geometry of special Lagrangian submanifolds in Calabi-Yau manifolds. This includes constructions of special Lagrangian submanifolds ([**417**] and others) and (special) Lagrangian fibrations by Gross [**293, 294**] and W.D. Ruan [**576**], mean curvature of Lagrangian submanifolds in Calabi–Yau manifolds by Thomas and Yau [**670**] [**671**], structures of singularities on such submanifolds by Joyce [**376**], and Fourier transformations along special Lagrangian fibration by Leung-Yau-Zaslow [**424**] and Leung [**422**].

The second direction is affine geometry with singularities. As explained in [**643**], the mirror transformation at the large structure limit corresponds to a Legendre transformation of the base of the special lagrangian fibration which carries a natural special affine structure with singularities. Solving these affine problems is not trivial in geometric analysis [**473**] [**474**] and much work is still needed to be done here.

The third direction is the geometry of special holonomy and duality and triality transformation in M-theory. In [**305**], Gukov, Yau and Zaslow proposed a similar picture to explain the duality in M-theory. The corresponding differential geometric structures are fibrations on G_2 manifolds by coassociative submanifolds. These structures are studied by Kovalev [**400**], Leung and others [**416**] [**423**].

> **Comment**: Although the first demonstration of the existence of Kähler Ricci flat metric was shown by me in 1976, it was not until the first revolution of string theory in 1984

> that a large group of researchers did extensive calculations and changed the face of the subject. It is a subject that provides a good testing ground for analysis, geometry, physics, algebraic geometry, automorphic forms and number theory.

5.6.2. *Kähler metric with harmonic Ricci form and stability.* The existence of a Kähler Einstein metric with negative scalar curvature was proved by Aubin [**23**] and me [**736**] independently. I [**735**] found many important applications for solving classical problems in algebraic geometry, e.g., the uniqueness of complex structure over $\mathbb{C}P^2$ [**735**], the Chern number inequality of Miyaoka [**520**]-Yau [**735**] and the rigidity of algebraic manifolds biholomorphic to Shimura varieties. The problem of existence of Kähler Einstein metrics with positive scalar curvature in the general case is not solved. However, my proof of the Calabi conjecture already provided all the necessary estimates except some integral estimate on the unknown. This of course can be turned into hypothesis. I conjectured that an integral estimate of this sort is related to the stability of manifolds. Tian [**678**] called it K-stability. Mabuchi's functional [**489**] made the integral estimate more intrinsic and it gave rise to a natural variational formulation of the problem. Siu has pointed out that the work of Tian [**677**] on two dimensional surfaces is not complete. The work of Nadel [**535**] on the multiplier ideal sheaf did give useful methods for the subject of the Kähler-Einstein metric.

For Kähler Einstein manifolds with positive scalar curvature, it is possible that they admit a continuous group of automorphisms. Matsushima [**494**] was the first one to observe that such a group must be reductive. Futaki [**242**] introduced a remarkable invariant for general Kähler manifolds and proved that it must vanish for such manifolds. In my seminars in the eighties, I proposed that Futaki's theorem should be generalized to understand the projective group acting on the embedding of the manifold by a high power of anti-canonical embedding and that Futaki's invariant should be relevant to my conjecture [**743**] relating the Kähler Einstein manifold to stability. Tian asked what happens when manifolds have no group actions. I explained that the shadow of the group action is there once it is inside the projective space and one should deform the manifold to a possibly singular variety to obtain more information. The connection of Futaki invariant to stability of manifolds has finally appeared in the recent work of Donaldson [**202, 203**]. Donaldson introduced a remarkable concept of stability based purely on concept of algebraic geometry. It is not clear that Donaldson's algebraic definition has anything to do with Tian's analytic definition of stability. Donaldson proved that the existence of Kähler-Einstein did imply his K-stability which in turn implies Hilbert stability and asymptotic Chow stability of the manifold. This theorem of Donaldson already gives nontrivial information for manifolds with negative first Chern class and Calabi-Yau manifolds, where existence of Kähler-Einstein metrics was known. Some part of the deep work of Gieseker [**253**] and Viehweg [**694**] can be recovered by these theorems. One should also mention the recent interesting work of Ross-Thomas [**572, 573**] on the stability of manifolds. Phong-Strum [**555**] also studied solutions of certain degenerate Monge-Ampére equations and [**556**] the convergence of the Kähler-Ricci flow.

A Kähler metric with constant scalar curvature is equivalent to the harmonicity of the first Chern form. The uniqueness theorem for harmonic Kähler metric was due to X. Chen [**135**], Donaldson [**202**] and Mabuchi for various cases. (Note that the most important case of the uniqueness of the Kähler Einstein metric with positive scalar curvature was due to the remarkable argument of Bando-Mabuchi [**34**].) My general conjecture for existence of harmonic Kähler manifolds based on stability of such manifolds is still largely unknown. In my seminar in the mid-eighties, this problem was discussed extensively. Several students of mine, including Tian [**676**], Luo [**481**] and Wang [**709**] had written a thesis related to this topic. Prior to them, my former students Bando [**31**] and Cao [**100**] had made attempts to study the problem of constructing Kähler-Einstein metrics by Ricci flow. The fundamental curvature estimate was due to Cao [**101**]. The Kähler Ricci flow may either converge to Kähler Einstein metric or Kähler solitons. Hence in order for the approach, based on Ricci flow, to be successful, stability of the projective manifold should be related to such Kähler solitons. The study of harmonic Kähler metrics with constant scalar curvature on toric variety was initiated by S. Donaldson [**203**], who proposed to study the existence problem via the real Monge-Ampère equation. This problem of Donaldson in the Kähler-Einstein case was solved by Wang-Zhu [**708**]. LeBrun and his coauthors [**382**] also have found special constructions, based on twistor theory, for harmonic Kähler surfaces. Bando was also interested in Kähler manifolds with harmonic i-th Chern form. (There should be an analogue of stability of algebraic manifolds associated to manifolds with harmonic i-th Chern form.)

In the early 90's, S.W. Zhang [**754**] studied heights of manifolds. By comparing metrics on Deligne pairings, he found that a projective variety is Chow semistable if and only if it can be mapped by an element of a special linear group to a balanced subvariety. (Note that a subvariety in $\mathbb{CP}^N$ is called balanced if the integral of the moment map with respect to $SU(N+1)$ is zero, where the measure for the integral is induced from the Fubini-Study metric.) Zhang communicated his results to me. It is clearly related to Kähler-Einstein metric and I urged my students, including Tian, to study this connection.

Zhang's work has a nontrivial consequence on the previous mentioned development of Donaldson [**202, 203**]. Assume the projective manifold is embedded by an ample line bundle L into projective space. If the manifold has a finite automorphism group and admits a harmonic Kähler metric in $c_1(L)$, then Donaldson showed that for k large, L^k gives rise to an embedding which is balanced. Furthermore, the induced Fubini-Study form divided by

k will converge to the harmonic Kähler form. Combining the work of Zhang and Luo, he then proved that the manifold given by the embedding of L^k is stable in the sense of geometric invariant theory. Recently, Mabuchi generalized Donaldson's theorem to certain case which allow nontrivial projective automorphism.

Donaldson considered the problem from the point of view of symplectic geometry (Kähler form is a natural symplectic form). The Hamiltonian group then acts on the Hilbert space H of square integrable sections of the line bundle L where the first Chern class is the Kähler form. For each integrable complex structure on the manifold compatible with the symplectic form, the finite dimensional space of holomorphic sections gives a subspace of H. The Hamiltonian group acts on the Grassmannian of such subspaces. The moment map can be computed to be related to the Bergman kernel $\sum_\alpha s_\alpha(x) \otimes s_\alpha^*(y)$ where s_α form an orthonormal basis of the holomorphic sections. On the other hand, Fujiki [**236**] and Donaldson [**200**] computed the moment map for the Hamiltonian group action on the space of integrable complex structure, which turns out to be the scalar curvature of the Kähler metric. These two moment maps may not match, but for the line bundle L^k with large k, one can show that they converge to each other after normalization. Lu [**479**] has shown the first term of the expansion (in terms of $1/k$) of the Bergman kernel gives rise to scalar curvature. Hence we see the relevance of constant scalar curvature for a Kähler metric to these with a constant Bergman kernel function. S.W. Zhang's result says that the manifold is Chow semistable if and only if it is balanced. The balanced condition implies that there is a Kähler metric where the Bergman kernel is constant. With the work of Zhang and Donaldson, what remains to settle my conjecture is the convergence of the balanced metric when k is large. In general, we should not expect this to be true. However, for toric manifolds, this might be the case.

It may be noted that in my paper with Bourguignon and P. Li [**74**] on giving an upper estimate of the first eigenvalue of an algebraic manifold, this balanced condition also appeared. Perhaps the first eigenfunction may play a role for questions of stability.

> **Comment**: Kähler metrics with constant scalar curvature is a beautiful subject as it is related to structure questions of algebraic varieties including the concept of stability of manifolds. The most effective application of such metrics to algebraic geometry are still restricted to the Kähler-Einstein metric. The singular Kähler-Einstein metric as was initiated by my paper on Calabi conjecture should be studied further in application to algebraic geometry.

5.7. Manifolds with special holonomy group. Besides Kähler manifolds, there are manifolds with special holonomy groups. Holonomy groups of Riemannian manifolds were classified by Berger [**48**]. The most important ones are $O(n)$, $U(n)$, $SU(n)$, G_2 and $Spin(7)$. The first two groups correspond to Riemannian and Kähler geometry respectively. $SU(n)$ corresponds to Calabi-Yau manifolds. A G_2 manifold is seven dimensional and a $Spin(7)$ is eight dimensional (assuming they are irreducible manifolds). These last three classes of manifolds have zero Ricci curvature. It may be noted that before I [**736**] proved the Calabi conjecture in 1976, there was no known nontrivial compact Ricci flat manifold. Manifolds with a special holonomy group admit nontrivial parallel spinors and they correspond to supersymmetries in the language of physics. The input of ideas from string theory did give a lot of help to understand these manifolds. However, the very basic question of constructing these structures on a given topological space is still not well understood. In the case of G_2 and $Spin(7)$, it was initiated by Bryant (see [**84, 86**]). The first set of compact examples was given by Joyce [**373, 374, 375**]. Recently Dai-Wang-Wei [**183**] proved the stability of manifolds with parallel spinors.

The nice construction of Joyce was based on a singular perturbation which is similar to the construction of Taubes [**655**] on anti-self-dual connections. However, it is not global enough to give a good parametrization of G_2 or $Spin(7)$ structures. A great deal more work is needed. The beautiful theory of Hitchin [**337, 338**] on three forms and four forms may lead to a resolution of these important problems.

> **Comment**: Recent interest in M-theory has stimulated a lot of activities on manifolds with special holonomy group. We hope a complete structure theorem for such manifolds can be found.

5.8. Geometric structures by reduction. One can also obtain new geometric structures by imposing some singular structures on a manifold with a special holonomy group. For example, if we require a metric cone to admit a G_2, $Spin(7)$ or Calabi-Yau structure, the link of the cone will be a compact manifold with special structures. They give interesting Einstein metrics. When the cone is Calabi-Yau, the structure on the odd dimensional manifold is called Sasakian Einstein metric.

There is a natural Killing field called the Reeb vector field defined on a Sasakian Einstein manifold. If it generates a circle action, the orbit space gives rise to a Kähler Einstein manifold with positive scalar curvature. However, it need not generate a circle action and J. Sparks, Gauntlelt, Martelli and Waldram [**247**] gave many interesting explicit examples of non-regular Sasakian Einstein structures. They have interesting properties related to conformal field theory. For quasi-regular examples, there was work by Boyer, Galicki and Kollár [**76**]. The procedure gave many interesting examples of Einstein metrics on odd dimensional manifolds.

Sparks, Matelli and I have been pursuing general theory of Sasakian Einstein manifolds. I would like to consider them as a natural generalization of Kähler manifolds.

Comment: The recent development of the Sasakian Einstein metric shows that it gives a natural generalization of the Kähler-Einstein metric. Its relation with the recent activities on ADS/CFT theory is exciting.

5.9. Obstruction for existence of Einstein metrics on general manifolds. The existence of Einstein metrics on a fixed topological manifold is clearly one of the most important questions in geometry. Any metrics with a compact special holonomy group are Einstein. Besides Kähler geometry, we do not know much of their moduli space. For an Einstein metric with no special structures, we know only some topological constraints on four dimensional manifolds. There is work by Berger [**49**], Gray [**269**] and Hitchin [**332**] in terms of inequalities linking a Euler number and the signature of the manifold. (This is of course based on Chern's work [**149**] on the representation of characteristic classes by curvature forms.) Gromov [**285**] made use of his concept of Gromov volume to give further constraint. LeBrun [**415**] then introduced the ideas from Seiberg-Witten invariants to enlarge such classes and gave beautiful rigidity theorems on Einstein four manifolds. Unfortunately it is very difficult to understand moduli space of Einstein metrics when they admit no special structures. For example, it is still an open question of whether there is only one Einstein metric on the four dimensional sphere. M. Wang and Ziller [**707**] and C. Boehm [**58**] did use symmetric reductions to give many examples of Einstein metrics for higher dimensional manifolds. There may be much more examples of Einstein manifolds with negative Ricci curvature than we expected. This is certainly true for compact manifolds, with negative Ricci curvature. Gao-Yau [**243**] was the first one to demonstrate that such a metric exists on the three sphere. A few years later, Lokhamp [**475**] used the h-principle of Gromov to prove such a metric exists on any manifold with a dimension greater than three. It would be nice to prove that every manifold with a dimension greater than 4 admits an Einstein metric with negative Ricci curvature.

Comment: The Einstein manifold without extra special structures is a difficult subject. Do we expect a general classification for such an important geometric structure?

5.10. Metric Cobordism. In the last five years, a great deal of attention was addressed by physicists on the holographic principle: boundary geometry should determine the geometry in the interior. The ADS/CFT correspondence studies the conformal boundary of the Einstein manifold which is asymptotically hyperbolic. Gauge theory on the boundary is supposed to be dual to the theory of gravity in the bulk. Much fascinating work was done in this direction. Manifolds with positive scalar curvature appeared as conformal boundary are important for physics. Graham-Lee [**267**] have studied a perturbation problem near the standard sphere which bounds the hyperbolic manifold. Witten-Yau [**722**] proved that for a manifold with positive scalar curvature to be a conformal boundary, it must be connected. It is not known whether there are further obstructions.

Cobordism theory had been a powerful tool in the classification of the topology of manifolds. The first fundamental work was done by Thom who determined the cobordism group. Characteristic numbers play important roles. When two manifolds are cobordant to each other, the theory of surgery helps us to deform one manifold to another. It is clear that any construction of surgery that may preserve geometric structures would play a fundamental role in the future of geometry.

There are many geometric structures that are preserved under a connected sum construction. This includes the category of conformally flat structures, metrics with positive isotropic curvature and metrics with positive scalar curvature. For the last category, there was work by Schoen-Yau-Gromov-Lawson where they perform surgery on spheres with a codimension greater than or equal to 3. A key part of the work of Hamilton-Perelman is to find a canonical neighborhood to perform surgery. If we can deform the spheres in the above SYGL construction to a more canonically defined position, one may be able able to create an extra geometric structure for the result of SYGL. In fact, the construction of Schoen-Yau did provide some information about the conformal structures of the manifold. In complex geometry, there are two important canonical neighborhoods given by the log transform of Kodaira and the operation of flop. There should be similar constructions for other geometric structures.

The theory of quasi-local mass mentioned in Section 4.4 is another example of how boundary geometry can be controlled by the geometry in the bulk. The work of Choi-Wang [**155**] on the first eigenvalue is also based on the manifold that it bounds. There can be interesting theory of metric cobordism.

In the other direction, there are also beautiful rigidity of inverse problems for metric geometry by Gerver-Nadirashvili [**251**] and Pestov-Uhlmann [**554**] on recovering a Riemannian metric when one knows the distance functions between a pair of points on the boundary, if the Riemannian manifold is reasonably convex.

> **Comment**: There should be a mathematical foundation of the holographic principle of physicists. Good understanding of metric cobordism may be useful.

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