Recent results on the moduli space of Riemann surfaces

Kefeng Liu

Abstract. In this paper we briefly discuss some of our recent results in the study of moduli space of Riemann surfaces. It is naturally divided into two parts, one about the differential geometric aspect, another on the topological aspect of the moduli spaces. To understand the geometry of the moduli spaces we introduced new metrics, studied in detail all of the known classical complete metrics, especially the Kähler-Einstein metric. As a corollary we proved that the logarithmic cotangent bundle of the moduli space is strictly stable in the sense of Mumford. The topological results we obtained were motivated by conjectures from string theory. We will describe in this part our proofs by localization method of the Mariño-Vafa formula, its two partition analogue as well as the theory of topological vertex and the simple localization proofs of the ELSV formula and the Witten conjecture. The applications of these formulas in Gromov-Witten theory and string duality will also be mentioned.

1. Introduction

The study of moduli space and Teichmüller space has a long history. These two spaces lie in the intersections of researches in many areas of mathematics and physics. Many deep results have been obtained in history by many famous mathematicians. Moduli spaces and Teichmüller spaces of Riemann surfaces have been studied for many many years since Riemann, by Ahlfors, Bers, Royden, Deligne, Mumford, Yau, Siu, Thurston, Faltings, Witten, Kontsevich, McMullen, Gieseker, Mazur, Harris, Wolpert, Bismut, Sullivan, Madsen and many others including a young generation of mathematicians. Many aspects of the moduli spaces have been understood, but there are still many unsolved problems. Riemann was the first who considered the space $\mathcal{M}$ of all complex structures on an orientable surface modulo the action of orientation preserving diffeomorphisms. He derived the dimension of this space $\dim_{\mathbb{R}} \mathcal{M} = 6g - 6$, where $g \geq 2$ is the genus of the topological surface.

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The moduli space appears in many subjects of mathematics, from geometry, topology, algebraic geometry to number theory. For example, Faltings' proof of the Mordell conjecture depends heavily on the moduli space which can be defined over the integer ring. Moduli space also appears in many areas of theoretical physics. In string theory, many computations of path integrals are reduced to integrals of Chern classes on the moduli space. Based on conjectural physical theories, physicists have made several amazing conjectures about generating series of Hodge integrals for all genera and all marked points on the moduli spaces. The mathematical proofs of these conjectures supply strong evidences to their theories.

This article surveys two types of results; the first is on the geometric aspect of moduli spaces, and the second is on the topological aspect, in particular the computations of Hodge integrals. The first part is based on our joint work with X. Sun and S.-T. Yau. The main results are in [**34**], [**35**] and [**36**]. The second part is based on our joint work with C.-C. Liu, J. Zhou, J. Li and Y.-S. Kim. The main results are contained in [**27**], [**28**], [**21**] and [**14**]. Now we briefly describe some background and statements of the main results.

Our goal of the geometric project with Sun and Yau is to understand the geometry of the moduli spaces. More precisely, we want to understand the relationships among all of the known canonical complete metrics introduced in history on the moduli and the Teichmüller spaces, and by using them to derive geometric consequences about the moduli spaces. More importantly, we introduce and study certain new complete Kähler metrics: the Ricci metric and the perturbed Ricci metric. Through a detailed study we proved that these new metrics have very good curvature properties and Poincaré-type growth near the compactification divisor [**34**], [**35**]. In particular we proved that the perturbed Ricci metric has bounded negative Ricci and holomorphic sectional curvature and has bounded geometry. To the knowledge of the authors this is the first known such metric on the moduli space and the Teichmüller space with such good properties. We know that the Weil-Petersson metric has negative Ricci and holomorphic sectional curvature, but it is incomplete and its curvatures are not bounded from below. Also note that one has no control on the signs of the curvatures of the other complete Kähler metrics mentioned above.

We have obtained a series of results in this direction. In [**34**] and [**35**] we have proved that all of these known complete metrics are actually equivalent, and as a consequence we proved two old conjectures of Yau about the equivalence between the Kähler-Einstein metric and the Teichmüller metric and also its equivalence with the Bergman metric. In [**57**] and [**46**], which were both written in early 1980s, Yau raised various questions about the Kähler-Einstein metric on the Teichmüller space. By using the curvature properties of these new metrics, we obtained good understanding of the Kähler-Einstein metric such as its boundary behavior and the strongly bounded geometry.

As one consequence we proved the stability of the logarithmic extension of the cotangent bundle of the moduli space [**35**]. Note that the major parts of our papers were to understand the Kähler-Einstein metrics and the two new metrics. One of our goals is to find a good metric with the best possible curvature property. The perturbed Ricci metric is close to being such a metric. The most difficult part of our results is the study of the curvature properties and the asymptotic behavior of the new metrics near the boundary, only from which we can derive geometric applications such as the stability of the logarithmic cotangent bundle. The comparisons of those classical metrics as well as the two new metrics are quite easy and actually simple corollaries of the study and the basic definitions of those metrics. In particular the argument we used to prove the equivalences of the Bergman metric, the Kobayashi metric and the Carathéodory metric is rather simple from basic definitions and Yau's Schwarz lemma, and is independent of the other parts of our works.

Our results on the topological aspect of the moduli spaces are all motivated by string theory. This project on the topological aspect of the moduli spaces was jointly carried out with C.-C. Liu, J. Zhou, J. Li and Y.-S. Kim. According to string theorists, string theory, as the most promising candidate for the grand unification of all fundamental forces in nature, should be the final theory of the world, and should be unique. But now there are five different-looking string theories. As argued by the physicists, these theories should be equivalent, in a way dual to each other. On the other hand, all previous theories like the Yang-Mills and the Chern-Simons theory should be parts of string theory. In particular their partition functions should be equal or equivalent to each other in the sense that they are equal after certain transformations. To compute partition functions, physicists use localization technique, a modern version of residue theorem, on infinite dimensional spaces. More precisely they apply localization formally to path integrals, which is not well-defined yet in mathematics. In many cases such computations reduce the path integrals to certain integrals of various Chern classes on various finite dimensional moduli spaces, such as the moduli spaces of stable maps and the moduli spaces of vector bundles. The identifications of these partition functions among different theories have produced many surprisingly beautiful mathematical formulas like the famous mirror formula [**24**], as well as the Mariño-Vafa formula [**39**].

The mathematical proofs of these conjectural formulas from string duality also depend on localization techniques on these various finite dimensional moduli spaces. In this part I will briefly discuss the proof of the Mariño-Vafa formula, its generalizations and the related topological vertex theory [**1**]. More precisely we will use localization formulas in various forms to compute the integrals of Chern classes on moduli spaces, and to prove those conjectures from string duality. For the proof of the Mariño-Vafa formula and the theory of topological vertex, we note that many aspects of mathematics are involved, such as the Chern-Simons knot invariants, combinatorics of symmetric groups, representations of Kac-Moody algebras, Calabi-Yau manifolds, geometry and topology of moduli space of stable maps, etc.

We remark that localization technique has been very successful in proving many conjectures from physics, see my ICM 2002 lecture [**31**] for more examples. One of our major tools in the proofs of these conjectures is the functorial localization formula which is a variation of the classical localization formula: it transfers computations on complicated spaces to simple spaces, and connects computations of mathematicians and physicists.

Starting from the proof of the Mariño-Vafa formula [**28**], we have proved a series of results about Hodge integrals on the moduli spaces of stable curves. Complete closed formulas for the Gromov-Witten invariants of open toric Calabi-Yau manifolds are given, and their relationships with equivariant indices of elliptic operators on the moduli spaces of framed stable bundles on the projective plane are found and proved. Simple localization proofs of the ELSV formula and the Witten conjecture are discovered through this project. Here we can only give a brief overview of the results and the main ideas of their proofs. For the details see [**27**], [**22**], [**28**], [**29**], [**30**], [**21**]. While the Marinõ-Vafa formula gives a close formula for the generating series of triple Hodge integrals on the moduli spaces of all genera and any number marked points, the mathematical theory of topological vertex [**21**] gives the most effective ways to compute the Gromov-Witten invariants of any open toric Calabi-Yau manifolds. Recently Pan Peng was able to use our results on topological vertex to give a complete proof of the Gopakumar-Vafa integrality conjecture for any open toric Calabi-Yau manifolds [**45**]. Kim also used our technique to derive new effective recursion formulas for Hodge integrals on the moduli spaces of stable curves [**13**]. Together we were able to give a very simple direct proof of the Witten conjecture by using localization [**14**].

The spirit of our topological results is the duality between gauge theory, Chern-Simons theory and the Calabi-Yau geometry in string theory. One of our observations about the geometric structure of the moduli spaces is the convolution formula which is encoded in the moduli spaces of relative stable maps [**17**], [**18**], and also in the combinatorics of symmetric groups, [**28**], [**21**]. This convolution structure implies the differential equation which we called the cut-and-join equation. The cut-and-join equation arises from both representation theory and geometry. The verification of the cut-and-join equation in combinatorics is a direct computation through character formulas, while its proof in geometry is quite subtle and involves careful analysis of the fixed points on the moduli spaces of relative stable maps, see [**27**]-[**30**] and [**21**] for more details. The coincidence of such a kind of equation in both geometry and combinatorics is quite remarkable.

The mathematical theory of topological vertex was motivated by the physical theory as first developed by the Vafa group [**1**], who has been working on string duality for the past several years. Topological vertex theory is a high point of their work starting from their geometric engineering theory and Witten's conjecture that Chern-Simons theory is a string theory [**51**].

The Gopakumar-Vafa integrality conjecture is a very interesting conjecture in the subject of Gromov-Witten invariants. It is rather surprising that for some cases such invariants can be interpreted as the indices of elliptic operators in gauge theory in [**27**]. A direct proof of the conjecture for open toric Calabi-Yau manifolds was given recently by Peng [**45**], by using the combinatorial formulas for the generating series of all genera and all degree Gromov-Witten invariants of open toric Calabi-Yau. These closed formulas are derived from the theory of topological vertex through the gluing property.

This note is based on my lecture in May 2005, at the Journal of Differential Geometry Conference in memory of the late great geometer Prof. S.-S. Chern. It is essentially a combination of a survey article by Xiaofeng Sun, Shing-Tung Yau and myself on the geometric aspect of the modulis spaces [**37**] with another survey by myself on localization and string duality [**33**]. Through my research career I have been working in geometry and topology on problems related to Chern classes. Twenty years ago, at his Nankai Institute of Mathematics, a lecture of S.-S. Chern on the Atiyah-Singer index formula introduced me to the beautiful subject of geometry and topology. He described Chern classes and the Atiyah-Singer index formula and its three proofs. That is the first seminar on modern mathematics I had ever attended. It changed my life. I would like to dedicate this note to Prof. Chern for his great influence in my life and in my career.

Part I: The Geometric Aspect

2. Basics on Moduli and the Techmüller Spaces

In this section, we recall some basic facts in Teichmüller theory and introduce various notations for the following discussions.

Let Σ be an orientable surface with genus $g \geq 2$. A complex structure on Σ is a covering of Σ by charts such that the transition functions are holomorphic. By the uniformization theorem, if we put a complex structure on Σ, then it can be viewed as a quotient of the hyperbolic plane $\mathbb{H}^2$ by a Fuchsian group. Thus there is a unique Kähler-Einstein metric, or the hyperbolic metric on Σ.

Let $\mathcal{C}$ be the set of all complex structures on Σ. Let $\mathrm{Diff}^+(\Sigma)$ be the group of orientation preserving diffeomorphisms and let $\mathrm{Diff}_0^+(\Sigma)$ be the subgroup of $\mathrm{Diff}^+(\Sigma)$ consisting of those elements which are isotopic to identity.

The groups $\mathrm{Diff}^+(\Sigma)$ and $\mathrm{Diff}_0^+(\Sigma)$ act naturally on the space $\mathcal{C}$ by pull-back. The Teichmüller space is a quotient of the space $\mathcal{C}$

$$\mathcal{T}_g = \mathcal{C}/\mathrm{Diff}_0^+(\Sigma).$$

From the famous Bers embedding theorem, now we know that $\mathcal{T}_g$ can be embedded into $\mathbb{C}^{3g-3}$ as a pseudoconvex domain and is contractible. Let

$$\mathrm{Mod}_g = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0^+(\Sigma)$$

be the group of isotopic classes of diffeomorphisms. This group is called the (Teichmüller) moduli group or the mapping class group. Its representations are of great interest in topology and in quantum field theory.

The moduli space $\mathcal{M}_g$ is the space of distinct complex structures on Σ and is defined to be

$$\mathcal{M}_g = \mathcal{C}/\mathrm{Diff}^+(\Sigma) = \mathcal{T}_g/\mathrm{Mod}_g.$$

The moduli space is a complex orbifold.

For any point $s \in \mathcal{M}_g$, let $X = X_s$ be a representative of the corresponding class of Riemann surfaces. By the Kodaira-Spencer deformation theory and the Hodge theory, we have

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X.

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X)$ is the space of holomorphic quadratic differentials on X.

Pick $\mu \in HB(X)$ and $\varphi \in Q(X)$. If we fix a holomorphic local coordinate z on X, we can write $\mu = \mu(z)\frac{\partial}{\partial z} \otimes d\overline{z}$ and $\varphi = \varphi(z)dz^2$. Thus the duality between $T_X\mathcal{M}_g$ and $T_X^*\mathcal{M}_g$ is

$$[\mu : \varphi] = \int_X \mu(z)\varphi(z)dzd\overline{z}.$$

By the Riemann-Roch theorem, we have

$$\dim_{\mathbb{C}} HB(X) = \dim_{\mathbb{C}} Q(X) = 3g - 3,$$

which implies

$$\dim_{\mathbb{C}} \mathcal{T}_g = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3.$$

3. Classical Metrics on the Moduli Spaces

In 1940s, Teichmüller considered a cover of $\mathcal{M}$ by taking the quotient of all complex structures by those orientation preserving diffeomorphims which are isotopic to the identity map. The Teichmüller space $\mathcal{T}_g$ is a contractible set in $\mathbb{C}^{3g-3}$. Furthermore, it is a pseudoconvex domain. Teichmüller also introduced the Teichmüller metric by first taking the L^1 norm on the cotangent space of $\mathcal{T}_g$ and then taking the dual norm on the tangent space. This is a Finsler metric. Two other interesting Finsler metrics are the Carathéodory metric and the Kobayashi metric. These Finsler metrics have been powerful tools in the study of the hyperbolic property of the moduli and the Teichmüller spaces and the mapping class groups. For example, in the 1970s Royden proved that the Teichmüller metric and the Kobayashi metric are the same, and as a corollary he proved the famous result that the holomorphic automorphism group of the Teichmüller space is exactly the mapping class group.

Based on the Petersson pairing on the spaces of automorphic forms, Weil introduced the first Hermitian metric on the Teichmüller space, the Weil-Petersson metric. It was shown by Ahlfors that the Weil-Petersson metric is Kähler and its holomorphic sectional curvature is negative. The work of Ahlfors and Bers on the solutions of Beltrami equation put a solid foundation of the theory of Teichmüller space and moduli space [**3**]. Wolpert studied in detail the Weil-Petersson metric including the precise upper bound of its Ricci and holomorphic sectional curvature. From these one can derive interesting applications in algebraic geometry. For example, see [**32**].

Moduli spaces of Riemann surfaces have also been studied in detail in algebraic geometry since 1960. The major tool is the geometric invariant theory developed by Mumford. In the 1970s, Deligne and Mumford studied the projective property of the moduli space and showed that the moduli space is quasi-projective and can be compactified naturally by adding in the stable nodal surfaces [**6**]. Fundamental work has been done by Gieseker, Harris and many other algebraic geometers. Note that the compactification in algebraic geometry is the same as the differential geometric compactification by using the Weil-Petersson metric.

The work of Cheng-Yau [**5**] in the early 1980s showed that there is a unique complete Kähler-Einstein metric on the Teichmüller space and is invariant under the moduli group action. Thus it descends to the moduli space. As it is well-known, the existence of the Kähler-Einstein metric gives deep algebraic geometric results, so it is natural to understand its properties like the curvature and the behaviors near the compactification divisor. In the early 1980s, Yau conjectured that the Kähler-Einstein metric is equivalent to the Teichmüller metric and the Bergman metric [**4**], [**57**], [**46**].

In 2000, McMullen introduced a new metric, the McMullen metric, by perturbing the Weil-Petersson metric to get a complete Kähler metric which is complete and Kähler hyperbolic. Thus the lowest eigenvalue of the Laplace operator is positive and the L^2-cohomology is trivial except for the middle dimension [**41**].

So there are many very famous classical metrics on the Teichmüller and the moduli spaces, and they have been studied independently by many famous mathematicians. Each metric has played an important role in the study of the geometry and topology of the moduli and Teichmüller spaces. There are three Finsler metrics: the Teichmüller metric $\|\cdot\|_T$, the Kobayashi metric $\|\cdot\|_K$ and the Carathéodory metric $\|\cdot\|_C$. They are all complete metrics on the Teichmüller space and are invariant under the moduli group action. Thus they descend down to the moduli space as complete Finsler metrics.

There are seven Kähler metrics: the Weil-Petersson metric ω_{WP} which is incomplete, the Cheng-Yau's Kähler-Einstein metric ω_{KE}, the McMullen metric ω_C, the Bergman metric ω_B, the asymptotic Poincaré metric on the moduli space ω_P, the Ricci metric ω_τ and the perturbed Ricci metric $\omega_{\tilde{\tau}}$. The last six metrics are complete. The last two metrics are new metrics studied in details in [**34**] and [**35**].

Now let us give the precise definitions of these metrics and state their basic properties.

The Teichmüller metric was first introduced by Teichmüller as the L^1 norm in the cotangent space. For each $\varphi = \varphi(z)dz^2 \in Q(X) \cong T^*_X\mathcal{M}_g$, the Teichmüller norm of φ is

$$\|\varphi\|_T = \int_X |\varphi(z)|\ dzd\overline{z}.$$

By using the duality, for each $\mu \in HB(X) \cong T_X\mathcal{M}_g$,

$$\|\mu\|_T = \sup\{Re[\mu;\varphi] \mid \|\varphi\|_T = 1\}.$$

It is known that Teichmüller metric has constant holomorphic sectional curvature -1.

The Kobayashi and the Carathéodory metrics can be defined for any complex space in the following way: Let Y be a complex manifold of dimension n. Let Δ_R be the disk in $\mathbb{C}$ with radius R. Let $\Delta = \Delta_1$ and let ρ be the Poincaré metric on Δ. Let $p \in Y$ be a point and let $v \in T_pY$ be a holomorphic tangent vector. Let $\mathrm{Hol}(Y,\Delta_R)$ and $\mathrm{Hol}(\Delta_R, Y)$ be the spaces of holomorphic maps from Y to Δ_R and from Δ_R to Y respectively. The Carathéodory norm of the vector v is defined to be

$$\|v\|_C = \sup_{f\in \mathrm{Hol}(Y,\Delta)} \|f_*v\|_{\Delta,\rho}$$

and the Kobayashi norm of v is defined to be

$$\|v\|_K = \inf_{f\in \mathrm{Hol}(\Delta_R,Y),\ f(0)=p,\ f'(0)=v} \frac{2}{R}.$$

The Bergman (pseudo) metric can also be defined for any complex space Y provided the Bergman kernel is positive. Let K_Y be the canonical bundle of Y and let W be the space of L^2 holomorphic sections of K_Y in the sense that if $\sigma \in W$, then

$$\|\sigma\|_{L^2}^2 = \int_Y (\sqrt{-1})^{n^2} \sigma \wedge \overline{\sigma} < \infty.$$

The inner product on W is defined to be

$$(\sigma, \rho) = \int_Y (\sqrt{-1})^{n^2} \sigma \wedge \overline{\rho}$$

for all $\sigma, \rho \in W$. Let $\sigma_1, \sigma_2, \ldots$ be an orthonormal basis of W. The Bergman kernel form is the non-negative (n,n)-form

$$B_Y = \sum_{j=1}^{\infty} (\sqrt{-1})^{n^2} \sigma_j \wedge \overline{\sigma}_j.$$

With a choice of local coordinates $z_i, \ldots, z_n$, we have

$$B_Y = BE_Y(z,\overline{z})(\sqrt{-1})^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n$$

where $BE_Y(z,\overline{z})$ is called the Bergman kernel function. If the Bergman kernel B_Y is positive, one can define the Bergman metric

$$B_{i\overline{j}} = \frac{\partial^2 \log BE_Y(z,\overline{z})}{\partial z_i \partial \overline{z}_j}.$$

The Bergman metric is well-defined and is nondegenerate if the elements in W separate points and the first jet of Y. In this case, the Bergman metric is a Kähler metric.

Remark 3.1. Both the Teichmüller space and the moduli space are equipped with the Bergman metrics. However, the Bergman metric on the moduli space is different from the metric induced from the Bergman metric of the Teichmüller space. The Bergman metric defined on the moduli space is incomplete due to the fact that the moduli space is quasi-projective and any L^2 holomorphic section of the canonical bundle can be extended over. However, the induced one is complete as we shall see later.

The basic properties of the Kobayashi, Carathéodory and Bergman metrics are stated in the following proposition. Please see [**15**] for the details.

Proposition 3.1. *Let X be a complex space. Then*

(1) $\|\cdot\|_{C,X} \leq \|\cdot\|_{K,X}$;
(2) *Let Y be another complex space and $f : X \to Y$ be a holomorphic map. Let $p \in X$ and $v \in T_pX$. Then $\|f_*(v)\|_{C,Y,f(p)} \leq \|v\|_{C,X,p}$ and $\|f_*(v)\|_{K,Y,f(p)} \leq \|v\|_{K,X,p}$;*

(3) *If $X \subset Y$ is a connected open subset and $z \in X$ is a point, then with any local coordinates we have $BE_Y(z) \leq BE_X(z)$;*
(4) *If the Bergman kernel is positive, then at each point $z \in X$, a peak section σ at z exists. Such a peak section is unique up to a constant factor c with norm 1. Furthermore, with any choice of local coordinates, we have $BE_X(z) = |\sigma(z)|^2$;*
(5) *If the Bergman kernel of X is positive, then $\|\cdot\|_{C,X} \leq 2\|\cdot\|_{B,X}$;*
(6) *If X is a bounded convex domain in $\mathbb{C}^n$, then $\|\cdot\|_{C,X} = \|\cdot\|_{K,X}$;*
(7) *Let $|\cdot|$ be the Euclidean norm and let B_r be the open ball with center 0 and radius r in $\mathbb{C}^n$. Then for any holomorphic tangent vector v at 0,*
$$\|v\|_{C,B_r,0} = \|v\|_{K,B_r,0} = \frac{2}{r}|v|,$$
where $|v|$ is the Euclidean norm of v.

The three Finsler metrics have been very powerful tools in understanding the hyperbolic geometry of the moduli spaces, and the mapping class group. It has also been known since the 1970s that the Bergman metric on the Teichmüller space is complete.

The Weil-Petersson metric is the first Kähler metric defined on the Teichmüller and the moduli space. It is defined by using the L^2 inner product on the tangent space in the following way:
Let $\mu, \nu \in T_X\mathcal{M}_g$ be two tangent vectors and let λ be the unique Kähler-Einstein metric on X. Then the Weil-Petersson metric is
$$h(\mu,\nu) = \int_X \mu\overline{\nu}\, dv$$
where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form. Details can be found in [**34**], [**40**] and [**54**].

The curvatures of the Weil-Petersson metric have been well-understood due to the works of Ahlfors, Royden and Wolpert. Its Ricci and holomorphic sectional curvature are all negative with negative upper bound, but with no lower bound. Its boundary behavior is understood, from which it is not hard to see that it is an incomplete metric.

The existence of the Kähler-Einstein metric was given by the work of Cheng-Yau [**4**]. Its Ricci curvature is -1. Namely,
$$\partial\overline{\partial}\log\omega_{KE}^n = \omega_{KE},$$
where $n = 3g-3$. They actually proved that a bounded domain in $\mathbb{C}^n$ admits a complete Kähler-Einstein metric if and only if it is pseudoconvex.

The McMullen $1/l$ metric defined in [**41**] is a perturbation of the Weil-Petersson metric by adding a Kähler form whose potential involves the short geodesic length functions on the Riemann surfaces. For each simple closed curve γ in X, let $l_\gamma(X)$ be the length of the unique geodesic in the homotopy class of γ with respect to the unique Kähler-Einstein metric. Then the

McMullen metric is defined as

$$\omega_{1/l} = \omega_{WP} - i\delta \sum_{l_\gamma(X)<\epsilon} \partial\overline{\partial}\mathrm{Log}\frac{\epsilon}{l_\gamma}$$

where ϵ and δ are small positive constants and $\mathrm{Log}(x)$ is a smooth function defined as

$$\mathrm{Log}(x) = \begin{cases} \log x, & x \geq 2, \\ 0, & x \leq 1. \end{cases}$$

This metric is Kähler hyperbolic, which means it satisfies the following conditions:

(1) $(\mathcal{M}_g, \omega_{1/l})$ has finite volume;
(2) The sectional curvature of $(\mathcal{M}_g, \omega_{1/l})$ is bounded above and below;
(3) The injectivity radius of $(\mathcal{T}_g, \omega_{1/l})$ is bounded below;
(4) On $\mathcal{T}_g$, the Kähler form $\omega_{1/l}$ can be written as $\omega_{1/l} = d\alpha$ where α is a bounded 1-form.

An immediate consequence of the Kähler hyperbolicity is that the L^2-cohomology is trivial except for the middle dimension.

The asymptotic Poincaré metric can be defined as a complete Kähler metric on a complex manifold M which is obtained by removing a divisor Y with only normal crossings from a compact Kähler manifold $(\overline{M}, \omega)$.

Let $\overline{M}$ be a compact Kähler manifold of dimension m. Let $Y \subset \overline{M}$ be a divisor of normal crossings and let $M = \overline{M} \setminus Y$. Cover $\overline{M}$ by coordinate charts $U_1, \ldots, U_p, \ldots, U_q$ such that $(\overline{U}_{p+1} \cup \cdots \cup \overline{U}_q) \cap Y = \Phi$. We also assume that for each $1 \leq \alpha \leq p$, there is a constant n_α such that $U_\alpha \setminus Y = (\Delta^*)^{n_\alpha} \times \Delta^{m-n_\alpha}$ and on U_α, Y is given by $z_1^\alpha \cdots z_{n_\alpha}^\alpha = 0$. Here Δ is the disk of radius $\frac{1}{2}$ and Δ^* is the punctured disk of radius $\frac{1}{2}$. Let $\{\eta_i\}_{1\leq i\leq q}$ be the partition of unity subordinate to the cover $\{U_i\}_{1\leq i\leq q}$. Let ω be a Kähler metric on $\overline{M}$ and let C be a positive constant. Then for C large, the Kähler form

$$\omega_p = C\omega + \sum_{i=1}^{p} \sqrt{-1}\partial\overline{\partial}\left(\eta_i \log\log \frac{1}{z_1^i \cdots z_{n_i}^i}\right)$$

defines a complete metric on M with finite volume since on each U_i with $1 \leq i \leq p$, ω_p is bounded from above and below by the local Poincaré metric on U_i. We call this metric the asymptotic Poincaré metric.

The signs of the curvatures of the above metrics are all unknown. We actually only know that the Kähler-Einstein metric has constant negative Ricci curvature and that the McMullen metric has bounded geometry. Also, except the asymptotic Poincaré metric, the boundary behaviors of the other metrics are unknown before our works [**34**], [**35**]. It is interesting that to understand them we need to introduce new metrics.

Now we define the Ricci metric and the perturbed Ricci metric. The curvature properties and asymptotics of these two new metrics are understood by us and will be stated in the following sections. Please also see [**34**] and [**35**] for details.

With the works of Ahlfors, Royden and Wolpert we know that the Ricci curvature of the Weil-Petersson metric has a negative upper bound. Thus we can use the negative Ricci form of the Weil-Petersson metric as the Kähler form of a new metric. We call this metric the Ricci metric and denote it by τ. That is

$$\omega_\tau = \partial\overline{\partial} \log \omega_{WP}^n.$$

Through the careful analysis, we now understand that the Ricci metric is a natural canonical complete Kähler metric with many good properties. However, its holomorphic sectional curvature is only asymptotically negative. To get a metric with good sign on its curvatures, we introduced the perturbed Ricci metric $\omega_{\widetilde{\tau}}$ as a combination of the Ricci metric and the Weil-Petersson metric:

$$\omega_{\widetilde{\tau}} = \omega_\tau + C\omega_{WP}$$

where C is a large positive constant. As we shall see later, the perturbed Ricci metric has desired curvature properties so that we can put it either on the target or on the domain manifold in Yau's Schwarz lemma, from which we can compare the above metrics.

4. The Curvature Formulas

In this section we describe the harmonic lift of a vector field on the moduli space to the universal curve due to Royden, Siu [**48**] and Schumacher [**47**]. Details can also be found in [**34**]. We then use this method to derive the curvature formula for the Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric.

To compute the curvature of a metric on the moduli space, we need to take derivatives of the metric in the direction of the moduli space. However, it is quite difficult to estimate the curvature by using a formula obtained in such a way. The central idea is to obtain a formula where the derivatives are taken in the fiber direction. We can view the deformation of complex structures on a topological surface as the deformation of the Kähler-Einstein metrics.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, where $g \geq 2$. Let $n = 3g - 3$ be the complex dimension of $\mathcal{M}_g$. Let $\mathfrak{X}$ be the total space and let $\pi : \mathfrak{X} \to \mathcal{M}_g$ be the projection map.

Let $s_1, \ldots, s_n$ be holomorphic local coordinates near a regular point $s \in \mathcal{M}_g$ and assume that z is a holomorphic local coordinate on the fiber $X_s = \pi^{-1}(s)$. For holomorphic vector fields $\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_n}$, there are vector fields $v_1, \ldots, v_n$ on $\mathfrak{X}$ such that

(1) $\pi_*(v_i) = \frac{\partial}{\partial s_i}$ for $i = 1, \ldots, n$;

(2) $\overline{\partial} v_i$ are harmonic TX_s-valued $(0,1)$ forms for $i = 1, \ldots, n$.

The vector fields $v_1, \ldots, v_n$ are called the harmonic lift of the vectors $\frac{\partial}{\partial s_1}$, $\ldots, \frac{\partial}{\partial s_n}$. The existence of such harmonic vector fields was pointed out by Siu. Schumacher in his work gave an explicit construction of such lift. We now describe it.

Since $g \geq 2$, we can assume that each fiber is equipped with the Kähler-Einstein metric $\lambda = \frac{\sqrt{-1}}{2}\lambda(z,s)dz \wedge d\overline{z}$. The Kähler-Einstein condition gives the following equation:

$$\partial_z \partial_{\overline{z}} \log \lambda = \lambda. \tag{4.1}$$

For the rest of this paper we denote $\frac{\partial}{\partial s_i}$ by ∂_i and $\frac{\partial}{\partial z}$ by ∂_z. Let

$$a_i = -\lambda^{-1} \partial_i \partial_{\overline{z}} \log \lambda$$

and let

$$A_i = \partial_{\overline{z}} a_i.$$

Then the harmonic horizontal lift of ∂_i is

$$v_i = \partial_i + a_i \partial_z.$$

In particular

$$B_i = A_i \partial_z \otimes d\overline{z} \in H^1(X_s, T_{X_s})$$

is harmonic. Furthermore, the lift $\partial_i \mapsto B_i$ gives the Kodaira-Spencer map $T_s\mathcal{M}_g \to H^1(X_s, T_{X_s})$. Thus the Weil-Petersson metric on $\mathcal{M}_g$ is

$$h_{i\overline{j}}(s) = \int_{X_s} B_i \cdot \overline{B_j}\, dv = \int_{X_s} A_i \overline{A_j}\, dv,$$

where $dv = \frac{\sqrt{-1}}{2}\lambda dz \wedge d\overline{z}$ is the volume form on the fiber X_s.

Let $R_{i\overline{j}k\overline{l}}$ be the curvature tensor of the Weil-Petersson metric. Here we adopt the following notation for the curvature of a Kähler metric:
For a Kähler metric (M, g), the curvature tensor is given by

$$R_{i\overline{j}k\overline{l}} = \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z}_l} - g^{p\overline{q}} \frac{\partial g_{i\overline{q}}}{\partial z_k} \frac{\partial g_{p\overline{j}}}{\partial \overline{z}_l}.$$

In this case, the Ricci curvature is given by

$$R_{i\overline{j}} = -g^{k\overline{l}} R_{i\overline{j}k\overline{l}}.$$

By using the curvature of the Weil-Petersson metric, we can define the Ricci metric:

$$\tau_{i\overline{j}} = h^{k\overline{l}} R_{i\overline{j}k\overline{l}}$$

and the perturbed Ricci metric:

$$\widetilde{\tau}_{i\overline{j}} = \tau_{i\overline{j}} + C h_{i\overline{j}},$$

where C is a positive constant.

Before we present the curvature formulas for the above metrics, we need to introduce the Maass operators and norms on a Riemann surface [**54**].

Let X be a Riemann surface and let κ be its canonical bundle. For any integer p, let $S(p)$ be the space of smooth sections of $(\kappa \otimes \overline{\kappa}^{-1})^{\frac{p}{2}}$. Fix a conformal metric $ds^2 = \rho^2(z)|dz|^2$. In the following, we will take ds^2 to be the Kähler-Einstein metric although the following definitions work for all metrics.

The Maass operators K_p and L_p are defined to be the metric derivatives $K_p : S(p) \to S(p+1)$ and $L_p : S(p) \to S(p-1)$ given by

$$K_p(\sigma) = \rho^{p-1}\partial_z(\rho^{-p}\sigma)$$

and

$$L_p(\sigma) = \rho^{-p-1}\partial_{\overline{z}}(\rho^p\sigma)$$

where $\sigma \in S(p)$.

The operators $P = K_1K_0$ and $\Box = -L_1K_0$ will play important roles in the curvature formulas. Here the operator $\Box$ is just the Laplace operator. We also let $T = (\Box + 1)^{-1}$ be the Green operator.

Each element $\sigma \in S(p)$ has a well-defined absolute value $|\sigma|$ which is independent of the choice of local coordinate. We define the C^k norm of σ: Let Q be an operator which is a composition of operators K_* and L_*. Denote by $|Q|$ the number of factors. For any $\sigma \in S(p)$, define

$$\|\sigma\|_0 = \sup_X |\sigma|$$

and

$$\|\sigma\|_k = \sum_{|Q|\le k} \|Q\sigma\|_0.$$

We can also localize the norm on a subset of X. Let $\Omega \subset X$ be a domain. We can define

$$\|\sigma\|_{0,\Omega} = \sup_\Omega |\sigma|$$

and

$$\|\sigma\|_{k,\Omega} = \sum_{|Q|\le k} \|Q\sigma\|_{0,\Omega}.$$

We let $f_{i\overline{j}} = A_i\overline{A}_j$ and $e_{i\overline{j}} = T(f_{i\overline{j}})$. These functions will be the building blocks for the curvature formulas.

The trick of converting derivatives from the moduli directions to the fiber directions is the following lemma due to Siu and Schumacher:

Lemma 4.1. *Let η be a relative $(1,1)$-form on the total space $\mathfrak{X}$. Then*

$$\frac{\partial}{\partial s_i}\int_{X_s} \eta = \int_{X_s} L_{v_i}\eta.$$

The curvature formula of the Weil-Petersson metric was first established by Wolpert by using a different method [**52**] and later was generalized by Siu [**48**] and Schumacher [**47**] by using the above lemma:

THEOREM 4.2. *The curvature of the Weil-Petersson metric is given by*

$$R_{i\bar{j}k\bar{l}} = \int_{X_s} (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}})\, dv. \tag{4.2}$$

For the proof, please see [**34**]. From this formula it is rather easy to show that the Ricci and the holomorphic sectional curvature have explicit negative upper bound.

To establish the curvature formula of the Ricci metric, we need to introduce more operators. Firstly, the commutator of the operator v_k and $(\Box+1)$ will play an important role. Here we view the vector field v_k as a operator acting on functions. We define

$$\xi_k = [\Box + 1, v_k].$$

A direct computation shows that

$$\xi_k = -A_k P.$$

Also we can define the commutator of $\overline{v_l}$ and ξ_k. Let

$$Q_{k\bar{l}} = [\overline{v_l}, \xi_k].$$

We have

$$Q_{k\bar{l}}(f) = \overline{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\Box f + \lambda^{-1}\partial_z f_{k\bar{l}}\partial_{\bar{z}} f$$

for any smooth function f.

To simplify the notation, we introduce the symmetrization operator of the indices. Let U be any quantity which depends on indices $i, k, \alpha, \bar{j}, \bar{l}, \bar{\beta}$. The symmetrization operator σ_1 is defined by taking summation of all orders of the triple (i, k, α). That is

$$\begin{aligned}\sigma_1(U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta})) = {} & U(i,k,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(i,\alpha,k,\bar{j},\bar{l},\bar{\beta}) \\ & + U(k,i,\alpha,\bar{j},\bar{l},\bar{\beta}) + U(k,\alpha,i,\bar{j},\bar{l},\bar{\beta}) \\ & + U(\alpha,i,k,\bar{j},\bar{l},\bar{\beta}) + U(\alpha,k,i,\bar{j},\bar{l},\bar{\beta}).\end{aligned}$$

Similarly, σ_2 is the symmetrization operator of $\bar{j}$ and $\bar{\beta}$ and $\widetilde{\sigma_1}$ is the symmetrization operator of $\bar{j}$, $\bar{l}$ and $\bar{\beta}$.

In [**34**] the following curvature formulas for the Ricci and perturbed Ricci metric were proved:

THEOREM 4.3. *Let $s_1,\dots,s_n$ be local holomorphic coordinates at $s\in M_g$. Then at s, we have*

$$
\begin{aligned}
(4.3)\quad \widetilde{R}_{i\bar{j}k\bar{l}} = h^{\alpha\overline{\beta}}\Big\{\sigma_1\sigma_2\int_{X_s} &\Big\{(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}) \\
&+(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\Big\}\,dv\Big\} \\
&+h^{\alpha\overline{\beta}}\Big\{\sigma_1\int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\Big\} \\
&-\tau^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\Big\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\Big\}\Big\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\Big\} \\
&+\tau_{p\bar{j}}h^{p\overline{q}}R_{i\overline{q}k\bar{l}}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.4)\quad P_{i\bar{j}k\bar{l}} = h^{\alpha\overline{\beta}}\Big\{\sigma_1\sigma_2\int_{X_s} &\Big\{(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_l(e_{\alpha\overline{\beta}}) \\
&+(\Box+1)^{-1}(\xi_k(e_{i\bar{j}}))\overline{\xi}_\beta(e_{\alpha\bar{l}})\Big\}\,dv\Big\} \\
&+h^{\alpha\overline{\beta}}\Big\{\sigma_1\int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}})e_{\alpha\overline{\beta}}\,dv\Big\} \\
&-\widetilde{\tau}^{p\overline{q}}h^{\alpha\overline{\beta}}h^{\gamma\overline{\delta}}\Big\{\sigma_1\int_{X_s}\xi_k(e_{i\overline{q}})e_{\alpha\overline{\beta}}\,dv\Big\}\Big\{\widetilde{\sigma}_1\int_{X_s}\overline{\xi}_l(e_{p\bar{j}})e_{\gamma\overline{\delta}})\,dv\Big\} \\
&+\tau_{p\bar{j}}h^{p\overline{q}}R_{i\overline{q}k\bar{l}}+CR_{i\bar{j}k\bar{l}}
\end{aligned}
$$

where $R_{i\bar{j}k\bar{l}}$, $\widetilde{R}_{i\bar{j}k\bar{l}}$, and $P_{i\bar{j}k\bar{l}}$ are the curvature of the Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric, respectively.

Unlike the curvature formula of the Weil-Petersson metric, from which we can see the sign of the curvature directly, the above formulas are too complicated and we cannot see the sign. So we need to study the asymptotic behaviors of these curvatures, and first the metrics themselves.

5. The Asymptotics

To compute the asymptotics of these metrics and their curvatures, we first need to find a canonical way to construct local coordinates near the boundary of the moduli space. We first describe the Deligne-Mumford compactification of the moduli space and introduce the pinching coordinate and the plumbing construction according to Earle and Marden. Please see [**40**], [**54**], [**49**] and [**34**] for details.

A point p in a Riemann surface X is a node if there is a neighborhood of p which is isometric to the germ $\{(u,v)\mid uv=0,\ |u|,|v|<1\}\subset\mathbb{C}^2$. Let $p_1,\dots,p_k$ be the nodes on X. X is called stable if each connected component of $X\backslash\{p_1,\dots,p_k\}$ has negative Euler characteristic. Namely, each connected component has a unique complete hyperbolic metric.

Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus $g \geq 2$. The Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ is the union of $\mathcal{M}_g$ and corresponding stable nodal surfaces [**6**]. Each point $y \in \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ corresponds to a stable noded surface X_y.

We recall the rs-coordinate on a Riemann surface defined by Wolpert in [**54**]. There are two cases: the puncture case and the short geodesic case. For the puncture case, we have a noded surface X and a node $p \in X$. Let a, b be two punctures which are paired to form p.

Definition 5.1. The local coordinate chart (U, u) near a is called rs-coordinate if $u(a) = 0$, u maps U to the punctured disc $0 < |u| < c$ with $c > 0$ and the restriction to U of the Kähler-Einstein metric on X can be written as $\frac{1}{2|u|^2(\log|u|)^2}|du|^2$. The rs-coordinate (V, v) near b is defined in a similar way.

For the short geodesic case, we have a closed surface X, a closed geodesic $\gamma \subset X$ with length $l < c_*$, where c_* is the collar constant.

Definition 5.2. The local coordinate chart (U, z) is called rs-coordinate at γ if $\gamma \subset U$, z maps U to the annulus $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$ and the Kähler-Einstein metric on X can be written as

$$\frac{1}{2}\left(\frac{\pi}{\log|t|}\frac{1}{|z|}\csc\frac{\pi\log|z|}{\log|t|}\right)^2|dz|^2.$$

Remark 5.1. We put the factor $\frac{1}{2}$ in the above two definitions to normalize such that (4.1) holds.

By Keen's collar theorem, we have the following lemma:

Lemma 5.3. *Let X be a closed surface and let γ be a closed geodesic on X such that the length l of γ satisfies $l < c_*$. Then there is a collar Ω on X with holomorphic coordinate z defined on Ω such that*

(1) *z maps Ω to the annulus $\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c$ for $c > 0$;*
(2) *the Kähler-Einstein metric on X restricted to Ω is given by*

$$(\frac{1}{2}u^2r^{-2}\csc^2\tau)|dz|^2, \tag{5.1}$$

where $u = \frac{l}{2\pi}$, $r = |z|$ and $\tau = u\log r$;

(3) *the geodesic γ is given by $|z| = e^{-\frac{\pi^2}{l}}$.*

We call such a collar Ω a genuine collar.

We notice that the constant c in the above lemma has a lower bound such that the area of Ω is bounded from below. Also, the coordinate z in the above lemma is rs-coordinate. In the following, we will keep the notation u, r and τ.

Now we describe the local manifold cover of $\overline{\mathcal{M}}_g$ near the boundary. We take the construction of Wolpert [**54**]. Let $X_{0,0}$ be a noded surface corresponding to a codimension m boundary point. $X_{0,0}$ have m nodes $p_1, \ldots, p_m$. $X_0 = X_{0,0} \setminus \{p_1, \ldots, p_m\}$ is a union of punctured Riemann surfaces. Fix rs-coordinate charts (U_i, η_i) and (V_i, ζ_i) at p_i for $i = 1, \ldots, m$ such that all the U_i and V_i are mutually disjoint. Now pick an open set $U_0 \subset X_0$ such that the intersection of each connected component of X_0 and U_0 is a nonempty relatively compact set and the intersection $U_0 \cap (U_i \cup V_i)$ is empty for all i. Now pick Beltrami differentials $\nu_{m+1}, \ldots, \nu_n$ which are supported in U_0 and span the tangent space at X_0 of the deformation space of X_0. For $s = (s_{m+1}, \ldots, s_n)$, let $\nu(s) = \sum_{i=m+1}^n s_i \nu_i$. We assume $|s| = (\sum |s_i|^2)^{\frac{1}{2}}$ is small enough such that $|\nu(s)| < 1$. The noded surface $X_{0,s}$ is obtained by solving the Beltrami equation $\overline{\partial} w = \nu(s) \partial w$. Since $\nu(s)$ is supported in U_0, (U_i, η_i) and (V_i, ζ_i) are still holomorphic coordinates on $X_{0,s}$. Note that they are no longer rs-coordinates. By the theory of Alhfors and Bers [**3**] and Wolpert [**54**] we can assume that there are constants $\delta, c > 0$ such that when $|s| < \delta$, η_i and ζ_i are holomorphic coordinates on $X_{0,s}$ with $0 < |\eta_i| < c$ and $0 < |\zeta_i| < c$. Now we assume $t = (t_1, \ldots, t_m)$ has small norm. We do the plumbing construction on $X_{0,s}$ to obtain $X_{t,s}$. Remove from $X_{0,s}$ the discs $0 < |\eta_i| \le \frac{|t_i|}{c}$ and $0 < |\zeta_i| \le \frac{|t_i|}{c}$ for each $i = 1, \ldots, m$. Now identify $\frac{|t_i|}{c} < |\eta_i| < c$ with $\frac{|t_i|}{c} < |\zeta_i| < c$ by the rule $\eta_i \zeta_i = t_i$. This defines the surface $X_{t,s}$. The tuple $(t_1, \ldots, t_m, s_{m+1}, \ldots, s_n)$ are the local pinching coordinates for the manifold cover of $\overline{\mathcal{M}}_g$. We call the coordinates η_i (or ζ_i) the plumbing coordinates on $X_{t,s}$ and the collar defined by $\frac{|t_i|}{c} < |\eta_i| < c$ the plumbing collar.

Remark 5.2. By the estimate of Wolpert [**53**], [**54**] on the length of short geodesic, the quantity $u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log |t_i|}$.

Now we describe the estimates of the asymptotics of these metrics and their curvatures. The principle is that, when we work on a nearly degenerated surface, the geometry focuses on the collars. Our curvature formulas depend on the Kähler-Einstein metrics of the family of Riemann surfaces near a boundary points. One can obtain an approximate Kähler-Einstein metric on these collars by the graft construction of Wolpert [**54**] which is done by gluing the hyperbolic metric on the nodal surface with the model metric described above.

To use the curvature formulas (4.2), (4.3) and (4.4) to estimate the asymptotic behavior, one also needs to analyze the transition from the plumbing coordinates on the collars to the rs-coordinates. The harmonic Beltrami differentials were constructed by Masur [**40**] by using the plumbing coordinates, and it is easier to compute the integration by using rs-coordinates. Such computation was done in [**49**] by using the graft metric of Wolpert and the maximum principle. A clear description can be found in [**34**]. We have the following theorem:

THEOREM 5.4. *Let (t,s) be the pinching coordinates on $\overline{\mathcal{M}}_g$ near $X_{0,0}$ which corresponds to a codimension m boundary point of $\overline{\mathcal{M}}_g$. Then there exist constants $M, \delta > 0$ and $0 < c < 1$ such that if $|(t,s)| < \delta$, then the j-th plumbing collar on $X_{t,s}$ contains the genuine collar Ω_c^j. Furthermore, one can choose rs-coordinate z_j on the collar Ω_c^j properly such that the holomorphic quadratic differentials $\psi_1, \dots, \psi_n$ corresponding to the cotangent vectors $dt_1, \dots, ds_n$ have form $\psi_i = \varphi_i(z_j)dz_j^2$ on the genuine collar Ω_c^j for $1 \le j \le m$ where*

(1) $\varphi_i(z_j) = \frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $i \ge m+1$;
(2) $\varphi_i(z_j) = (-\frac{t_j}{\pi})\frac{1}{z_j^2}(q_j(z_j) + \beta_j)$ *if* $i = j$;
(3) $\varphi_i(z_j) = (-\frac{t_i}{\pi})\frac{1}{z_j^2}(q_i^j(z_j) + \beta_i^j)$ *if* $1 \le i \le m$ *and* $i \ne j$.

Here β_i^j and β_j are functions of (t,s), q_i^j and q_j are functions of (t,s,z_j) given by

$$q_i^j(z_j) = \sum_{k<0} \alpha_{ik}^j(t,s)t_j^{-k}z_j^k + \sum_{k>0} \alpha_{ik}^j(t,s)z_j^k$$

and

$$q_j(z_j) = \sum_{k<0} \alpha_{jk}(t,s)t_j^{-k}z_j^k + \sum_{k>0} \alpha_{jk}(t,s)z_j^k$$

such that

(1) $\sum_{k<0} |\alpha_{ik}^j| c^{-k} \le M$ *and* $\sum_{k>0} |\alpha_{ik}^j| c^k \le M$ *if* $i \ne j$;
(2) $\sum_{k<0} |\alpha_{jk}| c^{-k} \le M$ *and* $\sum_{k>0} |\alpha_{jk}| c^k \le M$;
(3) $|\beta_i^j| = O(|t_j|^{\frac{1}{2}-\epsilon})$ *with* $\epsilon < \frac{1}{2}$ *if* $i \ne j$;
(4) $|\beta_j| = (1 + O(u_0))$

where $u_0 = \sum_{i=1}^m u_i + \sum_{j=m+1}^n |s_j|$.

By definition, the metric on the cotangent bundle induced by the Weil-Petersson metric is given by

$$h^{i\overline{j}} = \int_{X_{t,s}} \lambda^{-2} \varphi_i \overline{\varphi}_j \, dv.$$

We then have the following series of estimates, see [**34**]. First by using this formula and taking inverse, we can estimate the Weil-Petersson metric.

THEOREM 5.5. *Let (t,s) be the pinching coordinates. Then*

(1) $h^{i\overline{i}} = 2u_i^{-3}|t_i|^2(1+O(u_0))$ *and* $h_{i\overline{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1+O(u_0))$ *for* $1 \le i \le m$;
(2) $h^{i\overline{j}} = O(|t_it_j|)$ *and* $h_{i\overline{j}} = O(\frac{u_i^3 u_j^3}{|t_it_j|})$ *if* $1 \le i,j \le m$ *and* $i \ne j$;
(3) $h^{i\overline{j}} = O(1)$ *and* $h_{i\overline{j}} = O(1)$ *if* $m+1 \le i,j \le n$;
(4) $h^{i\overline{j}} = O(|t_i|)$ *and* $h_{i\overline{j}} = O(\frac{u_i^3}{|t_i|})$ *if* $i \le m < j$ *or* $j \le m < i$.

Then we use the duality to construct the harmonic Beltrami differentials. We have

LEMMA 5.6. *On the genuine collar Ω_c^j for c small, the coefficient functions A_i of the harmonic Beltrami differentials have the form:*

(1) $A_i = \frac{z_j}{\overline{z_j}} \sin^2 \tau_j \big(\overline{p_i^j(z_j)} + \overline{b_i^j}\big)$ *if* $i \neq j$;
(2) $A_j = \frac{z_j}{\overline{z_j}} \sin^2 \tau_j (\overline{p_j(z_j)} + \overline{b_j})$

where

(1) $p_i^j(z_j) = \sum_{k\leq -1} a_{ik}^j \rho_j^{-k} z_j^k + \sum_{k\geq 1} a_{ik}^j z_j^k$ *if* $i \neq j$;
(2) $p_j(z_j) = \sum_{k\leq -1} a_{jk} \rho_j^{-k} z_j^k + \sum_{k\geq 1} a_{jk} z_j^k$.

In the above expressions, $\rho_j = e^{-\frac{2\pi^2}{l_j}}$ *and the coefficients satisfy the following conditions:*

(1) $\sum_{k\leq -1} |a_{ik}^j| c^{-k} = O(u_j^{-2})$ *and* $\sum_{k\geq 1} |a_{ik}^j| c^k = O(u_j^{-2})$ *if* $i \geq m+1$;
(2) $\sum_{k\leq -1} |a_{ik}^j| c^{-k} = O(u_j^{-2}) O\big(\frac{u_i^3}{|t_i|}\big)$ *and* $\sum_{k\geq 1} |a_{ik}^j| c^k = O(u_j^{-2}) O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \leq m$ *and* $i \neq j$;
(3) $\sum_{k\leq -1} |a_{jk}| c^{-k} = O(\frac{u_j}{|t_j|})$ *and* $\sum_{k\geq 1} |a_{jk}| c^k = O(\frac{u_j}{|t_j|})$;
(4) $|b_i^j| = O(u_j)$ *if* $i \geq m+1$;
(5) $|b_i^j| = O(u_j) O\big(\frac{u_i^3}{|t_i|}\big)$ *if* $i \leq m$ *and* $i \neq j$;
(6) $b_j = -\frac{u_j}{\pi \overline{t_j}}(1 + O(u_0))$.

To use the curvature formulas to estimate the Ricci metric and the perturbed Ricci metric, one needs to find accurate estimate of the operator $T = (\Box + 1)^{-1}$. More precisely, one needs to estimate the functions $e_{i\bar{j}} = T(f_{i\bar{j}})$. To avoid writing down the Green function of T, we construct approximate solutions and localize on the collars in [**34**]. Pick a positive constant $c_1 < c$ and define the cut-off function $\eta \in C^\infty(\mathbb{R}, [0,1])$ by

$$\begin{cases} \eta(x) = 1, & x \leq \log c_1, \\ \eta(x) = 0, & x \geq \log c, \\ 0 < \eta(x) < 1, & \log c_1 < x < \log c. \end{cases} \tag{5.2}$$

It is clear that the derivatives of η are bounded by constants which only depend on c and c_1. Let $\widetilde{e_{i\bar{j}}}(z)$ be the function on X defined in the following way, where z is taken to be z_i on the collar Ω_c^i:

(1) if $i \leq m$ and $j \geq m+1$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2} \sin^2 \tau_i \overline{b_i} b_j^i, & z \in \Omega_{c_1}^i, \\ (\frac{1}{2} \sin^2 \tau_i \overline{b_i} b_j^i) \eta(\log r_i), & z \in \Omega_c^i \text{ and } c_1 < r_i < c, \\ (\frac{1}{2} \sin^2 \tau_i \overline{b_i} b_j^i) \eta(\log \rho_i - \log r_i), & z \in \Omega_c^i \text{ and} \\ & c^{-1}\rho_i < r_i < c_1^{-1} \rho_i, \\ 0, & z \in X \setminus \Omega_c^i. \end{cases}$$

(2) if $i, j \le m$ and $i \neq j$, then

$$\widetilde{e_{i\bar{j}}}(z) = \begin{cases} \frac{1}{2}\sin^2 \tau_i \overline{b_i} b^i_j, & z \in \Omega^i_{c_1}, \\ (\frac{1}{2}\sin^2 \tau_i \overline{b_i} b^i_j)\eta(\log r_i), & z \in \Omega^i_c \text{ and } c_1 < r_i < c, \\ (\frac{1}{2}\sin^2 \tau_i \overline{b_i} b^i_j)\eta(\log \rho_i - \log r_i), & z \in \Omega^i_c \text{ and} \\ & c^{-1}\rho_i < r_i < c_1^{-1}\rho_i, \\ \frac{1}{2}\sin^2 \tau_j \overline{b^j_i} b_j, & z \in \Omega^j_{c_1}, \\ (\frac{1}{2}\sin^2 \tau_i \overline{b^j_i} b_j)\eta(\log r_j), & z \in \Omega^j_c \text{ and } c_1 < r_j < c, \\ (\frac{1}{2}\sin^2 \tau_i \overline{b^j_i} b_j)\eta(\log \rho_j - \log r_j), & z \in \Omega^j_c \text{ and} \\ & c^{-1}\rho_j < r_j < c_1^{-1}\rho_j, \\ 0, & z \in X \setminus (\Omega^i_c \cup \Omega^j_c). \end{cases}$$

(3) if $i \le m$, then

$$\widetilde{e_{i\bar{i}}}(z) = \begin{cases} \frac{1}{2}\sin^2 \tau_i |b_i|^2, & z \in \Omega^i_{c_1}, \\ (\frac{1}{2}\sin^2 \tau_i |b_i|^2)\eta(\log r_i), & z \in \Omega^i_c \text{ and } c_1 < r_i < c, \\ (\frac{1}{2}\sin^2 \tau_i |b_i|^2)\eta(\log \rho_i - \log r_i), & z \in \Omega^i_c \text{ and} \\ & c^{-1}\rho_i < r_i < c_1^{-1}\rho_i, \\ 0, & z \in X \setminus \Omega^i_c. \end{cases}$$

Also, let $\widetilde{f_{i\bar{j}}} = (\Box+1)\widetilde{e_{i\bar{j}}}$. It is clear that the supports of these approximation functions are contained in the corresponding collars. We have the following estimates:

LEMMA 5.7. *Let $\widetilde{e_{i\bar{j}}}$ be the functions constructed above. Then*

(1) $e_{i\bar{i}} = \widetilde{e_{i\bar{i}}} + O\big(\frac{u_i^4}{|t_i|^2}\big)$ *if $i \le m$;*
(2) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3 u_j^3}{|t_i t_j|}\big)$ *if $i, j \le m$ and $i \neq j$;*
(3) $e_{i\bar{j}} = \widetilde{e_{i\bar{j}}} + O\big(\frac{u_i^3}{|t_i|}\big)$ *if $i \le m$ and $j \ge m+1$;*
(4) $\|e_{i\bar{j}}\|_0 = O(1)$ *if $i, j \ge m+1$.*

Now we use the approximation functions $\widetilde{e}_{i\bar{j}}$ in the formulas (4.2), (4.3) and (4.4). The following theorems were proved in [**34**] and [**35**]. We first have the asymptotic estimate of the Ricci metric:

THEOREM 5.8. *Let (t, s) be the pinching coordinates. Then we have*

(1) $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1 + O(u_0))$ *and* $\tau^{i\bar{i}} = \frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1 + O(u_0))$ *if $i \le m$;*
(2) $\tau_{i\bar{j}} = O\left(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j)\right)$ *and* $\tau^{i\bar{j}} = O(|t_i t_j|)$ *if $i, j \le m$ and $i \neq j$;*
(3) $\tau_{i\bar{j}} = O\big(\frac{u_i^2}{|t_i|}\big)$ *and* $\tau^{i\bar{j}} = O(|t_i|)$ *if $i \le m$ and $j \ge m+1$;*
(4) $\tau_{i\bar{j}} = O(1)$ *if $i, j \ge m+1$.*

By the asymptotics of the Ricci metric in the above theorem, we have

Corollary 5.1. *There is a constant $C > 0$ such that*

$$C^{-1}\omega_{P} \leq \omega_{\tau} \leq \omega_{P}.$$

Next we estimate the holomorphic sectional curvature of the Ricci metric:

Theorem 5.9. *Let $X_0 \in \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ be a codimension m point and let $(t_1, \ldots, t_m, s_{m+1}, \ldots, s_n)$ be the pinching coordinates at X_0 where $t_1, \ldots, t_m$ correspond to the degeneration directions. Then the holomorphic sectional curvature is negative in the degeneration directions and is bounded in the non-degeneration directions. Precisely, there is a $\delta > 0$ such that if $|(t,s)| < \delta$, then*

$$\widetilde{R}_{i\bar{i}i\bar{i}} = \frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0)) > 0 \tag{5.3}$$

if $i \leq m$ and

$$\widetilde{R}_{i\bar{i}i\bar{i}} = O(1) \tag{5.4}$$

if $i \geq m+1$.

Furthermore, on $\mathcal{M}_g$, the holomorphic sectional curvature, the bisectional curvature and the Ricci curvature of the Ricci metric are bounded from above and below.

This theorem was proved in [**34**] by using the formula (4.3) and estimating error terms. However, the holomorphic sectional curvature of the Ricci metric is not always negative. We need to introduce and study the perturbed Ricci metric. We have

Theorem 5.10. *For a suitable choice of positive constant C, the perturbed Ricci metric $\widetilde{\tau}_{i\bar{j}} = \tau_{i\bar{j}} + Ch_{i\bar{j}}$ is complete and comparable with the asymptotic Poincaré metric. Its bisectional curvature is bounded. Furthermore, its holomorphic sectional curvature and Ricci curvature are bounded from above and below by negative constants.*

Remark 5.3. The perturbed Ricci metric is the first complete Kähler metric on the moduli space with bounded curvature and negatively pinched holomorphic sectional curvature and Ricci curvature.

By using the minimal surface theory and Bers' embedding theorem, we have also proved the following theorem in [**35**]:

Theorem 5.11. *The moduli space equipped with either the Ricci metric or the perturbed Ricci metric has finite volume. The Teichmüller space equipped with either of these metrics has bounded geometry.*

6. The Equivalence of the Complete Metrics

In this section we describe our arguments that all of the complete metrics on the Teichmüller space and moduli space discussed above are equivalent.

With the good understanding of the Ricci and the perturbed Ricci metrics, the results of this section are quite easy consequences of Yau's Schwarz lemma and the basic properties of these metrics. We first give the definition of equivalence of metrics:

DEFINITION 6.1. Two Kähler metrics g_1 and g_2 on a manifold X are equivalent or two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the tangent bundle of X are equivalent if there is a constant $C>0$ such that

$$C^{-1}g_1 \leq g_2 \leq Cg_1$$

or

$$C^{-1}\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1.$$

We denote this by $g_1 \sim g_2$ or $\|\cdot\|_1 \sim \|\cdot\|_2$.

The main result of this section that we want to discuss is the following theorem proved in [**34**] and [**35**]:

THEOREM 6.2. *On the moduli space $\mathcal{M}_g$ $(g \geq 2)$, the Teichmüller metric $\|\cdot\|_T$, the Carathéodory metric $\|\cdot\|_C$, the Kobayashi metric $\|\cdot\|_K$, the Kähler-Einstein metric ω_{KE}, the induced Bergman metric ω_B, the McMullen metric ω_M, the asymptotic Poincaré metric ω_P, the Ricci metric ω_τ, and the perturbed Ricci metric $\omega_{\tilde{\tau}}$ are equivalent. Namely*

$$\omega_{KE} \sim \omega_{\tilde{\tau}} \sim \omega_\tau \sim \omega_P \sim \omega_B \sim \omega_M$$

and

$$\|\cdot\|_K = \|\cdot\|_T \sim \|\cdot\|_C \sim \|\cdot\|_M.$$

As a corollary we proved the following conjecture of Yau made in the early 1980s [**57**], [**46**]:

THEOREM 6.3. *The Kähler-Einstein metric is equivalent to the Teichmüller metric on the moduli space: $\|\cdot\|_{KE} \sim \|\cdot\|_T$.*

Another corollary was also conjectured by Yau as one of his 120 famous problems [**57**], [**46**]:

THEOREM 6.4. *The Kähler-Einstein metric is equivalent to the Bergman metric on the Teichmüller space: $\omega_{KE} \sim \omega_B$.*

Now we briefly describe the idea of proving the comparison theorem. To compare two complete metrics on a noncompact manifold, we need to write down their asymptotic behavior and compare near infinity. However, if one cannot find the asymptotics of these metrics, the only tool we have is the following Yau's Schwarz lemma [**55**]:

THEOREM 6.5. *Let $f:(M^m,g)\to(N^n,h)$ be a holomorphic map between Kähler manifolds, where M is complete and $Ric(g)\geq -c\,g$ with $c\geq 0$.*

(1) *If the holomorphic sectional curvature of N is bounded above by a negative constant, then $f^*h \leq \tilde{c}\,g$ for some constant $\tilde{c}$.*

(2) *If $m = n$ and the Ricci curvature of N is bounded above by a negative constant, then $f^*\omega_h^n \leq \tilde{c}\,\omega_g^n$ for some constant $\tilde{c}$.*

We briefly describe the proof of the comparison theorem by using Yau's Schwarz lemma and the curvature computations and estimates.

Sketch of proof. To use this result, we take $M = N = \mathcal{M}_g$ and let f be the identity map. We know the perturbed Ricci metric is obtained by adding a positive Kähler metric to the Ricci metric. Thus it is bounded from below by the Ricci metric.

Consider the identity map

$$id : (\mathcal{M}_g, \omega_\tau) \to (\mathcal{M}_g, \omega_{WP}).$$

Yau's Schwarz Lemma implies $\omega_{WP} \leq C_0\omega_\tau$. So

$$\omega_\tau \leq \omega_{\tilde{\tau}} = \omega_\tau + C\omega_{WP} \leq (CC_0+1)\omega_\tau.$$

Thus $\omega_\tau \sim \omega_{\tilde{\tau}}$.

To control the Kähler-Einstein metric, we consider

$$id : (\mathcal{M}_g, \omega_{KE}) \to (\mathcal{M}_g, \omega_{\tilde{\tau}})$$

and

$$id : (\mathcal{M}_g, \omega_{\tilde{\tau}}) \to (\mathcal{M}_g, \omega_{KE}).$$

Yau's Schwarz Lemma implies

$$\omega_{\tilde{\tau}} \leq C_0\omega_{KE}$$

and

$$\omega_{KE}^n \leq C_0\omega_{\tilde{\tau}}^n.$$

The equivalence follows from linear algebra.

Thus by Corollary 5.1 we have

$$\omega_{KE} \sim \omega_{\tilde{\tau}} \sim \omega_\tau \sim \omega_P.$$

By using a similar method we have $\omega_\tau \leq C\omega_M$. To show the other side of the inequality, we have to analyze the asymptotic behavior of the geodesic length functions. We showed in [**34**] that

$$\omega_\tau \sim \omega_M.$$

Thus by the work of McMullen [**41**] we have

$$\omega_\tau \sim \omega_M \sim \|\cdot\|_T.$$

The work of Royden showed that the Teichmüller metric coincides with the Kobayashi metric. Thus we need to show that the Carathéodory metric and the Bergman metric are comparable with the Kobayashi metric. This was done in [**35**] by using Bers' Embedding Theorem. The idea is as follows:

By the Bers' Embedding Theorem, for each point $p \in \mathcal{T}_g$, there is a map $f_p : \mathcal{T}_g \to \mathbb{C}^n$ such that $f_p(p) = 0$ and

$$B_2 \subset f_p(\mathcal{T}_g) \subset B_6$$

where B_r is the open ball in $\mathbb{C}^n$ centered at 0 with radius r. Since both Carathéodory metric and Kobayashi metric have the restriction property and can be computed explicitly on balls, we can use these metrics defined on B_2 and B_6 to pinch these metrics on the Teichmüller space. We can also use this method to estimate peak sections of the Teichmüller space at point p. A careful analysis shows

$$\|\cdot\|_C \sim \|\cdot\|_K \sim \omega_B.$$

The argument is quite easy. Please see [**35**] for details. $\square$

7. Bounded Geometry of the Kähler-Einstein Metric

The comparison theorem gives us some control on the Kähler-Einstein Metric. Especially we know that it has Poincaré growth near the boundary of the moduli space and is equivalent to the Ricci metric which has bounded geometry. In this section we sketch our proof that the Kähler-Einstein metric also has bounded geometry. Precisely we have

THEOREM 7.1. *The curvature of the Kähler-Einstein metric and all of its covariant derivatives are uniformly bounded on the Teichmüller spaces, and its injectivity radius has lower bound.*

Now we briefly describe the proof. Please see [**35**] for details.

Sketch of proof. We follow Yau's argument in [**56**]. The first step is to perturb the Ricci metric using Kähler-Ricci flow

$$\begin{cases} \frac{\partial g_{i\bar{j}}}{\partial t} = -(R_{i\bar{j}} + g_{i\bar{j}}), \\ g(0) = \tau \end{cases}$$

to avoid complicated computations of the covariant derivatives of the curvature of the Ricci metric.

For $t > 0$ small, let $h = g(t)$ and let g be the Kähler-Einstein metric. We have

(1) h is equivalent to the initial metric τ and thus is equivalent to the Kähler-Einstein metric,
(2) the curvature and its covariant derivatives of h are bounded.

Then we consider the Monge-Amperé equation

$$\log\det(h_{i\bar{j}} + u_{i\bar{j}}) - \log\det(h_{i\bar{j}}) = u + F$$

where $\partial\overline{\partial}u = \omega_g - \omega_h$ and $\partial\overline{\partial}F = Ric(h) + \omega_h$.

The curvature of $P_{i\bar{j}k\bar{l}}$ of the Kähler-Einstein metric is given by

$$P_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} + u_{p\bar{j}}h^{p\bar{q}}R_{i\bar{q}k\bar{l}} + u_{;i\bar{j}k\bar{l}} - g^{p\bar{q}}u_{;i\bar{q}k}u_{;\bar{j}p\bar{l}}.$$

The comparison theorem implies $\partial\overline{\partial}u$ has C^0-bound and the strong bounded geometry of h implies $\partial\overline{\partial}F$ has C^k-bound for $k \geq 0$. Also, the equivalence of h and g implies $u + F$ is bounded.

So we need the C^k-bound of $\partial\overline{\partial}u$ for $k \geq 1$. Let

$$S = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} u_{;i\bar{q}k} u_{;\bar{j}p\bar{l}}$$

and

$$V = g^{i\bar{j}} g^{k\bar{l}} g^{p\bar{q}} g^{m\bar{n}} \left(u_{;i\bar{q}k\bar{n}} u_{;\bar{j}p\bar{l}m} + u_{;i\bar{n}kp} u_{;\bar{j}m\bar{l}\bar{q}} \right),$$

where the covariant derivatives of u were taken with respect to the metric h.

Yau's C^3 estimate in [**56**] implies S is bounded. Let $f = (S+\kappa)V$ where κ is a large constant. The inequality

$$\Delta' f \geq C f^2 + (\text{ lower order terms })$$

implies f is bounded and thus V is bounded. So the curvature of the Kähler-Einstein metric is bounded. The same method can be used to derive boundedness of higher derivatives of the curvature. □

Actually we have also proved that all of these complete Kähler metrics have bounded geometry, which should be useful in understanding the geometry of the moduli and the Teichmüller spaces.

8. Application to Algebraic Geometry

The existence of the Kähler-Einstein metric is closely related to the stability of the tangent and cotangent bundle. In this section we review our results that the logarithmic extension of the cotangent bundle of the moduli space is stable in the sense of Mumford. We first recall the definition.

Definition 8.1. Let E be a holomorphic vector bundle over a complex manifold X and let Φ be a Kähler class of X. The (Φ-)degree of E is given by

$$\deg(E) = \int_X c_1(E)\Phi^{n-1}$$

where n is the dimension of X. The slope of E is given by the quotient

$$\mu(E) = \frac{\deg(E)}{\operatorname{rank}(E)}.$$

The bundle E is Mumford (Φ-)stable if for any proper coherent subsheaf $\mathcal{F} \subset E$, we have

$$\mu(\mathcal{F}) < \mu(E).$$

Now we describe the logarithmic cotangent bundle. Let U be any local chart of $\mathcal{M}_g$ near the boundary with pinching coordinates $(t_1, \ldots, t_m, s_{m+1}, \ldots, s_n)$ such that $(t_1, \ldots, t_m)$ represent the degeneration directions. Let

$$e_i = \begin{cases} \frac{dt_i}{t_i} & i \leq m; \\ ds_i & i \geq m+1. \end{cases}$$

The logarithmic cotangent bundle E is the extension of $T^*\mathcal{M}_g$ to $\overline{\mathcal{M}}_g$ such that on U, $e_1, \ldots, e_n$ is a local holomorphic frame of E. One can write down the transition functions and check that there is a unique bundle over $\overline{\mathcal{M}}_g$ satisfying the above condition.

To prove the stability of E, we need to specify a Kähler class. It is natural to use the polarization of E. The main theorem of this section is the following:

THEOREM 8.2. *The first Chern class $c_1(E)$ is positive and E is stable with respect to $c_1(E)$.*

We briefly describe here the proof of this theorem. Please see [**35**] for details.

Sketch of the proof. Since we only deal with the first Chern class, we can assume the coherent subsheaf $\mathcal{F}$ is actually a subbundle F.

Since the Kähler-Einstein metric induces a singular metric g^*_{KE} on the logarithmic extension bundle E, our main job is to show that the degree and slope of E and any proper subbundle F defined by the singular metric are finite and are equal to the genuine ones. This depends on our estimates of the Kähler-Einstein metric which are used to show the convergence of the integrals defining the degrees.

More precisely we need to show the following:

(1) As a current, ω_{KE} is closed and represents the first Chern class of E, that is
$$[\omega_{KE}] = c_1(\overline{E}).$$
(2) The singular metric g^*_{KE} on E induced by the Kähler-Einstein metric defines the degree of E
$$\deg(E) = \int_{\mathcal{M}_g} \omega^n_{KE}.$$
(3) The degree of any proper holomorphic sub-bundle F of E can be defined by using $g^*_{KE}\mid_F$:
$$\deg(F) = \int_{\mathcal{M}_g} -\partial\overline{\partial}\log\det\left(g^*_{KE}\mid_F\right)\wedge\omega^{n-1}_{KE}.$$

These three steps were proved in [**35**] by using the Poincaré growth property of the Kähler-Einstein metric together with a special cut-off function. This shows that the bundle E is semi-stable.

To get the strict stability, we proceeded by contradiction. If E is not stable, then E, thus $E\mid_{\mathcal{M}_g}$, split holomorphically. This implies a finite smooth cover of the moduli space splits, which implies a finite index subgroup of the mapping class group splits as a direct product of two subgroups. This is impossible by a topological fact. Again, the detailed proof can be found in [**35**].

Part II: The Topological Aspect

9. The Physics of the Mariño-Vafa Conjecture

Our original motivation to study Hodge integrals was to find a general mirror formula for counting higher genus curves in Calabi-Yau manifolds. To generalize the mirror principle to count the number of higher genus curves, we need to first compute Hodge integrals, i.e., the intersection numbers of the λ classes and ψ classes on the Deligne-Mumford moduli space of stable curves $\overline{\mathcal{M}}_{g,h}$. This moduli space is possibly the most famous and most interesting orbifold. It has been studied since Riemann, and by many Fields medalists for the past 50 years, from many different points of view. Still many interesting and challenging problems about the geometry and topology of these moduli spaces remain unsolved. String theory has motivated many fantastic conjectures about these moduli spaces, including the famous Witten conjecture which is about the generating series of the integrals of the ψ-classes. We start with the introduction of some notations.

Recall that a point in $\overline{\mathcal{M}}_{g,h}$ consists of $(C, x_1, \ldots, x_h)$, a (nodal) curve C of genus g, and n distinguished smooth points on C. The Hodge bundle $\mathbb{E}$ is a rank g vector bundle over $\overline{\mathcal{M}}_{g,h}$ whose fiber over $[(C, x_1, \ldots, x_h)]$ is $H^0(C, \omega_C)$, the complex vector space of holomorphic one forms on C. The λ classes are the Chern Classes of $\mathbb{E}$,

$$\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

On the other hand, the cotangent line $T^*_{x_i}C$ of C at the i-th marked point x_i induces a line bundle $\mathbb{L}_i$ over $\overline{\mathcal{M}}_{g,h}$. The ψ classes are the Chern classes:

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,h}; \mathbb{Q}).$$

Introduce the total Chern class

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g.$$

The Mariño-Vafa formula is about the generating series of the triple Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,h}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^h (1 - \mu_i \psi_i)},$$

where τ is considered as a parameter here. Later we will see that it actually comes from the weight of the group action, and also from the framing of the knot. Taking Taylor expansions in τ or in μ_i one can obtain information on the integrals of the Hodge classes and the ψ-classes. The Marinõ-Vafa conjecture asserts that the generating series of such triple Hodge integrals for all genera and any numbers of marked points can be expressed by a close formula which is a *finite* expression in terms of representations of symmetric groups, or Chern-Simons knot invariants.

We remark that the moduli spaces of stable curves have been the source of many interests from mathematics to physics. Mumford has computed some low genus numbers. The Witten conjecture, proved by Kontsevich, is about the integrals of the ψ-classes.

Let us briefly recall the background of the conjecture. Mariño and Vafa [**39**] made this conjecture based on the large N duality between Chern-Simons and string theory. It starts from the conifold transition. We consider the resolution of singularity of the conifold X defined by

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = 0 \right\}$$

in two different ways:

(1) Deformed conifold T^*S^3

$$\left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbf{C}^4 : xw - yz = \epsilon \right\}$$

where ϵ a real positive number. This is a symplectic resolution of the singularity.

(2) Resolved conifold by blowing up the singularity, which gives the total space

$$\tilde{X} = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$$

which is explicitly given by

$$\left\{ \left([Z_0, Z_1], \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) \in \mathbf{P}^1 \times \mathbf{C}^4 : \begin{array}{l} (x, y) \in [Z_0, Z_1] \\ (z, w) \in [Z_0, Z_1] \end{array} \right\}$$

$$\begin{array}{ccc} \tilde{X} & \subset & \mathbf{P}^1 \times \mathbf{C}^4 \\ \downarrow & & \downarrow \\ X & \subset & \mathbf{C}^4 \end{array}$$

The brief history of the development of the conjecture is as follows. In 1992 Witten first conjectured that the open topological string theory on the deformed conifold T^*S^3 is equivalent to the Chern-Simons gauge theory on S^3. This idea was pursued further by Gopakumar and Vafa in 1998, and then by Ooguri and Vafa in 2000. Based on the above conifold transition, they conjectured that the open topological string theory on the deformed conifold T^*S^3 is equivalent to the closed topological string theory on the resolved conifold $\tilde{X}$. Ooguri-Vafa only considered the zero framing case. Later Marinõ-Vafa generalized the idea to the non-zero framing case and discovered the beautiful formula for the generating series of the triple Hodge integrals. Recently Vafa and his collaborators systematically developed the theory, and for the past several years, they developed these duality ideas into the most effective tool to compute Gromov-Witten invariants on toric Calabi-Yau manifolds. The high point of their work is the theory of topological vertex. We refer to [**39**] and [**1**] for the details of the physical theory and the history of the development.

Starting with the proof of the Marinõ-Vafa conjecture [**28**], [**29**], we have developed a rather complete mathematical theory of topological vertex [**21**]. Many interesting consequences have been derived during the past year. Now let us see how the string theorists derived mathematical consequence from the above naive idea of string duality. First the Chern-Simons partition function has the form

$$\langle Z(U,V)\rangle = \exp(-F(\lambda,t,V)),$$

where U is the holonomy of the $U(N)$ Chern-Simons gauge field around the knot $K \subset S^3$, and V is an extra $U(M)$ matrix. The partition function $\langle Z(U,V)\rangle$ gives the Chern-Simons knot invariants of K.

String duality asserts that the function $F(\lambda,t,V)$ should give the generating series of the open Gromov-Witten invariants of $(\tilde{X}, L_K)$, where L_K is a Lagrangian submanifold of the resolved conifold $\tilde{X}$ canonically associated to the knot K. More precisely, by applying the t'Hooft large N expansion, and the "canonical" identifications of parameters similar to mirror formula, which at level k are given by

$$\lambda = \frac{2\pi}{k+N}, \quad t = \frac{2\pi i N}{k+N},$$

we get the partition function of the topological string theory on conifold $\tilde{X}$, and then on $\mathbf{P}^1$, which is just the generating series of the Gromov-Witten invariants. This change of variables is very striking from the point of view of mathematics.

The special case when K is the unknot is already very interesting. In non-zero framing it gives the Mariño-Vafa conjectural formula. In this case $\langle Z(U,V)\rangle$ was first computed in the zero framing by Ooguri-Vafa and in any framing $\tau \in \mathbb{Z}$ by Mariño-Vafa [**39**]. Comparing with Katz-Liu's computations of $F(\lambda,t,V)$, Mariño-Vafa conjectured the striking formula about the generating series of the triple Hodge integrals for all genera and any number of marked points in terms of the Chern-Simons invariants, or equivalently in terms of the representations and combinatorics of symmetric groups. It is interesting to note that the framing in the Mariño-Vafa's computations corresponds to the choice of lifting of the circle action on the pair $(\tilde{X}, L_{\mathrm{unknot}})$ in Katz-Liu's localization computations. Both choices are parametrized by an integer τ which will be considered as a parameter in the triple Hodge integrals. Later we will take derivatives with respect to this parameter to get the cut-and-join equation.

It is natural to ask what mathematical consequence we can have for general duality, that is for general knots in general three manifolds; a first naive question is what kind of general Calabi-Yau manifolds will appear in the duality, in place of the conifold. Some special cases corresponding to the Seifert manifolds are known by gluing several copies of conifolds.

10. The Proof of the Mariño-Vafa Formula

Now we give the precise statement of the Mariño-Vafa conjecture, which is an identity between the geometry of the moduli spaces of stable curves and

Chern-Simons knot invariants, or the combinatorics of the representation theory of symmetric groups.

Let us first introduce the geometric side. For every partition $\mu = (\mu_1 \geq \cdots \mu_{l(\mu)} \geq 0)$, we define the triple Hodge integral to be

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

where the coefficient

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\operatorname{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1}\prod_{i=1}^{l(\mu)}\frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

The expressions, although very complicated, arise naturally from localization computations on the moduli spaces of relative stable maps into $\mathbf{P}^1$ with ramification type μ at ∞.

We now introduce the generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0}\lambda^{2g-2+l(\mu)}G_{g,\mu}(\tau).$$

The special case when $g=0$ is given by

$$\int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{\Lambda_0^\vee(1)\Lambda_0^\vee(-\tau-1)\Lambda_0^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)} = \int_{\overline{\mathcal{M}}_{0,l(\mu)}} \frac{1}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

which is known to be equal to $|\mu|^{l(\mu)-3}$ for $l(\mu)\geq 3$, and we use this expression to extend the definition to the case $l(\mu)<3$.

Introduce formal variables $p=(p_1,p_2,\ldots,p_n,\ldots)$, and define

$$p_\mu = p_{\mu_1}\cdots p_{\mu_{l(\mu)}}$$

for any partition μ. These p_{μ_j} correspond to $\operatorname{Tr} V^{\mu_j}$ in the notations of string theorists. The generating series for all genera and all possible marked points are defined to be

$$G(\lambda;\tau;p) = \sum_{|\mu|\geq 1} G_\mu(\lambda;\tau)p_\mu,$$

which encode complete information of the triple Hodge integrals we are interested in.

Next we introduce the representation theoretical side. Let χ_μ denote the character of the irreducible representation of the symmetric group $S_{|\mu|}$, indexed by μ with $|\mu| = \sum_j \mu_j$. Let $C(\mu)$ denote the conjugacy class of $S_{|\mu|}$ indexed by μ. Introduce

$$\mathcal{W}_\mu(\lambda) = \prod_{1\leq a<b\leq l(\mu)}\frac{\sin[(\mu_a-\mu_b+b-a)\lambda/2]}{\sin[(b-a)\lambda/2]}\cdot\frac{1}{\prod_{i=1}^{l(\nu)}\prod_{v=1}^{\mu_i}2\sin[(v-i+l(\mu))\lambda/2]}.$$

This has an interpretation in terms of quantum dimension in Chern-Simons knot theory.

We define the following generating series

$$R(\lambda;\tau;p) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \cdot \sum_{\mu} \left[\sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^{n} \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda) \right] p_\mu$$

where μ^i are sub-partitions of μ, $z_\mu = \prod_j \mu_j! j^{\mu_j}$ and

$$\kappa_\mu = |\mu| + \sum_i (\mu_i^2 - 2i\mu_i)$$

for a partition μ which is also standard for representation theory of symmetric groups. There is the relation $z_\mu = |\operatorname{Aut}(\mu)|\mu_1 \cdots \mu_{l(\mu)}$.

Finally we can give the precise statement of the Mariño-Vafa formula:

Conjecture: *We have the identity*

$$G(\lambda;\tau;p) = R(\lambda;\tau;p).$$

Before discussing the proof of this conjecture, we first give several remarks. This conjecture is a formula: G : Geometry = R : Representations, and the representations of symmetric groups are essentially combinatorics. We note that each $G_\mu(\lambda,\tau)$ is given by a finite and closed expression in terms of the representations of symmetric groups:

$$G_\mu(\lambda,\tau) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \cdot \sum_{\cup_{i=1}^n \mu^i = \mu} \prod_{i=1}^{n} \sum_{|\nu^i|=|\mu^i|} \frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2} \mathcal{W}_{\nu^i}(\lambda).$$

The generating series $G_\mu(\lambda,\tau)$ gives the values of the triple Hodge integrals for moduli spaces of curves of all genera with $l(\mu)$ marked points. Finally we remark that an equivalent expression of this formula is the following non-connected generating series. In this situation we have a relatively simpler combinatorial expression:

$$\begin{aligned} G(\lambda;\tau;p)^\bullet &= \exp\left[G(\lambda;\tau;p)\right] \\ &= \sum_{|\mu|\geq 0} \left[\sum_{|\nu|=|\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_\nu\lambda/2} \mathcal{W}_\nu(\lambda) \right] p_\mu. \end{aligned}$$

According to Mariño and Vafa, this formula gives values for all Hodge integrals up to three Hodge classes. Lu proved that this is right if we combine with some previously known simple formulas about Hodge integrals.

By taking Taylor expansion in τ on both sides of the Mariño-Vafa formula, we have derived various Hodge integral identities in [**30**].

For examples, as easy consequences of the Mariño-Vafa formula and the cut-and-join equation as satisfied by the above generating series, we have unified simple proofs of the λ_g conjecture by comparing the coefficients in τ in the Taylor expansions of the two expressions,

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \ldots, k_n} \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!},$$

for $k_1 + \cdots + k_n = 2g-3+n$, and the following identities for Hodge integrals:

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_{g-1}\lambda_g = \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g},$$

where B_{2g} are Bernoulli numbers. And

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{\lambda_{g-1}}{1-\psi_1} = b_g \sum_{i=1}^{2g-1} \frac{1}{i} - \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ g_1,g_2>0}} \frac{(2g_1-1)!(2g_2-1)!}{(2g-1)!} b_{g_1} b_{g_2},$$

where

$$b_g = \begin{cases} 1, & g = 0, \\ \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0. \end{cases}$$

Now let us look at how we proved this conjecture. This is joint work with Chiu-Chu Liu, Jian Zhou. See [**27**] and [**28**] for details.

The first proof of this formula is based on the *Cut-and-Join* equation which is a beautiful match of combinatorics and geometry. The details of the proof is given in [**27**] and [**28**]. First we look at the combinatorial side. Denote by $[s_1, \ldots, s_k]$ a k-cycle in the permutation group. We have the following two obvious operations:

Cut: a k-cycle is cut into an i-cycle and a j-cycle:

$$[s, t] \cdot [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j] = [s, s_2, \ldots, s_i][t, t_2, \ldots, t_j].$$

Join: an i-cycle and a j-cycle are joined to an $(i+j)$-cycle:

$$[s, t] \cdot [s, s_2, \ldots, s_i][t, t_2, \ldots t_j] = [s, s_2, \ldots, s_i, t, t_2, \ldots, t_j].$$

Such operations can be organized into differential equations which we call the cut-and-join equation.

Now we look at the geometry side. In the moduli spaces of stable maps, cut and join have the following geometric meaning: *Cut*: one curve splits into two lower degree or lower genus curves. *Join*: two curves are joined together to give a higher genus or higher degree curve.

The combinatorics and geometry of cut-and-join are reflected in the following two differential equations, which look like a heat equation. It is easy to show that such an equation is equivalent to a series of systems of linear ordinary differential equations by comparing the coefficients on p_μ. These equations are proved either by easy and direct computations in combinatorics or by localizations on moduli spaces of relative stable maps in geometry. In combinatorics, the proof is given by direct computations and was explored in combinatorics in the mid '80s and by Zhou [**27**] for this case. The differential operator on the right hand side corresponds to the cut-and-join operations which we also simply denote by (CJ).

LEMMA 10.1.

$$\frac{\partial R}{\partial \tau} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\left((i+j)p_i p_j \frac{\partial R}{\partial p_{i+j}} + ijp_{i+j}\left(\frac{\partial R}{\partial p_i}\frac{\partial R}{\partial p_j} + \frac{\partial^2 R}{\partial p_i \partial p_j}\right)\right).$$

On the geometry side the proof of such equation is given by localization on the moduli spaces of relative stable maps into the the projective line $\mathbf{P}^1$ with fixed ramifications at ∞:

LEMMA 10.2.

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\sqrt{-1}\lambda \sum_{i,j=1}^{\infty}\left((i+j)p_i p_j \frac{\partial G}{\partial p_{i+j}} + ijp_{i+j}\left(\frac{\partial G}{\partial p_i}\frac{\partial G}{\partial p_j} + \frac{\partial^2 G}{\partial p_i \partial p_j}\right)\right).$$

The proof of the above equation is given in [**27**]. Together with the following

Initial Value: $\tau = 0$,

$$G(\lambda, 0, p) = \sum_{d=1}^{\infty} \frac{p_d}{2d \sin\left(\frac{\lambda d}{2}\right)} = R(\lambda, 0, p)$$

which is precisely the Ooguri-Vafa formula and which has been proved previously for example in [**58**], we therefore obtain the equality which is the Mariño-Vafa conjecture by the uniqueness of the solution:

THEOREM 10.3. *We have the identity*

$$G(\lambda;\tau;p) = R(\lambda;\tau;p).$$

During the proof we note that the cut-and-join equation is encoded in the geometry of the moduli spaces of stable maps. In fact we later find the convolution formula of the following form, which is a relation for the disconnected version $G^\bullet = \exp G$,

$$G^\bullet_\mu(\lambda,\tau) = \sum_{|\nu|=|\mu|} \Phi^\bullet_{\mu,\nu}(-\sqrt{-1}\tau\lambda) z_\nu K^\bullet_\nu(\lambda)$$

where $\Phi^\bullet_{\mu,\nu}$ is the generating series of double Hurwitz numbers, and z_ν is the combinatorial constant that appeared in the previous formulas. Equivalently this gives the explicit solution of the cut-and-join differential equation with initial value $K^\bullet(\lambda)$, which is the generating series of the integrals of certain Euler classes on the moduli spaces of relative stable maps to $\mathbf{P}^1$. See [**26**] for the derivation of this formula, and see [**29**] for the two partition analogue.

The Witten conjecture as proved by Kontsevich states that the generating series of the ψ-class integrals satisfy an infinite number of differential equations. The remarkable feature of Mariño-Vafa formula is that it gives a finite close formula. In fact, by taking limits in τ and μ_i's one can obtain the Witten conjecture as argued by Okounkov-Pandhrapande. But the combinatorics involved is non-trivial. A much simpler direct proof of the Witten conjecture was obtained recently by Kim and myself. We directly derived the recursion formula which implies both the Virasoro relations and the KdV equations. We will discuss this proof later.

The same argument as our proof of the conjecture gives a simple and geometric proof of the ELSV formula for Hurwitz numbers. It reduces to the fact that the push-forward of 1 is a constant in equivariant cohomology for a generically finite-to-one map. This will also be discussed in a later section. See [**28**] for more details.

We would like to briefly explain the technical details of the proof of the Mariño-Vafa formula. The proof of the combinatorial cut-and-join formula is based on the Burnside formula and various simple results in symmetric functions. See [**58**], [**22**] and [**28**].

The proof of the geometric cut-and-join formula used the functorial localization formula in [**24**] and [**25**]. Here we only state its simple form for manifolds as used in [**24**]; the virtual version of this formula is proved and used in [**25**].

Given X and Y two compact manifolds with torus action. Let $f : X \to Y$ be an equivariant map. Let $F \subset Y$ be a fixed component, and let $E \subset f^{-1}(F)$ denote the fixed components lying inside $f^{-1}(F)$. Let $f_0 = f|_E$; then we have

Functorial Localization Formula: *For $\omega \in H_T^*(X)$ an equivariant cohomology class, we have the identity on F:*

$$f_{0*}\left[\frac{i_E^*\omega}{e_T(E/X)}\right] = \frac{i_F^*(f_*\omega)}{e_T(F/Y)}.$$

This formula, which is a generalization of the Atiyah-Bott localization formula to relative setting, has been applied to various settings to prove many interesting conjectures from physics. It was discovered and effectively used in [**24**]. A virtual version which was first applied to the virtual fundamental cycles in the computations of Gromov-Witten invariants was first proved and used in [**25**].

This formula is very effective and useful because we can use it to push computations on complicated moduli space to simpler moduli space. The moduli spaces used by mathematicians are usually the correct but complicated moduli spaces like the moduli spaces of stable maps, while the moduli spaces used by physicists are usually the simple but wrong ones like the projective spaces. This functorial localization formula has been used successfully in the proof of the mirror formula [**24**], [**25**], the proof of the Hori-Vafa formula [**23**], and the easy proof of the ELSV formula [**28**]. Our first proof of the Mariño-Vafa formula also used this formula in a crucial way.

More precisely, let $\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$ denote the moduli space of relative stable maps from a genus g curve to $\mathbf{P}^1$ with fixed ramification type μ at ∞, where μ is a fixed partition. We apply the functorial localization formula to the divisor morphism from the relative stable map moduli space to the projective space,

$$\mathrm{Br}: \quad \overline{\mathcal{M}}_g(\mathbf{P}^1, \mu) \to \mathbf{P}^r,$$

where r denotes the dimension of $\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$. This is similar to the set-up of mirror principle, only with a different linearized moduli space, but in both cases the target spaces are projective spaces.

We found that the fixed points of the target $\mathbf{P}^r$ precisely label the cut-and-join operations of the triple Hodge integrals. Functorial localization reduces the problem to the study of polynomials in the equivariant cohomology group of $\mathbf{P}^r$. We were able to squeeze out a system of linear equations which implies the cut-and-join equation. Actually we derived a stronger relation than the cut-and-join equation, while the cut-and-join equation we need for the Mariño-Vafa formula is only the very first of such kind of relations. See [**28**] for higher order cut-and-join equations.

As was known in infinite Lie algebra theory, the cut-and-join operator is closely related to and more fundamental than the Virasoro algebras in some sense.

Recently there have appeared two different approaches to the Mariño-Vafa formula. The first one is a direct derivation of the convolution formula which was discovered during our proof of the two partition analogue of the formula [**29**]. See [**26**] for the details of the derivation in this case. The second is by Okounkov-Pandhripande [**44**]; they gave a different approach by using the ELSV formula as initial value, as well as the λ_g conjecture and other recursion relations from localization on the moduli spaces of stable maps to $\mathbf{P}^1$.

11. Two Partition Generalization

The two partition analogue of the Mariño-Vafa formula naturally arises from the localization computations of the Gromov-Witten invariants of the open toric Calabi-Yau manifolds, as explained in [**59**].

To state the formula we let μ^+, μ^- be any two partitions. Introduce the Hodge integrals involving these two partitions:

$$G_{\mu^+,\mu^-}(\lambda;\tau) = B(\tau;\mu^+,\mu^-) \cdot \sum_{g\geq 0} \lambda^{2g-2} A_g(\tau;\mu^+,\mu^-)$$

where

$$A_g(\tau;\mu^+,\mu^-) = \int_{\overline{\mathcal{M}}_{g,l(\mu^+)+l(\mu^-)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{i=1}^{l(\mu^+)}\left(1-\mu_i^+\psi_i\right)\prod_{j=1}^{l(\mu^-)}\tau\left(\tau-\mu_i^-\psi_{j+l(\mu^+)}\right)}$$

and

$$\begin{aligned}
&B(\tau;\mu^+,\mu^-)\\
&= -\frac{(\sqrt{-1}\lambda)^{l(\mu^+)+l(\mu^-)}}{|\operatorname{Aut}(\mu^+)||\operatorname{Aut}(\mu^-)|}\left[\tau(\tau+1)\right]^{l(\mu^+)+l(\mu^-)-1}\\
&\quad\cdot\prod_{i=1}^{l(\mu^+)}\frac{\prod_{a=1}^{\mu_i^+-1}\left(\mu_i^+\tau+a\right)}{(\mu_i^+-1)!}\cdot\prod_{i=1}^{l(\mu^-)}\frac{\prod_{a=1}^{\mu_i^--1}\left(\mu_i^-\frac{1}{\tau}+a\right)}{(\mu_i^--1)!}.
\end{aligned}$$

These complicated expressions naturally arise in open string theory, as well as in the localization computations of the Gromov-Witten invariants on open toric Calabi-Yau manifolds.

We introduce two generating series, first on the geometry side,

$$G^\bullet(\lambda;p^+,p^-;\tau)=\exp\left(\sum_{(\mu^+,\mu^-)\in\mathcal{P}^2}G_{\mu^+,\mu^-}(\lambda,\tau)p^+_{\mu^+}p^-_{\mu^-}\right),$$

where $\mathcal{P}^2$ denotes the set of pairs of partitions and $p^\pm_{\mu^\pm}$ are two sets of formal variables associated to the two partitions as in the last section.

On the representation side, we introduce

$$\begin{aligned}
&R^\bullet(\lambda;p^+,p^-;\tau)\\
&=\sum_{|\nu^\pm|=|\mu^\pm|\geq 0}\frac{\chi_{\nu^+}(C(\mu^+))}{z_{\mu^+}}\frac{\chi_{\nu^-}(C(\mu^-))}{z_{\mu^-}}\\
&\qquad\cdot e^{\sqrt{-1}(\kappa_{\nu^+}\tau+\kappa_{\nu^-}\tau^{-1})\lambda/2}\mathcal{W}_{\nu^+,\nu^-}p^+_{\mu^+}p^-_{\mu^-}.
\end{aligned}$$

Here

$$\begin{aligned}
\mathcal{W}_{\mu,\nu}&=q^{l(\nu)/2}\mathcal{W}_\mu\cdot s_\nu(\mathcal{E}_\mu(t))\\
&=(-1)^{|\mu|+|\nu|}q^{\frac{\kappa_\mu+\kappa_\nu+|\mu|+|\nu|}{2}}\sum_\rho q^{-|\rho|}s_{\mu/\rho}(1,q,\dots)s_{\nu/\rho}(1,q,\dots)
\end{aligned}$$

in terms of the skew Schur functions s_μ [**38**]. They appear naturally in the Chern-Simons invariant of the Hopf link.

THEOREM 11.1. *We have the identity:*

$$G^\bullet(\lambda;p^+,p^-;\tau)=R^\bullet(\lambda;p^+,p^-;\tau).$$

The idea of the proof is similar to that of the proof of the Mariño-Vafa formula. We prove that both sides of the above identity satisfy the same cut-and-join equation of the following type:

$$\frac{\partial}{\partial\tau}H^\bullet=\frac{1}{2}(CJ)^+H^\bullet-\frac{1}{2\tau^2}(CJ)^-H^\bullet,$$

where $(CJ)^\pm$ denote the cut-and-join operator, the differential operator with respect to the two set of variables $p^\pm$. We then prove that they have the same initial value at $\tau = -1$:

$$G^\bullet(\lambda; p^+, p^-; -1) = R^\bullet(\lambda; p^+, p^-; -1),$$

which is again given by the Ooguri-Vafa formula [**29**], [**59**].

The cut-and-join equation can be written in a linear matrix form, and such equation follows from the convolution formula of the form

$$\begin{aligned} &K^\bullet_{\mu^+,\mu^-}(\lambda) \\ &= \sum_{|\nu^\pm|=\mu^\pm} G^\bullet_{\mu^+,\mu^-}(\lambda;\tau) z_{\nu^+} \Phi^\bullet_{\nu^+,\mu^+}(-\sqrt{-1}\lambda\tau) z_{\nu^-} \Phi^\bullet_{\nu^-,\mu^-}\left(\frac{-\sqrt{-1}}{\tau}\lambda\right) \end{aligned}$$

where $\Phi^\bullet$ denotes the generating series of double Hurwitz numbers, and K_{μ^+,μ^-} is the generating series of certain integrals on the moduli spaces of relative stable maps. For more details see [**29**].

This convolution formula arises naturally from localization computations on the moduli spaces of relative stable maps to $\mathbf{P}^1 \times \mathbf{P}^1$ with the point (∞, ∞) blown up. So it reflects the geometric structure of the moduli spaces. Such a convolution type formula was actually discovered during our search for a proof of this formula, both on the geometric and the combinatorial side; see [**29**] for the detailed derivations of the convolution formulas in both geometry and combinatorics.

The proof of the combinatorial side of the convolution formula is again a direct computation. The proof of the geometric side for the convolution equation is to reorganize the generating series from localization contributions on the moduli spaces of relative stable maps into $\mathbf{P}^1 \times \mathbf{P}^1$ with the point (∞, ∞) blown up, in terms of the double Hurwitz numbers. It involves careful analysis and computations.

12. Theory of Topological Vertex

When we worked on the Mariño-Vafa formula and its generalizations, we were simply trying to generalize the method and the formula to involve more partitions, but it turned out that in the three partition case, we naturally met the theory of topological vertex. Topological vertex was first introduced in string theory by Vafa et al in [**1**]; it can be deduced from a three partition analogue of the Mariño-Vafa formula in a highly nontrivial way. From this we were able to give a rigorous mathematical foundation for the physical theory. Topological vertex is a high point of the theory of string duality as developed by Vafa and his group for the past several years, starting from Witten's conjectural duality between Chern-Simons and open string theory. It gives the most powerful and effective way to compute the Gromov-Witten invariants for all open toric Calabi-Yau manifolds. In physics it is rare to have two theories agree up to all orders, and topological vertex theory gives a very significant example. In mathematics the theory of topological vertex already has many interesting applications. Here we only briefly sketch the rough idea for the three partition analogue of the Mariño-Vafa formula. For its relation to the theory of topological vertex, we refer the reader to [**21**] for the details.

Given any three partitions $\overrightarrow{\mu} = \{\mu^1, \mu^2, \mu^3\}$, the cut-and-join equation in this case, for both the geometry and representation sides, has the form:

$$\frac{\partial}{\partial \tau} F^\bullet(\lambda;\tau;\mathbf{p}) = (CJ)^1 F^\bullet(\lambda;\tau;\mathbf{p}) + \frac{1}{\tau^2}(CJ)^2 F^\bullet(\lambda;\tau;\mathbf{p}) + \frac{1}{(\tau+1)^2}(CJ)^3 F^\bullet(\lambda;\tau;\mathbf{p}).$$

The cut-and-join operators $(CJ)^1$, $(CJ)^2$ and $(CJ)^3$ are with respect to the three partitions. More precisely they correspond to the differential operators with respect to the three groups of infinite numbers of variables $\mathbf{p} = \{p^1, p^2, p^3\}$.

The initial value for this differential equation is taken at $\tau = 1$, which is then reduced to the formulas of the two partition case. The combinatorial, or the Chern-Simons invariant side is given by $\mathcal{W}_{\overrightarrow{\mu}} = \mathcal{W}_{\mu^1,\mu^2,\mu^3}$ which is a combination of the $\mathcal{W}_{\mu,\nu}$ as in the two partition case. See [**21**] for its explicit expression.

On the geometry side,

$$G^\bullet(\lambda;\tau;\mathbf{p}) = \mathtt{exp}(G(\lambda;\tau;\mathbf{p}))$$

is the non-connected version of the generating series of the triple Hodge integral. More precisely,

$$G(\lambda;\tau;\mathbf{p}) = \sum_{\overrightarrow{\mu}} \left[\sum_{g=0}^{\infty} \lambda^{2g-2+l(\overrightarrow{\mu})} G_{g,\overrightarrow{\mu}}(\tau)\right] p^1_{\mu^1} p^2_{\mu^2} p^3_{\mu_3}$$

where $l(\overrightarrow{\mu}) = l(\mu^1) + l(\mu^2) + l(\mu^3)$ and $G_{g,\overrightarrow{\mu}}(\tau)$ denotes the Hodge integrals of the following form,

$$A(\tau) \int_{\overline{\mathcal{M}}_{g,l_1+l_2+l_3}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)}{\prod_{j=1}^{l_1}(1-\mu_j^1\psi_j)\prod_{j=1}^{l_2}\tau(\tau-\mu_j^2\psi_{l_1+j})} \cdot \frac{(\tau(\tau+1))^{l_1+l_2+l_3-1}}{\prod_{j=1}^{l_3}(\tau+1)(\tau+1+\mu_j^3\psi_{l_1+l_2+j})},$$

where

$$A(\tau) = \frac{-(\sqrt{-1}\lambda)^{l_1+l_2+l_3}}{|\operatorname{Aut}(\mu^1)||\operatorname{Aut}(\mu^2)||\operatorname{Aut}(\mu^3)|} \prod_{j=1}^{l_1} \frac{\prod_{a=1}^{\mu_j^1-1}(\tau\mu_j^1 + a)}{(\mu_j^1-1)!} \cdot \prod_{j=1}^{l_2} \frac{\prod_{a=1}^{\mu_j^1-1}((-1-1/\tau)\mu_j^2 + a)}{(\mu_j^2-1)!} \prod_{j=1}^{l_3} \frac{\prod_{a=1}^{\mu_j^1-1}(-\mu_j^3/(\tau+1)+a)}{(\mu_j^3-1)!}.$$

In the above expression, $l_i = l(\mu^i)$, $i = 1,2,3$. Despite its complicated coefficients, these triple integrals naturally arise from localizations on the moduli spaces of relative stable maps into the blow-up of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ along certain divisors. It also naturally appears in open string theory computations [**1**]. See [**21**] for more details.

One of our results in [**21**] states that $G^\bullet(\lambda;\tau;\mathbf{p})$ has a combinatorial expression $R^\bullet(\lambda;\tau;\mathbf{p})$ in terms of the Chern-Simons knot invariants $W_{\overrightarrow{\mu}}$, and it is a closed combinatorial expression. More precisely it is given by

$$\begin{aligned} &R^\bullet(\lambda;\tau;\mathbf{p}) \\ &= \sum_{\overrightarrow{\mu}} \left[\sum_{|\nu^i|=|\mu^i|} \prod_{i=1}^{3} \frac{\chi_{\nu^i}(\mu^i)}{z_{\mu^i}} q^{\frac{1}{2}(\sum_{i=1}^{3} \kappa_{\nu^i} \frac{w_{i+1}}{w_i})} W_{\overrightarrow{\nu}}(q) \right] p^1_{\mu^1} p^2_{\mu^2} p^3_{\mu^3}. \end{aligned}$$

Here $w_4 = w_1$ and $w_3 = -w_1 - w_2$ and $\tau = \frac{w_2}{w_1}$. Due to the complicated combinatorics in the initial values, the combinatorial expression $W_{\overrightarrow{\mu}}$ we obtained is different from the expression $\mathcal{W}_{\overrightarrow{\mu}}$ obtained by Vafa et al. Actually our expression is even simpler than theirs in some sense. The expression we obtained is more convenient for mathematical applications such as the proof of the Gopakumar-Vafa conjecture for open toric Calabi-Yau manifolds. It should be possible to identify the two combinatorial expressions by using the classical theory of symmetric functions, as pointed out to us by R. Stanley.

Theorem 12.1. *We have the equality:*

$$G^\bullet(\lambda;\tau;\boldsymbol{p}) = R^\bullet(\lambda;\tau;\boldsymbol{p}).$$

The key point to prove that the above theorem is still the proof of convolution formulas for both sides which imply the cut-and-join equation. The proof of the convolution formula for $G^\bullet(\lambda;\tau;\mathbf{p})$ is much more complicated than the one and two partition cases. See [**21**] for details.

The above theorem is crucial for us to establish the theory of topological vertex in [**21**], which gives the most powerful way to compute the generating series of all genera and all degree Gromov-Witten invariants for open formal Calabi-Yau manifolds. The most useful property of topological vertex is its gluing property induced by the orthogonal relations of the characters of the symmetric group. This is very close to the situation of two dimensional gauge theory. In fact string theorists consider topological vertex as a kind of lattice theory on Calabi-Yau manifolds. By using the gluing formula we can easily obtain closed formulas for generating series of Gromov-Witten invariants of all genera and all degrees, open or closed, for all open toric Calabi-Yau manifolds, in terms of the Chern-Simons knot invariants. Such formulas are always given by finite sum of products of those Chern-Simons type invariants $\mathcal{W}_{\mu,\nu}$'s. The magic of topological vertex is that, by simply looking at the moment map graph of the toric surfaces in the open toric Calabi-Yau, we can immediately write down the closed formula for the generating series for all genera and all degree Gromov-Witten invariants, or more precisely the Euler numbers of certain bundles on the moduli space of stable maps.

Here we only give one example to describe the topological vertex formula for the generating series of the all degree and all genera Gromov-Witten invariants for the open toric Calabi-Yau 3-folds. We write down the explicit close formula of the generating series of the Gromov-Witten invariants for $O(-3) \longrightarrow \mathbf{P}^2$ in terms of the Chern-Simons invariants.

Example: The complete generating series of Gromov-Witten invariants of all degree and all genera for $O(-3) \longrightarrow \mathbf{P}^2$ s given by

$$\begin{aligned}&\exp\left(\sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t)\right)\\ &= \sum_{\nu_1,\nu_2,\nu_3}\mathcal{W}_{\nu_1,\nu_2}\mathcal{W}_{\nu_2,\nu_3}\mathcal{W}_{\nu_3,\nu_1}(-1)^{\sum_{j=1}^3|\nu_j|}q^{\frac{1}{2}\sum_{i=1}^3\kappa_{\nu_i}}\,e^{t(\sum_{j=1}^3|\nu_j|)}\end{aligned}$$

where $q = e^{\sqrt{-1}\lambda}$. The precise definition of $F_g(t)$ will be given in the next section.

For general open toric Calabi-Yau manifolds, the expressions are just similar. They are all given by finite and closed formulas, which are easily read out from the moment map graphs associated to the toric surfaces, with the topological vertex associated to each vertex of the graph.

In [**1**] Vafa and his group first developed the theory of topological vertex by using string duality between Chern-Simons and Calabi-Yau, which is a physical theory. In [**21**] we established the mathematical theory of the topological vertex, and derived various mathematical corollaries, including the relation of the Gromov-Witten invariants to the equivariant index theory as motivated by the Nekrasov conjecture in string duality [**27**].

13. Gopakumar-Vafa Conjecture and Indices of Elliptic Operators

Let $N_{g,d}$ denote the so-called Gromov-Witten invariant of genus g and degree d of an open toric Calabi-Yau 3-fold. $N_{g,d}$ is defined to be the Euler number of the obstruction bundle on the moduli space of stable maps of degree $d \in H_2(S,\mathbb{Z})$ from genus g curve into the surface base S. The open toric Calabi-Yau manifold associated to the toric surface S is the total space of the canonical line bundle K_S on S. More precisely

$$N_{g,d} = \int_{[\overline{\mathcal{M}}_g(S,d)]^v} e(V_{g,d})$$

with $V_{g,d} = R^1\pi_* u^* K_S$ a vector bundle on the moduli space induced by the canonical bundle K_S. Here $\pi: U \to \overline{\mathcal{M}}_g(S,d)$ denotes the universal curve and u can be considered as the evaluation or universal map. Let us write

$$F_g(t) = \sum_{d\geq 0} N_{g,d}\, e^{-d\cdot t}.$$

The Gopakumar-Vafa conjecture is stated as follows:

Gopakumar-Vafa Conjecture: *There exists an expression:*

$$\sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \sum_{k=1}^{\infty} \sum_{g,d\geq 0} n_d^g \frac{1}{d} (2sin \frac{d\lambda}{2})^{2g-2} e^{-kd\cdot t},$$

such that n_d^g are integers, called instanton numbers.

Motivated by the Nekrasov duality conjecture between the four dimensional gauge theory and string theory, we are able to interpret the above integers n_d^g as equivariant indices of certain elliptic operators on the moduli spaces of anti-self-dual connections [**27**]:

THEOREM 13.1. *For certain interesting cases, these n_d^g's can be written as equivariant indices on the moduli spaces of anti-self-dual connections on $\mathbf{C}^2$.*

For more precise statement, we refer the reader to [**27**]. The interesting cases include open toric Calabi-Yau manifolds when S is Hirzebruch surface. The proof of this theorem is to compare fixed point formula expressions for equivariant indices of certain elliptic operators on the moduli spaces of anti-self-dual connections with the combinatorial expressions of the generating series of the Gromov-Witten invariants on the moduli spaces of stable maps. They both can be expressed in terms of Young diagrams of partitions. We find that they agree up to certain highly non-trivial "mirror transformation", a complicated variable change. This result is not only interesting for the index formula interpretation of the instanton numbers, but also for the fact that it gives the first complete examples that the Gopakumar-Vafa conjecture holds for all genera and all degrees.

Recently P. Peng [**45**] has given the proof of the Gopakumar-Vafa conjecture for all open toric Calabi-Yau 3-folds by using the Chern-Simons expressions from the topological vertex. His method is to explore the property of the Chern-Simons expression in great detail with some clever observation about the form of the combinatorial expressions. On the other hand, Kim in [**13**] has derived some remarkable recursion formulas for Hodge integrals of all genera and any number of marked points, involving one λ-classes. His method is to add marked points in the moduli spaces and then follow the localization argument we used to prove the Mariño-Vafa formula.

14. Simple Localization Proofs of the ELSV Formula

Given a partition μ of length $l(\mu)$, denote by $H_{g,\mu}$ the Hurwitz numbers of almost simple Hurwitz covers of $\mathbb{P}^1$ of ramification type μ by connected genus g Riemann surfaces. The ELSV formula [**8, 10**] states:

$$H_{g,\mu} = (2g-2+|\mu|+l(\mu))! I_{g,\mu}$$

where

$$I_{g,\mu} = \frac{1}{|\operatorname{Aut}(\mu)|} \prod_{i=1}^{l(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^{\vee}(1)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)}.$$

Define generating functions

$$
\begin{aligned}
\Phi_\mu(\lambda) &= \sum_{g\geq 0} H_{g,\mu}\frac{\lambda^{2g-2+|\mu|+l(\mu)}}{(2g-2+|\mu|+l(\mu))!},\\
\Phi(\lambda;p) &= \sum_{|\mu|\geq 1}\Phi_\mu(\lambda)p_\mu,\\
\Psi_\mu(\lambda) &= \sum_{g\geq 0} I_{g,\mu}\lambda^{2g-2+|\mu|+l(\mu)},\\
\Psi(\lambda;p) &= \sum_{|\mu|\geq 1}\Psi_\mu(\lambda)p_\mu.
\end{aligned}
$$

In terms of generating functions, the ELSV formula reads

$$\Psi(\lambda;p) = \Phi(\lambda;p).$$

It was known that $\Phi(\lambda;p)$ satisfies the following cut-and-join equation:

$$\frac{\partial\Theta}{\partial\lambda} = \frac{1}{2}\sum_{i,j\geq 1}\left(ijp_{i+j}\frac{\partial^2\Theta}{\partial p_i\partial p_j} + ijp_{i+j}\frac{\partial\Theta}{\partial p_i}\frac{\partial\Theta}{\partial p_j} + (i+j)p_ip_j\frac{\partial\Theta}{\partial p_{i+j}}\right).$$

This formula was first proved in [**7**]. Later this equation was reproved by sum formula of symplectic Gromov-Witten invariants [**20**].

The calculations in Section 7 and Appendix A of [**27**] show that

$$\tilde{H}_{g,\mu} = (2g-2+|\mu|+l(\mu))!I_{g,\mu}$$

$$
\begin{aligned}
\tilde{H}_{g,\mu} = (2g-3+|\mu|+l(\mu))!\Bigg(&\sum_{\nu\in J(\mu)} I_{g,\nu} + \sum_{\nu\in C(\mu)} I_2(\nu)I_{g-1,\nu}\\
&+\sum_{g_1+g_2=g}\ \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)I_{g_1,\nu_1}I_{g_2,\nu_2}\Bigg)
\end{aligned}
$$

where

$$\tilde{H}_{g,\mu} = \int_{[\overline{\mathcal{M}}_{g,0}(\mathbf{P}^1,\mu)]^{\mathrm{vir}}} \mathrm{Br}^* H^r$$

is some relative Gromov-Witten invariant of $(\mathbf{P}^1,\infty)$, and $C(\mu), J(\mu), I_1$, I_2, I_3 are defined as in [**20**]. In fact, as proved in [**27**], this is double Hurwitz numbers. So we have

$$
\begin{aligned}
(2g-2+|\mu|+l(\mu))I_{g,\mu} = &\sum_{\nu\in J(\mu)} I_{g,\nu} + \sum_{\nu\in C(\mu)} I_2(\nu)I_{g-1,\nu}\\
&+\sum_{g_1+g_2=g}\ \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)I_{g_1,\nu_1}I_{g_2,\nu_2},
\end{aligned}
$$

which is equivalent to the statement that the generating function $\Psi(\lambda;p)$ of $I_{g,\mu}$ also satisfies the cut-and-join equation.

Any solution $\Theta(\lambda; p)$ to the cut-and-join equation (14) is uniquely determined by its initial value $\Theta(0; p)$, so it remains to show that $\Psi(0; p) = \Phi(0; p)$. Note that $2g - 2 + |\mu| + l(\mu) = 0$ if and only if $g = 0$ and $\mu = (1)$, so

$$\Psi(0; p) = H_{0,(1)} p_1, \quad \Phi(0; p) = I_{0,(1)} p_1.$$

It is easy to see that $H_{0,(1)} = I_{0,(1)} = 1$, so

$$\Psi(0; p) = \Phi(0; p).$$

One can see geometrically that the relative Gromov-Witten invariant $\tilde{H}_{g,\mu}$ is equal to the Hurwitz number $H_{g,\mu}$. This together with (14) gives a proof of the ELSV formula presented in [**27**, Section 7] in the spirit of [**10**]. Note that $\tilde{H}_{g,\mu} = H_{g,\mu}$ is not used in the proof described above.

On the other hand we can deduce the ELSV formula as the limit of the Mariño-Vafa formula. By the Burnside formula, one easily gets the following expression (see e.g., [**29**]):

$$\begin{aligned}\Phi(\lambda; p) &= \log \left(\sum_{\mu} \left(\sum_{|\nu|=|\mu|} \frac{\chi_\nu(\mu)}{z_\mu} e^{\kappa_\nu \lambda/2} \frac{\dim R_\nu}{|\nu|!} \right) p_\mu. \right) \\ &= \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\mu} \sum_{\cup_{i=1}^n \mu_i = \mu} \prod_{i=1}^n \sum_{|\nu_i|=|\mu_i|} \frac{\chi_{\nu_i}(\mu_i)}{z_{\mu_i}} e^{\kappa_{\nu_i}\lambda/2} \frac{\dim R_{\nu_i}}{|\nu_i|!} p_\mu.\end{aligned}$$

The ELSV formula reads

$$\Psi(\lambda; p) = \Phi(\lambda; p)$$

where the left hand side is a generating function of Hodge integrals $I_{g,\mu}$, and the right hand side is a generating function of representations of symmetric groups. So the ELSV formula and the MV formula are of the same type.

Actually, the ELSV formula can be obtained by taking a particular limit of the MV formula $G(\lambda; \tau; p) = R(\lambda; \tau; p)$. More precisely, it is straightforward to check that

$$\begin{aligned}&\lim_{\tau \to 0} G(\lambda\tau; \frac{1}{\tau}; (\lambda\tau)p_1, (\lambda\tau)^2 p_2, \ldots) \\ &= \sum_{|\mu| \neq 0} \sum_{g=0}^{\infty} \sqrt{-1}^{2g-2+|\mu|+l(\mu)} I_{g,\mu} \lambda^{2g-2+|\mu|+l(\mu)} p_\mu \\ &= \Psi(\sqrt{-1}\lambda; p)\end{aligned}$$

and

$$
\begin{aligned}
&\lim_{\tau\to 0} R(\lambda\tau; \frac{1}{\tau}; (\lambda\tau)p_1, (\lambda\tau)^2 p_2, \ldots) \\
&= \log\left(\sum_{\mu}\left(\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}\kappa_\nu\lambda/2}\lim_{t\to 0}(t^{|\nu|}V_\nu(t))\right)p_\mu\right) \\
&= \log\left(\sum_{\mu}\left(\sum_{|\nu|=|\mu|}\frac{\chi_\nu(C(\mu))}{z_\mu}e^{\sqrt{-1}\kappa_\nu\lambda/2}\frac{1}{\prod_{x\in\nu}h(x)}\right)p_\mu\right) \\
&= \Phi(\sqrt{-1}\lambda; p)
\end{aligned}
$$

where we have used

$$\frac{1}{\prod_{x\in\nu}h(x)} = \frac{\dim R_\nu}{|\nu|!}.$$

See [**30**] for more details. In this limit, the cut-and-join equation of $G(\lambda;\tau;p)$ and $R(\lambda;\tau;p)$ reduces to the cut-and-join equation of $\Psi(\lambda;p)$ and $\Phi(\lambda;p)$, respectively.

15. A Localization Proof of the Witten Conjecture

The Witten conjecture for moduli spaces states that the generating series F of the integrals of the ψ classes for all genera and any number of marked points satisfies the KdV equations and the Virasoro constraint. For example, the Virasoro constraint states that F satisfies

$$L_n \cdot F = 0, \;\; n \geq -1$$

where L_n denote certain Virasoro operators as given below.

Witten conjecture was first proved by Kontsevich [**16**] using a combinatorial model of the moduli space and matrix model, with later approaches by Okounkov-Pandhripande [**43**] using ELSV formula and combinatorics, and by Mirzakhani [**42**] using Weil-Petersson volumes on moduli spaces of bordered Riemann surfaces.

I will present a much simpler proof by using functorial localization and asymptotics. This was done [**14**] jointly with Y.-S. Kim. This is also motivated by methods in proving conjectures from string duality. It should have more applications.

The basic idea of our proof is to directly prove the following recursion formula which, as derived in physics by Dijkgraaf, Verlinde and Verlinde by using quantum field theory, implies the Virasoro and the KdV equation for the generating series F of the integrals of the ψ classes:

THEOREM 15.1. *We have identity*

$$\left\langle \tilde{\sigma}_n \prod_{k\in S} \tilde{\sigma}_k \right\rangle_g = \sum_{k\in S}(2k+1)\left\langle \tilde{\sigma}_{n+k-1} \prod_{l\neq k} \tilde{\sigma}_l \right\rangle_g$$
$$+\frac{1}{2}\sum_{a+b=n-2}\left\langle \tilde{\sigma}_a\tilde{\sigma}_b \prod_{l\neq a,b} \tilde{\sigma}_l \right\rangle_{g-1}$$
$$+\frac{1}{2}\sum_{\substack{S=X\cup Y,\\ a+b=n-2,\\ g_1+g_2=g}}\left\langle \tilde{\sigma}_a \prod_{k\in X} \tilde{\sigma}_k \right\rangle_{g_1}\left\langle \tilde{\sigma}_b \prod_{l\in Y} \tilde{\sigma}_l \right\rangle_{g_2}.$$

Here $\tilde{\sigma}_n = (2n+1)!!\psi^n$ and

$$\left\langle \prod_{j=1}^n \tilde{\sigma}_{k_j} \right\rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \tilde{\sigma}_{k_j}.$$

The notation $S = \{k_1, \ldots, k_n\} = X \cup Y$.

To prove the above recursion relation, we first apply the functorial localization to the natural branch map from moduli space of relative stable maps $\overline{\mathcal{M}}_g(\mathbf{P}^1, \mu)$ to projective space $\mathbf{P}^r$ where $r = 2g-2+|\mu|+l(\mu)$ is the dimension of the moduli. Since the push-forward of 1 is a constant in this case, we easily get the cut-and-join equation for one Hodge integral

$$I_{g,\mu} = \frac{1}{|\mathrm{Aut}\ \mu|}\prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^\vee(1)}{\prod(1-\mu_i\psi_i)}.$$

As given in the previous section, we have

$$\begin{aligned}
&(2g-2+|\mu|+l(\mu))I_{g,\mu}\\
&= \sum_{\nu\in J(\mu)} I_{g,\nu} + \sum_{\nu\in C(\mu)} I_2(\nu)I_{g-1,\nu}\\
&\quad + \sum_{g_1+g_2=g}\ \sum_{\nu^1\cup\nu^2\in C(\mu)} I_3(\nu^1,\nu^2)I_{g_1,\nu_1}I_{g_2,\nu_2}.
\end{aligned}$$

Write $\mu_i = Nx_i$. Let N goes to infinity and expand in x_i, and we get:

$$\begin{aligned}
&\sum_{i=1}^{n}\Big[\frac{(2k_i+1)!!}{2^{k_i+1}k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-\frac{1}{2}}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n}}\prod\psi_j^{k_j}\\
&\quad-\sum_{j\neq i}\frac{(x_i+x_j)^{k_i+k_j-\frac{1}{2}}}{\sqrt{2\pi}}\prod_{l\neq i,j}\frac{x_l^{k_l-\frac{1}{2}}}{\sqrt{2\pi}}\int_{\overline{\mathcal{M}}_{g,n-1}}\psi^{k_i+k_j-1}\prod\psi_l^{k_l}\\
&-\frac{1}{2}\sum_{k+l=k_i-2}\frac{(2k+1)!!(2l+1)!!}{2^{k_i}\,k_i!}x_i^{k_i}\prod_{j\neq i}\frac{x_j^{k_j-\frac{1}{2}}}{\sqrt{2\pi}}\Big[\int_{\overline{\mathcal{M}}_{g-1,n+1}}\psi_1^k\psi_2^l\prod\psi_j^{k_j}\\
&+\sum_{\substack{g_1+g_2=g,\\ \nu_1\cup\nu_2=\nu}}\int_{\overline{\mathcal{M}}_{g_1,n_1}}\psi_1^k\prod\psi_j^{k_j}\int_{\overline{\mathcal{M}}_{g_2,n_2}}\psi_1^l\prod\psi_j^{k_j}\Big]\Big]=0.
\end{aligned}$$

Performing Laplace transforms on the x_i's, we get the recursion formula which implies both the KdV equations and the Virasoro constraints. For example, the Virasoro constraints state that the generating series

$$\tau(\tilde{t}) = \exp\sum_{g=0}^{\infty}\left\langle \exp\sum_n \tilde{t}_n\tilde{\sigma}_n\right\rangle_g$$

satisfies the equations:

$$L_n\cdot\tau = 0, \qquad (n\geq -1)$$

where L_n denote the Virasoro differential operators

$$\begin{aligned}
L_{-1} &= -\frac{1}{2}\frac{\partial}{\partial\tilde{t}_0}+\sum_{k=1}^{\infty}\left(k+\frac{1}{2}\right)\tilde{t}_k\frac{\partial}{\partial\tilde{t}_{k-1}}+\frac{1}{4}\tilde{t}_0^2\\
L_0 &= -\frac{1}{2}\frac{\partial}{\partial\tilde{t}_1}+\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\tilde{t}_k\frac{\partial}{\partial\tilde{t}_k}+\frac{1}{16}\\
L_n &= -\frac{1}{2}\frac{\partial}{\partial\tilde{t}_{n-1}}+\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)\tilde{t}_k\frac{\partial}{\partial\tilde{t}_{k+n}}+\frac{1}{4}\sum_{i=1}^{n}\frac{\partial^2}{\partial\tilde{t}_{i-1}\partial\tilde{t}_{n-i}}.
\end{aligned}$$

We remark that the same method can be used to derive very general recursion formulas in Hodge integrals and general Gromov-Witten invariants. We hope to report these results on a later occasion.

16. Final Remarks

We have briefly reviewed our recent results on both the geometric and the topological aspect of the moduli spaces of Riemann surfaces. Although significant progress has been made in understanding the geometry and topology of the moduli spaces of Riemann surfaces, there are still many problems that remain to be solved in both aspects.

For the geometric aspect, it will be interesting to understand the convergence of the Ricci flow starting from the Ricci metric to the Kähler-Einstein metric, the representations of the mapping class group on the middle dimensional L^2-cohomology of these metrics, and the index theory associated to these complete Kähler metrics. Recently, we showed that the metrics on the logarithm cotangent bundle induced by the Weil-Petersson metric, the Ricci metric and the perturbed Ricci metric are good in the sense of Mumford [**36**]. Also the perturbed Ricci metric is the first complete Kähler metric on the moduli spaces with bounded negative Ricci and holomorphic sectional curvature and bounded geometry, and we believe this metric must have more interesting applications. Another question is which of these metrics are actually identical. We hope to report on the progress of the study of these problems on a later occasion.

For the topological aspect it will be interesting to have closed formulas to compute Hodge integrals involving more Hodge classes, and to use our complete understanding of the Gromov-Witten theory in the open formal toric Calabi-Yau manifolds to understand the compact Calabi-Yau case. We strongly believe that there is a more interesting and grand duality picture between Chern-Simons invariants for three dimensional manifolds and the Gromov-Witten invariants for open toric Calabi-Yau manifolds. Our proofs of the Mariño-Vafa formula, and the setup of the mathematical foundation for topological vertex theory and the results of Peng and Kim all together have just opened a small window for a more splendid picture.

Finally, although we have worked on two quite different aspects of the moduli spaces, we strongly believe that the methods and results we have developed and obtained in these seemingly unrelated aspects will eventually merge together to give us a completely clear understanding of the moduli spaces of Riemann surfaces.

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Center of Mathematical Sciences, Zhejiang University, Hangzhou, China
E-mail address: `liu@cms.zju.edu.cn`

and

Department of Mathematics, Univ. of California at Los Angeles
Los Angeles, CA 90095-1555
E-mail address: `liu@math.ucla.edu`