

The Ehrhart function for symbols

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ABSTRACT. We derive an Ehrhart function for symbols from the Euler-MacLaurin formula with remainder.

In memory of Prof. S.S. Chern, our teacher and mentor.

1. Introduction

Let $\Delta \subset \mathbb{R}^n$ be a convex polytope whose vertices are in \mathbb{Z}^n and such that the origin 0 is in the interior of Δ . Consider the expanded polytope $N \cdot \Delta$.

Ehrhart's theorem [**Ehr**] asserts that for N a positive integer, the number of lattice points in the expanded polytope, i.e.,

$$\#((N \cdot \Delta) \cap \mathbb{Z}^n), \quad N \in \mathbb{Z}_+$$

is a polynomial in N . More generally, suppose that f is a polynomial, and let

$$(1) \quad p(N, f) := \sum_{\ell \in N \cdot \Delta \cap \mathbb{Z}^n} f(\ell).$$

Then Ehrhart's theorem asserts that $p(N, f)$ is a polynomial in N .

In the case that Δ is a simple polytope (meaning that n edges emanate from each vertex) Ehrhart's theorem is a consequence of the Euler-MacLaurin formula, [**Kh**, **KP**, **CS1**, **CS2**, **Gu**, **BV**, **DR**] and one can be more explicit about the nature of the polynomial $p(N, f)$.

Let us explain how this works in the more restrictive case where Δ is not only simple but is regular, meaning that the local cone at each vertex can be transformed by an integral unimodular affine transformation into a

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neighborhood of the origin in the standard orthant \mathbb{R}_+^n .¹ In this case we can apply the formula of Khovanskii-Pukhlikov [KP], which reads as follows: The polytope Δ can be described by a set of inequalities

$$x \cdot u_i + a_i \geq 0, \quad i = 1, \dots, m$$

where m is the number of facets of Δ , where the u_i are primitive lattice vectors, and where the a_i are positive integers. Then for any positive number t , the expanded polytope $t \cdot \Delta$ is described by the inequalities

$$x \cdot u_i + ta_i \geq 0, \quad i = 1, \dots, m.$$

Let $\Delta_{t,h}$, $h = (h_1, \dots, h_m)$ be the polytope defined by

$$(2) \quad x \cdot u_i + ta_i + h_i \geq 0, \quad i = 1, \dots, m.$$

Then the function

$$(3) \quad \tilde{p}(t, h, f) := \int_{\Delta_{t,h}} f(x) dx$$

is a polynomial in t and h .

The formula of Khovanskii-Pukhlikov (applied to $N \cdot \Delta$) expresses $p(N, f)$ in terms of a differential operator applied to $\tilde{p}(t, h, f)$: Explicitly, consider the infinite order constant coefficient differential operator

$$(4) \quad \text{Todd} \left(\frac{\partial}{\partial h} \right) = \sum_{\gamma} b_{\gamma} \left(\frac{\partial}{\partial h} \right)^{\gamma}$$

where $\sum_{\gamma} b_{\gamma} x^{\gamma}$ is the Taylor series expansion at the origin of the Todd function

$$\text{Todd}(x) = \prod_{i=1}^m \frac{x_i}{1 - e^{-x_i}}.$$

The Khovanskii-Pukhlikov formula says that

$$p(N, f) = \text{Todd} \left(\frac{\partial}{\partial h} \right) \tilde{p}(N, h, f) \Big|_{h=0}.$$

Note that since \tilde{p} is a polynomial in h the right hand side really involves only a finite order differential operator.

For purposes below it will be convenient to write the Khovanskii-Pukhlikov formula in the form

$$(5) \quad p(N, f) - \tilde{p}(N, 0, f) = \left(\text{Todd} \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0}.$$

¹Of course, this use of the word “regular” has nothing to do with the term denoting Platonic solids. There are other names in the literature for the property we are describing, such as “smooth”, “Delzant”, “torsionfree”, “unimodular” etc.. It is unfortunate that the nomenclature for polytopes with this property has not yet been standardized.

For simple polytopes there is a more general formula due to [CS2, Gu, BV]. Our goal in this paper is to prove an analogue of (5) and its generalizations when the polynomial f is replaced by a “symbol”, a term whose definition we recall from the theory of partial differential equations: A smooth function $f \in C^\infty(\mathbb{R}^n)$ is called a **symbol of order r** if for every n -tuple of non-negative integers $a := (a_1, \dots, a_n)$, there exists a constant C_a such that

$$|\partial_1^{a_1} \dots \partial_n^{a_n} f(x)| \leq C_a (1 + |x|)^{r-|a|}$$

where $|a| = \sum_i a_i$. In particular, a polynomial of degree r is a symbol of order r . Note that if f is a symbol of order r on \mathbb{R}^n then its derivatives of order a are in L^1 if $r < |a| - n$.

For simplicity, we will restrict ourselves in this paper to **polyhomogeneous symbols**, meaning functions $f \in C^\infty(\mathbb{R}^n)$ which admit asymptotic expansions of the form:

$$(6) \quad f(x) \sim \sum_{-\infty}^r f_\ell(x)$$

for $\|x\| \gg 0$ where the f_ℓ are homogeneous symbols of degree ℓ meaning that each f_ℓ is a symbol of order ℓ with the property that for $\|x\|$ sufficiently large and $t \geq 1$ we have $f_\ell(tx) = t^\ell f_\ell(x)$. The sum in (6) is over a discrete sequence of numbers tending to $-\infty$. “Asymptotic” means that for any j

$$(7) \quad f(x) - \sum_{\ell=j}^r f_\ell(x) = o(\|x\|^j)$$

as $\|x\| \rightarrow \infty$. The number r occurring in (6) is again called the **order** of the asymptotic series and the collection of functions satisfying (6) will be called polyhomogeneous symbols of order r and denoted by S^r .

For the sake of exposition in this introduction, we will continue to discuss the case where Δ is regular.

We will show that if f is a polyhomogeneous symbol then the function $p(N, f)$ given by (1) is a polyhomogeneous symbol in N and its asymptotic expansion in powers of N is given by a formula similar to (5) with two key differences:

- (1) For symbols, an infinite number of differentiations occur on the right hand side of (5), i.e., the whole Todd operator must be applied. So (5) must be understood as an asymptotic series, not as an equality.
- (2) The formula (5) has to be corrected by adding a constant term C to the right hand side, a constant which is zero for the case of a polynomial.

More precisely, we will prove:

THEOREM 1.1. *Let Δ be a regular polytope whose vertices lie in \mathbb{Z}^n with 0 in the interior of Δ . Let $f \in S^r$ and $N \in \mathbb{Z}_+$ and let*

$$p(N, f) := \sum_{\ell \in N \cdot \Delta \cap \mathbb{Z}^n} f(\ell).$$

Let $\tilde{p}(t, h, f)$ be defined by (3) so that

$$\tilde{p}(N, 0, f) = \int_{N \cdot \Delta} f(x) dx,$$

Then $p(N, f) - \tilde{p}(N, 0, f)$ is a symbol in N and has the asymptotic expansion

$$p(N, f) - \tilde{p}(N, 0, f) \sim \left(\text{Todd} \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0} + C$$

where C is a constant.

The constant C is of interest in its own right. It can be thought of as a “regularized” version of the difference

$$(8) \quad \sum_{\ell \in \mathbb{Z}^n} f(\ell) - \int_{\mathbb{R}^n} f(x) dx.$$

Of course there is no reason why either the sum or the integral in (8) should converge. But we can “regularize” both as follows: Define the function $\langle x \rangle$ by

$$\langle x \rangle^2 := 1 + \|x\|^2.$$

For $s \in \mathbb{C}$ let

$$f(x, s) := f(x) \langle x \rangle^s.$$

We will show that

$$(9) \quad C(s) := \sum_{\ell \in \mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x, s) dx,$$

which is holomorphic for $\text{Re } s \ll 0$, has an analytic continuation to the entire complex plane and that the missing constant C on the right hand side of (5) is exactly $C(0)$. In particular, the constant C is independent of the particular polytope in question.

This result is related to, and inspired by, a result of Friedlander-Guillemin [FG] on “Szego regularizations” of determinants of pseudodifferential operators. In their result, as in ours, there is a missing constant which also has to be computed by the above process of “zeta regularization”.

Our result is somewhat insensitive to the mode of regularization. In fact, it can be generalized as follows: Define a “gauged symbol” [Gugau] to be a function $f(x, s) \in C^\infty(\mathbb{R}^n \times \mathbb{C})$ which depends holomorphically on s and for fixed s is a symbol of order $\text{Re } s + r$. For example, if f is a symbol, the function $f(x) \langle x \rangle^s$ introduced above is such a gauged symbol. We make a similar definition of “gauged polyhomogenous symbols”. We will prove that if $f(x, s)$ is a gauged polyhomogenous symbol with

$$f(x) = f(x, 0)$$

then the function given by (9) with this more general definition of $f(x, s)$ again extends holomorphically from $\operatorname{Re} s \ll 0$ to the entire plane and $C = C(0)$.

The above results will be proved for the more general case of simple integral polytopes in §2. The proof is largely based on the Euler-MacLaurin formula with remainder as proved in [KSW] and motivated by an argument of Hardy on “Ramanujan regularization” [Hardy]. Ramanujan’s key idea was to use the classical Euler-MacLaurin formula in one variable to regularize (8) by providing “counter terms” in passing to infinity in the difference between sum and integral in one dimension. The origin of this method goes back to Euler’s continuation of the zeta function past the pole at $z = 1$ and his introduction of what is known today as “Euler’s constant”.

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2. An Ehrhart formula for simple polytopes

We continue with the notation of §1. So $\Delta \subset \mathbb{R}^n$ is a convex polytope whose vertices are in \mathbb{Z}^n and such that the origin 0 is in the interior of Δ . We assume in this section that Δ is simple, which means that for every vertex p there are exactly n edges emanating from p so they lie on rays

$$(10) \quad p + t\alpha_{i,p}$$

where the $\alpha_{i,p}$, $i = 1, \dots, n$ form a basis of \mathbb{R}^n .

2.1. The Euler-MacLaurin formula for symbols. We want to apply the Euler-MacLaurin formula with remainder, [KSW]. In [KSW] one dealt with a weighted sum where points in the interior of the polytope are given weight 1, points on the relative interior of a facet are given weight $w(x) := \frac{1}{2}$, and, more generally, points in the relative interior of a face of codimension k are given weights $w(x) := \frac{1}{2^k}$. The weighted sum $p_{\frac{1}{2}}(N, f)$ is then defined as

$$p_{\frac{1}{2}}(N, f) := \sum_{\ell \in (N \cdot \Delta) \cap \mathbb{Z}^n} w(\ell) f(\ell).$$

Theorem 3 of [KSW] gives an Euler-MacLaurin formula with remainder for weighted sums of symbols.

More generally, [AW] consider the Euler-MacLaurin formula with remainder for more general weightings including the unweighted sum we considered in §1. We refer to equations (28) and (29) in [AW] for the definition of a general weighting, w , and we will denote the corresponding weighted sum here by $p_w(N, f)$. They stated their formula with remainder for smooth functions of compact support, but the passage from the case of smooth functions of compact support to that of symbols is exactly the same as in [KSW].

To formulate the Euler-MacLaurin formula with remainder we fix a vector $\xi \in \mathbb{R}^n$ such that

$$(11) \quad \alpha_{i,p} \cdot \xi \neq 0 \quad \forall p \text{ and } i.$$

We then define

$$(12) \quad \alpha_{i,p}^\# := \begin{cases} \alpha_{i,p} & \text{if } \alpha_{i,p} \cdot \xi > 0 \\ -\alpha_{i,p} & \text{if } \alpha_{i,p} \cdot \xi < 0 \end{cases}$$

$$(13) \quad (-1)^p := \prod_{i=1}^n \frac{\alpha_{i,p}^\# \cdot \xi}{\alpha_{i,p} \cdot \xi}$$

and

$$(14) \quad C_{p,t} := \left\{ tp + \sum_{i=1}^n t_i \alpha_{i,p}^\#, \quad t_i \geq 0 \right\}.$$

There is a certain infinite order differential operator \mathbf{M} (depending on the weighting) in the variables h_1, \dots, h_m with constant term 1 whose truncation of order k is given by the sum in equation (89) of [KSW] (for weight $\frac{1}{2}$) and the sum in equation (56) in [AW] (for general weights) such that for any symbol f of order r and $k > n + r$

$$(15) \quad p_w(N, f) - \tilde{p}(N, 0, f) = \left((\mathbf{M}^{[k]}) \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0} + R^k(f, N)$$

where $\mathbf{M}^{[k]}$ denotes the truncation of \mathbf{M} at order k and

$$(16) \quad R^k(f, N) = \sum_p (-1)^p \int_{C_{p,N}} \left(\sum_{\substack{|\gamma|=nk \\ |\gamma|=k}} \phi_{\gamma,k}^p D^\gamma f \right) dx$$

where the $\phi_{\gamma,k}^p$ are bounded piecewise smooth periodic functions. For an explicit expression for these functions see [KSW] or [AW]. We will not need this here. If the polytope is regular, and we use unweighted sums, the operator \mathbf{M} is exactly the operator Todd of §1.

Notice that $\mathbf{M}^{[k]} - 1$ has no constant term, so

$$\left((\mathbf{M}^{[k]}) \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0}$$

involves integration of derivatives of f over faces of the polytope. These faces are moving out to infinity as $N \rightarrow \infty$. A derivative of a homogeneous summand in (6) is itself a homogeneous function. So as $N \rightarrow \infty$, the integral of this derivative over a face of the polytope is a homogeneous function in N whose degree depends on the degree of this derivative and the dimension of the face. So the contributions of the polyhomogeneous terms with sufficiently negative degree in (6) will be homogeneous terms of high negative degree in N in (15). By the same token, each homogeneous summand in (6) will yield

a finite number of terms to each order in (15). So the left hand side of (15) will be polyhomogenous symbol in N .

2.2. The key idea. We want to investigate the behavior of the remainder (16) as $N \rightarrow \infty$.

Since the origin 0 is in the interior of the polytope, it is in the interior of the cone generated by the $\alpha_{i,p}$ for any vertex p . This means that when we use the $\alpha_{i,p}$ as a basis of \mathbb{R}^n (based at the origin), the point p has strictly negative coefficients relative to this basis. When we flip the basis from $\alpha_{i,p}$ to $\alpha_{i,p}^\sharp$, the coefficients of those edges which are actually flipped become positive.

Condition (11) implies that the function $x \mapsto \xi \cdot x$ has a unique minimum on the polytope, and that this minimum is achieved at a single vertex, q . At this vertex, no edges are flipped, and at any other vertex $p \neq q$ at least one edge is flipped. So for $p \neq q$ we have

$$(17) \quad p = \sum_{i=1}^n a_{i,p} \alpha_{i,p}^\sharp,$$

where at least one $a_{i,p} > 0$. Then the cone $C_{p,t}$ is contained in the half space

$$\left\{ x = \sum_i x_i \alpha_{i,p}^\sharp, \quad x_i \geq a_{i,p} t \right\}$$

and so the p -th summand in (16) tends to zero as $N \rightarrow \infty$ for $p \neq q$.

At the vertex q we have we have $q = \sum_i a_{i,q} \alpha_{i,q} = \sum_i a_{i,q} \alpha_{i,q}^\sharp$ with all the $a_{i,q} < 0$ and so the cone $C_{q,t}$ tends to the entire space \mathbb{R}^n as $t \rightarrow \infty$. Thus (16) tends to

$$(18) \quad C_k = \int_{\mathbb{R}^n} \left(\sum_{\substack{|\gamma|=nk \\ |\gamma|=k}} \phi_{\gamma,k}^q D^\gamma f \right) dx.$$

It follows from (15) that this limiting value C_k is independent of k for k sufficiently large. Let us call this common limit $C = C(f)$. It also follows from (15) that $C(f)$ is independent of the choice of the polarizing vector ξ .

We shall interpret this limiting value C using regularization in the next section, and we will find that C is also independent of the particular polytope we are expanding.

If f is a polynomial, so that we choose k to be greater than the degree of f , we see from (18) that $C = 0$, as it must be from the classical Ehrhart theorem.

2.3. Application to polyhomogeneous symbols. Now suppose that f is a polyhomogenous symbol. We can apply the above to each summand in the asymptotic series (6). But notice that if we choose j sufficiently

negative, the function

$$g(x) = g_j(x) = f(x) - \sum_{\ell=j}^r f_\ell(x)$$

occurring on the left hand side of (7) will have the property that both

$$\sum_{\ell \in \mathbb{Z}^n} g(\ell) \quad \text{and} \quad \int_{\mathbb{R}^n} g(x) dx$$

are absolutely convergent. Furthermore, given any negative number m we can arrange, by choosing j sufficiently negative, that

$$p_w(N, g) - \sum_{\ell \in \mathbb{Z}^n} g(\ell) = o(N^m)$$

and

$$\tilde{p}(N, 0, g) - \int_{\mathbb{R}^n} g(x) dx = o(N^m)$$

so

$$[p_w(N, g) - \tilde{p}(N, 0, g)] - \left[\sum_{\ell \in \mathbb{Z}^n} g(\ell) - \int_{\mathbb{R}^n} g(x) dx \right] = o(N^m).$$

So if we define

$$(19) \quad C(f) = \sum_{\ell=j}^r C(f_\ell) + \left[\sum_{\ell \in \mathbb{Z}^n} g_j(\ell) - \int_{\mathbb{R}^n} g_j(x) dx \right]$$

for j sufficiently negative, then $C(f)$ is independent of the choice of j . Furthermore, we see that if f is a polyhomogeneous symbol, we get an asymptotic expansion of the form

$$p_w(N, f) - \tilde{p}(N, 0, f) \sim \sum_{\ell=-\infty}^r \left(\mathbf{M} \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f_\ell) \Big|_{h=0} + C(f),$$

where each level in the asymptotic expansion in N involves only finitely many f_ℓ . By abuse of language, we shall denote this equation as

$$(20) \quad p_w(N, f) - \tilde{p}(N, 0, f) \sim \left(\mathbf{M} \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0} + C(f).$$

2.4. Regularization. Suppose that we replace f by a gauged polyhomogeneous symbol $f(x, s)$ with $f(x, 0) \in S^r$. Then the remainder term (16) applied to a summand in the asymptotic expansion of $f_s = f(\cdot, s)$ is well defined if $\text{Re } s < -r - n + k$. Moreover, if $p \neq q$ so that $a_{i,p} > 0$ for some i the p -th summand on the right of (16) is of order $O(N^{\text{Re } s + r + n - k})$.

At the unique vertex q where no edges are flipped q -th summand of (16) differs from the integral

$$(21) \quad \int_{\mathbb{R}^n} \left(\sum_{\substack{|\gamma|=nk \\ |\gamma|=k}} \phi_{\gamma,k}^q D_x^\gamma f_\ell(x, s) \right) dx$$

by a term of order $O(N^{\operatorname{Re} s + \ell + n - k})$. Thus the gauged version of (20) is

$$(22) \quad p_w(N, f_s) - \tilde{p}(N, 0, f_s) = \left(\mathbf{M}^{[k]} - \operatorname{Id} \right) \left(\frac{\partial}{\partial h} \right) \tilde{p}(N, h, f_s) \Big|_{h=0} + C_k(s) \\ + O(N^{\operatorname{Re} s + r + n - k})$$

for

$$\operatorname{Re} s < -n - r + k$$

where $f_s = f(\cdot, s)$ where $C_k(s)$ is (20) with f replaced by f_s , and we have computed $C(s)$ by going out to level k in the Euler-MacLaurin expansion.

All the terms on the right of (22) are holomorphic on the half-plane $\operatorname{Re} s < -n - r + k$.

Letting $k \rightarrow \infty$ we conclude that on this half-plane we have, in the notation of (20),

$$(23) \quad p_w(N, f_s) - \tilde{p}(N, 0, f_s) \sim \left(\mathbf{M} \left(\frac{\partial}{\partial h} \right) - \operatorname{Id} \right) \tilde{p}(N, h, f_s) \Big|_{h=0} + C(s)$$

where

$$(24) \quad C(s) - C_k(s) = O(N^{\operatorname{Re} s + r + n - k}).$$

Since the $C_k(s)$ are holomorphic of the half-plane $\operatorname{Re} s < -n - r + k$, it follows that $C(s)$ is holomorphic on the whole plane.

Moreover, in the asymptotic series on the right of (23) all the terms are of order at most $\operatorname{Re} s + r + n$. Hence for $\operatorname{Re} s < -r - n$ these terms tend to zero and we get

$$C(s) = \lim_{N \rightarrow \infty} (p(N, f_s) - \tilde{p}(N, 0, f_s)) \\ = \sum_{\ell \in \mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x, s) dx,$$

and both the sum and the integral converge absolutely. So if we set $s = 0$ we obtain

$$(25) \quad p_w(N, f) - \tilde{p}(N, 0, f) \sim \left(\mathbf{M} \left(\frac{\partial}{\partial h} \right) - \operatorname{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0} + C$$

where $f(x) = f(x, 0)$ and $C = C(0)$. So we can think of C as a “regularization” of (8). To summarize: We have proved

THEOREM 2.1. *Let Δ be a simple polytope whose vertices lie in \mathbb{Z}^n with 0 in the interior of Δ . Let $f \in S^r$ and $N \in \mathbb{Z}_+$ and let*

$$p_w(N, f) := \sum_{\ell \in N \cdot \Delta \cap \mathbb{Z}^n} w(\ell) f(\ell).$$

Let $\tilde{p}(t, h, f)$ be defined by (3) so that

$$\tilde{p}(N, 0, f) = \int_{N \cdot \Delta} f(x) dx,$$

Then $p(N, f) - \tilde{p}(N, 0, f)$ is a symbol in N and has the asymptotic expansion

$$p_w(N, f) - \tilde{p}(N, 0, f) \sim \left(\mathbf{M} \left(\frac{\partial}{\partial h} \right) - \text{Id} \right) \tilde{p}(N, h, f) \Big|_{h=0} + C$$

where C is a constant. Furthermore, if $f(x, s)$ is a gauged polyhomogenous symbol with $f(x, 0) = f(x)$ then $C = C(0)$ where $C(s)$ is the entire function given by (23) and (24). For $\text{Re } s < -r - n$

$$C(s) = \sum_{\ell \in \mathbb{Z}^n} f(\ell, s) - \int_{\mathbb{R}^n} f(x, s) dx.$$

Hence $C(s)$ and in particular $C = C(0)$ is independent of the polytope.

REMARKS.

1. In the course of the discussion we have proved a similar theorem with polyhomogeneous symbols replaced by symbols and gauged polyhomogenous symbols replaced by gauged symbols.
2. Since the initial posting of this paper we have received the interesting paper [MP].

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