Analysis on Riemannian co-compact covers

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ABSTRACT. This is a survey on analysis on non-compact co-compact Riemannian covers and how it relates to random walks on finitely generated groups. The focus is on the long time behavior of the heat kernel and related topics such as Liouville theorems, scale invariant Harnack inequalities and isoperimetric profiles.

CONTENTS

1. Introduction

This article surveys results concerning the large scale potential theory of non-compact co-compact Riemannian regular covers. Here, "large scale potential theory" means the large scale analysis of the Laplace operator and heat equation. This includes heat kernel estimates, Liouville theorems and related topics. The fundamental idea behind most of the results described in this survey is that the large scale potential theory of a co-compact Riemannian cover is determined by the behavior of some basic random walks on its deck transformation group. Thus this article also reviews the relevant results concerning random walks on finitely generated groups. Of course, what is reviewed in this survey reflects in parts the author's personal taste and limitations. For pointers to certain aspects (e.g., Novikov-Shubin invariants) that are not covered here, see [**20, 41, 81**].

1.1. Acknowledgments. This surveys owes a lot to contacts with colleagues and friends over the years. Many thanks to Alano Ancona, Philippe Bougerol, Yves Guivarc'h, V. Kaimanovich, François Ledrappier, Terry Lyons, and Wolfgang Woess. Thanks to T. Coulhon, A. Grigor'yan, W. Hebisch and C. Pittet for sharing their ideas with me. One of the well-known open questions recorded in this survey concerns the stability of the Liouville property, say, on Cayley graphs. I remember very well being introduced to this question and many related ideas present in this survey by N. Varopoulos, just after I finished my Ph.D, twenty years ago.

In August 2003, I sent a first draft of this survey to a few friends, including Martine Babillot, asking for help and comments. I learned that Martine would not respond. She had left us. I miss her.

1.2. Manifolds. Let (M, g) be a non-compact complete Riemannian manifold. Let $\mathcal{C}_0^\infty(M)$ be the space of smooth compactly supported functions on M. Denote by Δ the Laplace-Beltrami operator and by μ the Riemannian measure. Thus, in local coordinates,

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \left(\sqrt{|g|}\, g^{i,j} f_j \right)_i, \quad d\mu(x) = \sqrt{|g|(x)} dx,$$

where $|g|$ is the determinant of the metric tensor $(g_{i,j})$ and $(g^{i,j}) = (g_{i,j})^{-1}$. Let $B(x, r)$ denote the geodesic ball of radius r around x and set $V(x, r) = \mu(B(x, r))$.

A smooth function u is harmonic in an open set Ω if $\Delta u = 0$ in Ω. The heat kernel of M is the minimal positive solution $(t, x, y) \mapsto p(t, x, y)$ of

$$\begin{cases} \partial_t u = \Delta u \text{ on } (0, \infty) \times M \\ u(0, x) = \delta_y(x). \end{cases}$$

It is a smooth function of the three variables (t, x, y) on $(0, \infty) \times M \times M$. For instance, if $M = \mathbf{R}^n$ equipped with its Euclidean structure,

$$p(t, x, y) = \left(\frac{1}{4\pi t} \right)^{n/2} \exp\left(-\frac{\|x - y\|^2}{4t} \right).$$

Although we will not use it explicitly in the sequel, recall that there exists a Markov process $(X_t)_{t \geq 0}$ having continuous paths and such that, for any bounded continuous function f,

$$e^{t\Delta} f(x) = \int_M p(t, x, y) f(y) d\mu(y) = \mathbb{E}_x(f(X_t)).$$

In particular, $\int_M p(t, x, y) d\mu(y) \leq 1$ (in fact, for the class of manifolds of interest to us here, $\int_M p(t, xy) d\mu(y) = 1$). The process $(X_t)_{t \geq 0}$ is called the Brownian motion on M. For background information, see the excellent survey [**56**].

1.3. Graphs. For our purpose, a graph is a pair (X, E) where $E \subset X \times X$ is symmetric (i.e., $(x, y) \in E$ if and only if $(y, x) \in E$). The set X is the vertex set and the elements of E are called edges. Two vertices $x, y \in X$ are neighbors $(x \sim y)$ if and only if $(x, y) \in E$. Our graphs are oriented but symmetric so that orientation is merely a notational convention. There is a loop at x if $(x, x) \in E$. We do not consider multiple edges. For $x \in X$, set

$$N(x) = \#\{y \in X : (x, y) \in E\}.$$

The integer $N(x)$ is the degree of the vertex x (we will only consider locally finite graphs, i.e., $N < \infty$. In fact, most of our graphs will have uniformly bounded

degree). For a subset $A \subset X$, set $N(A) = \sum_{x \in A} N(x)$. Thus the function N defines the "volume" on the graph (X, E). For any edge $e = (x, y)$ and any function f on X, set $df(e) = f(y) - f(x)$. The graph distance $d(x, y)$ between two points x and y in X is the minimal number of edges one has to cross to move from x to y. A graph is connected if $d(x, y) < \infty$ for all x, y. Set $B(x, r) = \{y : d(x, y) \le r\}$ and $V(x, r) = N(B(x, r))$. Denote by $\mathcal{C}_0(X)$ the set of all finitely supported functions on X.

The simple random walk on (X, E) is a stochastic process $(X_n)_0^\infty$ with values in X evolving as follows. If $X_n = x$ then X_{n+1} is one of the $N(x)$ neighbors of x chosen uniformly at random. This Markov process is associated with the kernel

$$K(x, y) = \begin{cases} 1/N(x) & \text{if } y \sim x \\ 0 & \text{otherwise.} \end{cases}$$

We can also view K as a self-adjoint bounded operator acting on $L^2(X, N)$ by

$$Kf(x) = \frac{1}{N(x)} \sum_{y \sim x} f(y).$$

The iterated kernel $K_n(x, y)$ is defined inductively by

$$K_1 = K, \quad K_n(x, y) = \sum_z K_{n-1}(x, z) K(z, y).$$

In this setting, a harmonic function in a set Ω is a function u defined on $\Omega_1 = \{x : d(x, \Omega) \le 1\}$ and such that $Ku = u$ in Ω.

1.4. Cayley graphs. Let Γ be a finitely generated group equipped with a finite symmetric generating set S. The Cayley graph (G, S) is the oriented symmetric graph with vertex set G and an edge from x to y if and only if $y = xs$ for some $s \in S$. Thus, the edge set E is

$$E = \{(x, xs) : x \in \Gamma, \ s \in S\}.$$

The distance $d(x, y)$ from x to y is the smallest k such that $y = xs_1 \ldots s_k$ with $s_i \in S$, $i = 1, \ldots, k$. Set $B(x, r) = \{y \in \Gamma : d(x, y) \le r\}$ and $V(x, r) = \#B(x, r)$. Obviously, these objects depend on the choice of S. Note that, to be consistent with the notation introduced above for general graphs, we should have set $V(x, r) = |S| \# B(x, r)$ instead.

Given a probability measure q on Γ, the left-invariant random walk on Γ driven by q is the discrete Markov process (X_n) which evolves as follows. If the position at time n is $X_n = x$ then pick s in Γ with probability $q(s)$ and move to $X_{n+1} = xs$. Thus, if the process starts from x at time 0, the probability that $X_n = y$ is $q^{(n)}(x^{-1}y)$ where $q^{(n)}$ is the n-fold convolution of q with itself (recall that $u * v(x) = \sum_y u(y)v(y^{-1}x)$). A function u is q-harmonic on Γ if $u * q = u$. When $q = q_S = (\#S)^{-1} \mathbf{1}_S$, the corresponding random walk is called the simple random walk on (G, S). For instance, if $\Gamma = \mathbb{Z}$ and $S = \{+1, -1\}$ then

$$q_S^n(x) = 2^{-n} \binom{n}{(n-x)/2}$$

if $n - x$ is even and $q_S^{(n)}(x) = 0$ otherwise. In particular,

$$q_S^{(2n)}(0) = 2^{-2n} \binom{2n}{n} \sim (\pi n)^{-1/2}.$$

On the d-dimensional square lattice $\mathbb{Z}^d$ with $S = \{\pm e_i, 1 \leq i \leq d\}$, we have

$$q_S^{(2n)}(0) \sim c_d n^{-d/2}, \quad c_d = 2(d/4\pi)^{d/2}.$$

For background on random walks, see Spitzer's book [**114**] and [**78**, **131**].

1.5. Algebraic properties. Throughout this survey, we will encounter several classes of groups defined by certain algebraic properties. For the convenience of the reader, we gather here the main definitions. See, e.g., [**107**] for details.

Solvable groups. A group G is solvable if it admits a descending normal series $G = H_1 \supset H_2 \supset \cdots \supset H_{k+1} = \{e\}$ such that H_i/H_{i+1} is abelian. It is polycyclic if there is such a series with H_i/H_{i+1} cyclic. Polycyclic groups are always finitely generated hence countable. They are obviously solvable. One of the essential differences between polycyclic groups and general solvable groups is that subgroups of a polycyclic group are always finitely generated whereas a non-polycyclic finitely generated solvable groups always has (abelian) subgroups that are not finitely generated. In fact, polycyclic groups are exactly those solvable groups all of whose subgroups (equivalently, abelian subgroups) are finitely generated.

The lower central series of a group G is the non-increasing sequence of subgroups defined by $G = G_1$, $G_{i+1} = [G_i, G]$, $i = 2, \ldots$. A group is nilpotent if there is a k such that $G_{k+1} = \{e\}$. Finitely generated nilpotent groups are always polycyclic.

Examples of solvable groups. A typical finitely generated nilpotent group is the group $\mathrm{Up}_n(\mathbb{Z})$ of all $n \times n$ upper-triangular matrices with integer coefficients and diagonal entries all equal to 1 ("Up" stands for upper and for unipotent!).

The semidirect products $G = \mathbb{Z} \ltimes_A \mathbb{Z}^2$, $A \in SL_2(\mathbb{Z})$, where the product is given by

$$(k, x)(k', x') = (k + k', x + A^k x')$$

for $k, k' \in \mathbb{Z}$ and $x, x' \in \mathbb{Z}^2$ are polycyclic groups. For instance, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ gives a polycyclic group having exponential volume growth. This group is isomorphic to a lattice in the three dimensional Lie group which defines the so-called "Sol" geometry in the theory of 3-manifolds [**116**].

Examples of solvable groups that are not polycyclic can be obtained as follows. Consider the affine group of the real line $\mathrm{Aff}(\mathbb{R})$, i.e., the group of all affine transformations $x \mapsto ax + b$, $a > 0$, $b \in \mathbb{R}$, under composition. Fix a real $\lambda > 1$ and consider the subgroup $\mathbb{A}_\lambda$ of $\mathrm{Aff}(\mathbb{R})$ generated by the transformations $x \mapsto x + 1$ and $x \mapsto \lambda x$. This group is solvable but, for most values of λ, it is not polycyclic. In fact, it is polycyclic if and only if the group T_λ of all translations contained in $\mathbb{A}_\lambda$ is finitely generated. For instance, if $\lambda = k > 1$ is an integer then T_λ is the ring $\mathbb{Z}[1/k] \subset \mathbb{Q}$ which, as a group, is not finitely generated.

Wreath products. The following construction is known to play an important role in the theory of solvable groups. Let A and B be two finitely generated groups. Consider the algebraic direct sum $A_B = \sum_{b \in B} A_b$ of a countable number of copies of A indexed by B. Thus A_B is the set of all sequences $(a_b)_B$ where all but a finite number of a_b are trivial (i.e., equal to the neutral element in A). The group law in A_B is product coordinate by coordinate. Now, define the wreath product $A \wr B$ as the semidirect product $A_B \rtimes_\tau B$ where the action τ is given by $\tau_c((a_b)) = (a_{c^{-1}b})$ for all $(a_b) \in A_B$ and $c \in B$. Thus the product in $A \wr B$ is given by

$$(u, c).(v, d) = (u\tau_c(v), cd), \quad u, v \in A_B, \ c, d \in B.$$

If A and B are finitely generated, so is $A \wr B$ although it contains the subgroup A_B which is not finitely generated unless B is finite. The reader should work out this definition for $A = \{0,1\} = \mathbb{Z}_2$ and $B = \mathbb{Z}$. The group $\mathbb{Z}_2 \wr \mathbb{Z}$ (also known as the lamplighter group) is an example of a solvable group which is not polycyclic. When iterating this construction, parentheses should be used with care. Moreover, iterated wreath products obtained through the above definition are distinct from the groups obtained by the more general iterating procedure described in [**107**].

Amenability. Recall that a "mean" on a discrete group G is a continuous linear functional ν defined on the space of all bounded functions such that $f \geq 0 \Rightarrow \nu(f) \geq 0$ and $\nu(1) = 1$. A mean is invariant if for any $g \in G$, $\nu(f_g) = \nu(f)$ where $f_g(x) = f(gx)$. A group is amenable if it admits an invariant mean. All abelian groups are amenable and so are all solvable groups because amenability is preserved by quotient, extension, passage to a subgroup and increasing limit. Note however that the existence of a mean on $\mathbb{Z}$ requires the use of the axiom of choice. See e.g., [**95**, Chapter 0].

The free group $\mathbb{F}_r$ on $r \geq 2$ generators is an example of a group that is not amenable and so is $SL_2(\mathbb{Z})$. See [**95**, Chapter 0]. The surface group S_g, i.e., the fundamental group of a compact surface of genus $g > 1$ is non-amenable. May be the most surprising non-amenable groups are the Burnside groups $B(r,n)$ (exponent n and r generators) for large enough odd exponent n and $r \geq 2$ [**1**] (Adian's proof uses the co-growth criteria of Grigorchuk [**51**, **53**]).

It should also be noted that it is not always easy to decide whether a group is amenable or not. One currently popular example is Thompson's group F defined by the presentation (see, e.g., [**19**])

$$\langle x_0, x_1, \ldots | x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \rangle.$$

The group F is generated by x_0, x_1. In fact, it is finitely presented and has several interesting realizations. It has exponential growth. Whether this group is amenable or not is an open problem.

1.6. Regular coverings. Let M be a complete Riemannian manifold. Assume that there exists a discrete subgroup Γ of the group of isometries of M such that $N = M/\Gamma$ is a compact Riemannian manifold (we write M/Γ even so we always think of Γ as acting on the left). Such a manifold M is called a regular covering (or cover) of N with deck transformation group Γ. If we consider the fundamental groups $\pi_1(M), \pi_1(N)$ then $\pi_1(M)$ is a normal subgroup of $\pi_1(N)$ and $\Gamma \cong \pi_1(N)/\pi_1(M)$. We can realize Γ as net in M by picking an arbitrary origin $o \in M$ and considering $\Gamma o = \{\gamma o : \gamma \in \Gamma\}$. One calls this "a net" because there are positive constants c, C such that any point $x \in M$ is at distance at most C of Γo and any two points in Γo are at least distance c apart.

By construction, the local geometry of a Riemannian co-compact regular cover is uniformly under control. More precisely, fix $r_0 > 1$. Then there are positive finite constants c, C such that $c \leq \mu(B) \leq C$ for all balls of radius between r_0 and $1/r_0$. Moreover, for all $x \in M$ and $0 < r \leq r_0$, $V(x,r) \leq CV(x,2r)$. Any co-compact regular cover satisfies uniform local Harnack inequalities, both elliptic and parabolic. In particular, for all $t \geq 1/r_0$ and for all x, y, z with $d(y,z) \leq r_0$,

$$cp(t,x,x) \leq p(t,y,z) \leq Cp(t,x,x).$$

Of course, any co-compact regular cover has curvature bounded below and positive injectivity radius.

1.7. Quasi-isometries. The following definition is useful to capture in a very general form the idea that a co-compact regular cover and its deck transformation group are similar. See, e.g., [**22, 58, 59, 74**]. It owes its fame to the work and ideas of Gromov.

Definition 1.1. Let (X, d), (X', d') be two metric spaces. A map $\psi : X \to X'$ is a quasi-isometry from X to X' if there are constants $C_1, \ldots, C_5$ such that:

(a) For all $x' \in X'$, there exists $x \in X$ such that $d'(x', \psi(x)) \leq C_1$;
(b) For all $x, y \in X$, $C_2 d(x, y) - C_3 \leq d'(\psi(x), \psi(y)) \leq C_4 d(x, y) + C_5$.

Property (a) says that no points in X' are very far from $\psi(X)$. Property (b) says that distances are roughly preserved at large scale. If ψ is as in Definition 1.1, there is a quasi-isometry ψ' from X' to X such that $\sup_{x \in X} d(\psi' \circ \psi(x), x) < \infty$.

It is not hard to check that a regular Riemannian covering M of a compact manifold with deck transformation group Γ is quasi-isometric to any fixed Cayley graph (Γ, S). Also, any two Cayley graphs of a given finitely generated group are quasi-isometric (with the identity map as quasi-isometry). More generally, if (Γ_i, S_i), $i \in \{1, 2\}$, are two Cayley graphs and Γ_2 is either a subgroup of Γ_1 with finite index or a quotient of Γ_1 by a finite subgroup then these two Cayley graphs are quasi-isometric.

Definition 1.2. Let u, v be two positive functions defined on a neighborhood of ∞ in $\mathbb{R}_+$ or $\mathbb{N}$. We say that u dominates v ($u \gg v$) if there are positive finite constants a, b, c, $a \leq b$, such that, for all t large enough,

$$v(t) \leq c \sup_{at \leq s \leq bt} u(t), \qquad \inf_{at \leq s \leq bt} v(t) \leq cu(t).$$

We say that u and v are equivalent ($u \approx v$) if u dominates v and vice versa.

This definition is useful to construct quasi-isometric invariants. Note that it simplifies considerably if one of the two functions u, v is monotone. Indeed, if either u or v is monotone then $u \gg v$ is equivalent to say that there exist two positive finite constants c, b such that, for all t large enough, $v(t) \leq cu(bt)$.

Example 1.3. For any metric space (X, d) equipped with a measure μ, define the volume growth of (X, d, μ) as the $\approx$-equivalence class of $v(r) = \mu(B(o, r))$ where o is an arbitrary fixed point in X. In general, the volume growth is not a quasi-isometric invariant. However, it is preserved under quasi-isometry if we restrict our attention to spaces such that

$$\sup_{x \in X} \frac{\mu(B(x, 2r))}{\mu(B(x, r))} = C(r)$$

is finite for each $r > 0$ and to quasi-isometries $\psi : (X, d, \mu) \to (X', d', \mu')$ such that

$$\forall\, x \in X, \;\; c \leq \frac{\mu(B(x, 1))}{\mu'(B(\psi(x), 1))} \leq C$$

for some finite positive constants c, C.

2. Some quasi-isometric invariants

2.1. Bounded geometry. It will be convenient to consider the collection **BG** (bounded geometry) of metric measure spaces which we now describe. By definition, any space in **BG** either is a complete Riemannian manifold equipped

with its Riemannian distance and measure or is a connected graph equipped with the graph distance and the measure N ($N(x)$ is the degree of x). The manifolds in **BG** are exactly those with Ricci curvature bounded below and with a uniform lower bound on the volume of balls of radius 1. The graphs in **BG** are those with uniformly bounded degree.

If one thinks in terms of metric measure spaces, the above definition is obviously very narrow minded but it will serve our purpose for the present survey.

It is not hard to see that any manifold in **BG** is quasi-isometric to some graph(s) in **BG** (and vice-versa). From a local viewpoint, manifolds in **BG** are similar to regular coverings in that their local geometry is uniformly under control. The volume function is uniformly doubling as long as the radius stays bounded above. Balls of radius 1 all have comparable volume. Uniform local Harnack inequalities, elliptic and parabolic are satisfied thanks to the groundbreaking work of S-T. Yau and his collaborators on analysis on manifolds with Ricci curvature bounded below, see, e.g., [**24, 80, 134**] and [**22, 111**].

2.2. Volume growth. The following simple result goes back to [**44, 113**] and follows from the discussion in Example 1.3. Indeed, for manifolds with Ricci curvature bounded below, the Bishop–Gromov volume comparison theorem gives (see, e.g., [**22**, Theorem 3.10])

$$\sup_{x \in M} V(x, 1) < \infty \ \ \text{and} \ \ \sup_{\substack{x \in M \\ 0 < r \leq R}} \frac{V(x, 2r)}{V(x, r)} < \infty.$$

PROPOSITION 2.1. *In **BG**, the volume growth is a quasi-isometric invariant. In particular, a regular Riemannian covering of a compact manifold has the same volume growth as its deck transformation group.*

Note in particular that the volume growth of a finitely generated group is independent of the choice of a symmetric finite generating set.

Example 2.2. The fundamental group of a compact manifold N with negative sectional curvature has exponential volume growth because its volume growth is comparable to that of the universal cover of N which is exponential (see, e.g., [**22**, Theorem 3.7]).

Example 2.3. The fundamental group of a compact n-manifold N with non-negative Ricci curvature has volume growth $\ll r^n$. Indeed, if $V(x, r)$ is the volume function on the universal cover of N and $V_n(r) = c_n r^n$ is the n-dimesnional Euclidean volume, Bishop's volume comparison theorem (e.g., [**22**, Theorem 3.9]) gives $V(x, r) \leq V_n(r)$.

Proposition 2.1 shows that the study of the volume growth of finitely generated groups is of great importance for the development of the large scale analysis on Riemannian co-compact covers. A good source of information on this subject is [**63**]. We recall the following results.

(0) Finitely generated groups have at most exponential volume growth.
(1) Non-amenable groups have exponential growth ([**48**]).
(2) For finitely generated discrete subgroups of almost connected Lie groups, the volume is either exponential or polynomial r^d for some $d = 0, 1, 2, \ldots$ ([**13, 61, 92, 117, 133**]).

(2) For finitely generated solvable groups, the volume is either exponential or polynomial r^d for some $d = 0, 1, 2, \ldots$ ([**13, 61, 90, 133**]). This led Milnor to ask whether the growth function of any group is either exponential or polynomial. Grigorchuk surprised many by showing that the answer is no. See (4) below.

(3) A nilpotent group H has polynomial volume growth r^d with

$$(2.1) \qquad\qquad d = \sum_1^k i \ \mathrm{rank}(H_i/H_{i+1})$$

where $H_1 = H$, $H_i = [H_{i-1}, H]$, k is the smallest integer such that $H_{k+1} = \{e\}$, and $\mathrm{rank}(A)$ denotes the torsion free rank of the abelian group A. See [**13, 61**] and the discussion in [**63**].

(4) There are many finitely generated groups of intermediate growth, that is, whose volume growth is slower than exponential but is faster than polynomial. Such groups where first discovered by Grigorchuk. See [**52**] and the references therein.

(5) A group such that $\liminf_{r\to\infty} r^{-D} V(r) < \infty$ for some finite D contains a nilpotent subgroup of finite index ([**58, 118**]). This is Gromov's celebrated result on groups with polynomial growth.

Specialists of volume growth conjecture that there are no groups with volume growth faster than any polynomial but slower than $\exp(r^{1/2})$ (see, e.g., [**63**, p. 203] and [**52**]). Not only did Grigorchuk give examples of groups with intermediate volume growth, he also proved that the set of all possible growths has the cardinality of the continuum and that there are groups G_1, G_2 with volume growths v_1, v_2 such that neither $v_1 \ll v_2$ nor $v_2 \ll v_1$ hold true. See [**52**]. At this writing, there are no examples of groups of intermediate growth for which the ($\approx$-equivalence class of) the volume growth is precisely known.

2.3. Parabolicity. On any complete Riemannian manifold M, set

$$G(x,y) = \int_0^\infty p(t,x,y)dt.$$

If, for $x \neq y$, $G(x,y) < \infty$ then the function G is called the Green function on M. It is the minimal positive solution of $\Delta_y G(x,y) = -\delta_x(y)$. When $G(x,y) = \infty$ for some (equivalently any) $x \neq y$, one says that M is parabolic. A manifold M is parabolic if and only if the Brownian motion $(X_t)_{t\geq0}$ on M is recurrent, i.e., almost surely, $(X_t)_{t\geq0}$ visits any fixed non-empty relatively compact open set infinitely often (see, e.g., [**56**]). It is proved in [**24**] that Riemannian manifolds with $\liminf_{r\to\infty} r^{-2} V(x_0, r) < \infty$ are parabolic (see the discussion in [**33**] for further references).

On a connected graph (X, E), the simple random walk is recurrent (i.e., almost surely (X_n) visits any fixed point infinitely often) if and only if $\sum_n K^n(x, x) = \infty$. When $\sum_n K^n(x, x) = \infty$, we say that (X, E) is recurrent or, equivalently, parabolic. For a finitely generated group, we say that Γ is recurrent (equivalently parabolic) if and only if the Cayley graph (Γ, S) is recurrent for at least one symmetric finite generating set S. A walk that is not recurrent is called transient. For the following theorem, see [**3, 73, 119, 120, 125**].

THEOREM 2.4. *In* **BG**, *parabolicity is a quasi-isometric invariant. In particular, a co-compact regular cover is parabolic if and only if its deck transformation group is parabolic.*

SKETCH OF THE PROOF. It turns out that parabolicity can be described in many different ways. One useful way is that a graph is non-parabolic if and only if, for some/any point $o \in X$ there exists a constant $C(o)$ such that

$$\forall f \in \mathcal{C}_0(\Gamma), \ |f(o)|^2 \leq C(o) \sum_E |df|^2.$$

Similarly, a manifold is non-parabolic if and only if

$$\forall f \in \mathcal{C}_0^\infty(M), \ \int_U |f|^2 d\mu \leq C(U) \int_M |\nabla f|^2 d\mu$$

for one/any non-empty open precompact set U. These criteria go back to Deny [**39**]. See also [**83, 121**].

Consider for instance two distinct Cayley graphs (Γ, S_1), (Γ, S_2) of a given finitely generated group Γ. Then it is not hard to see that there are finite positive constants c, C such that

$$c \sum_{E_1} |df|^2 \leq \sum_{E_2} |df|^2 \leq C \sum_{E_1} |df|^2$$

where E_1, E_2 are the corresponding edge sets. It follows from the criterion above that if the simple random walk on a particular Cayley graph of a finitely generated group Γ is recurrent then the simple random walk on any of its Cayley graphs is recurrent. Thus recurrence (i.e., parabolicity) is a property of the group Γ, i.e., it does not depend on the choice of the finite symmetric generating set S. Theorem 2.4 extends this to show that parabolicity is a quasi-isometric invariant in **BG**. To see how to pass from a manifold to a graph and vice versa, see the proof of Theorem 2.8 below. □

Pòlya was the first to realize that the simple random walk on the square lattice $\mathbb{Z}^d$ is recurrent in dimension $d = 1, 2$ and transient otherwise. The question of finding exactly which finitely generated groups are recurrent emerged from the work of Kesten in the sixties and was solved around 1985 by Varopoulos. Varopoulos' proof uses Gromov's theorem on groups of polynomial volume growth. See Section 4.2 below.

THEOREM 2.5. *A finitely generated group is recurrent (i.e., parabolic) if and only if it is a finite extention of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$, that is, if and only if it contains one of group $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$ as a subgroup with finite index.*

COROLLARY 2.6. *Let M be a regular Riemannian covering of a compact manifold and assume that M is parabolic. Then M covers a (possibly different) compact manifold with deck transformation group equal to either $\{e\}, \mathbb{Z},$ or $\mathbb{Z}^2$.*

PROOF. Let M cover the compact manifold N with deck transformation group Γ. If M is parabolic, so is Γ by Theorem 2.4. By Theorem 2.5, Γ contains a subgroup H equals to $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$ and with finite index in Γ. Taking the quotient of M by H gives a compact Riemannian manifold covered by M with deck transformation group H. □

Let us quote another corollary (due to G. Mess). For the definition of quasi-conformal maps, we refer the reader to [**4, 65**] and the references therein.

COROLLARY 2.7. *Let M be a* 2-*dimensional Riemannian manifold in* **BG**. *Assume that M is quasi-isometric to a finitely generated group and quasi-conformally equivalent to the plane* $\mathbb{R}^2$. *Then M is quasi-isometric to* $\mathbb{R}^2$.

SKETCH OF THE PROOF. In dimension 2, parabolicity is a quasi-conformal invariant. Thus M is parabolic. Let Γ be the finitely generated group which is quasi-isometric to M. By Theorem 2.4, Γ is parabolic. By Theorem 2.5, Γ is a finite extention of $\{e\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. It is easy to rule out $\{e\}$ and $\mathbb{Z}$ because M is quasi-conformal to the plane. □

The question naturally arises of generalizing this result. For instance, the author asked several years ago the following question.

QUESTION 1. *Let M be a* 3 *dimensional manifold in* **BG** *quasi-isometric to a finitely generated group and quasi-conformally equivalent to* $\mathbb{R}^3$. *Is M quasi-isometric to* $\mathbb{R}^3$?

Recently, Maillot made interesting advances in this direction [**87**]. Let M_0 be a regular co-compact Riemannian covering. Say that M_0 is "large-scale conformally rigid" if, for any finitely generated group Γ which is quasi-isometric to a manifold conformally equivalent to M_0 and of bounded geometry, Γ is quasi-isometric to M_0 (Maillot's definition of bounded geometry is more restrictive than the one used in the definition of **BG**). It is proved in [**87**] that $\mathbb{R}^3$ (in fact, any 3-manifold M_0 which is conformally flat and homeomorphe to $\mathbb{R}^3$) is large scale conformally rigid. This gives a positive answer to Question 1 (up to the different definitions of bounded geometry, a technicality).

Say a manifold is not p-parabolic if for one/any relatively compact non-empty open set U, there is a constant $C(U)$ such that

$$\forall\, f \in \mathcal{C}_0^\infty(M), \quad \int_U |f|^p d\mu \le C(U) \int |\nabla f|^p d\mu.$$

This notion of p-parabolicity is equivalent to others found in the literature on quasi-regular maps and non-linear potential theory. See, e.g., [**33**]. By using p-parabolicity with $p = 3$, it is easy to see that the manifold M in Question 1 is quasi-isometric to $\mathbb{Z}^d$ with $d = 0, 1, 2$ or 3. See the sketch of the proof of Theorem 2.5 and [**128**, Chapter X]. As before, it is not very hard to eliminate $\{e\}$ and $\mathbb{Z}$. I did not know how to eliminate $\mathbb{Z}^2$ but Maillot is able to use the simple connectedness at infinity of $\mathbb{R}^3$ to conclude that $\mathbb{Z}^3$ is the only possibility.

One can ask the same question in higher dimension. If a manifold is quasi-isometric to a group and is n-parabolic then the group must be of polynomial growth of degree at most n. In dimension 4, the possible groups are (up to finite extentions) $\mathbb{Z}^d$ with $d = 0, 1, 2, 3, 4$ and the discrete Heisenberg group $\mathrm{Up}_3(\mathbb{Z})$.

2.4. Spectral gaps. On a Riemannian manifold M, for any $1 \le p < \infty$, let $\lambda_p(M)$ be the largest real λ such that

(2.2) $$\lambda \int |f|^p d\mu \le \int |\nabla f|^p d\mu$$

for all smooth functions f with compact support. For $p = 2$, $\lambda_2(M)$ is the bottom of the L^2-spectrum of $-\Delta$ and, for any $x \in M$,

$$\lambda_2(M) = -\lim_{t \to \infty} t^{-1} \log p(t, x, x).$$

See [**37**, Prop.12] and the references therein. For $p = 1$, $\lambda_1(M)$ is an isoperimetric constant. Indeed, by the familiar co-area formula, (2.2) with $p = 1$ is equivalent to

$$\lambda \mu(A) \leq \mu'(\partial A)$$

for all compact sets A with smooth boundary ∂A. Here μ' denotes the induced Riemannian measure on the hypersurface ∂A.

On a graph (X, E) with vertex set X and (symmetric oriented) edge set E, let $\lambda_p(X)$ be the largest real λ such that

$$(2.3) \qquad\qquad 2\lambda \sum_X |f|^p N \leq \sum_E |df|^p$$

When $p = 2$, $\lambda_2(X)$ is the bottom of the L^2-spectrum of the operator $(I - K)$. Moreover, for any $x \in X$,

$$1 - \lambda_2(X) = \lim_{n \to \infty} K^{2n}(x, x)^{1/2n}.$$

If (X, E) is a Cayley graph (Γ, S) then $K(x, y) = q_S(x^{-1}y)$, the operator K is convolution by q_S and $K^{2n}(x, x) = q^{(2n)}(e)$.

On graphs, the discrete version of the co-area formula asserts that

$$\sum_E |df| = 2 \int_0^\infty \#\partial\{f > t\}dt$$

for any function $f \geq 0$ with finite support. Thus, when $p = 1$, (2.3) is equivalent to

$$\lambda N(A) \leq \#\partial A$$

for all finite sets $A \subset X$. Here $N(A) = \sum_A N$ and ∂A is the boundary of A defined as the set of those oriented edges $e = (x, y)$ in E such that $x \in A$ and $y \in X \setminus A$.

THEOREM 2.8. *In* **BG**, *for each $p \in [1, \infty)$, the property "$\lambda_p > 0$" is a quasi-isometric invariant. In particular, for a regular covering M of a compact manifold with deck transformation group Γ, $\lambda_p(M) > 0$ if and only if $\lambda_p(\Gamma) > 0$.*

SKETCH OF THE PROOF. (See, e.g., [**3**, **36**, **74**]) We only discuss the case of a co-compact regular cover M with deck transformation group Γ. Identify Γ with the subset $\{\gamma o : \gamma \in \Gamma\}$ of M and consider a partition of unity χ_γ, $\sum_\Gamma \chi_\gamma \equiv 1$, such that each χ_γ is a smooth compactly supported bump function centered around $\gamma o \in M$ and whose "profile" is essentially independent of γ. Then, we can consider the maps:

$$\mathbf{rst} \quad : \quad \mathcal{C}_0^\infty(M) \to \mathcal{C}_0(\Gamma), \quad f \mapsto \mathbf{rst}(f) = \sum_\gamma \left(\int f\chi_\gamma d\mu \right) \mathbf{1}_\gamma$$

$$\mathbf{ext} \quad : \quad \mathcal{C}_0(\Gamma) \to \mathcal{C}_0^\infty(M), \quad f \mapsto \mathbf{ext}(f) = \sum_\gamma f(\gamma)\chi_\gamma.$$

These two maps are sort of "inverse" of each others. The maps **rst** "restricts" smooth functions defined on M to functions on Γ whereas **ext** "extends" functions defined on Γ to smooth functions on M. Moreover, if we restrict our attention to

non-negative functions, these two maps essentially preserve the size of the function and the size of its gradient. More precisely, for non-negative functions,

$$\sum_{\Gamma} |f|^p \leq C \int_M |\mathbf{ext}(f)|^p d\mu, \quad \int_M |\mathbf{ext}(f)|^p d\mu \leq C \sum_{\Gamma} |f|^p,$$

$$\int_M |\nabla(\mathbf{ext}(f))|^p d\mu \leq C \sum_E |df|^p$$

and

$$\sum_{\Gamma} |\mathbf{rst}(f)|^p \leq C \left(\int_M f^p d\mu + \int_M |\nabla f|^p d\mu \right),$$

$$\int_M f^p d\mu \leq C \left(\sum_{\Gamma} |\mathbf{rst}(f)|^p + \int_M |\nabla f|^p d\mu \right),$$

$$\sum_E |d(\mathbf{rst}(f))|^p \leq C \int_M |\nabla f|^p d\mu.$$

where $E = \{(\gamma, \gamma s) : \gamma \in \Gamma, s \in S\}$, S being a fixed symmetric finite generating set of Γ. As λ_p can be computed by using only non-negative functions, the theorem follows. The two crucial properties of manifolds in **BG** used in this discretization procedure are that, for any $r > 0$, (a) all balls of radius in $(r, 2r)$ have comparable volumes and (b) all balls of radius in $(r, 2r)$ satisfy the Poincaré inequality

$$\int_B |f - f_B|^p d\mu \leq C(p, r) \int_{2B} |\nabla f|^p d\mu$$

where f_B is the mean value of f over B.   □

PROPOSITION 2.9. *In* **BG**, *for* $1 \leq p, q < \infty$, *the properties "$\lambda_p > 0$" and "$\lambda_q > 0$" are equivalent.*

SKETCH OF THE PROOF. (See, e.g., [**3, 36, 40, 74, 121**]) Using the change of function $f \to |f|^s$ for an appropriate $s > 1$ and Hölder inequality, it is easy to see that $\lambda_p > 0$ implies $\lambda_q > 0$ for all $q \geq p \geq 1$. For instance, on a Riemannian manifold, for non-negative functions and $s > 1$, we have $|\nabla f^s| = s f^{s-1} |\nabla f|$. Hence, if $q > p$, $s = q/p$ and $s' = s/(s-1)$ then

$$
\begin{aligned}
\int_M |\nabla f^s|^p d\mu &= s^p \int_M f^{(s-1)p} |\nabla f|^p d\mu \\
&\leq s^p \left(\int f^{(s-1)ps'} d\mu \right)^{1/s'} \left(\int_M |\nabla f|^{ps} d\mu \right)^{1/s} \\
&\leq s^p \left(\int f^q d\mu \right)^{1-1/s} \left(\int_M |\nabla f|^q d\mu \right)^{1/s}.
\end{aligned}
$$

It follows that $\lambda_q \geq (p/q)^q \lambda_p^{q/p}$. Up to technical details, the same proof works on graphs.

To prove the converse, we will work on graphs and appeal to Theorem 2.8 to conclude in the case of manifolds (the proof given for graphs does not easily adapt to manifolds). Let (X, E) be a graph with $\lambda_p(X) > 0$. According to what we just proved, it suffices to show that $\lambda_1(X) > 0$. Setting $f = \mathbf{1}_A$ in (2.3) we obtain

$$(2.4) \qquad\qquad\qquad \lambda_p(X) N(A) \leq \#\partial A$$

for any finite set $A \subset X$ is a finite set. By the discrete co-area formula, (2.4) is equivalent to

$$2\lambda_p(X) \sum |f| N \leq \sum_E |df|.$$

This shows that $\lambda_p(X) \leq \lambda_1(X)$ as desired. It follows from this discussion that, for graphs, $\lambda_p > 0 \Leftrightarrow \lambda_q > 0$ for all $1 \leq q, p < \infty$. By Theorem 2.8, this is also true in **BG**. $\square$

Now we recall the following fundamental results of Følner [**48**] and Kesten [**76, 77**]. Without using the present terminology, Følner was working essentially with the isoperimetric constant λ_1. Kesten was interested in random walks and the spectral constant λ_2. At that time (1955-59), the relation between the two was not clearly established.

THEOREM 2.10. *A finitely generated group* Γ *satisfies* $\lambda_p > 0$ *for one/any* $p \in [1, \infty)$ *if and only if it is non-amenable.*

Together, the two theorems above give the following well-known statement due to Brooks [**18**] (see also [**3, 119**]; a different proof is in [**112**]).

THEOREM 2.11. *The covering M of a compact manifold satisfies $\lambda_p(M) > 0$ for one/any $p \in [1, \infty)$ if and only if its deck transformation group is non-amenable.*

Example 2.12. Let S_g be the fundamental group of a compact surface of genus $g > 1$. Then S_g is quasi-isometric to the hyperbolic plane. Hence $\lambda_p(S_g) > 0$ for all $p \in [1, \infty)$ and thus S_g is non-amenable. More generally, the fundamental group of a compact manifold admitting a metric of negative sectional curvature is non-amenable.

2.5. Isoperimetry. Let Ψ be an arbitrary positive function. We say that a graph (X, E) satisfies the Ψ-isoperimetric inequality if

(2.5) $$N(A) \leq \Psi(N(A)) \# \partial A.$$

If we set

$$I(t) = \inf\{\# \partial A : A \subset X, \ N(A) = t\}$$

then $t/I(t)$ is the best possible Ψ for the graph (X, E). These definitions should be handled with care because the so-called isoperimetric profile I is not an increasing function in general. Even for Cayley graphs, it is not known if I is $\approx$-equivalent to the monotone isoperimetric profile $I^\uparrow(k) = \inf\{\# \partial A : A \subset X, \ N(A) \geq k\}$. Moreover, $I^\uparrow$ is inappropriate for the discussion of inequalities of the form (2.5). Indeed, the smallest non-decreasing Ψ satisfying (2.5) is

$$J(k) = \sup \left\{ \frac{N(A)}{\# \partial A} : N(A) \leq k \right\}.$$

Even for Cayley graphs, it is not known if $J(k) \approx k/I(k)$ nor if $J(k) \approx k/I^\uparrow(k)$.

The Følner function of (G, S) is defined by setting

$$F(t) = \min\{N(A) : \text{ there exists } A \subset X \text{ such that } \# \partial A < t^{-1} N(A)\}.$$

This is a non-decreasing function. It is related to J by

$$F(t) > k \ \Leftrightarrow \ J(k) \leq t.$$

The following is an elementary result.

PROPOSITION 2.13. *For graphs in* **BG**, $I, I^\uparrow, J, F$ *are all* $\approx$-*invariant under quasi-isometry.*

On a complete Riemannian manifold M, we define the isoperimetric functions $I, I^\uparrow, J$ of M and the Følner function F of M by replacing $N(A)$ by $\mu(A)$ and $\#\partial A$ by $\mu'(\partial A)$, working with precompact sets A with smooth boundary instead of finite sets. In particular, we say that M satisfies the Ψ-isoperimetric inequality if

$$(2.6) \qquad\qquad \mu(A) \leq \Psi(\mu(A))\mu'(\partial A)$$

for all relatively compact open sets A with smooth boundary.

It is not entirely obvious how to distinguish between small scale and large scale isoperimetry on a manifold (see [**60**, Lemma 6.21] and [**28, 23**]). To state a precise and fairly general result, we will use Young conjugate functions and Orlicz norms (for a detailed treatment of Orlicz spaces, see [**75**]; for their use in the present context, see, e.g., [**9, 70, 89**]). Let P, Q be a pair of conjugate Young functions (this implies that P, Q are convexe increasing functions). Define the Orlicz P-norm of a function f by

$$\|f\|_P = \sup\left\{\int fg d\mu : \int Q(|g|)d\mu \leq 1\right\}.$$

The Orlicz P-norm of the caracteristic function of a set U is

$$\|\mathbf{1}_U\|_P = \mu(U)Q^{-1}(1/\mu(U))$$

where Q^{-1} is the inverse function of Q. It follows from the co-area formula that the isoperimetric inequality

$$\mu(A) \leq \Psi(\mu(A))\mu'(\partial A)$$

for relatively compact sets with smooth boundary is equivalent to the Sobolev inequality (See [**89**])

$$\forall f \in \mathcal{C}_0^\infty(M), \;\; \|f\|_P \leq \|\nabla f\|_1$$

as long as Ψ and P are related by $\Psi(t) = 1/Q^{-1}(1/t)$. As $1/Q^{-1}(1/t)$ is increasing we can state this equivalence as

$$\forall f \in \mathcal{C}_0^\infty(M), \;\; \|f\|_P \leq \|\nabla f\|_1 \;\; \Leftrightarrow \;\; J(t) \leq 1/Q^{-1}(1/t).$$

PROPOSITION 2.14. *Let* P, Q *be a pair of Young conjugate functions and assume that* P *satisfies* $P(2t) \leq CP(t)$ *for all* $t > 0$. *Set* $\Psi(t) = 1/Q^{-1}(1/t)$. *Let* $M_1, M_2 \in$ **BG** *(thus each* M_i *is either a graph or a manifold with bounded geometry). Assume that* M_1 *and* M_2 *satisfy the local isoperimetric inequality*

$$(2.7) \qquad\qquad \mu(A) \leq C\Psi(\mu(A))[\mu(A) + \mu'(\partial A)]$$

and that M_1 *and* M_2 *are quasi-isometric. Then* M_1 *satisfies* $J(t) \ll \Psi(t)$, *that is,*

$$\mu(A) \leq C\Psi(\mu(A))\mu'(\partial A)$$

if and only if M_2 *does.*

If M_i is a graph, $\mu(A)$ should be understood as $N(A)$ and $\mu'(A)$ as $\#\partial A$ and in this case the hypothesis that M_i satisfies $N(A) \leq C\Psi(\mu(A))[N(A) + \#\partial A]$ is trivially satisfied because ψ is increasing and $N(A)$ is bounded away form 0. It is also useful to note that the fact that P is doubling implies that P^{-1} and Q^{-1} (hence Ψ) are also doubling although Q may not be doubling.

SKETCH OF THE PROOF OF PROPOSITION 2.14. The proof of Proposition 2.14 is along the same lines as the proof of Theorem 2.8. If M_1 and M_2 are both graphs, Proposition 2.13 gives a more precise result. Thus its suffices to treat the case where $M_1 = X$ is a graph and $M_2 = M$ is a manifold. For simplicity, we outline the proof in the case when M is a regular covering of a compact manifold with deck transformation group Γ. We keep the notation introduced in the sketch of the proof of Theorem 2.8. In addition to the maps **rst** and **ext** introduced in that proof, we consider the regularization map $\mathbf{reg} : \mathcal{C}_0^\infty(M) \mapsto \mathcal{C}_0^\infty(M)$ defined by

$$\mathbf{reg}(f)(x) = \sum_\gamma \left(\frac{\int f \chi_\gamma d\mu}{\int \chi_\gamma d\mu} \right) \chi_\gamma.$$

It is plain that $\|\mathbf{reg}(f)\|_P \approx \|\mathbf{rst}(f)\|_P$. It is easy to show that a ψ-isoperimetric inequality on M implies the same type of inequality on Γ (more generaly, X). To go from Γ to M, write

$$\|f\|_P \leq \|\mathbf{reg}(f)\|_P + \|f - \mathbf{reg}(f)\|_P.$$

Observe that $\|\mathbf{reg}(f)\|_P \approx \|\mathbf{rst}(f)\|_P$ and estimate this term using the P-Sobolev inequality on Γ and $\sum_E |d(\mathbf{rst}(f))| \leq C \int_M |\nabla f| d\mu$. To estimate the second term $\|f - \mathbf{reg}(f)\|_P$, use the local P-Sobolev inequality $\|f\|_P \leq C[\|f\|_1 + \|\nabla f\|_1]$ (which is equivalent to (2.7)) and the inequality $\|f - \mathbf{reg}(f)\|_1 \leq C\|\nabla f\|_1$. $\square$

Let $V(t)$ be the volume growth function of a finitely generated group Γ (for some finite symmetric set of generators) and set $W(t) = \inf\{s > 0 : V(s) > t\}$. It is proved in [**35**] that Γ satisfies the isoperimetric inequality $J(t) \ll W(t)$, that is

(2.8) $\#A \leq CW(C\#A)\#\partial A.$

For instance, if $V(t) \gg t^d$ then $J(t) \ll t^{1/d}$ whereas $V(t) \gg \exp(t^\alpha)$ implies $J(t) \leq [\log(1+t)]^{1/\alpha}$. Comments are in order concerning (2.8) since in his wonderful and very influencial book [**60**, Chap. 6, E+], Gromov gives a very misleading and erroneous account of the discovery of this inequality. Varopoulos was the first to obtain valuable general isoperimetric inequalities on groups based on volume growth. He proved (2.8) for groups of polynomial growth [**122, 123, 124**] using several structure theorems such as Malcev's embedding of any torsion free finitely generated nilpotent group as a co-compact lattice in a nilpotent Lie group and Gromov's theorem on groups of polynomial volume growth. He also proved that groups of super-polynomial growth satisfy $J(t) \ll t^{1/d}$ for all $d > 0$ and that $J(t) \ll [\log(1+t)]^{2/\alpha}$ if $V(t) \gg \exp(t^\alpha)$ (see [**123, 126, 127**]). In [**126**], Varopoulos conjectured that $V(t) \gg \exp(t^\alpha) \implies J(t) \ll [\log(1+t)]^{1/\alpha}$ (as noted above, this follows immediately from (2.8)). The method used by Varopoulos to prove the partial results mentioned above are not suited to handle general volume growth functions and, as far as I know, they cannot be used to prove (2.8) or even the much simpler implication $V(t) \gg \exp(t^\alpha) \implies J(t) \ll [\log(1+t)]^{1/\alpha}$. The proof of (2.8) in [**35**] uses a variation on an idea used by D. W. Robinson in [**106**] to prove a Nash inequality on Lie groups.

By Proposition 2.14, we obtain the following result (see [**35**, Théorème 4] and [**96, 97, 128**]).

THEOREM 2.15. *Let M be a n-dimensional Riemannian regular covering of a compact manifold. Let $V(o, r)$ be the volume growth function from a fixed origin o.*

- If $V(o,r) \gg r^d$ for some $d > 0$ then M satisfies the Ψ-isoperimetric inequality (2.6) where $\Psi(t)$ is a concave function satisfying $\Psi(t) \approx t^{1/n}$ at $t = 0$ and $\Psi(t) \approx t^{1/d}$ at $t = \infty$.
- If $V(o,r) \gg \exp(r^\alpha)$ with $\alpha \in (0,1]$ then M satisfies the Ψ-isoperimetric inequality (2.6) where $\Psi(t)$ is a concave function satisfying $\Psi(t) \approx t^{1/n}$ at $t = 0$ and $\Psi(t) \approx (\log t)^{1/\alpha}$ at $t = \infty$.
- If the deck transformation group of M contains a nilpotent group H of finite index then
$$J(t) \approx \begin{cases} t^{1/n} & \text{at } 0 \\ t^{1/d} & \text{at } \infty \end{cases}$$
where d is given by (2.1) applied to the group H.
- If the deck transformation group of M contains a polycyclic group H of finite index and exponential volume growth then
$$J(t) \approx \begin{cases} t^{1/n} & \text{at } 0 \\ \log t & \text{at } \infty. \end{cases}$$
- If the deck transformation group of M is not amenable then
$$J(t) \approx \begin{cases} t^{1/n} & \text{at } 0 \\ 1 & \text{at } \infty. \end{cases}$$

For refined results concerning the very special case of periodic metrics on $\mathbb{R}^n$, see [**94**].

The results of [**45, 99**] and Proposition 2.14 give examples of regular coverings of compact manifolds whose volume growth is exponential and whose isoperimetric profile J at infinity is intermediate between the extreme case $J(t) \approx 1$ (non-amenable) and $J(t) \approx \log t$ (polycyclic of exponential growth). The simplest example is the wreath product $\mathbb{Z} \wr \mathbb{Z}$ which has $J(t) \approx (\log t)/(\log\log t)$. See [**45**] where there are also examples for which $J(t)$ behaves at infinity as $(\log t)^{1/k}$ with k an integer, or $(\log_m t)^{1/k}$ where m, k are integers and $\log_m$ denotes the m-iterated logarithm.

QUESTION 2 ([**103**]). *Let Γ be a solvable group with exponential volume growth and finite Prüfer rank (i.e., there exists k such that any finitely generated subgroup of Γ contains a generating set with atmost k elements). Prove that $J(t) \approx \log t$.*

Pittet and the author make the following conjecture.

CONJECTURE 1. *For any finitely generated torsion free solvable group with exponential growth, $J(t) \approx \log t$ if and only if the group has finite Prüfer rank.*

The group $\mathbb{Z}_2 \wr \mathbb{Z}$ is solvable with exponential growth. It is not of finite Prüfer rank and satisfies $J(t) \approx \log t$ but it is not torsion free.

3. Invariance of the heat decay

3.1. Questions concerning the invariance of the heat decay in BG. On a manifold, define
$$\phi_M(x,t) = p(t,x,x), \quad \phi_M(t) = \sup_{x \in M} p(t,x,x).$$

On a graph set
$$\phi_X(x,n) = K_{2n}(x,x), \quad \phi_X(n) = \sup_{x \in X} K_{2n}(x,x).$$

In what follows we will often write M for either a graph or a manifold. In great generality, thanks to the local parabolic Harnack inequality, we have

$$0 < c(x,y) \leq \lim_{t\to\infty} \frac{\phi_M(x,t)}{\phi_M(y,t)} \leq C(x,y) < \infty.$$

Thus $\phi_M(x,t) \approx \phi_M(y,t)$ for any pair x,y. Note that it may easily happen that the $\phi_M(x,t)$ is not $\approx$ equivalent to $\phi_M(t)$. For instance, one may have $\phi_M(x,t) \approx t^{-d/2}$ whereas $\phi_M(t) \approx t^{-D/2}$ with $0 < d < D < \infty$ arbitrary integers. See, e.g., [**37, 57**].

In view of the results presented in the previous sections, one may ask the following question.

QUESTION 3. *In* **BG**, *is the $\approx$-equivalence class of the functions $\phi_M(x,\cdot)$, $x \in M$, a quasi-isometric invariant? Is the $\approx$-equivalence class of the function ϕ_M a quasi-isometric invariant?*

A more precise form of these questions is as follows.

QUESTION 4. *Let M_1, M_2 be two objects in* **BG** *(thus, each is either a manifold or a graph). Let ψ be a quasi-isometry from M_1 to M_2. Are there constants $0 < c \leq 1 \leq C < \infty$ such that, for all $x \in M$ and all $t \geq 1$,*

$$c\phi_{M_1}(x, Ct) \leq \phi_{M_2}(\psi(x), t) \leq C\phi_{M_1}(x, ct)?$$

3.2. Invariance for Cayley graphs and regular covers. For a Cayley graph (Γ, S), we have

$$\phi_\Gamma(x,n) = \phi_\Gamma(n) = q_S^{(2n)}(e).$$

For a cover of a compact manifolds, there is a positive finite constant c such that

$$\forall\, x \in M, \quad c\phi_M(t) \leq \phi_M(x,t) \leq \phi_M(t).$$

Thus, in these cases, we consider only the functions $\phi_\Gamma(t), \phi_M(t)$. We call "heat decay" of Γ (resp. M) the $\approx$-equivalence class of ϕ_Γ (resp. ϕ_M). The following result is taken from [**100**].

THEOREM 3.1. *For Cayley graphs and regular covers of compact manifolds the heat decay is a quasi-isometric invariant. In particular, a regular cover of a compact manifold and its deck transformation group share the same heat decay.*

In more concrete terms, if M covers a compact manifold with deck transformation group Γ, and if S is any fixed symmetric finite set generating Γ, there are positive finite constants c, C such that

$$\forall\, x \in M, \ \forall\, n \in \{1, 2, \dots\}, \quad cp(Cn, x, x) \leq q_S^{(2n)}(e) \leq Cp(cn, x, x).$$

What is perhaps remarkable about this result is that, in the present generality, we know very little about the behavior of $p(n, x, x) \approx q^{(2n)}(e)$. In the next section we review what is known about $q^{(2n)}(e) = \phi_\Gamma(n)$ under various algebraic assumptions on Γ.

4. The heat decay for random walks

This section reviews what is known about the heat decay for finitely generated groups. Of course, by Theorem 3.1, most of the results described below can be translated in the context of regular covers of compact manifolds.

Fix a finitely generated group Γ. We assume that Γ is equipped with a finite symmetric generating set S. Distance and volume growth on Γ are computed

with respect to S. Theorem 3.1 shows that the $\approx$-equivalence class of $\phi_{(\Gamma,S)}$ is independent of the choice of S. By definition, it is the heat decay of Γ and we denote it by ϕ_Γ.

4.1. Further stability results. We start by collecting some results from [**100**] that complement Theorem 3.1.

THEOREM 4.1. *Let Γ be a finitely generated group. Let Γ' be a finitely generated subgroup of a quotient of Γ. Then $\phi_\Gamma \ll \phi_{\Gamma'}$.*

Note that this applies in particular to quotients and to subgroups of Γ.

THEOREM 4.2. *Let Γ be a finitely generated group. Let q be a symmetric probability measure on Γ.*

- *Assume that $\sum_\Gamma |\gamma|^2 q(\gamma) < \infty$. Then $\phi_\Gamma \ll \phi_q$.*
- *Assume that q has finite generating support. Then $\phi_q \ll \phi_\Gamma$.*

It may be surprising at first that such a wide variety of probability measures shares the same basic behavior. However, in $\mathbb{Z}^d$, it is well-known that the second moment condition $\sum_\gamma |\gamma|^2 q(\gamma) < \infty$ delimits the natural domain of application of the central limit theorem.

4.2. From volume growth to the heat decay. Suppose that M is a graph or manifold in **BG** and that we are given a uniform lower bound

$$\forall\, x, \forall\, r > 1,\ \ V(x,r) \geq v(r)$$

where v is a continuous positive increasing convex function. What can be said about the heat decay? The answer found in [**12**] is that ϕ_M is then bounded above by the inverse function of $sv^{-1}(s)$ where v^{-1} is the inverse function of v. Hence, if $v(r) = cr^d$ with $d \geq 1$ then $\phi_M(t) \leq Ct^{d/(1+d)}$. If $v(r) = \exp(cr)$ then $\phi_M(t) \leq Ct/\log t$. Moreover, examples show that these results are essentially sharp.

For Cayley graphs, the situation is much better thanks to the following result. The precise form given here is from [**27, 104**]. The two special cases emphasized in the statement below were obtained earlier in ground breaking works by Varopoulos. See [**127, 128**].

THEOREM 4.3. *Let Γ be a finitely generated group with finite symmetric generating set S. Let $w(t) = \inf\{n : V(n) > t\}$. Define ψ implicitly for t large enough by*

$$t = \int_1^{1/\rho(t)} \frac{w(s)^2 ds}{s}.$$

Then $\phi_\Gamma \ll \rho$. In particular:

- *if $V(n) \gg n^d$ then $\phi_\Gamma(t) \ll t^{-d/2}$.*
- *if $V(n) \gg \exp(n^\alpha)$ then $\phi_\Gamma(t) \ll \exp(-t^{\alpha/(\alpha+2)})$.*

The proof of this remarkable result although not very difficult is too long to be included here. The crucial ingredient that catches the difference between, say, general graphs of bounded degree and Cayley graphs is the following "Calculus-type" inequality. For any $f \in \mathcal{C}_0(\Gamma)$, we have

$$\forall\, y \in \Gamma,\ \ \sum_{x\in\Gamma} |f(xy) - f(x)| \leq |y|_S \sum_E |df|.$$

The proof of this inequality is the only place where the group structure is used in the proof of Theorem 4.3.

Example 4.4. Let Γ be a finitely generated group. Assume that Γ contains a infinite normal subgroup Z_1, the quotient Γ/Z_1 contains an infinite normal subgroup Z_2 and so on, k times. Then Γ has volume growth $V(n) \gg n^k$ (this simple result is noted in [**58**, p.59]). Thus $\phi_\Gamma(n) \ll n^{-k/2}$.

Theorem 4.3 is one of the keys to the proof of Theorem 2.5 (i.e., to Varopoulos' solution of Kesten's conjecture concerning recurrent groups). Indeed, Theorem 4.3 shows that a recurrent group must satisfy $\liminf r^{-\alpha} V_S(r) < \infty$ for each $\alpha > 2$ (if not, we would have $V(r) \gg r^\alpha$ for some $\alpha > 2$. By Theorem 4.3, this would give $\phi_\Gamma(n) \ll n^{-\alpha/2}$ and thus $\phi_\Gamma(n)$ would be summable, contradicting the hypothesis that Γ is recurrent). By Gromov's theorem on groups of polynomial volume growth, this implies that $V_S(n) \approx n^d$ with $d = 0, 1$ or 2 and it is not hard to show that the only groups with such growth are the finite extensions of $\{e\}$, $\mathbb{Z}$ and $\mathbb{Z}^2$.

Coulhon and Grigor'yan [**31**] give a simple and general proof that if $V_S(n) \approx r^d$ then $\phi_\Gamma(n) \gg n^{-d/2}$ (properly stated, their result is not restricted to groups). Alexopoulos [**2**] shows that polycyclic groups with exponential volume growth satisfy $\phi_\Gamma(n) \gg \exp(-n^{1/3})$. Thus the results stated in Theorem 4.3 are sharp on some examples.

4.3. Classical behavior and discrete linear groups. There are several equivalent definitions of polycyclic groups (see Section 1.5). In particular, polycyclic groups are exactly those countable solvable groups that can be realized as a closed subgroup of a Lie group with finitely many connected components. See [**105**]. Moreover, it turns out that any discrete subgroup of a connected Lie group is either non-amenable or contains a polycyclic subgroup of finite index. This follows from the work of Tits [**117**] and Mostow [**92**]. Thus, for discrete subgroups of connected Lie groups, we have the following (see, e.g., [**98**]).

THEOREM 4.5. *Let Γ be a discrete subgroup of a real Lie group with finitely many connected components. Then either (a) Γ is non-amenable or (b) Γ is finitely generated and contains a polycyclic subgroup of finite index. In the second case, either Γ has exponential volume growth and satisfies $\phi_\Gamma(n) \approx \exp(-n^{1/3})$ or Γ has polynomial volume growth $V(r) \approx r^d$ and satisfies $\phi_\Gamma(n) \approx n^{-d/2}$ for some $d = 0, 1, 2, \ldots$.*

Thus for a finitely generated discrete subgroup Γ of a connected Lie group there are only three cases: $\phi_\Gamma \approx \exp(-n)$, $\phi_\Gamma \approx \exp(-n^{1/3})$, $\phi_\Gamma \approx n^{-d/2}$ for some integer d. We call these three behaviors the classical behaviors.

4.4. Exotic behaviors for solvable groups. Because of the great variety of groups of intermediate growth discovered by Grigorchuk [**52**], it seems unlikely that one can obtain a complete classification of the possible behaviors of ϕ_Γ in full generality. One might, however, dream of a classification of all possible behaviors of ϕ_Γ for finitely generated solvable groups. Indeed, this would be very easy is ϕ_Γ was determined by the volume growth for solvable groups but it is not as the following examples show.

Example 4.6 ([**99, 101, 102**]). Let λ be a real that is not algebraic. Then the subgroup of $\mathbb{A}_\lambda$ of Aff$(\mathbb{R})$ generated by $x \mapsto x + 1$ and $x \mapsto \lambda x$ is isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$ and satisfies $\phi_{\mathbb{A}_\lambda}(n) \approx \exp(-n^{1/3}(\log n)^{2/3})$.

Not only is $\mathbb{Z} \wr \mathbb{Z}$ solvable, it is in fact metabelian and even abelian by cyclic. It is proved in [**102**] that, for any abelian by cyclic group Γ with exponential growth, either $\phi_\Gamma(n) \approx \exp(-n^{1/3})$ or $\mathbb{Z} \wr \mathbb{Z} \subset \Gamma$ and $\phi_\Gamma(n) \approx \exp(-n^{1/3}(\log n)^{2/3})$. Thus, the volume growth is not sufficient to determine the behavior of ϕ_Γ in the class of solvable groups. This fact is more dramatically illustrated by the following examples, all of exponential volume growth (each of the different classes of groups considered below contains solvable groups).

Example 4.7 ([**45, 97, 101, 104**]). Let A and B be two finitely generated groups and consider the wreath product $\Gamma = A \wr B$.

- Assume that B has polynomial growth of degree d (i.e., $V(r) \approx r^d$). Then

$$\phi_\Gamma(n) \approx \begin{cases} \exp(-n^{d/(d+2)}) & \text{if } A \text{ is finite and non trivial} \\ \exp(-n^{d/(d+2)}(\log n)^{2/(d+2)}) & \text{if } A \text{ is infinite with polynomial growth} \\ \exp(-n^{(d+1)/(d+3)}) & \text{if } A \text{ is polycyclic with exponential growth.} \end{cases}$$

- Assume that B is polycyclic with exponential volume growth. Then for any non-trivial A, finite or polycyclic, we have

$$\phi_{A \wr B}(n) \approx \exp(-n(\log n)^{-2}).$$

- Assume that $\Gamma = A_k \wr (A_{k-1} \wr (\cdots (A_1 \wr B) \cdots))$ where each A_i is non-trivial polycyclic. Set $\log_k(t) = \log(1 + \log_{k-1}(t))$ with $\log_0(t) = t$, $t > 1$.
    - If B is polycyclic with exponential growth then

    $$\phi_\Gamma(n) \approx \exp(-n(\log_k n)^{-2}).$$

    - If $k \geq 2$ and B has polynomial growth of degree $d \geq 1$, let V_1 be the volume growth function of A_1. Then

$$\phi_\Gamma(n) \approx \begin{cases} \exp(-n(\log_{k-1} n)^{-2/(d+1)} & \text{if } V_1 \text{ is exponential} \\ \exp(-n[(\log_{k-1} n)/\log_k n]^{-2/d}) & \text{if } V_1 \text{ is polynomial}, V_1 \not\approx 1 \\ \exp(-n(\log_{k-1} n)^{-2/d}) & \text{if } V_1 \approx 1, \text{ i.e., } A_1 \text{ is finite} \end{cases}$$

More examples can be constructed from the results in [**45, 101**]. In particular, Erschler's results from [**45**] represent a breakthrough and provide a variety of examples. See also [**47**].

As mentioned above, for abelian by cyclic groups of exponential growth, only two behaviors of ϕ_Γ are possible. The papers [**101, 103**] give two other results in this direction. Recall that a group Γ has finite Prüfer rank if there exists an integer k such that any finitely generated subgroup of Γ can be generated by k elements or less.

THEOREM 4.8. *Let Γ be a finitely generated solvable group.*
- *If Γ is metabelian then $\phi_\Gamma(n) \gg \exp(-n^{1-\epsilon})$ for some $\epsilon \in (0, 1)$.*
- *If Γ has finite Prüfer rank then $\phi_\Gamma(n) \gg \exp(-n^{1/3})$.*

Pittet and the author make the following conjecture.

CONJECTURE 2. *Let Γ be a finitely generated solvable torsion free group. Then $\phi_\Gamma(n) \gg \exp(-n^{1/3})$ if and only if Γ has finite Prüfer rank.*

The group $\mathbb{Z}_2 \wr \mathbb{Z}$ is solvable with exponential growth. It is not of finite Prüfer rank and satisfies $\phi_{\mathbb{Z} \wr \mathbb{Z}}(n) \approx \exp(-n^{1/3})$ but it is not torsion free.

QUESTION 5. *Is it true that if Γ is metabelian of exponential growth then there exist two rational numbers α, β, such that $\phi_\Gamma(n) \approx \exp(-n^\alpha(\log n)^\beta)$?*

QUESTION 6. *Let $\mathbb{S}_d^2 = \mathbb{F}_d / \mathbb{F}_d''$ be the free metabelian group with d generators ($\mathbb{F}_d''$ is the second derived group). What is the behavior of $\phi_{\mathbb{S}_d^2}$?*

The study of the group $\mathbb{S}_d^2$ is advocated in [**130**] where the Poisson boundary is studied (see also [**46**]). In preliminary work, Pittet and the author observed that

$$\exp(-n^{d/(d+2)}) \ll \phi_{\mathbb{S}_d^2}(n) \ll \exp(-n^{(d-1)/(d+1)}(\log n)^{2/(d+1)}).$$

QUESTION 7. *Is it true that for any step-three solvable group Γ there exists $\epsilon > 0$ such that*

$$\phi_\Gamma(n) \gg \exp(-n(\log n)^{-\epsilon})?$$

4.5. Isoperimetry and the heat decay on groups. It is well understood (e.g., [**30, 55, 104, 128**] and the different approach of [**91**]) that, in a very general setting, control of the isoperimetric profile yields control on the uniform heat kernel decay. In general, something is lost in this type of argument (see [**34**]).

In the case of a Cayley graph (Γ, S), one has the following precise statement, versions of which hold in much greater generality. Define $J_\Gamma = J$ by

$$J(k) = \sup\left\{ \frac{\#A}{\#\partial A} : \#A \leq k \right\}$$

and define the function $\rho = \rho_J$ implicitly by

$$t = \int_1^{1/\rho(t)} J(s)^2 \frac{ds}{s}.$$

THEOREM 4.9. *The heat decay ϕ_Γ of the Cayley graph (Γ, S) is bounded by*

$$\phi_\Gamma \ll \rho.$$

It is also possible to estimate J in terms the heat decay but the known results in this direction are less satisfactory and the following question is open.

QUESTION 8. *Let (Γ_i, S_i) be two Cayley graphs.*
- *Suppose $J_{\Gamma_1} \approx J_{\Gamma_2}$. Does it implies $\phi_{\Gamma_1} \approx \phi_{\Gamma_2}$?*
- *Suppose $\phi_{\Gamma_1} \approx \phi_{\Gamma_2}$. Does it implies $J_{\Gamma_1} \approx J_{\Gamma_2}$?*

That is, does J and ϕ contain the same information about the underlying group?

For further discussions and results concerning this question, see [**31, 32, 104**].

4.6. The heat decay and Sobolev inequality on regular covers. This section gather statements concerning the on-diagonal behavior of the heat kernel on co-compact Riemannian covers. These statements are simple consequences of the results reviewed earlier. First, as a consequence of Theorems 4.5 and 3.1, we have the following result.

THEOREM 4.10. *Let M be a co-compact Riemannian regular cover. Assume that the deck transformation group of M can be realized as a discrete subgroup of a real Lie group with finitely many connected components. Then the heat decay $\phi_M(t)$, $t \gg 1$, is given by one of the following three exclusive possibilities:*
1. *$\phi_M(t) \approx e^{-t}$;*
2. *$\phi_M(t) \approx e^{-t^{1/3}}$;*
3. *There exists an integer d such that $\phi_M(t) \approx t^{-d/2}$.*

Of course, the first case corresponds exactly to the case where the deck transformation group Γ is non-amenable, the second case to the case where Γ is amenable with exponential growth and the third case to the case where Γ has polynomial volume growth of degree d. The examples of Section 4.4 show that many other behaviors are possible when Γ cannot be realized as a discrete subgroup of a real Lie group with finitely many connected components. In such cases, the following corollary to Theorems 3.1, 4.3 may be useful.

THEOREM 4.11. *Let M be a co-compact Riemannian regular cover.*
- *Assume that $V(x,r) \geq cr^d$, $r > 1$. Then, for all $t > 1$, $\phi_M(t) \leq Ct^{-d/2}$.*
- *Assume that $V(x,r) \geq c\exp(cr^\alpha)$, $r > 1$. Then*

$$\forall t > 1, \quad \phi_M(t) \leq C_1 \exp(-c_1 t^{\alpha/(2+\alpha)}).$$

The next statement connects these heat kernel estimates to the most classical form of the Sobolev inequality. See, e.g., [**128**].

THEOREM 4.12. *Let M be a co-compact Riemannian regular cover of dimension n. Then M satisfies the Sobolev inequality*

$$\forall f \in \mathcal{C}_0(M), \quad \left(\int_M |f|^{2\nu/(\nu-2)} d\mu \right)^{(\nu-2)/2\nu} \leq C \int_M |\nabla f|^2 d\mu$$

if and only if $\nu \geq n$ and $V(x,r) \geq cr^\nu$, $r > 1$.

Note that the last volume condition above is satisfied if and only if the deck transformation group has volume growth at least r^ν. One can prove other versions of the above inequality (see, e.g., [**110, 111, 128**]). More precisely, assume that $V(x,r) \geq cr^\nu$ for all $r > 0$ (hence, $\nu > n$) and fix $1 \leq p < \nu$. Then there exists a constant C such that

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f\|_{p\nu/(\nu-p)} \leq C\|\nabla f\|_p.$$

Assume in addition that the volume growth is subexponential, i.e., $\lim_{n\to\infty} V(n)^{1/n} < 1$. Then there exists a constant C such that for any smooth function f on M satisfying $|\nabla f| \in L^p$ (not necessarily compactly supported), there exists a constant $C(f)$ for which

$$\forall f \in \mathcal{C}_0^\infty(M), \quad \|f - C(f)\|_{p\nu/(\nu-p)} \leq C\|\nabla f\|_p.$$

In cases where Γ has polynomial volume growth or degree d with $d < n$, no Sobolev inequalities of the form stated above can possibly hold but there are several ways to cope with this situation. Assume indeed that Γ has polynomial volume growth or degree d and set $\nu = \max\{n, d\}$. Then the families of local Sobolev inequalities

$$\forall B = B(x,r), \ \ \forall f \in \mathcal{C}_0^\infty(B), \ \ \|f\|_{B,p\nu/\nu-p)} \leq C_p \, r \, \mu(B)^{-1/\nu} \|\nabla f\|_{B,p}$$

and

$$\forall B = B(x,r), \ \ \forall f \in \mathcal{C}^\infty(B), \ \ \|f - f_B\|_{B,p\nu/(\nu-p)} \leq C_p \, r \, \mu(B)^{-1/\nu} \|\nabla f\|_{B,p},$$

hold for all $1 \leq p < \nu$ (see, e.g., [**111**]). Here, $\|\cdot\|_{B,p}$ is the L^p norm restricted to the set B and f_B denotes the mean of f over B. Alternatively, one can use the Faber-Krahn type inequality (see [**55**])

$$\forall B = B(x,r), \ \ \forall \Omega \subset B, \ \ \lambda(\Omega) \geq \frac{c}{r^2} \left(\frac{\mu(\Omega)}{\mu(B)} \right)^{2/\nu}$$

where $\lambda(\Omega)$ denotes the lowest eigenvalue of minus the Laplacian with Dirichlet boundary condition in the open set Ω.

5. Harmonic functions

5.1. Liouville properties. A complete Riemannian manifold is Liouville if any bounded harmonic function is constant. It has the strong Liouville property if any positive harmonic function is constant. The same definitions apply to graphs. A measure q on a group is Liouville (resp. strong Liouvillle) if any bounded (resp. positive) solution of $u * q = u$ is constant.

The study of Liouville's properties is really only the iceberg's top of the deeper problem of studying the space of all bounded harmonic functions or the cone of all positive harmonic functions through the construction and understanding of the Poisson and Martin boundaries. The Liouville property is equivalent to the fact that the Poisson boundary is reduced to a singleton. In the class **BG**, the strong Liouville property is equivalent to the fact that the Martin boundary is reduced to a singleton. We will not discuss Poisson and Martin boundaries here. Pointers to the literature can be found in [**3, 71, 72, 85, 129, 131**].

Blackwell [**16**] appears to be the first to have observed that any bounded harmonic functions on the square lattice $\mathbb{Z}^d$ must be constant. This is a special case of the celebrated Choquet-Deny theorem [**25**] which implies that, under mild conditions on the probability measure q, bounded q-harmonic functions on a locally compact abelian group are constants. Dynkin and Maljutov [**43**] observed that finitely generated nilpotent groups have the Liouville property. Later, the following three satisfactory results concerning the Liouville property were obtained:

- If q is a probability measure with finite generating support on a group Γ and Γ has subexponential volume growth (i.e., $\lim_{n\to\infty} V(n)^{1/n} = 1$) then any bounded q-harmonic function is constant. See [**5**].
- If q is a symmetric finitely supported measure on a polycyclic group then any bounded q-harmonic function is constant. See [**2, 68**].
- If the support of q generates Γ and Γ is non-amenable then there are non-trivial bounded q-harmonic functions on Γ. See [**6**] and also [**50**].

Concerning the strong Liouville property, Margulis [**88**] proved that if q is symmetric and Γ is nilpotent then any positive q-harmonic function is constant. More recently, Bougerol and Elie [**17**] proved the following theorem and corollary (their results are slightly more general than stated here).

THEOREM 5.1. *Let Γ be a finitely generated group. Assume that there exists a continuous homomorphism h from Γ into a group G having finitely many connected components such that the closure of $h(\Gamma)$ has exponential volume growth. Assume that q is symmetric with generating support and has a third moment (i.e., $\sum |\gamma|^3 q(\gamma) < \infty$). Then there are non-constant positive q-harmonic functions on Γ.*

COROLLARY 5.2. *If q is symmetric, with finite generating support and Γ is polycyclic then the positive q-harmonic functions are constant if and only if Γ has polynomial volume growth.*

Example 5.3. Theorem 5.1 applies to the subgroups $\mathbb{A}_\lambda$ of the affine group of the line (recall that $\mathbb{A}_\lambda$ is the group generated by $x \mapsto x + 1$ and $x \mapsto \lambda x$). For any

$\lambda > 1$, and for any symmetric probability measure q with finite generating support on $\mathbb{A}_\lambda$, there are non-constant positive q-harmonic functions.

5.2. Instability of the Liouville properties. In [**84**], Lyons gives examples of quasi-isometric pairs M_1, M_2 in **BG** such that M_1 admits non-constant bounded harmonic functions whereas M_2 has the strong Liouville property. He provides examples where both M_1 and M_2 are graphs, or both manifolds, or one is a graph and the other a manifold. Thus, each of the Liouville properties is unstable under quasi-isometry in **BG**. See also [**14**]. However, these works leave open the following basic questions.

QUESTION 9. *Let (Γ_1, S_1) and (Γ_2, S_2) be two quasi-isometric Cayley graphs. Is it true that (Γ_1, S_1) has the (strong) Liouville property if and only if (Γ_2, S_2) does?*

QUESTION 10. *Given two symmetric probability measure q_1, q_2 with finite generating supports on a group Γ, does the (strong) Liouville property for q_1 implies the same property for q_2?*

QUESTION 11. *Let M be a regular cocompact cover with deck transformation group Γ. Let g_1, g_2 be two Γ invariant Riemannian metrics on M. Is it true that (M, g_1) has the (strong) Liouville property if and only if (M, g_2) does?*

5.3. Passage from the deck transformation group to the cover. Despite the fact that we do not know if the Liouville properties are stable when passing from a covering to its deck transformation group (assuming the quotient is compact), there is a way to transport some information thanks to a discretization procedure introduced by Furstenberg [**49**] and refined successively by Lyons and Sullivan [**86**], Ancona [**3**], Kaimanovich [**69**], and Ballman and Ledrappier [**10**] (the result in [**10**] is more precise than the one stated below).

PROPOSITION 5.4. *Let M be a regular Riemannian covering of a compact manifold with deck transformation group Γ. There exists a symmetric probability measure q on Γ whose support is Γ, which satisfies $\sum_{\gamma \in \Gamma} e^{c|\gamma|} q(\gamma) < \infty$ for some $c > 0$, and such that:*

- *Any bounded harmonic function on M restricted to Γ is q-harmonic.*
- *any positive q-harmonic function on Γ can be extended to a positive harmonic function on M.*

Suppose we want to apply this proposition to show that a certain co-compact Riemannian covering has (does not have) the Liouville property by using information concerning the deck transformation group Γ. Then it is not enough to have information about a specific random walk on Γ, nor does it suffice to know about all random walks driven by finitely symmetric supported measures. In the form stated above, Proposition 5.4 requires dealing with symmetric measures μ having an exponential moment. This is in contrast with what happens when a property is stable under quasi-isometries.

Example 5.5. Let M be a regular co-compact cover with non-amenable deck transformation group. Then, by the aforementioned result of Azencott (any non-amenable group admits non trivial bounded harmonic functions for (essentially) any random walk) and Theorem 5.4, M admits non-constant bounded harmonic function.

Example 5.6. Let M be a regular covering of a compact manifold with deck transformation group $\Gamma = \mathbb{A}_\lambda$, $\lambda > 1$. (this group is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ when λ is not algebraic). Then, using Proposition 5.4 and Theorem 5.1, we obtain that M admits non-constant positive harmonic functions.

Example 5.7. Let M be a regular covering of a compact manifold with deck transformation group Γ having sub-exponential growth (i.e., $\lim V(n)^{1/n} < 1$). Then any bounded harmonic function on M is constant. See [**67**] and [**3**, Théorème 3.1].

Example 5.8. Consider the wreath products $\Gamma_k = \mathbb{Z}_2 \wr \mathbb{Z}^k$. Then the Cayley graphs of Γ_k have the Liouville property if and only if $k \le 2$ [**71**, **72**]. From Proposition 5.4 and [**71**, Theorem 3.3], it follows that a co-compact regular cover with deck transformation group Γ_k has the Liouville property if and only if $k \le 2$.

Further examples are dicussed in [**46**] where, in particular, a compact Riemannian manifold with amenable fundamental group whose universal cover does not have the Liouville property is constructed.

The following result is from [**17**]. See also [**3, 8, 62, 69**].

THEOREM 5.9. *Assume that M is a regular co-compact cover whose deck transfomation group is a closed subgroup of a group having finitely many connected components. Then M has non-trivial positive harmonic functions if and only if Γ has exponential volume growth. It has non-trivial bounded harmonic functions if and only if Γ is non-amenable.*

5.4. Harmonic functions with finite energy. On a manifold, consider the energy form $D(f) = \int_M |\nabla f|^2 d\mu$ whereas, on a graph, set $D(f) = \sum_{e \in E} |df(e)|^2$. Royden work [**108**] sparkled off interest on whether or not harmonic functions with finite energy are constant. Say that the "finite energy Liouville property" holds if any harmonic function u with $D(u) < \infty$ is constant.

The following result of Holopainen and Soardi [**66**] may be a little surprising in view of the instability of the Liouville and strong Liouville properties.

THEOREM 5.10. *The finite energy Liouville property is invariant under quasi-isometries in* **BG**. *In particular, a co-compact Riemannain cover has the finite energy Liouville property if and only if its deck transformation group does.*

There are several simple but not completely obvious facts concerning harmonic functions with finite eneregy that are worth noting (for graphs, a good reference is [**115**]). Any harmonic function of finite energy is the difference of two positive harmonic functions. Thus, if the strong Liouville property holds, so does the finite energy Liouville property. In fact, the finite energy Liouville property holds already if bounded harmonic functions with finite energy are constant. In particular, it holds if the Liouville property holds (see, e.g., [**66**] which treats p-harmonic functions).

Example 5.11. Let M be a regular Riemannian co-compact cover with deck transformation group Γ. Assume that Γ as subexponential volume growth. Then M has the finite energy Liouville property.

5.5. Harnack inequalities. Assume that M is a metric space equipped with a family of harmonic functions (for our purpose, it suffices to think of the examples above where M is a Riemannian manifold or a Cayley graph). We say that M

satisfies the elliptic Harnack inequality if there is a constant C such that for any ball B and any positive function u harmonic in $2B$, we have

$$(5.1) \qquad \sup_B\{u\} \le C\inf_B\{u\}.$$

Let us emphasize that C is independent of the ball B and of the function u. In particular, this is in an obvious sense a scale invariant property.

The Harnack inequality implies the strong Liouville property (the converse is false as will become clear from some of the examples discussed below). Classical harmonic functions in Euclidean space satisfy the Harnack inequality. For manifolds with non-negative Ricci curvature, (5.1) was obtained in [**24**] under the stronger form $|\nabla u| \le \frac{C}{r}u$ in B.

QUESTION 12. *Is the elliptic Harnack inequality* (5.1) *a quasi-isometric invariant in* **BG***?*

For simplicity, we define the parabolic Harnack inequality only on manifolds (see, e.g., [**29, 38**] for the graph version). Given a real s, $r \in (0, \infty)$ and $x \in M$, set $Q = (s, s + 4r^2) \times B(x, 2r)$ and

$$Q_- = (s + r^2, s + 2r^2) \times B(x, r), \quad Q_+ = (s + 3r^2, s + 4r^2) \times B(x, r).$$

Thus, Q_- and Q_+ are two sub-cylinders contained in Q with Q_+ sitting well above Q_- and Q_- floating in the bottom part of Q.

We say that a manifold M satisfies the parabolic Harnack inequality if there exists a constant C such that for any non-negative solution u of $(\partial_t + \Delta)u = 0$ in Q we have

$$(5.2) \qquad \sup_{Q_-}\{u\} \le C\inf_{Q_+}\{u\}.$$

It is proved in [**36**] that the parabolic Harnack inequality (5.2) is stable under quasi-isometries in **BG**. This is because it can be characterized by the volume doubling property and the scale invariant Poincaré inequality. See [**54, 109, 111**].

5.6. A necessary condition for Harnack inequality in BG. The following useful result is due to Barlow [**11**].

PROPOSITION 5.12. *Let M be a graph or a manifold in* **BG***. If M satisfies the Harnack inequality* (5.1) *then the volume growth of M is bounded above by a power function.*

SKETCH OF THE PROOF. In [**11**], Barlow proves this result (in a more general form) for graphs. Here, we adapt Barlow's proof to manifolds. In M, consider a maximal set of point $X = \{x_i, i = 1, 2, \ldots\}$ such that $d(x_i, x_j) > 16$ if $i \ne j$. Obviously, the (closed) ball $B(x_i, 8)$ are pairwise disjoint whereas the (closed) balls $B(x_i, 16)$ cover M. Fix $o \in M$. Let $F_n = \{i : B(x_i, 1) \cap B(o, n) \setminus B(o, n-1) \ne \emptyset\}$. We claim that $B(o, n)$ is covered by the balls $B(x_i, 64)$, $i \in F_n$. Indeed, let z be in $B(o, n)$. Let j be such that $z \in B(x_j, 16)$. Along a shortest path from x_j to o, let y be the first point such that $d(o, y) \le n - 16$ and let j_1 be such that $y \in B(x_{j_1}, 16)$. Then $B(x_{j_1}, 1)$ intersects $B(o, n)$. As $d(o, x_j) \le n + 16$, we must have $d(x_j, y) \le 32$ hence $d(x_j, x_{j_1}) \le 48$. Thus $d(x_{j_1}, z) \le 64$. As balls of a fixed radius all have comparable volume (because $M \in$ **BG**), there exists a constant C

$$\mu(B(o, n)) \le \sum_{k=0}^n \sum_{i \in F_k} \mu(B(x_i, 64)) \le C\sum_0^n \#F_k.$$

We now claim that there exist $C, \alpha > 0$ such that

(5.3) $$\#F_k \leq Ck^\alpha.$$

If this is the case then $\mu(B(o, n)) \leq C'n^{\alpha+1}$ which is the desired result. To prove this claim, we use the Brownian motion $(X_t)_{t>0}$ on M. Fix k and set

$$A = A_k = \cup_{i \in F_k} B(x_i, 1),$$

$$\tau = \inf\{t > 0 : X_t \in A_k\}$$

and

$$h_i(x) = \mathbb{P}_x(X_\tau \in B(x_i, 1)).$$

Thus $h_i(x)$ is the probability that, starting from x, one first enters A by entering $B(x_i, 1)$. By the strong Markov property, each h_i is harmonic on $M \setminus A$.

ASSERTION 5.13. *There exists a positive constant $c_0 > 0$ such that, for every k, $i \in F_k$, and x such that $d(x, x_i) = 2$, we have $h_i(x) \geq c_0$.*

PROOF. This assertion is satisfied because M has bounded geometry (i.e., $M \in$ **BG**). To see this, for $x \in B(x_i, 8)$, observe that

$$h_i(x) \geq u_i(x) = \mathbb{P}_x(X_{\tau_i} \in B(x_i, 1) \text{ and } \tau_i < \sigma_i)$$

where τ_i is the first hitting time of $B(x_i, 1)$ and σ_i is the first exit time from $B(x_i, 8)$. Then, for $x \in B(x_i, 8)$, we have

$$u_i(x) \approx \frac{g_i(x_i, x)}{g_i(x_i, y)}$$

where g_i is the Green function with Dirichlet boundary condition in $B(x_i, 8)$ and y is an arbitrary point on $\partial B(x_i, 1)$. By well-known Dirichlet heat kernel estimates, for z such that $1 \leq d(x_i, z) \leq 7$, we have $g_i(x_i, z) \approx 1$. $\square$

ASSERTION 5.14. *There are finite positive constants c, α such that, for every k and every $i \in F_k$, $h_i(o) \geq ck^{-\alpha}$.*

PROOF. Fix $i \in F_k$. Let γ_t, $0 \leq t \leq T$ be a shortest curve from x_i to o parametrized by the distance from x_i so that $d(x_i, \gamma_t) = t$, $0 \leq t \leq T$. We need to control de distance from γ_t to A. By construction, if $t \leq 8$, $d(\gamma_t, A) = d(\gamma_t, B(x_i, 1)) = t - 1$. Moreover, the triangle inequality shows that for any $j \in F_k$ with $j \neq i$, we have

$$d(\gamma_t, B(x_j, 1)) \geq d(o, B(x_j, 1)) - d(o, \gamma_t) \geq t - 3.$$

Thus, for $t \in [2, T]$,

$$d(\gamma_t, A) \geq t/2.$$

Now, consider the points $z_j = \gamma_{t_j}$ with $t_j = 2(1 + 1/8)^{j-1}$, $j = 1, 2, \ldots, j_0$ where j_0 is such that $d(o, x_{j_0}) \leq k/4$. It follows that $j_0 \approx \log k$. For each $j = 1, \ldots, j_0 - 1$, $d(x_j, x_{j+1}) = t_j/8$ and the function h_i is harmonic in $B(x_j, t_j/2)$. Thus the Harnack inequality (5.1) gives $h_i(z_{j+1}) \geq C^{-1}h_i(z_j)$ (C being the constant from (5.1)). It follows that

$$h_i(z_{j_0}) \geq C^{-j_0}h_i(z_1).$$

Applying (5.1) one more time to pass from z_{j_0} to o and using Assertion 5.13 to estimate $h_i(z_1)$, we obtain $h_i(o) \geq c_0 C^{-j_0} \geq ck^{-\alpha}$ for some $c, \alpha > 0$, as desired. $\square$

To finish the proof of Proposition 5.12, we show that Assertion 5.14 implies (5.3). Indeed, as the balls $B(x_i, 1)$, $i \in F_k$, form a partition of A_k, $\sum_{i \in F_k} h_i(x) = 1$. Hence

$$1 \geq \sum_{i \in F_k} h_i(o) \geq c \, \#F_k \, k^{-\alpha}$$

which gives (5.3).   □

5.7. Analysis on regular covers having polynomial volume growth. Using Proposition 5.12, Gromov's theorem on groups of polynomial growth, and the results of [**54, 109**], we can describe those co-compact regular covers satisfying the elliptic (and parabolic) Harnack inequality.

THEOREM 5.15. *Let M be a Riemannian co-compact regular cover with deck transformation group Γ. The following properties are equivalent.*

- *The elliptic Harnack inequality (5.1) is satisfied.*
- *The parabolic Harnack inequality (5.2) is satisfied.*
- *The group Γ has polynomial volume growth.*
- *The heat kernel $p(t, x, y)$ satisfies the two-sided estimate*

$$\frac{c_1}{V(x, \sqrt{t})} e^{-c_2 d(x,y)^2/t} \leq p(t, x, y) \leq \frac{c_3}{V(x, \sqrt{t})} e^{-c_4 d(x,y)^2/t}$$

for all $t > 0$ and $x, y \in M$.

If M is a co-compact Riemannian regular cover and its deck transformation group Γ has polynomial growth then M satisfies the Poincaré inequality

$$(5.4) \qquad \forall \, f \in \mathcal{C}^\infty(B), \quad \int_B |f - f_B|^p d\mu \leq C_p \, r(B)^p \int_B |\nabla f|^p d\mu$$

for each $p \in [1, \infty)$. Here $r(B)$ denotes the radius of that ball, f_B is the mean of f over the ball B, and the constant C_p is independent of B (see, e.g., [**36**]). Note that, by the results of [**54, 109, 111**], the two sided Gaussian heat kernel inequality of Theorem 5.15 holds on a co-compact cover M having polynomial volume growth as soon as the Riemannian metric is uniformly comparable to a metric lifted from the compact quotient. This type of global heat kernel estimate is similar to the estimates obtained by Li and Yau [**80**] for manifolds with non-negative Ricci curvature. Note however that, in general, the manifolds in Theorem 5.15 do not have non-negative Ricci curvature. For instance, Γ may have volume growth r^d with d strictly larger than the dimension of M, preventing the existence of a Riemannian metric of non-negative Ricci curvature on M.

As a corollary of Theorem 5.15, we see that a regular co-compact Riemannian cover M with deck transformation group of polynomial volume growth of degree $d \geq 3$ admits a Green function $G(x, y)$ which satisfies

$$c d(x, y)^{2-d} \leq G(x, y) \leq C d(x, y)^{2-d}$$

for all $x, y \in M$ with $d(x, y) \geq 1$. For the case of abelian cover, [**7**] gives the following more precise result (obtained by Fourier transform techniques).

THEOREM 5.16. *Let M be a regular Riemannian co-compact cover with deck transformation group equal to $\mathbb{Z}^d$, $d \geq 3$. Then there exists an Euclidean norm $\| \cdot \|$ on $\mathbb{R}^d$ such that the green function G satisfies*

$$\lim_{\gamma \to \infty} \|\gamma\|^{d-2} G(x, \gamma(y)) = c_d > 0$$

for all $x, y \in M$ *(in the limit,* γ *is an element of* $\Gamma = \mathbb{Z}^d$ *and* $\gamma(y)$ *is the image of* y *under the action of* γ*).*

A result that is similar in spirit to the above large scale asymptotic of the Green function and concerns the heat kernel is obtained in [**82**].

In a recent work, N. Dungey [**42**] has obtained gradient estimates that complement the results of Theorem 5.15.

THEOREM 5.17. *Let* M *be a co-compact Riemannian cover with deck transformation group* Γ *of polynomial volume growth. Let* S *be a symmetric finite generating set of* Γ*. Let* $p(t, x, y)$ *be the heat kernel on* M_0*. Then, for all* $t > 0$*, and* $x, y \in M_0$,

$$(5.5) \qquad |\nabla_y p(t, x, y)| \leq \frac{C}{t^{1/2} V(x, \sqrt{t})} \exp\left(-cd(x, y)^2/t\right).$$

Dungey uses (5.5) and further estimates obtained in [**42**] to prove that the Riesz transforms are bounded on $L^p(M, \mu)$, $1 < p < \infty$, that is,

$$\forall f \in \mathcal{C}_0^\infty(M), \quad c_p \|\nabla f\|_p \leq \|(-\Delta)^{1/2} f\|_p \leq C_p \|\nabla f\|_p.$$

Again, Dungey's gradient estimate is similar to the global gradient estimates that follows from [**80**] in the case of manifolds with non-negative Ricci curvature.

Another result worth mentioning here concerns harmonic functions of at most polynomial growth. See [**26, 79**].

THEOREM 5.18. *Let* M *be a co-compact Riemannian cover with deck transformation group* Γ *of polynomial volume growth of degree* d*. Then there exists a constant* C *such that the space of all harmonic functions on* M *satisfying*

$$\sup_{r > 1} \sup_{d(o, x) \leq r} \{r^{-\alpha} |u(x)|\} < \infty$$

has finite dimension bounded by $C\alpha^{d-1}$*.*

An interesting and non-trivial generalization of Theorem 5.15 concerns quotients of regular covering by subgroups that are not necessarily normal. See [**64, 109**].

THEOREM 5.19. *Let* M *be a Riemannian co-compact regular cover with deck transformation group* Γ*. Assume that* Γ *has polynomial volume growth. Let* Γ_0 *be a subgroup of* Γ *(not necessarily a normal subgroup) and let* $M_0 = M/\Gamma_0$ *be the Riemannian manifold obtained by taking the quotient of* M *by* Γ_0*. Then* M_0 *satisfies the elliptic and parabolic Harnack inequalities* (5.1), (5.2) *and the heat kernel* $p(t, x, y)$ *on* M_0 *satisfies the two-sided estimate*

$$\frac{c_1}{V(x, \sqrt{t})} e^{-c_2 d(x, y)^2/t} \leq p(t, x, y) \leq \frac{c_3}{V(x, \sqrt{t})} e^{-c_4 d(x, y)^2/t}$$

for all $t > 0$ *and* $x, y \in M_0$*.*

Note that in the setting of this theorem the volume function $V(x, r)$ will often exhibit a non-uniform behavior as x varies. The manifolds in Theorem 5.19 satisfy the Poincaré inequality (5.4). Dungey's gradient estimate (5.5) and the boundedness of the Riesz transforms also hold in this context.

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