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Anomalous diffusion and stability of Harnack inequalities

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Contents

1. Introduction.

Let (M, g) be a complete Riemannian manifold, with metric $d(x, y)$, and let $\Delta = \text{div}\nabla$ be the Laplace-Beltrami operator on M. Let $V(x, r)$ denote the volume of the ball $B(x, r)$ with centre x and radius r. In this paper we survey the stability of elliptic and parabolic Harnack inequalities on M.

A function $u = u(x)$ is *harmonic* in a domain $D \subset M$ if it is a solution of the Laplace equation:

$$\Delta u(x) = 0, \qquad x \in D.$$

M satisfies the *elliptic Harnack inequality* (EHI) if there exists a constant C_E such that, for any ball $B(x, R)$, whenever u is a non-negative harmonic function on $B(x, R)$ then

$$\sup_{B(x,R/2)} u \le C_E \inf_{B(x,R/2)} u.$$

The parabolic Harnack inequality (PHI) is a little more complicated to state. Let $x \in M$, $R > 0$, $T = R^2$, $\widetilde{D}(x_0, R) = (0, 4T) \times B(x_0, 2R)$, and

$$Q_- = (T, 2T) \times B(x_0, R), \qquad Q_+ = (3T, 4T) \times B(x_0, R).$$

M satisfies the PHI if there exists C_P such that, for any x_0 and R, if u is a non-negative solution of

$$\frac{\partial}{\partial t} u(x, t) = \Delta u(x, t), \qquad (x, t) \in \widetilde{D}(x_0, R) \tag{1.1}$$

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then

$$\sup_{Q_-} u \le C_P \inf_{Q_+} u.$$

Since any harmonic function solves (1.1), the PHI immediately implies the EHI.

DEFINITION 1.1. (a) M satisfies volume doubling (VD) if there exists a constant C_1 such that

$$V(x,2R) \le C_1 V(x,R) \quad \text{for all } x \in M,\ R \ge 0. \tag{1.2}$$

(b) M satisfies a Poincaré inequality, denoted PI(2), if there exists a constant C_2 such that, for any $B(x,R)$ and $f \in C^\infty(B(x,R))$,

$$\int_{B(x,R)} |f - \overline{f}_B|^2 d\mu \le C_2 R^2 \int_{B(x,R)} |\nabla f|^2 d\mu. \tag{1.3}$$

(c) M satisfies HK(2) if the heat kernel $p_t(x,y)$ on M satisfies the two-sided Gaussian bound

$$\frac{c_1}{V(x,t^{1/2})} e^{-c_2 d(x,y)^2/t} \le p_t(x,y) \le \frac{c_3}{V(x,t^{1/2})} e^{-c_4 d(x,y)^2/t}. \tag{1.4}$$

In [**47**] Li-Yau proved that if M has non-negative Ricci curvature then M satisfies HK(2). This result was refined by subsequent work of Grigoryan [**22**] and Saloff-Coste [**54**], who proved that PHI is equivalent to two conditions on M: volume doubling, and a family of Poincaré inequalities. The relation with HK(2) comes from [**44**].

THEOREM 1.2. *(*[**22**],[**54**],[**20**]*). Let M be a complete Riemannian manifold. The following conditions are equivalent:*
(a) The heat kernel $p_t(x,y)$ on M satisfies the two-sided Gaussian bound (1.4).
(b) M satisfies the parabolic Harnack inequality.
(c) M satisfies VD and PI(2).

The arguments used to prove Theorem 1.2 are quite general, and can be translated into other contexts: see [**18**] for graphs, and [**57**] for general metric spaces. Since the condition (c) is stable under rough isometries (see Section 5), it follows that HK(2) and the PHI are also stable under rough isometries (with suitable 'side conditions'). Theorem 1.2 gave a fairly complete characterization of the PHI, but left open two significant questions concerning the EHI:

1. Is the EHI equivalent to the PHI?
2. Is the EHI stable under rough isometries?

Of these, (2) is still open, while the (negative) answer to (1) had been for some time implicit in work on diffusions on fractals, before being made explicit in [**6**], following a conversation between the author and A. Grigoryan at MSRI in 1997.

In the early 1980s mathematical physicists became interested in the properties of random structures, such as percolation clusters, at criticality. Let us recall the definition of percolation on the Euclidean lattice $\mathbb{Z}^d$: one regards the edges $\{x,y\}$ (with $|x-y| = 1$) as *bonds*, and each bond is *open* with probability $p \in [0,1]$, independently of all other bonds – see [**28**]. The *open clusters* are the collections of points connected by paths consisting of open bonds. For small p all the open clusters are finite, while for p close to 1 there is a single giant open cluster with small holes. At a critical value $p_c = p_c(d) \in (0,1)$ there is a *phase transition*; all clusters are finite if $p < p_c$, while if $p > p_c$ there is a unique infinite cluster. (All statements of this kind are *almost sure*, that is, disregarding a set of probability zero.) At the critical value $p = p_c$ it is believed that all clusters are finite (this is known if $d = 2$ or $d \geq 19$), but nevertheless there are many large clusters – in fact (see [**14**]) a cube of side n will contain, with high probability, clusters of diameter n. In some cases (see [**41**], [**34**], [**35**]) it is possible to define an *incipient infinite cluster* $\widetilde{\mathcal{C}}$ – essentially the cluster containing 0 conditioned to be infinite. The local structure of $\widetilde{\mathcal{C}}$ should then be similar to that of large finite critical clusters.

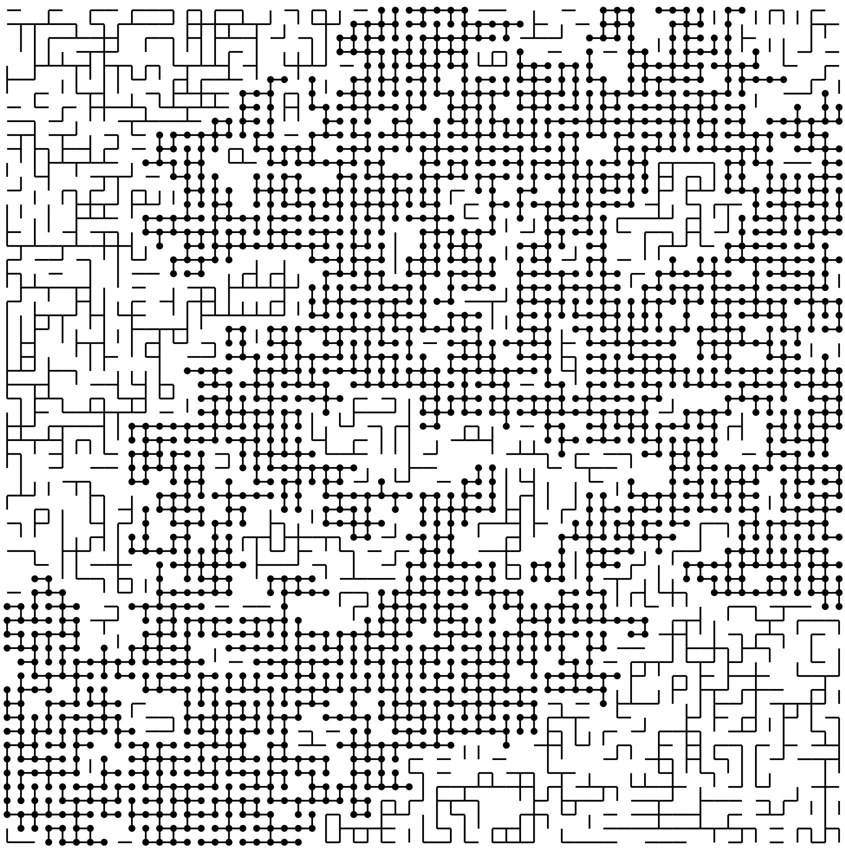

Figure 1.1. Percolation at criticality; the points in the largest cluster are marked with bullets.

The physicist's conjecture (confirmed in some cases by theorems in these papers) is that $\widetilde{\mathcal{C}}$ has a fractal structure. Physicists are interested in what are called *transport properties* of percolation clusters – that is, to mathematicians, the behaviour of solutions to the Laplace, heat and wave equations. In view of the connections between the heat equation and Markov processes, this motivated the study of random walks on percolation clusters: the 'ant in the labyrinth' of De Gennes [**17**]. (The wave equation is much harder – one reason being the difficulty in making a useful probabilistic connection.)

Since it is hard make exact calculations on sets such as $\widetilde{\mathcal{C}}$, physicists therefore proposed (see [**1**], [**53**]) that one should look random walks on regular, deterministic fractals such as the *Sierpinski gasket*. This idea has proved fruitful – not only is the study of random walks and diffusions on such sets interesting in its own right, but recent work indicates that random walks on fractals and critical percolation clusters have similar behaviour.

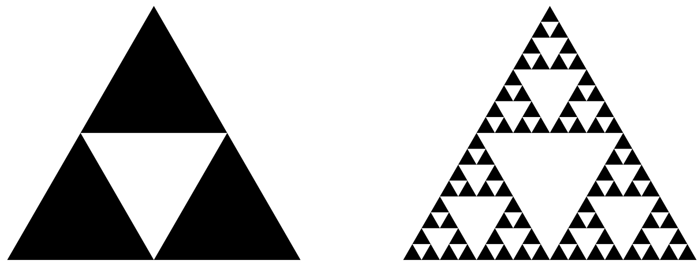

Figure 1.2. The first and fourth stages in the construction of the Sierpinski gasket.

For a brief survey of heat kernels and fractal sets see [**4**], and for more details on the various families of regular fractals which have been considered see the book [**42**], and lectures [**2**]. See also the survey [**56**], and for a review of the physics literature [**31**].

The limit of the construction illustrated in Figure 1.2 is a compact subset $F_{SG} \subset \mathbb{R}^2$. (We take the origin as the lower left hand corner of F_{SG}.) One can define an unbounded set $\widetilde{F}_{SG}$ with a local structure similar to F_{SG} by setting

$$\widetilde{F}_{SG} = \bigcup_{n=0}^{\infty} 2^n F_{SG}.$$

On certain sets $\widetilde{F}$ of this type, including the Sierpinski gasket, (see the references above for other examples) one can define a Laplacian operator $\mathcal{L}$, and the associated heat kernel $p_t(x,y)$ satisfies

$$\begin{aligned} c_1 t^{-\alpha/\beta} \exp(-c_2 (\frac{d(x,y)^\beta}{t})^{1/(\beta-1)}) &\le p_t(x,y) \\ &\le c_3 t^{-\alpha/\beta} \exp(-c_4 (\frac{d(x,y)^\beta}{t})^{1/(\beta-1)}), \qquad x,y \in \widetilde{F},\ t>0. \end{aligned} \tag{1.5}$$

Here α and β depend on the fractal F; α is the Hausdorff dimension of F while $\beta \ge 2$ is a number (called the *walk dimension* of F), which gives the space-time scaling of the heat equation on F. One finds that $2 \le \beta \le 1+\alpha$, and that these are the only constraints on α and β – see [**3**], [**33**], [**23**]. While fractal sets with $\beta = 2$ are known (see [**15**], [**46**]), the main families of regular fractals have $\beta > 2$: for example the Sierpinski gasket has $\alpha_{SG} = \log 3/\log 2$ and $\beta_{SG} = \log 5/\log 2$. Let X be the diffusion process associated with $p_t(x,y)$; then (1.5) leads easily to the bound

$$\mathbb{E}^x d(x, X_t)^2 \asymp t^{2/\beta}, \qquad t > 0. \tag{1.6}$$

The case $\beta \ne 2$ is called by physicists *anomalous diffusion*. The intuitive explanation for (1.6) for the Sierpinski gasket is that the motion of X is impeded by a sequence of successively larger obstacles.

It might be thought that this unusual scaling is due to the local fractal structure of $\widetilde{F}_{SG}$. However, $\widetilde{F}_{SG}$ is self-similar, and the large time behaviour of $p_t(x,y)$ is governed by the large scale structure of $\widetilde{F}_{SG}$. In fact the bound (1.5) (for $t \geq 1$ and $d(x,y) \leq t$) holds for various 'classical' sets with a large scale fractal structure. Thus Jones [**38**] proved that the random walk on the graphical Sierpinski gasket G_{SG} (see Figure 1.3) satisfies this bound, while Bass and I in [**7**] obtained an analogous result for graphical Sierpinski carpets. (These graphs are sometimes called 'pre-fractals'.)

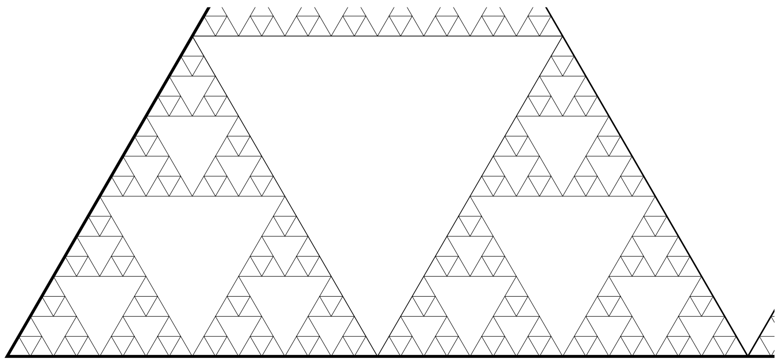

Figure 1.3. The graphical Sierpinski gasket G_{SG}: the small triangles have side 1.

One can also look at 'pre-fractal' domains in $\mathbb{R}^d$. These are open domains $D \subset \mathbb{R}^d$ with a large scale structure similar to some unbounded regular fractal – see [**6**], [**8**] for the pre-Sierpinski carpet. (Pre-fractal manifolds have also been considered in [**45**], [**11**].)

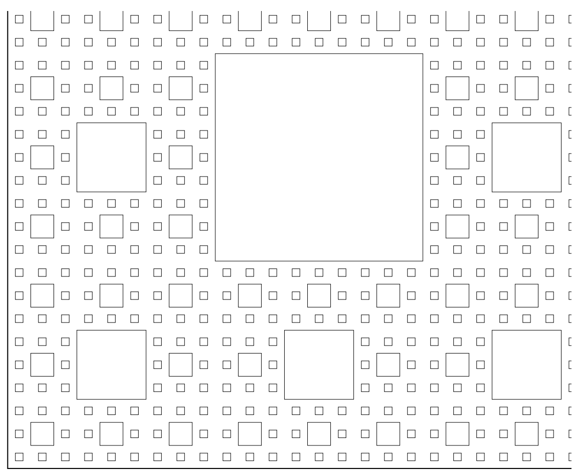

Figure 1.4. The pre-Sierpinski carpet: the small squares have side 1.

These sets satisfy the elliptic Harnack inequality. For the graphical Sierpinski gasket there is a very easy direct proof – see Theorem 2.6 of [**4**]. The proof for pre-Sierpinski carpets in dimensions $d \geq 3$ is quite long, and uses probabilistic coupling – see [**6**]. If $\beta > 2$ it is easy to deduce from (1.5) that PHI fails for these sets. The underlying reason is that in (1.5) the space-time scaling is given by the 'anomalous diffusion' coefficient β, rather than 2; the EHI contains no information about this scaling while the PHI does.

Given this, it is clear how the PHI should be generalized. Given $\beta \geq 2$ one replaces $T = R^2$ in the definition above by $T = R^\beta$; denote this condition PHI(β). It is immediate that PHI(β) implies EHI.

The argument in [**20**] used to prove the equivalence of PHI and (1.4) extends to the case $\beta > 2$: see Section 5 of [**32**]. Thus it follows that PHI(β) (with the appropriate β) holds for the Sierpinski gasket, carpet, and other families of regular fractals. One can also prove quite easily from (1.5) that these spaces satisfy a Poincaré inequality with anomalous scaling, denoted PI(β), and obtained by replacing R^2 by R^β in (1.3).

The proofs of (1.5) in the literature all use very strongly various symmetry properties of the spaces. In transferring results from one kind of pre-fractal object to another (for example, from the graphical Sierpinski gasket to a manifold made from it in the same way the 'jungle gym' is from the Euclidean lattice – see [**52**], [**40**]) one would wish for the same kind of stability in the case $\beta > 2$ as is given by Theorem 1.2 if $\beta = 2$.

An initial guess that Theorem 1.2 holds if one just replaces 2 by β is easily seen to be false – for example the product graph $G = \mathbb{Z} \times G_{SG}$ satisfies VD and PI(β_{SG}), but fails EHI – see the proof of Theorem 6 of [**3**]. (The reason is that the different space-time scaling in the two directions means that a random walk will with high probability leave a ball in the $\mathbb{Z}$ direction before it has moved very far in the G_{SG} direction.)

In [**9**] Bass and I, in the graph case, gave an additional condition, denoted CS(β), which, with VD and PI(β) is equivalent to PHI(β). This condition, which is unfortunately quite complicated, is described in Section 2 below. Given open sets $U_1 \subset \overline{U}_1 \subset U_2$, we will say a function f is a *cutoff function for* $U_1 \subset U_2$ if $f \geq 1$ on U_1 and $f = 0$ on U_2^c.

In Euclidean space (or a manifold satisfying VD) if we look for the lowest energy cutoff function f for $B(x,R) \subset B(x,2R)$ then this has energy $\mathcal{E}(f,f) = \int |\nabla f|^2 \asymp R^{-2}V(x,r)$. On a pre-fractal domain D, such as the pre-Sierpinski carpet (Figure 1.4), one can do better, and for the optimal f one obtains $\mathcal{E}(f,f) \asymp R^{-\beta}V(x,r)$. (One takes $|\nabla f|$ higher on shells where the set D is relatively thin.) The condition CS(β) is that there exists a large family of well-behaved cutoff functions for balls $B(x,R) \subset B(x,2R)$, with energy of order $R^{-\beta}V(x,R)$. Balls with radius less than 1 in a graph are trivial, but for manifolds we need to be able to treat balls of any size. Since a manifold is locally Euclidean, one expects the usual R^2 behaviour for small R. One therefore needs to introduce the function

$$\Psi(r) = \begin{cases} r^2 & \text{if} \quad r < 1, \\ r^\beta & \text{if} \quad r \geq 1. \end{cases}$$

and discuss PI(Ψ), PHI(Ψ) and CS(Ψ).

We have the following:

THEOREM 1.3. *Let M be a complete smooth Riemannian manifold. The following are equivalent:*
(a) M satisfies PHI(Ψ).

(b) M satisfies VD, PI(Ψ), and CS(Ψ).
(c) The heat kernel $p_t(x,y)$ on M satisfies a two-sided bound denoted $HK(\Psi)$.

A similar theorem for graphs was the main result of [**9**], and was based on methods used in [**8**] to study divergence from operators on the pre-Sierpinski carpet. The proof of Theorem 1.3 is very similar, and we will only sketch the main arguments. Full details will appear in [**10**], which will treat the general case of a measure metric space with a Dirichlet form: this covers pre-fractal domains and manifolds, as well as true fractals.

The conditions VD, PI(Ψ), and CS(Ψ) are all stable under rough isometries, given suitable 'side conditions' on the families of sets considered. (Since rough isometries only preserve global properties, it is clear that some conditions of this kind are necessary.)

While CS(Ψ) (with VD, PI(Ψ)) is therefore a necessary and sufficient for PHI(Ψ)), of course it may not be the simplest such condition. In fact, in the 'strongly recurrent' case ($\alpha < \beta$) a much simpler characterization of PHI(Ψ) is possible, just using VD and estimates on the electrical resistance: this will appear in [**12**]. (See also [**43**].)

One might hope that these stability results for PHI(Ψ) would lead to the stability of EHI. However, this still seems a hard problem. Delmotte [**19**] has shown, by joining two different graphs by one edge, that EHI can hold even if VD fails. Let G_i be pre-fractal graphs satisfying EHI with indices α_i and β_i, such that $\beta_2 - \alpha_2 = \beta_1 - \alpha_1 = \zeta > 0$. Then (see [**3**]) joining G_1 and G_2 by one edge gives a graph G satisfying EHI but with different space-time scaling in different regions. Thus any heat equation approach to the stability of the EHI would appear to need to deal with spaces with rather irregular properties. However, in this example the electrical resistance between points x and y decays as $d(x,y)^{-\zeta}$, so that, in terms of electrical resistance, the graph G is quite regular. This suggests that it may be possible to characterize EHI in terms of electrical resistance – see the open problem mentioned at the end of [**5**].

The structure of the remainder of this survey is as follows. In Section 2 we introduce precisely the main concepts and give our main results. In Section 3 we sketch the proof that PHI(Ψ) implies CS(Ψ); the argument uses Green's functions to build a suitable family of cutoff functions. Section 4 deals with the implication (b) $\Rightarrow$ (a). We begin by showing how Moser's iteration argument breaks down in the case $\beta > 2$ if we try to use standard cutoff functions. We then sketch how the difficulty can be overcome using a weighted Sobolev inequality derived from CS(Ψ). This proves that VD, PI(Ψ) and CS(Ψ) imply EHI; PHI(Ψ) then follows easily using the scaling information in PI(Ψ). The stability of CS(Ψ) under rough isometries is proved in Section 5.

We write c_i to denote positive constants which are constant within each argument, and $f \asymp g$ to mean there exist positive constants c_i such that $c_1 f \le g \le c_2 f$.

2. Stability theorems

Let M be a complete smooth non-compact Riemannian manifold, $\Delta = \mathrm{div}\nabla$ be the Laplace-Beltrami operator on M, d be the Riemannian metric and μ be the volume. We write $B(x,r)$ for open balls in M, and set $V(x,r) = \mu(B(x,r))$. We define the Dirichlet form $\mathcal{E}(f,f)$ on $L^2(M)$ with core $C_0^\infty(M)$ by taking

$$\mathcal{E}(f,f) = \int_M |\nabla f|^2 d\mu, \qquad f \in C_0^\infty(M).$$

(See [**21**] for more details.) Note (see [**54**]) that we have

$$d(x,y) = \sup\{|f(x) - f(y)| : f \in C_0^\infty(M),\, |\nabla f| \le 1\}.$$

We write $W = (W_t, t \ge 0, \mathbb{P}^x, x \in M)$ for the Brownian motion on M.

We now give a number of conditions M may or may not satisfy. Let $\beta \ge 2$ and define Ψ as in (1.7). Recall from Section 1 the definitions of volume doubling VD and EHI. It is easy to deduce from VD that there exists $\alpha < \infty$ such that if $x, y \in M$ and $0 < r < R$ then

$$\frac{V(x,R)}{V(y,r)} \le c_1 \Big(\frac{d(x,y) + R}{r} \Big)^\alpha. \tag{2.1}$$

Definition 2.1. M satisfies the *Poincaré inequality* PI(Ψ), if there exists a constant c_1 such that for any ball $B = B(x,R) \subset M$ and $f \in C^\infty(B)$

$$\int_B (f(x) - \overline{f}_B)^2 d\mu(x) \le c_1 \Psi(R) \int_B |\nabla f|^2 d\mu. \tag{2.2}$$

Here $\overline{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$.

Definition 2.2. M satisfies the *parabolic Harnack inequality* (PHI(Ψ)), if there exists a constant c_P such that the following holds. Let $x_0 \in M$, $R > 0$, $T = \Psi(R)$ and $u = u(t,x)$ be a non-negative solution of the heat equation $\partial_t u = \Delta u$ in $(0,4T) \times B(x_0, 2R)$. Let $Q_- = (T, 2T) \times B(x_0, R)$ and $Q_+ = (3T, 4T) \times B(x_0, r)$; then

$$\sup_{Q_-} u \le c_P \inf_{Q_+} u. \tag{2.3}$$

Definition 2.3. Let A, B be disjoint subsets of M. We define the effective resistance $R_{\rm eff}(A,B)$ by

$$R_{\rm eff}(A,B)^{-1} = \inf \Big\{ \int_M |\nabla f|^2 : f = 0 \text{ on } A \text{ and } f = 1 \text{ on B},\, f \in C^\infty(M) \Big\}.$$

M satisfies the condition RES(Ψ) if for any $x \in M$, $R \ge 0$,

$$R_{\rm eff}(B(x,R), B(x,2R)^c) \asymp \frac{\Psi(R)}{V(x,R)}. \tag{2.4}$$

We have by the same arguments as in Lemma 5.1 of [**9**]

Lemma 2.4. *Let M satisfy VD, EHI, and PI(Ψ). Then M satisfies RES(Ψ).*

Let $p_t(x,y)$ be the fundamental solution of the heat equation on M. Recall that we have a 'crossover' from the classical $t = r^2$ scaling when $r < 1$ to the anomalous scaling $t = r^\beta$ for $r \ge 1$. For $(t,r) \in (0,\infty) \times [0,\infty)$ we define the regions:

$$\Lambda_1 = \{(t,r) : t \le 1 \vee r\}, \qquad \Lambda_2 = \{(t,r) : t \ge 1 \vee r\}. \tag{2.5}$$

Let $\theta \ge 2$ and

$$h_\theta(r,t) = \exp(-(r^\theta/t)^{1/(\theta-1)}).$$

Definition 2.5. M satisfies the heat kernel bounds HK(Ψ), if, writing $r = d(x,y)$,

$$c_1 V(x,t^{1/2})^{-1} h_2(c_2 r,t) \le p_t(x,y) \le c_3 V(x,t^{1/2})^{-1} h_2(c_4 r,t) \tag{2.6}$$

for $x,y \in M$ and $t \in (0,\infty)$ with $(t,r) \in \Lambda_1$, and

$$c_1 V(x,t^{1/\beta})^{-1} h_\beta(c_2 r,t) \le p_t(x,y) \le c_3 V(x,t^{1/\beta})^{-1} h_\beta(c_4 r,t) \tag{2.7}$$

for $x,y \in M$ and $t \in (0,\infty)$ with $(t,r) \in \Lambda_2$.

Remarks 2.6.
1. In [**6**] it was proved that the pre-Sierpinski carpet satisfies HK(Ψ). We will see below that this also holds on sufficiently regular pre-fractal manifolds.
2. If $r > t$ then $h_\beta(r,t) \ge h_2(r,t)$.
3. To understand why the crossover in HK(Ψ) takes the form it does, it is useful to consider the contribution to $p_t(x,y)$ from various types of path in M. First, if $0 < t \le 1$ and $d(x,y) < 1$ then the behaviour is essentially local, and the locally Euclidean structure of M gives the usual type of bound (2.6).

If $r > t$ then we are in the 'large deviations' regime: the the main contribution to $p_t(x,y)$ is from those paths of the Brownian motion W which are within a distance $O(t/r)$ of a geodesic from x to y. So, once the length of the geodesic is given, only the local structure of M plays a role. Note that in this case the term in the exponential is smaller than e^{-ct}, so that the volume term $V(x,t^{1/2})^{-1}$ could be absorbed into the exponential with a suitable modification of the constants c_2 and c_4.

Finally, if $t > 1$ and $r < t$, then the paths which contribute to $p_t(x,y)$ fill out a much larger part of M: those which lie in $B(x,t^{1/\beta})$ if $r < t^{1/\beta}$, and those which are within a distance $O(t/r^{\beta-1})$ of a geodesic from x to y in the case when $t^{1/\beta} \le r \le t$.

We will also want to discuss local versions of these conditions. We say M satisfies $\mathrm{VD}_{\mathrm{loc}}$ if (1.2) holds for $x \in M$, $0 < R \le 1$. Similarly we define $\mathrm{PI}(\Psi)_{\mathrm{loc}}$, $\mathrm{EHI}_{\mathrm{loc}}$ and $\mathrm{PHI}(\Psi)_{\mathrm{loc}}$ by requiring the conditions only for $0 < R \le 1$. For $\mathrm{HK}_{\mathrm{loc}}$ we require the bounds only for $t \in (0,1)$ – so only (2.6) is involved. Note that these local conditions are all independent of the the parameter β. The value 1 here is just for simplicity: each of the local conditions implies an analogous local condition for $0 < R \le R_0$ for any (fixed) $R_0 > 1$ – see Section 2 of [**32**].

The following result of Hebisch and Saloff-Coste localizes Theorem 1.2.

Theorem 2.7. *([**32**], Theorem 2.7). The following are equivalent:*
(a) M satisfies VD_{loc} and $PI(\Psi)_{loc}$.
(b) M satisfies $PHI(\Psi)_{loc}$.
(c) M satisfies $HK(\Psi)_{loc}$.
If any of (a)–(c) hold then M satisfies EHI_{loc}, and $p_t(x,y)$ is continuous on $(0,\infty) \times M \times M$.

In addition we will need the following:

Theorem 2.8. *(See [**32**], Theorem 5.3, [**26**]). The following are equivalent:*
(a) M satisfies PHI(Ψ).
(b) M satisfies HK(Ψ).
(c) M satisfies VD, EHI and RES(Ψ).

Proof. The equivalence of (a) and (b) is given in [**32**]; and that these are equivalent to (c) is proved in [**26**]. (See [**25**] for the graph case.) □

We now introduce the condition CS(Ψ).

Definition 2.9. M satisfies CS(Ψ) for $\beta \geq 2$ if there exists $\theta \in (0,1]$ and constants c_1, c_2 such that the following holds. For every $x_0 \in M$, $R > 0$ there exists a cutoff function $\varphi (= \varphi_{x_0,R})$ with the following properties:
(a) $\varphi(x) \geq 1$ for $x \in B(x_0, R/2)$.
(b) $\varphi(x) = 0$ for $x \in B(x_0, R)^c$.
(c) $|\varphi(x) - \varphi(y)| \leq c_1 (d(x,y)/R)^\theta$ for all x,y.
(d) For any ball $B(x,s)$ with $0 < s \leq R$ and $f : B(x, 2s) \to \mathbb{R}$

$$\int_{B(x,s)} f^2 |\nabla\varphi|^2 d\mu \leq c_2 (s/R)^{2\theta} \Big(\int_{B(x,2s)} |\nabla f|^2 d\mu + \Psi(s)^{-1} \int_{B(x,2s)} f^2 d\mu \Big). \tag{2.8}$$

Remarks 2.10.
1. We call (2.8) a weighted Sobolev inequality. It is clear that to prove (2.8) it is enough to consider nonnegative f.
2. Suppose CS(Ψ) holds for M, but with (a) above replaced by

$$\varphi(x) \geq 1 \text{ for } x \in B(x_0, \delta R), \tag{2.9}$$

for some $\delta < \frac{1}{2}$. Then an easy covering argument (using VD) gives CS(Ψ) with $\delta = \frac{1}{2}$.
3. Let $\lambda > 1$. Suppose that CS(Ψ) holds, except that instead of (2.8) we have

$$\int_{B(x,s)} f^2 |\nabla\varphi|^2 d\mu \leq c_2 (s/R)^{2\theta} \Big(\int_{B(x,\lambda s)} |\nabla f|^2 d\mu + \Psi(s)^{-1} \int_{B(x,\lambda s)} f^2 d\mu \Big). \tag{2.10}$$

Then once again it is easy to obtain CS(Ψ) with $\lambda = 2$ by a covering argument.
4. Any operation on φ which reduces $|\nabla\varphi|$ while keeping properties (a), (b) and (c) of Definition 2.9 will generate a new cutoff function which still satisfies (2.8). We can therefore assume that any cutoff function φ satisfies the following:
(a) $0 \leq \varphi \leq 1$.
(b) For each $t \in (0,1)$ the set $\{x : \varphi(x) > t\}$ is connected and contains $B(x_0, R/2)$.
(c) Each connected component A of $\{x : \varphi(x) < t\}$ intersects $B(x_0, R)^c$.
5. CS(2) always holds since one can take $\varphi(x) = (2/R) d(x, B(x_0, R)^c)$. Then $|\nabla\varphi| \leq 2/R$ and (2.8) (with $\theta = 1$) follows easily.
6. Let $2 \leq \beta \leq \beta'$, and write Ψ, Ψ' for the functions given by (1.7). Then PI(Ψ) implies PI(Ψ'), while CS(Ψ') implies CS(Ψ). Further (see Lemma 2.14 below) if PI(Ψ) holds, then CS(Ψ') cannot hold for any $\beta' > \beta$.
7. If M_i, $i = 1, 2$, are manifolds satisfying PHI(Ψ_i), respectively, with $\beta_1 < \beta_2$, then the product $M = M_1 \times M_2$ satisfies PI(Ψ_2). However, since M does not satisfy PHI(Ψ_2) it cannot satisfy CS(Ψ_2). Thus the conditions PI(Ψ) and CS(Ψ) are independent.

The following theorem gives a characterization of PHI(Ψ) in terms of conditions which have good stability properties.

Theorem 2.11. *The following are equivalent:*
(a) M satisfies VD, PI(Ψ) and CS(Ψ).
(b) M satisfies PHI(Ψ).

This was proved in the graph case in [**9**]. The extension to manifolds is sketched below in Sections 3 and 4. Full details, and the extension to a general class of measure metric spaces will be given in [**10**].

We will see in Section 5 that the conditions VD, PI(Ψ) and CS(Ψ) are stable under rough isometries between manifolds and graphs, provided these have the appropriate local regularity.

THEOREM 2.12. *Let G be a graph with bounded vertex degree. Let M be a manifold with Ricci curvature bounded below, and positive injectivity radius, which is roughly isometric to G. The following are equivalent:*
(a) G satisfies PHI(Ψ).
(b) M satisfies PHI(Ψ).

See [**40**] for the definition of injectivity radius.

EXAMPLES 2.13. Given a graph G with bounded vertex degree one can create a manifold M satisfying the conditions of Theorem 2.12 by replacing the edges of G by tubes of length 1, (and diameter say 1/10) and gluing these together smoothly at the vertices.

Using Theorem 2.12 with Theorems 2.8 and 2.11 we see that the pre-Sierpinksi gasket manifolds defined in [**45**] satisfy HK(Ψ) with $\beta = \log 5/\log 2$. We also deduce that the manifolds based on the family of Vicsek fractals studied in [**11**] satisfy HK(Ψ), where $\Psi(r) = r^2 \vee r^\beta$, and β is the 'walk dimension' of the associated graph.

We conclude this section by discussing the relation of PI(Ψ) and CS(Ψ) with the spectral gap of balls. Let $B = B(x, R)$ be an open ball in M. Let $\mathcal{M}(B) = \{f \in C^\infty(B) : \int_B f = 0, ||f 1_B||_2 \neq 0\}$; then the smallest non-zero Neumann eigenvalue of $-\Delta$ on B is given by

$$\lambda_1(B) = \inf_{f \in \mathcal{M}(B)} \frac{\int_B |\nabla f|^2}{\int_B |f|^2}. \tag{2.11}$$

LEMMA 2.14.
(a) If M satisfies PI(Ψ) then

$$\lambda_1(B(x,R)) \geq c\Psi(R)^{-1}, \qquad x \in M,\ R > 0.$$

(a) If M satisfies VD and CS(Ψ) then

$$\lambda_1(B(x,R)) \leq c\Psi(R)^{-1}, \qquad x \in M,\ R > 0.$$

PROOF. (a) is immediate from the definition of PI(Ψ) and (2.11).
(b) Let γ be a geodesic from x_0 to $y \in \partial B$ and $x_1 \in \gamma$ with $d(x_0, x_1) = 2R/3$. For $i = 0, 1$ let $B_i = B(x_i, R/8)$ and $B_i^* = B(x_i, R/4)$. Using VD we have

$$c_1 \mu(B_1^*) \leq \mu(B_2) \leq \mu(B_2^*) \leq c_2 \mu(B_1).$$

By CS(Ψ) there exist cutoff functions φ_i for $B_i \subset B_i^*$. Let $g = \varphi_0 - \varphi_1 - c_3$, where c_3 is chosen so that $\int_B g = 0$. It is easy to check that

$$\int_B g^2 d\mu \geq c\mu(B).$$

By (2.8) applied to the constant function 1

$$\int_{B_i^*} |\nabla \varphi_i|^2 d\mu \le c_2 \Psi(R/4)^{-1} \int_{B(x_i,R/2)} d\mu.$$

Hence

$$\int_B |\nabla g|^2 \le c\Psi(R)^{-1}\mu(B) \le c'\Psi(R)^{-1} \int_B g^2,$$

so that $\lambda_1(B) \le c'\Psi(R)^{-1}$. □

3. Construction of cutoff functions

Throughout this section we assume that M satisfies HK(Ψ) (and hence VD and PHI(Ψ)). We will sketch the proof that M satisfies CS(Ψ); the argument, which runs along the same lines as that in [**9**], uses the Greens functions $g_\lambda(x,y)$, given by

$$g_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t(x,y) dt,$$

to build a cutoff function φ. The main difference here is that as [**9**] used $g(x,y)$ rather than $g_\lambda(x,y)$, a strong transience condition (called (FVG)) was needed in the initial arguments. (This was then removed using a standard trick with product spaces.)

Lemma 3.1. *Let $x_0 \in M$, $R \ge 1$. Then there exists $\delta > 0$ such that if $\lambda = cR^{-\beta}$*

$$g_\lambda(x_0,y) \le c_2 \frac{R^\beta}{V(x_0,R)}, \qquad y \in B(x_0,R)^c, \tag{3.1}$$

$$g_\lambda(x_0,y) \ge 2c_2 \frac{R^\beta}{V(x_0,R)}, \qquad y \in B(x_0,2\delta R). \tag{3.2}$$

Proof. This follows easily from HK(Ψ) by integration. □

Lemma 3.2. *Let $x_0 \notin B(x_1,r)$. Then there exists $\theta > 0$ such that*

$$|g_\lambda(x_0,x) - g_\lambda(x_0,y)| \le (\frac{d(x,y)}{r})^\theta \sup_{B(x_1,r)} g_\lambda(x_0,.). \tag{3.3}$$

Proof. The Hölder continuity of p_t is given by PHI(Ψ). Integrating we obtain (3.3). □

We begin the proof of CS(Ψ) with the special case when we only require the weighted Sobolev inequality for $I = B$.

Proposition 3.3. *Let $x_1 \in M$, $r > 0$. There exists $\delta > 0$ and a cutoff function φ for $B(x_1,\delta r) \subset B(x_1,r)$ such that, writing $B' = B(x_1,\delta r)$, $B = B(x_1,r)$,*

$$\int_{B'} f^2 |\nabla\varphi|^2 d\mu \le c \int_B |\nabla f|^2 d\mu + c\Psi(r)^{-1} \int_{B'} f^2. \tag{3.4}$$

Proof. If $r \le c$ then we can take φ to be a local cutoff function for $B' \subset B$, so suppose $r > c$.

Let $D = B(x_1, r-\varepsilon)$ where $\varepsilon < r/10$, and $\lambda > 0$. Let G_λ^D be the resolvent associated with the process W killed on exiting D, that is,

$$G_\lambda^D f(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\lambda t} f(W_t) dt,$$

for bounded measurable f, where

$$\tau_D = \inf\{t : W_t \in M - D\}.$$

Let $h = 1_{B'}$, and let $v = G_\lambda^D h$. Using the estimates on the heat kernel of W, as in Lemma 4.7 of [**9**], we can take λ small enough so that

$$v(x) \geq c_1 \Psi(r), \qquad x \in B',$$

and

$$v(x) \leq c_2 \Psi(r), \qquad x \in D.$$

Let $f \in C_0^\infty(M)$. Then

$$\int_B f^2 |\nabla v|^2 d\mu \leq \int_M f^2 |\nabla v|^2 d\mu = \int_M (\nabla(f^2 v).\nabla v) d\mu - \int_M 2fv\nabla f.\nabla v d\mu.$$

By Gauss-Green

$$\int_M (\nabla(f^2 v).\nabla v) d\mu = -\int_M f^2 v \Delta v d\mu \leq c\Psi(r) \int_{B'} f^2 d\mu.$$

Using Cauchy-Schwarz we obtain

$$\begin{aligned} |\int_M 2fv(\nabla f.\nabla v) d\mu| &\leq c\Big(\int_M v^2 |\nabla f|^2 d\mu\Big)^{1/2} \Big(\int_M f^2 |\nabla v|^2 d\mu\Big)^{1/2} \\ &\leq c\Psi(r)\Big(\int_B |\nabla f|^2 d\mu\Big)^{1/2} \Big(\int_M f^2 |\nabla v|^2 d\mu\Big)^{1/2}. \end{aligned}$$

So, writing $I = \int_M f^2 |\nabla v|^2 d\mu$, $J = \int_B |\nabla f|^2 d\mu$, $K = \int_B f^2 d\mu$, we have

$$I \leq c\Psi(r) K + c\Psi(r) J^{1/2} I^{1/2},$$

from which it follows that $I \leq c\Psi(r)K + c\Psi(r)^2 J$. Setting $\varphi = 1 \wedge (c\Psi(r)^{-1} v)$, where c is chosen so that φ is a cutoff function for $B' \subset B$, (3.4) follows. ☐

Now fix $x_0 \in M$ and $R \geq 1$. Let $\lambda = cR^{-\beta}$, δ, c_2 be as in Lemma 3.1, and

$$h = c_2 R^\beta / V(x_0, R).$$

We now define

$$Q(b) = Q(x_0, b) = \{y : g_\lambda(x_0, y) > b\}.$$

As in [**9**] we can approximate $Q(b)$ by balls. Let ρ be an approximate identity with support $B(x_0, \delta^2 R)$, and

$$\omega_0(x) = G_\lambda \rho(x), \qquad \omega(x) = (2h \wedge \omega(x) - h)^+.$$

Thus $h^{-1}\omega$ is a cutoff function for $B = B(x_0, \delta R) \subset B(x_0, R)$.

Proposition 3.4. *Let $x_0 \in M$, $R > 0$, ω be as above. Let $I = B(x_1, \delta s)$, with $s \leq R$, and $I^* = B(x_1, s)$. Suppose that either*

$$I^* \subset Q(2h) \tag{3.5}$$

or

$$I^* \cap B(x_0, \delta R) = \emptyset. \tag{3.6}$$

Let $f \in C^\infty(M)$. There exists $c_1 < \infty$ such that

$$h^{-2} \int_I f^2 |\nabla \omega|^2 \leq c_1 (s/R)^{2\theta} \Big(\int_{I^*} |\nabla f|^2 d\mu + c\Psi(r)^{-1} \int_I f^2\Big). \tag{3.7}$$

Proof. The argument follows the lines of [**9**], Proposition 4.10. As ω is constant on $Q(2h)$ it is enough to consider the case when (3.6) holds. Let v be a cutoff function for $I \subset I^*$ given by Proposition 3.3. Let $\omega_1(x) = \omega_0(x) - c_3$, where c_3 is chosen so that $\omega_1 \geq 0$ on I^* and

$$\omega_1(x) \leq ch(s/R)^\theta = L, \qquad x \in I^*;$$

this is possible by Lemma 3.2. Let

$$\begin{aligned} A &= \int_I f^2 |\nabla \omega|^2 d\mu, \\ D &= \int_{I^*} |\nabla f|^2 d\mu + \Psi(r)^{-1} \int_I f^2 d\mu, \\ F &= \int_{I^*} f^2 v^2 |\nabla \omega_1|^2. \end{aligned}$$

Now

$$A \leq F = \int_{I^*} f^2 v^2 \nabla \omega_1 . \nabla \omega_0 = \int_{I^*} \nabla(f^2 v^2 \omega_1).\nabla \omega_0 - \int_{I^*} \omega_1 \nabla (f^2 v^2).\nabla \omega_0. \tag{3.8}$$

For the first term in (3.8), by Gauss-Green

$$\begin{aligned} \int_{I^*} \nabla(f^2 v^2 \omega_1).\nabla \omega_0 &= - \int_M (f^2 v^2 \omega_1) \Delta \omega_0 \\ &= - \int_M (f^2 v^2 \omega_1)(\lambda \omega_0 - \rho) \\ &\leq \int_M (f^2 v^2 \omega_1) \rho = 0. \end{aligned}$$

Here we used the fact that $\omega_1 \geq 0$ on I^*, that v has support I^*, and that v and ρ have disjoint supports. The second term in (3.8) is handled exactly as in [**8**] and [**9**]. That is, using Cauchy-Schwarz,

$$\begin{aligned} &\Big| \int_{I^*} \omega_1 \nabla (f^2 v^2).\nabla \omega_0 \Big| \\ &\quad \leq c\Big(\big(\int_{I^*} v^2 |\nabla f|^2 d\mu \big)^{1/2} + \big(\int_{I^*} f^2 |\nabla v|^2 d\mu \big)^{1/2} \Big) \Big(\int_{I^*} \omega_1^2 f^2 v^2 |\nabla \omega_0|^2 d\mu \Big)^{1/2} \\ &\quad \leq cLD^{1/2} F^{1/2}, \end{aligned}$$

where we used Proposition 3.3 in the final line. Thus we obtain $F \leq cL^2 D$. □

Proposition 3.5. *Let M satisfy PHI(Ψ). Then M satisfies VD, PI(Ψ) and CS(Ψ).*

Proof. The arguments that M satisfies VD and PI(Ψ) are standard. CS(Ψ) follows from Proposition 3.4 and an easy covering argument just as in Corollary 4.11 of [**9**]. □

4. Proof of Harnack inequalities

We begin by explaining the necessity of CS(Ψ) in the anomalous diffusion case. Let M be a manifold satisfying VD, PI(Ψ) and having regular volume growth

$$V(x,r) \asymp r^\alpha, \qquad x \in M,\ r \geq 1.$$

Since M need not satisfy CS(Ψ), any argument to prove EHI must fail, but it is still instructive to see where the problem arises. Let us try to follow Moser's proof of the EHI in [**49**]; a similar obstacle would arise if one tried other approaches, such as that in [**20**]. The difficulty is with the first ('easy') part of Moser's argument.

Write

$$\fint_B f = \mu(B)^{-1} \int_B f d\mu.$$

From PI(Ψ) one obtains (see [**54**], [**55**], Section 5.2) the Sobolev inequality

$$\Big(\fint_B |f|^{2\kappa}\Big)^{1/\kappa} \le cR^\beta \fint_B |\nabla f|^2, \tag{4.1}$$

for $f \in C_0^\infty(B)$, where B has radius $R \ge 1$ and $\kappa > 1$.

Let $u > 0$ be harmonic in $B = B(x,R)$; as M is locally regular we have that u is continuous. Let $\frac12 < a_2 < a_1 < 1$, $B_i = B(x, a_i R)$, $p > 2$, $\psi \in C_0^\infty(B_1)$ be a cutoff function for $B_2 \subset B_1$ and $v = u^p$. Moser's argument (see [**55**], p. 121) gives

$$\int_{B_1} |\nabla(\psi v)|^2 \le c||\nabla\psi||_\infty^2 \int_{B_1} v^2. \tag{4.2}$$

By (4.1) applied to $f = v\psi = u^p\psi$

$$\Big(\fint_{B_2} u^{2\kappa p}\Big)^{1/\kappa} \le \Big(\fint_{B_1} (v\psi)^{2\kappa}\Big)^{1/\kappa} \le cR^\beta \fint_{B_1} |\nabla(v\psi)|^2. \tag{4.3}$$

Since we can find ψ such that $||\nabla\psi||_\infty \le 2R^{-2}(a_1 - a_2)^{-1}$ it follows that

$$\Big(\fint_{B_2} u^{2\kappa p}\Big)^{1/\kappa} \le cR^{\beta-2}(a_1 - a_2)^{-2} \fint_{B_1} u^{2p}. \tag{4.4}$$

Now let $a_k = \frac12(1 + 2^{-k})$, $p_k = p_0\kappa^k$ where $p_0 > 2$, and $B_k = B(x, a_k R)$. Then, if

$$I_k = \Big(\fint_{B_k} u^{2p_k}\Big)^{1/2p_k},$$

(4.4) implies that

$$I_{k+1} \le (cR^{\beta-2}2^{2k})^{1/2p_k} I_k, \quad k \ge 0. \tag{4.5}$$

If $\beta = 2$ this leads, by iteration as in [**49**], to the bound

$$u(y) \le C \fint_B u^{2p_0}, \qquad y \in B(x, \tfrac12 R). \tag{4.6}$$

If $\beta > 2$ one still obtains an L^∞ bound on u in $B(x, \frac12 R)$, but the constant C now depends on R, so that the final constant in the EHI will also depend on R.

Inspecting the argument above, the crucial loss is in using the bound (4.2) to go from (4.3) to (4.4); one needs a cutoff function ψ such that the final term in (4.3) can be controlled by a term of order $R^{-\beta}$.

We shall now see how CS(Ψ) enables one to do this. As the arguments in Section 5 of [**9**] can be repeated in this more general context with minor changes, we only sketch the main ideas. Full details will be given in [**10**].

Fix $x_0 \in M$, let $R \ge 0$, and φ be a cutoff function for $B(x_0, R/2) \subset B(x_0, R)$ given by CS(Ψ). We regularize φ so that it satisfies conditions (a)–(c) of Remark 2.10.4. We define the measure γ by

$$d\gamma = d\mu + R^\beta |\nabla\varphi|^2 d\mu.$$

We do not know if this measure satisfies volume doubling, but using CS(Ψ) we do have $\gamma(B(x_0,R)) \le cV(x_0,R)$.

This first step is to use CS(Ψ) (and PI(Ψ)) to obtain the following weighted Sobolev inequality, which will replace (4.1) in the iteration argument. We write $J^{(s)} = \{x : d(x,J) \le s\}$.

PROPOSITION 4.1. *(See Theorem 5.4 of [9].) Let $s \le R$ and $J \subset B(x_0,R)$ be a finite union of balls of radius s. There exist $\kappa > 1$ and $c_1 < \infty$ such that*

$$\Big(\mu(J)^{-1}\int_J |f|^{2\kappa} d\gamma\Big)^{1/\kappa} \le c_1\Big(R^\beta \mu(J)^{-1}\int_{J^s} |\nabla f|^2 d\mu + (s/R)^{-2\theta}\mu(J)^{-1}\int_J f^2 d\gamma\Big). \tag{4.7}$$

Now let u be harmonic in a domain $D \subset X$. By Theorem 2.7 u satisfies a local Harnack inequality, so is Hölder continuous. As in [49] we have

LEMMA 4.2. *Let D be a domain in M, let u be positive and harmonic in D, $v = u^k$, where $k \in \mathbb{R}$, $k \ne \frac{1}{2}$, and let $\eta \in C_0^\infty(D)$. Then*

$$\int_D \eta^2 |\nabla v|^2 d\mu \le c_1 \Big(\frac{2k}{2k-1}\Big)^2 \int_D v^2 |\nabla \eta|^2 d\mu.$$

Now let $u > 0$ be harmonic in $B(x_0,R)$, and γ be as above.

PROPOSITION 4.3. *Let v be either u or u^{-1}. There exists c_1, $\delta > 0$ such that if $0 < q < 2$, then*

$$\sup_{B(x,\delta R)} v^{2q} \le c_1 V(x,R)^{-1} \int_{B(x,R)} (R^\beta |\nabla v^q|^2 + v^{2q}) d\mu. \tag{4.8}$$

PROOF. If $R < 1$ this follows from the local Harnack inequality, so suppose $R \ge c$. Let φ, γ be as above. Let $h_n = 1 - 2^{-n}$, $0 \le n \le \infty$, so that $0 = h_0 < h_\infty = 1$. For $k \ge 0$ set

$$\varphi_k(x) = (\varphi(x) - h_k)^+, \qquad A_k = \{x : \varphi(x) > h_k\},$$

and note that $B(x,R/2) \subset A_n \subset A_0 \subset B(x,R)$ for every $n \ge 0$. We therefore have, writing $V = V(x,R)$,

$$c_2 V \le \mu(A_k) \le V, \quad k \ge 0.$$

The Hölder condition on φ given by CS(Ψ) implies that if $x \in A_{k+1}$ and $y \in A_k^c$, then $d(x,y) \ge c_3 2^{-k/\theta} R$. Set $s_k = \frac{1}{2} c_3 2^{-k/\theta} R$, and note that $\varphi_k > c_4 2^{-k}$ on $A_{k+1}^{(s_k)}$. Let $\{B_i\}$ be a cover of A_{k+1} by balls of radius $s_k/2$, and let $J_{k+1} = \cup B_i$. Write $J'_{k+1} = J_{k+1}^{(s_k/2)}$ $A'_{k+1} = A_{k+1}^{(s_k)}$, and note that $A_{k+1} \subset J_{k+1} \subset J'_{k+1} \subset A'_{k+1} \subset A_k$.

From Proposition 4.1 with $f = v^p$ and s replaced by $s_k/2$,

$$\begin{aligned}
\Big(V^{-1}\int_{A_{k+1}} f^{2\kappa} d\gamma\Big)^{1/\kappa} &\le \Big(V^{-1}\int_{J_{k+1}} f^{2\kappa} d\gamma\Big)^{1/\kappa} \\
&\le c_5 V^{-1}\Big[R^\beta \int_{J'_{k+1}} |\nabla f|^2 d\mu + (R/s_k)^{2\theta} \int_{J'_{k+1}} f^2 d\gamma\Big] \\
&\le c_6 V^{-1}\Big[R^\beta \int_{A'_{k+1}} |\nabla f|^2 d\mu + 2^{2k} \int_{A_k} f^2 d\gamma\Big].
\end{aligned} \tag{4.9}$$

The first term in (4.9) is controlled by Lemma 4.2:

$$\begin{aligned}R^\beta \int_{A'_{k+1}} |\nabla f|^2 &\le R^\beta (c_7 2^{-k})^{-2} \int_{A'_{k+1}} \varphi_k^2 |\nabla f|^2 \\ &\le c_8 2^{2k} R^\beta \int_{A_k} \varphi_k^2 |\nabla f|^2 \\ &\le c_9 2^{2k} R^\beta \Big(\frac{2p}{2p-1}\Big)^2 \int_{A_k} f^2 |\nabla \varphi_k|^2 \\ &\le c_{10} 2^{2k} \Big(\frac{2p}{2p-1}\Big)^2 \int_{A_k} f^2 d\gamma.\end{aligned}$$

We therefore deduce that

$$\Big(V^{-1} \int_{A_{k+1}} f^{2\kappa} d\gamma\Big)^{1/\kappa} \le c_{11} \Big(\frac{2p}{2p-1}\Big)^2 2^{2k} V^{-1} \int_{A_k} f^2 d\gamma. \tag{4.10}$$

Given this, the usual iteration argument, with $p_n = q_0 \kappa^n$ for an appropriate q_0, leads to

$$\sup_{B(x,r/2)} v \le c \Big(V^{-1} \int_{B(x,r)} v^{2q_0} d\gamma\Big)^{1/(2q_0)}. \tag{4.11}$$

(4.8) now follows using Hölder's inequality and CS(Ψ) – see [**9**] for details. □

While the right hand side of (4.8) is a little different from that in (4.6), one can still use the ideas of [**13**] and [**51**] to complete the proof of the EHI – see [**9**].

THEOREM 4.4. *Let M satisfy VD, PI(Ψ) and CS(Ψ). Then M satisfies EHI.*

It is possible that the same arguments could also be used to prove PHI(Ψ) directly. But, in view of Theorem 2.8, and Lemma 2.4 the EHI is enough: VD plus EHI plus PI(Ψ) implies RES(Ψ), and hence M satisfies PHI(Ψ).

5. Stability under rough isometries.

We will need to consider two types of space: weighted graphs, and manifolds.

DEFINITION 5.1. Let (G, E) be an infinite locally finite connected graph. Define edge weights (conductances) $\nu_{xy} = \nu_{yx} \ge 0$, $x, y \in G$, and assume that ν is adapted to the graph structure by requiring that $\nu_{xy} > 0$ if and only if $x \sim y$. Let $\nu_x = \sum_y \nu_{xy}$, and extend to a measure ν on G. We call (G, ν) a *weighted graph*.

We write $d(x, y)$ for the graph distance, and define the balls

$$B(x, r) = \{y : d(x, y) < r\}.$$

Given $A \subset G$ write $\partial A = \{y \in A^c : d(x, y) = 1 \text{ for some } x \in A\}$ for the exterior boundary of A, and let $\overline{A} = A \cup \partial A$.

DEFINITION 5.2. A weighted graph (G, ν) has *controlled weights* if there exists $p_0 > 0$ such that for all $x \in G$

$$\frac{\nu_{xy}}{\nu_x} \ge p_0, \qquad x \sim y. \tag{5.1}$$

This was called the p_0-condition in [**25**].

The Laplacian is defined on (G,ν) by

$$\Delta f(x) = \frac{1}{\nu_x} \sum_y \nu_{xy}(f(y) - f(x)).$$

We also define a Dirichlet form $(\mathcal{E},\mathcal{F})$ by taking $\mathcal{F} = L^2(G,\nu)$, and

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))(g(x) - g(y))\nu_{xy}, \quad f,g \in \mathcal{F}.$$

If $f \in \mathcal{F}$ we define the measure $\Gamma(f,f)$ on G by setting

$$\Gamma(f,f)(x) = \sum_{y\sim x} (f(x) - f(y))^2 \nu_{xy}. \tag{5.2}$$

The conditions VD, EHI and PHI(Ψ) for graphs are defined in exactly the same way as for manifolds. The definitions of PI(Ψ) and RES(Ψ) are also the same if we replace $|\nabla f|^2 d\mu$ by $d\Gamma(f,f)$ in (2.2) and (2.4). For the bound HK(Ψ) we only require (2.7). The condition CS(Ψ) is also the same; the weighted Sobolev inequality (2.8) takes the form

$$\begin{aligned} &\sum_{x\in B(x_1,s)} f(x)^2 \Gamma(\varphi,\varphi)(x) \\ &\qquad \le c_2 \Big(\frac{s}{R}\Big)^{2\theta} \Big(\sum_{x\in B(x_1,2s)} \Gamma(f,f)(x) + s^{-\beta} \sum_{x\in B(x_1,2s)} \nu_x f(x)^2 \Big). \end{aligned} \tag{5.3}$$

Definition 5.3. Let (X_i, d_i, μ_i), $i = 1,2$ be complete measure metric spaces; that is each (X_i, d_i) is a complete metric space and μ_i is a measure such that $\mu_i(B) \in (0,\infty)$ for each ball B in X_i. A map $\varphi : X_1 \to X_2$ is a *rough isometry* if there exist constants $C_1 - C_4$ such that

$$C_1^{-1}(d_1(x,y) - C_2) \le d_2(\varphi(x),\varphi(y)) \le C_1(d_1(x,y) + C_2), \tag{5.4}$$

$$\bigcup_{x\in X_1} B_{d_2}(\varphi(x), C_2) = X_2, \tag{5.5}$$

$$C_3^{-1}\mu_1(B_1(x,C_2)) \le \mu_2(B_{d_2}(\varphi(x),C_2)) \le C_3\mu_1(B_1(x,C_2)). \tag{5.6}$$

If there exists a rough isometry between two spaces they are said to be *roughly isometric*. (One can check this is an equivalence relation.)

This concept was introduced (for manifolds) by Kanai in [**39**]–[**40**], but without the condition (5.6). In those papers both manifolds were assumed to have Ricci curvature bounded below and positive injectivity radius; this leads to volume bounds which imply (5.6) – see p. 394 of [**40**].

A rough isometry between X_1 and X_2 means that the global structure of the two spaces is the same. For example, it is easy to prove that VD is stable under rough isometries. However, to have stability of Harnack inequalities, we also require some control over the local structure. In the case of graphs it is enough to have controlled weights, but for general measure metric spaces more regularity is needed.

Our main stability result is the following.

THEOREM 5.4. *For $i = 1, 2$ let X_i be either a manifold satisfying VD$_{loc}$ and PI$_{loc}$ or a graph with controlled weights. Suppose there exists a rough isometry $\varphi : X_1 \to X_2$.*
(a) If X_1 satisfies VD and PI(Ψ) then X_2 satisfies VD and PI(Ψ).
(b) If X_1 satisfies VD and CS(Ψ) then X_2 satisfies VD and CS(Ψ).

This result is proved in the graph case in [**29**]. Since rough isometry is an equivalence relation, to prove Theorem 5.4 it is enough to prove that if M is a manifold (satisfying VD$_{\text{loc}}$ and PI$_{\text{loc}}$) and G is a graph constructed by taking an appropriate net of M, (so that M and G are roughly isometric), then CS(Ψ) (resp. PI(Ψ)) holds for M if and only if it holds for G.

We will only prove (b) here. But note that since balls in the two metrics are deformations of each other, the initial argument for (a) only gives stability of weak Poincaré inequalities. An argument such as that in Jerison [**37**] (see also [**30**] for a more general formulation) is then needed to derive the (strong) Poincaré inequality from VD and the weak PI.

Let M be a manifold satisfying VD$_{\text{loc}}$. Let $G \subset M$ be a maximal set such that

$$d(x, y) \geq 1 \text{ for } x, y \in G, x \neq y.$$

Thus $B(x, \frac{1}{2})$, $x \in G$ are disjoint, and $\cup_{x \in G} B(x, 1) = M$. Give G a graph structure by letting $x \sim y$ if $d(x, y) \leq 3$. Let d_G be the usual graph distance on G, and write $B_G(x, r)$ for balls in G. It is straightforward to check that G is connected, and that

$$\frac{1}{3} d(x, y) \leq d_G(x, y) \leq d(x, y) + 1, \quad x, y \in G. \tag{5.7}$$

Since M satisfies VD$_{\text{loc}}$ we have, as in Lemma 2.3 of [**39**], that the vertex degree in G is uniformly bounded.

For each $x \sim y$ in G let z_{xy} be the midpoint of a geodesic connecting x and y, and $A_{xy} = B(z_{xy}, 5/2)$, so that $B(x, 1) \subset A_{xy} \subset B(x, 4)$. Let $\nu_{xy} = 0$ if $x \not\sim y$, and if $x \sim y$ let

$$\nu_{xy} = \mu(A_{xy}).$$

As usual we set $\nu_x = \sum_{y \sim x} \nu_{xy}$. Write $A_x = \cup_{y \sim x} A_{xy}$. Since M satisfies VD$_{\text{loc}}$, we have

$$\mu(B(x, 1)) \leq \nu_x \leq c\mu(A_x) \leq c\mu(B(x, 4)) \leq c' \mu(B(x, 1)), \tag{5.8}$$

and using (2.1) it is easy to verify that (G, ν) has controlled weights.

Define $\imath : G \to M$ by $\imath(x) = x$. We have

PROPOSITION 5.5. *Let M be a manifold satisfying VD$_{loc}$. Then the weighted graph (G, ν) has controlled weights and $\imath$ is a rough isometry.*

To prove Theorem 5.4 we will need to transfer functions between $C(G, \mathbb{R}_+)$ and $C(M, \mathbb{R}_+)$. Let $f \in C(M, \mathbb{R}_+)$. Define

$$\widehat{f}(x) = \mu(B(x, 1))^{-1} \int_{B(x,1)} f d\mu, \quad x \in G. \tag{5.9}$$

The transfer in the other direction requires a bit more care. Using the fact that M satisfies VD$_{\text{loc}}$ we can find a partition of unity (ψ_x), $x \in G$, with the following properties:

(i) $\psi_x(z) = 1$ for $z \in B(x, \frac{1}{4})$,
(ii) $\psi_x(z) = 0$ for $z \in B(x, \frac{3}{2})^c$,

(iii) $\psi_x \in C^\infty(M)$ and $|\nabla \psi_x| \le C_1$ for each $x \in G$.

Now if $g : G \to \mathbb{R}_+$ define $\widetilde{g} \in C^\infty(M)$ by

$$\widetilde{g}(z) = \sum_{x\in G} g(x)\psi_x(z), \quad z \in M. \tag{5.10}$$

Set also, if $f : G \to \mathbb{R}$, $k \in \mathbb{N}$,

$$V_k f(x) = \sup_{z:d_G(z,x)\le k} |f(x) - f(z)|, \quad x \in G.$$

LEMMA 5.6. *Let M satisfy VD_{loc}. Let $f : G \to \mathbb{R}_+$, and $x \in G$.*
(a) If $\psi_z(w) > 0$ for some $w \in B(x_0, 1)$ then $d(x, z) < 3$ and $x \sim z$.
(b) If $\psi_z(w) > 0$ for some $w \in A_x$ then $d_G(x, z) \le 4$, and

$$|f(x) - \widetilde{f}(w)| \le V_4 f(x), \quad w \in A_x.$$

(c) Let $A \subset G$, and $A' = \{y : d_G(y, A) \le 4\}$. Then

$$\sum_{z\in A} V_4 f(z)^2 \nu_z \le c \sum_{y,z\in A'} (f(y) - f(z))^2 \nu_{yz}.$$

(d)

$$\int_{B(x,r)} \widetilde{f}(w)^2 d\mu(w) \le \sum_{y\in G\cap B(x,r+2)} f(y)^2 \nu_y.$$

(e) On $B(x, 1)$, $|\nabla \widetilde{f}|^2 \le c_1 V_1 f(x)^2$.

PROOF. (a) If $w \in B(x, 1)$ and $\psi_z(w) > 0$ then $d(x, z) < 1 + \frac{3}{2} < 3$, so $x \sim z$.
(b) Suppose $w \in A_{xy}$ and $\psi_z(w) > 0$. Then $w \in B(x', 1)$ for some $x' \in G$, and $d(x, x') < 5$. Then $x' \sim z$ and it is easy to check that $d_G(x, x') \le 3$. Since

$$\widetilde{f}(w) = f(x) + \sum_z (f(z) - f(x))\psi_z(w),$$

and only those z with $d(z, x) \le 4$ contribute to the sum, the second part is immediate.
(c) For $x \in G$ we can, by (b), find a path $y_i(x)$, $0 \le i \le k(x) \le 4$, such that $x = y_0(x)$, and $V_4 f(x) = |f(x) - f(y_{k(x)})(x)|$. Then

$$\sum_{x\in A} V_4 f(x)^2 \nu_x \le 4 \sum_{x\in A} \sum_{i=1}^{k(x)} (f(y_i(x)) - f(y_{i-1}(x)))^2 \nu_x,$$

and using the fact that (G, ν) has controlled weights, (c) now follows.
(d) Since $\widetilde{f}(w)^2 \le \sum_z f(z)^2 \psi_z(w)$,

$$\begin{aligned}\int_{B(x_0,r)} \widetilde{f}(w)^2 d\mu &\le \sum_z f(z)^2 \int_B \psi_z(w) d\mu \\ &\le c \sum_{z\in G\cap B(x_0,r+2)} f(z)^2 \nu_z.\end{aligned}$$

(e) By (a) we can write, for $w \in B(x, 1)$,

$$\widetilde{f}(w) = \widetilde{f}(x) + \sum_{z\sim x} \psi_z(w)(f(z) - f(x)).$$

Hence

$$\begin{aligned} |\nabla \widetilde{f}|^2 &= \sum_{z\sim x}\sum_{z'\sim x}(f(z)-f(x))(f(z')-f(x))(\nabla\psi_z.\nabla\psi_{z'}) \\ &\le \sum_{z\sim x}(f(z)-f(x))^2|\nabla\psi_z|^2 \\ &\le C_1^2\sum_{z\sim x}V_1 f(x)^2 \le c' V_1 f(x)^2. \end{aligned}$$

□

LEMMA 5.7. *Let M satisfy VD_{loc} and PI_{loc}. Let $g : X \to \mathbb{R}_+$, $x \in G$, and $y \sim x$. Then*

$$(\widehat{g}(x)-\widehat{g}(y))^2\nu_{xy} \le c\int_{A_{xy}}|\nabla g|^2 d\mu.$$

PROOF. Write

$$\overline{g} = \mu(A_{xy})^{-1}\int_{A_{xy}} g(w)d\mu(w).$$

Then we have $(\widehat{g}(x)-\widehat{g}(y))^2 \le 2(\widehat{g}(x)-\overline{g})^2 + 2(\widehat{g}(y)-\overline{g})^2$. It is enough to bound the first term:

$$\begin{aligned} (\widehat{g}(x)-\overline{g})^2\nu_{xy} &= \frac{\mu(A_{xy})}{\mu(B(y,1))}\int_{B(x,1)}(\widehat{g}(x)-\overline{g})^2 d\mu \\ &\le c\int_{B(x,1)}(g(w)-\overline{g})^2 d\mu(w) \\ &\le c\int_{A_{xy}}(g(w)-\overline{g})^2 d\mu(w) \le c\int_{A_{xy}}|\nabla g|^2 d\mu, \end{aligned}$$

where we used $\mathrm{PI}_{\mathrm{loc}}$ in the final line. □

In the arguments that follow, we will use the fact, given in Remarks 2.10, that to verify CS(Ψ) it is enough to do so for any $\delta > 0$ in (2.9) and $\lambda > 0$ in (2.10).

PROPOSITION 5.8. *Let M satisfy VD_{loc} and PI_{loc}. Suppose that M satisfies VD and CS(Ψ). Then (G,ν) satisfies VD and CS(Ψ).*

PROOF. Let $B_G(x_0,R)$ be a ball in G; we need to construct a cutoff function $\widehat{\varphi}$ satisfying (a)–(d) of Definition 2.9. If $R \le c$ then it is easy to check that we can take $\widehat{\varphi}(x)$ be the indicator of $B_G(x_0,R/2)$.

So assume $R > c$. We can find a constant c_1 such that

$$B_G(x_0,c_1R) \subset G\cap B(x_0,R/8-6) \subset G\cap B(x_0,R/4+6) \subset B_G(x_0,R).$$

It is enough to construct a cutoff function $\widehat{\varphi}$ for $B_G(x_0,c_1R) \subset B_G(x_0,R)$. Let φ be a cutoff function for $B(x_0,R/8) \subset B(x_0,R/4)$, and let $\widehat{\varphi}$ be given by (5.9). Properties (a)–(c) of Definition 2.9 are easily checked, and it remains to verify the weighted Sobolev inequality (5.3).

Let $x_1 \in G$, $1 \le s \le R$, and $A_G = B_G(x_1,s)$. Choose c_2, c_3 so that

$$A_G \subset B(x_1,c_2s-6)\cap G \subset B(x_1,2c_2s)\cap G \subset B_G(x_1,c_3s-6).$$

Write $A'_G = B(x_1,c_3s)$, and let $f : A'_G \to \mathbb{R}_+$. We extend f to G by taking f to be zero outside A'_G, and define $\widetilde{f}$ by (5.10).

Let $x \in G$, and $y \sim x$. Then by Lemma 5.6(b) and Lemma 5.7

$$\begin{aligned} f(x)^2(\widehat{\varphi}(x)-\widehat{\varphi}(y))^2\nu_{xy} &\le c\int_{A_{xy}} f(x)^2|\nabla\varphi|^2 d\mu(w) \\ &\le 2c\int_{A_{xy}} \widetilde{f}(w)^2|\nabla\varphi|^2 d\mu(w) + 2c\int_{A_{xy}} V_4 f(x)^2|\nabla\varphi|^2 d\mu(w). \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{x\in A_G}\sum_{y\sim x} f(x)^2(\widehat{\varphi}(x)-\widehat{\varphi}(y))^2\nu_{xy} \\ &\quad\le c\sum_{x\in A_G}\sum_{y\sim x}\int_{A_{xy}} \widetilde{f}(w)^2|\nabla\varphi|^2 d\mu(w) + c\sum_{x\in A_G}\sum_{y\sim x}\int_{A_{xy}} V_4 f(x)^2|\nabla\varphi|^2 d\mu(w) \\ &\quad\le c\int_{B(x_1,c_2 s)} \widetilde{f}(w)^2|\nabla\varphi|^2 d\mu(w) + c\sum_{x\in A_G}\sum_{y\sim x} V_4 f(x)^2 \int_{A_{xy}} |\nabla\varphi|^2 d\mu(w). \end{aligned} \tag{5.11}$$

Applying (2.8) to the ball A_{xy} gives

$$\int_{A_{xy}} |\nabla\varphi|^2 d\mu \le cR^{-2\theta}\mu(B(z_{xy},5)) \le c'R^{-2\theta}\nu_{xy}. \tag{5.12}$$

Therefore, using Lemma 5.6(c), the second term in the final line of (5.11) is bounded by

$$cR^{-2\theta}\sum_{x\in A_G}\sum_{y\sim x} V_4 f(x)^2\nu_{xy} \le cR^{-2\theta}\sum_{x\in A'_G}\Gamma(f,f)(x). \tag{5.13}$$

Using (2.8) again

$$\begin{aligned} &\int_{B(x_1,c_2 s)} \widetilde{f}(w)^2|\nabla\varphi|^2 d\mu(w) \\ &\qquad\le c(s/R)^{2\theta}\Big(\int_{B(x_1,2c_2 s)} |\nabla\widetilde{f}|^2 + \Psi(s)^{-1}\int_{B(x_1,2c_2 s)} \widetilde{f}^2 d\mu\Big). \end{aligned} \tag{5.14}$$

By Lemma 5.6(e)

$$\begin{aligned} \int_{B(x_1,2c_2 s)} |\nabla\widetilde{f}|^2 &\le \sum_{x\in G\cap B(x_1,2c_2 s+1)} V_1 f(x)^2\mu(B(x,1)) \\ &\le \sum_{x,y\in A'_G} (f(x)-f(y))^2\nu_{xy}, \end{aligned} \tag{5.15}$$

while by Lemma 5.6(d)

$$\int_{B(x_1,2c_2 s)} \widetilde{f}(w)^2 d\mu(w) \le \sum_{x\in B_G(x_1,c_3 s)} f(x)^2\nu_x. \tag{5.16}$$

Combining the estimates (5.11)–(5.16) completes the proof. □

PROPOSITION 5.9. *Let M satisfy VD_{loc} and PI_{loc}. Suppose that (G,ν) satisfies VD and CS(Ψ). Then M satisfies VD and CS(Ψ).*

PROOF. Let $B = B(x_0, R)$ be a ball in M. If $R \le c_1$ then we can use the local regularity to construct a cutoff function φ for B. So assume $R \ge c_1$. We can therefore assume that $x_0 \in G$.

Given $A \subset G$ write $A^{(1)} = \cup_{x\in A} B(x, 1)$. We can find c_i such that

$$B(x_0, c_1 R) \subset B_G(x_0, c_2 R - 6)^{(1)} \subset B_G(x_0, 2c_2 R + 6)^{(1)} \subset B(x_0, R).$$

Let φ_G be a cutoff function for $B_G(x_0, c_2 R) \subset B_G(x_0, 2c_2 R)$, and let

$$\varphi(w) = \widetilde{\varphi}_G(w) = \sum_{z\in G} f(x)\psi_z(w).$$

Properties (a)–(c) of φ follow easily from those of φ_G, and it remains to verify (2.8).

Let $B_1 = B(x_1, s)$ with $s \in (0, R)$. If $s \le c_3$ then, as $V_1\varphi(x) \le cR^{-2\theta}$,

$$\int_{B(x_1,s)} g^2 |\nabla\varphi|^2 d\mu \le cR^{-2\theta} \int_{B(x_1,s)} g^2 d\mu.$$

Now suppose $s \ge c_3$. Then we can assume $x_1 \in G$, and there exist c_i so that

$$B(x_1, s) \subset B_G(x_1, c_4 s - 6)^{(1)} \subset B_G(x_1, 2c_4 s + 6)^{(1)} \subset B(x_1, c_5 s - 6).$$

Let $g : B(x_1, c_5 s) \to \mathbb{R}_+$. Define $\widehat{g}$ on $B_G(x_1, 2c_4 s + 6)$ by (5.9). Then

$$\begin{aligned}
\int_{B_1} g^2 |\nabla\varphi|^2 d\mu &\le \sum_{x\in B_G(x_1,c_4 s)} \int_{B(x,1)} g(w)^2 |\nabla\varphi|^2 d\mu(w) \\
&\le 2 \sum_{x\in B_G(x_1,c_4 s)} \int_{B(x,1)} (g(w) - \widehat{g}(x))^2 |\nabla\varphi|^2 d\mu(w) \qquad (5.17)\\
&\quad + 2 \sum_{x\in B_G(x_1,c_4 s)} \int_{B(x,1)} \widehat{g}(x)^2 |\nabla\varphi|^2 d\mu(w).
\end{aligned}$$

By Lemma 5.6(e) the first term above is bounded by

$$cR^{-2\theta} \sum_{x\in B_G(x_1,c_4 s)} \int_{B(x,1)} (g(w) - \widehat{g}(x))^2 d\mu,$$

and using $\mathrm{PI}_{\mathrm{loc}}$ this is bounded by

$$cR^{-2\theta} \sum_{x\in B_G(x_1,c_4 s)} \int_{B(x,1)} |\nabla g|^2 d\mu \le c' R^{-2\theta} \int_{B(x_1,c_5 s)} |\nabla g|^2 d\mu. \qquad (5.18)$$

For the final term in (5.17), by Lemma 5.6(e) and (2.8) for φ_G,

$$\begin{aligned}
&\sum_{x\in B_G(x_1,c_4 s)} \widehat{g}(x)^2 \int_{B(x,1)} |\nabla\varphi|^2 d\mu(w) \\
&\qquad \le \sum_{x\in B_G(x_1,c_4 s)} \widehat{g}(x)^2 V_1\varphi_G(x)^2 \mu(B(x,1)) \\
&\qquad \le c \sum_{x\in B_G(x_1,c_4 s)} \widehat{g}(x)^2 \Gamma(\varphi_G, \varphi_G)(x). \\
&\qquad \le c(s/R)^{2\theta} \Big(\sum_{x\in B_G(x_1,2c_4 s)} \Gamma(\widehat{g}, \widehat{g})(x) + \Psi(s)^{-1} \sum_{x\in B_G(x_1,2c_4 s)} \widehat{g}(x)^2 \nu_x \Big)
\end{aligned}$$

Using Lemma 5.7 for the first term, and an easy bound for the second, (2.8) now follows. $\square$

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