

Discreteness of minimal models of Kodaira dimension zero and subvarieties of moduli stacks

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1. Introduction

Let $f : V \rightarrow U$ be a smooth projective morphism with connected fibres over a complex quasi-projective manifold U .

DEFINITION 1.1.

- i. $\text{Var}(f)$ is the smallest integer η for which there exists a finitely generated subfield K of $\overline{\mathbb{C}(U)}$ of transcendence degree η over \mathbb{C} , a variety F' defined over K , and a birational equivalence

$$V \times_U \text{Spec}(\overline{\mathbb{C}(U)}) \sim F' \times_{\text{Spec}(K)} \text{Spec}(\overline{\mathbb{C}(U)}).$$

- ii. $f : V \rightarrow U$ is birationally isotrivial if $\text{Var}(f) = 0$, hence if there exists some generically finite covering $U' \rightarrow U$, a projective manifold F' , and a birational map

$$V \times_U U' \sim U' \times F'.$$

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- iii. $f : V \rightarrow U$ is (biregularly) isotrivial if there exists a generically finite covering $U' \rightarrow U$, a projective manifold F and an isomorphism

$$V \times_U U' \simeq U' \times F.$$

So the variation of a morphisms counts the number of parameters controlling the birational structure of the fibres of F .

Maehara has shown in [6] that under the assumption that ω_F is semi-ample and big for a general fibre F of f , a family is birationally isotrivial, if and only if it is biregularly isotrivial. In different terms, for families of minimal models of complex manifolds of general type, $\text{Var}(f)$ measures the number of directions where the structure of F varies.

We will slightly extend the methods used to prove [8], Theorem 6.24, to show that for families with $\omega_F^\delta = \mathcal{O}_F$, for some $\delta > 0$, the same holds true. We will show that for a given projective manifold F' the set of minimal models is discrete, hence that there are no non-trivial families of minimal models.

Let us fix some polarization \mathcal{L} of $f : V \rightarrow U$, with Hilbert polynomial h . If $\omega_{V/U}$ is f -ample we will choose $\mathcal{L} = \omega_{V/U}^\rho$, for some $\rho > 0$. By [8] there exists a quasi-projective moduli schemes M_h , parameterizing polarized manifolds (F, \mathcal{L}) with ω_F semiample and with $h(\nu) = \chi(\mathcal{L}^\nu)$. The family $f : V \rightarrow U$ together with \mathcal{L} induces a map

$$\varphi : U \rightarrow M_h$$

Since we require $\varphi : U \rightarrow M_h$ to be induced by a family it factors through the moduli stack \mathcal{M}_h .

THEOREM 1.2. *Let $f : V \rightarrow U$ be a family of polarized manifolds. Assume that $\omega_F^\delta = \mathcal{O}_F$, for some $\delta > 0$ (or that all fibres F of f are canonically polarized). Let $\varphi : U \rightarrow M_h$ be the induced morphism to the moduli scheme. Then $\text{Var}(f) = \dim(\varphi(U))$.*

If in Theorem 1.2 the morphism $f : V \rightarrow U$ is birationally isotrivial, $\varphi(U)$ must be zero dimensional, hence f is biregularly isotrivial.

It seems reasonable to conjecture that Theorem 1.2 remains true for families of polarized manifolds with $\omega_{X/Y}$ f -semiample. Here however one should replace the moduli scheme M_h by the moduli scheme P_h of polarized manifolds, up to numerical equivalence (see [8]).

PROBLEM 1.3. Let $f : V \rightarrow U$ be a family of polarized manifolds with $\omega_{V/U}$ f -semiample, and let $\psi : U \rightarrow P_h$ be the morphism to the moduli scheme of polarized manifolds up to numerical equivalence. Does this imply $\text{Var}(f) = \dim(\psi(U))$?

Theorem 1.2 allows to extends some of the results obtained in [10] for canonically polarized manifolds to families of manifolds F with $\omega_F^\delta = \mathcal{O}_F$ (see Theorem 6.3, i) and ii) and Section 7). This is done in the second half of this article, a continuation of [10]. What methods are concerned, the reader familiar with [10] will find nothing new. In fact we just sketch the changes needed to extend some of the results to this case.

In the final Section 8 we will state a criterion for the rigidity of non-isotrivial families over curves, and its translation to curves in the moduli stack of minimal polarized manifolds of Kodaira dimension zero, or of canonically polarized manifolds. This criterion is implicitly used in [10], Proof of 6.4 and 6.5, but it was not explicitly stated there.

A slightly weaker statement (8.2) extends to all families with ω_F semiample. A similar criterion has been shown by S. Kovács and, for families of Calabi-Yau manifolds by K. Liu, A. Todorov, S.-T. Yau and the second named author in [5]. As a corollary one obtains (see 8.4):

COROLLARY 1.4. *Let M_h be either the moduli scheme of canonically polarized manifolds or the moduli scheme of polarized manifolds F with $\omega_F^\delta = \mathcal{O}$ for some $\delta > 0$. There are only finitely many morphisms $\varphi : U \rightarrow M_h$ which are induced by a smooth family $f : V \rightarrow U$ with:*

For a general fibre F of f the n -th wedge product

$$0 \neq \wedge^n \xi \in H^n(F, \omega_F^{-1}),$$

where $\xi \in H^1(F, T_F)$ denotes the Kodaira Spencer class corresponding to the deformation $f : V \rightarrow U$ of F .

The way 1.2 is formulated, it is a trivial statement for canonically polarized manifolds. Maehara showed in [6] that problem 1.3 has an affirmative answer if the fibres of f are of maximal Kodaira dimension. This implies, that geometric properties of submanifolds of moduli stacks M_h of canonically polarized manifolds carry over to those of the moduli scheme P_h of polarized minimal manifolds of general type. To make notations not more complicated than they are already, we leave the necessary changes to the reader.

The first half of this article presents a proof of Theorem 1.2, hopefully of interest independently of the applications to subvarieties of moduli stacks. In the first section, we will show, that the proof of Theorem 1.2 can be reduced to families over a curve. Next we recall and strengthen a Positivity Theorem from [8]. It allows to reprove Maehara’s Result and thereby the first part of 1.2 in section 4. The case of minimal models of Kodaira dimension zero is handled in Section 5.

2. Reduction to families over curves

In Definition 1.1, i), we may choose a finitely generated subfield L of $\overline{\mathbb{C}(U)}$ which contains $\mathbb{C}(U)$ and K . Let U' be the normalization of U in L , and let T be a smooth quasi-projective variety with function field $\mathbb{C}(T) = K$. Replacing T by some open subscheme, we may assume that there exists a smooth projective morphism $g : Z \rightarrow T$ with general fibre F' , and replacing U' by some open subscheme one finds morphisms $\tau : U' \rightarrow U$ and $\pi : U' \rightarrow T$ fitting into a diagram

$$\begin{array}{ccccccc} V & \longleftarrow & V' & \xrightarrow{\sim} & Z' & \longrightarrow & Z \\ f \downarrow & & f' \downarrow & & g' \downarrow & & g \downarrow \\ U & \xleftarrow{\tau} & U' & \xrightarrow{=} & U' & \xrightarrow{\pi} & T, \end{array}$$

where $V' \rightarrow Z'$ is a birational equivalence, and where the right and left hand squares are fibre products. For a point $t \in T$ in general position, one has

$$\dim(\pi^{-1}(t)) = \dim(U') - \dim(T) = \dim(U) - \text{Var}(f).$$

If under the assumptions made in Theorem 1.2 $\text{Var}(f) < \dim(\varphi(U))$, for a point $\eta \in \varphi(U)$ in general position,

$$\dim(\tau^{-1}\varphi^{-1}(\eta)) = \dim(U') - \dim(\varphi(U)) < \dim(\pi^{-1}(t)),$$

hence there exists a curve C in $\pi^{-1}(t)$ with $\tau \circ \varphi|_C$ finite. In order to prove Theorem 1.2 one just has to show that such a curve can not exist. Theorem 1.2 follows from

PROPOSITION 2.1. *Let U be a non-singular irreducible curve and let F' be a projective manifold. Let $f : V \rightarrow U$ be a family of polarized manifolds. Assume that there exists a birational equivalence $V \sim U \times F'$ over U . If either the fibres F of f are canonically polarized, or if $\omega_F^\delta = \mathcal{O}_F$, for some $\delta > 0$, then the induced morphism $\varphi : U \rightarrow M_h$ is constant.*

In particular, $f : V \rightarrow U$ is biregularly isotrivial.

In order to prove Proposition 2.1 we may replace U by a finite covering. Doing so one can assume that $f : V \rightarrow U$ extends to a semistable morphism $f : X \rightarrow Y$ of projective manifolds, hence that $\Delta = f^{-1}(S)$ is a reduced normal crossing divisor, for $S = Y \setminus U$.

Moreover, we may replace the given polarization by some power. In fact, the corresponding map of the moduli schemes M_h is a finite map. This allows to assume that for all fibres F of $V \rightarrow U$ the polarization \mathcal{L} is very ample, and without higher cohomology.

3. Positivity of direct image sheaves

Recall that a locally free sheaf \mathcal{E} on a projective non-singular curve Y is numerically effective (nef), if for all finite morphisms $\tau : Z \rightarrow Y$ and for all invertible quotients \mathcal{L} of $\tau^*(\mathcal{E})$ the degree $\text{deg}(\mathcal{L}) \geq 0$.

Fujita's positivity theorem (today an easy corollary of Kollár's vanishing theorem) says that $f_*\omega_{X/Y}$ is nef. By [9], 2.3, one obtains as a direct consequence.

LEMMA 3.1. *Let $f : X \rightarrow Y$ be a morphism from a normal projective variety X to a curve Y , with connected fibres. Assume that X has at most rational double points as singularities. Let \mathcal{N} be an invertible sheaf on X and Γ an effective divisor. Assume that for some $N > 0$ there exists a nef locally free sheaf \mathcal{E} on Y and a surjection*

$$f^*\mathcal{E} \longrightarrow \mathcal{N}^N(-\Gamma).$$

Then

$$f_* \left(\mathcal{N} \otimes \omega_{X/Y} \left\{ -\frac{\Gamma}{N} \right\} \right)$$

is nef.

Here $\omega_{X/Y} \left\{ -\frac{\Gamma}{N} \right\}$ denotes the (algebraic) multiplier sheaf (see for example [3], 7.4, or [8], section 5.3). If $\tau : X' \rightarrow X$ is any blowing up with $\Gamma' = \tau^*\Gamma$ a normal crossing divisor, then

$$\omega_{X/Y} \left\{ -\frac{\Gamma}{N} \right\} = \tau_* \left(\omega_{X'/Y} \left(-\left[\frac{\Gamma'}{N} \right] \right) \right).$$

As in [3], § 7 and [8], section 5.3, we are mainly interested in the case where the multiplier sheaf on a general fibre F is isomorphic to ω_F . The corresponding threshold is defined for any effective divisor Π or any invertible sheaf \mathcal{L} on F with $H^0(F, \mathcal{L}) \neq 0$.

$$e(\Pi) = \text{Min} \left\{ N \in \mathbb{N} - \{0\}; \omega_F \left\{ -\frac{\Pi}{N} \right\} = \omega_F \right\} \quad \text{and}$$

$$e(\mathcal{L}) = \text{Max} \left\{ e(\Pi); \Pi \text{ the zero set of } \sigma \in H^0(F, \mathcal{L}) - \{0\} \right\}.$$

For smooth morphisms $f : X \rightarrow Y$ and for an f -ample sheaf \mathcal{L} on X we obtained in [8], 6.24 and 7.20, strong positivity theorems. Their proof, in case Y is a curve, can easily be extended to semistable morphisms $f : X \rightarrow Y$.

THEOREM 3.2. *Let Y be a curve, let $f : X \rightarrow Y$ be a semistable morphism between projective manifolds with connected fibres, and let \mathcal{M} be an invertible sheaf on X . Let $U \subset Y$ be an open dense subscheme with $V = f^{-1}(U) \rightarrow U$ smooth. Assume that for all fibres F of $V \rightarrow U$ the canonical sheaf ω_F is semiample, that $\mathcal{M}|_F$ is very ample and without higher cohomology. Then for*

$$e \geq c_1(\mathcal{M}|_F)^{\dim(F)} + 2, \quad r = \text{rank}(f_*\mathcal{M})$$

$$\text{and } r(\nu) = \text{rank}(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) :$$

a. For all $\nu > 0$

$$\left(\bigotimes^r f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu}) \right) \otimes \det(f_*\mathcal{M})^{-\nu}$$

is nef.

b. If the invertible sheaf $\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) \otimes \det(f_*\mathcal{M})^{-\nu r(\nu)}$ is ample for some $\nu > 0$,

$$\left(\bigotimes^r f_*(\mathcal{M} \otimes \omega_{X/Y}^e) \right) \otimes \det(f_*\mathcal{M})^{-1}$$

is ample.

c. If for all $\nu > 0$ the degree of

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) \otimes \det(f_*\mathcal{M})^{-\nu r(\nu)}$$

is zero, then $f : V \rightarrow U$ is biregular isotrivial as a family of polarized manifolds, i.e. there exists some finite covering $U' \rightarrow U$, a projective manifold F' , invertible sheaves \mathcal{L}' on F' and \mathcal{B} on U' , and an isomorphism

$$\pi : V' = X \times_Y U' \rightarrow F' \times U'$$

with

$$pr_1^*\mathcal{M} = \pi^*(pr_1^*\mathcal{L} \otimes pr_2^*\mathcal{B}).$$

PROOF. As indicated already, the proof of parts a) and b) will follow the arguments used in [8], 194–196, to prove 6.20. We just have to take care, that for a semistable family over a curve the sheaves are at most getting larger. So we repeat the arguments.

For a) let us fix some $\nu > 0$. For b) we assume that

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) \otimes \det(f_*\mathcal{M})^{-\nu \cdot r(\nu)}$$

is ample.

The semicontinuity of the threshold, shown in [8], 5.17 for example, allows to find some $\gamma \geq e \cdot \nu$ with

$$(3.2.1) \quad e(\mathcal{M}|_F^{\nu \cdot e} \otimes \omega_F^{e \cdot \nu \cdot (e-1)}) \leq \gamma$$

for all fibres F of $V \rightarrow U$.

For b) we will show that

$$S^\gamma \left(\left(\bigotimes^{r \cdot r(\nu)} f_*(\mathcal{M} \otimes \omega_{X/Y}^e) \right) \otimes \det(f_*\mathcal{M})^{-r(\nu)} \right) \otimes$$

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu}))^{-r \cdot (e-1)} \otimes \det(f_*\mathcal{M})^{\nu \cdot r \cdot (e-1) \cdot r(\nu)}$$

is nef. Hence for both, a) or b), it is sufficient to prove the corresponding statements for the pullback of the sheaves to any finite covering Y' of Y . Since we assumed $X \rightarrow Y$ to be semistable, the fibre product $X' = X \times_Y Y'$ is a normal variety with at most rational double points. Flat base change allows to replace Y by such a covering and $(f : X \rightarrow Y, \mathcal{M})$ by a desingularization of the pullback family.

Doing so, we may assume that $\det(f_*\mathcal{M})$ is the r -th power of an invertible sheaf, and since all the sheaves occurring in a), b) or c) are compatible with changing the polarization by the pullback of an invertible sheaf on Y , we can as well assume that $\det(f_*\mathcal{M}) = \mathcal{O}_Y$. Under this additional assumption we have to verify in a) that

$$f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})$$

is nef. For part b) we may assume in addition that

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu}))^{r \cdot (e-1)} = \mathcal{O}_Y(\gamma \cdot H)$$

for some effective divisor H supported in U . We have to prove that

$$\left(\bigotimes^{\nu \cdot r} f_*(\mathcal{M} \otimes \omega_{X/Y}^e) \right) \otimes \mathcal{O}_Y(-H)$$

is nef.

Let $f^s : X^s \rightarrow Y$ be the s -fold fibre product. X^s is normal with at most rational double points (see [7], page 291, for example). Consider

$$\mathcal{P} = \bigotimes_{i=1}^s pr_i^* \mathcal{M}.$$

By flat base change one obtains

$$f_*^s(\mathcal{P}^\alpha \otimes \omega_{X^s/Y}^\beta) = \bigotimes^s f_*(\mathcal{M}^\alpha \otimes \omega_{X/Y}^\beta)$$

for all α, β . The restriction of $\mathcal{P}^\nu \otimes \omega_{X^s/Y}^{e \cdot \nu - \iota}$ to $f^{s-1}(U) = V^s$ is f^s -ample for all $\iota \leq e \cdot \nu$. Let us write $\epsilon = e \cdot \nu$ or $\epsilon = e \cdot \nu - 1$, where ν may be any positive integer.

If Γ is the zero divisor of a section of \mathcal{P} , which does not contain any fibre F^s of $V^s \rightarrow U$ the compatibility of the threshold with products and its semicontinuity imply (see [8], 5.14 and 5.21)

$$(3.2.2) \quad e(\Gamma|_{F^s}) \leq e(\mathcal{P}|_{F^s}) = e(\mathcal{M}|_F) < e \quad \text{and} \quad e(\Gamma|_{V^s}) < e.$$

In fact, as shown in [8], 5.11), one has

$$e(\mathcal{M}|_F) \leq c_1(\mathcal{M}|_F)^{\dim(F)} + 1.$$

Moreover, by the choice of ϵ

$$(3.2.3) \quad e(\nu \cdot \Gamma|_{F^s}) \leq \nu \cdot e(\Gamma|_{F^s}) \leq \nu \cdot (c_1(\mathcal{M}|_F)^{\dim(F)} + 1) \leq \epsilon.$$

Let \mathcal{H} be an ample invertible sheaf on Y .

CLAIM 3.3. Assume that for some $\rho \geq 0, N > 0, M_0 > 0$ and for all multiples M of M_0 , the sheaf

$$f_*((\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^{M \cdot N}) \otimes \mathcal{H}^{\rho \cdot \epsilon \cdot N \cdot M}$$

is nef. Then

$$f_*((\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^N) \otimes \mathcal{H}^{\rho \cdot (\epsilon \cdot N - 1)}$$

is nef.

PROOF. Let us choose $s = r$. The determinant gives an inclusion

$$\det(f_*\mathcal{M}) = \mathcal{O}_Y \longrightarrow f_*^r \mathcal{P} = \bigotimes^r f_* \mathcal{M},$$

which splits locally. Hence the zero divisor Γ of the induced section of \mathcal{P} does not contain any fibre of $V^r \rightarrow U$. For

$$\mathcal{N} = \mathcal{P}^{\nu \cdot N} \otimes \omega_{X^r/Y}^{\epsilon \cdot N - 1} \otimes f^{r*} \mathcal{H}^{\rho \cdot (\epsilon \cdot N - 1) \cdot r}$$

one obtains that the restriction of

$$\mathcal{N}^\epsilon(-\nu \cdot \Gamma) = (\mathcal{P}^\nu \otimes \omega_{X^r/Y}^\epsilon \otimes f^{r*} \mathcal{H}^{\rho \cdot \epsilon \cdot r})^{(\epsilon \cdot N - 1)}$$

to V^s is f^r -ample. If M' is a positive integer, divisible by $M_0 \cdot N$ the sheaf

$$f_*^r(\mathcal{N}^\epsilon(-\nu \cdot \Gamma)^{M'}) = \bigotimes^r (f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^{(\epsilon \cdot N - 1) \cdot M'} \otimes \mathcal{H}^{\rho \cdot \epsilon \cdot r(\epsilon \cdot N - 1) \cdot M'})$$

is nef. Choose M' such that

$$f^* f_*((\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^{(\epsilon \cdot N - 1) \cdot M'}) \longrightarrow (\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^{(\epsilon \cdot N - 1) \cdot M'}$$

is surjective over U .

3.1 implies that the subsheaf $f_*^r(\mathcal{N} \otimes \omega_{X^r/Y} \{-\frac{\nu \cdot \Gamma}{\epsilon}\})$ of

$$f_*^r(\mathcal{N} \otimes \omega_{X^r/Y}) = \bigotimes^r (f_*(\mathcal{M}^{\nu \cdot N} \otimes \omega_{X/Y}^{\epsilon \cdot N}) \otimes \mathcal{H}^{\rho \cdot (\epsilon \cdot N - 1)})$$

is nef. On the other hand, (3.2.2) and (3.2.3) imply that both sheaves coincide on U . \square

Choose some $N_0 > 0$ such that for all multiples N of N_0 and for all $M > 0$ the multiplication maps

$$m : S^M(f_*(\mathcal{M}^{\nu \cdot N} \otimes \omega_{X/Y}^{\epsilon \cdot N})) \longrightarrow f_*(\mathcal{M}^{\nu \cdot N \cdot M} \otimes \omega_{X/Y}^{\epsilon \cdot N \cdot M})$$

are surjective over U . Define

$$\rho = \text{Min}\{\mu > 0; f_*(\mathcal{M}^{\nu \cdot N} \otimes \omega_{X/Y}^{\epsilon \cdot N}) \otimes \mathcal{H}^{\mu \cdot \epsilon \cdot N} \text{ is nef}\}.$$

The surjectivity of m implies that

$$f_*(\mathcal{M}^{\nu \cdot N \cdot M} \otimes \omega_{X/Y}^{\epsilon \cdot N \cdot M}) \otimes \mathcal{H}^{\rho \cdot \epsilon \cdot N \cdot M}$$

is nef for all $M > 0$. By 3.3

$$f_*(\mathcal{M}^{\nu \cdot N} \otimes \omega_{X/Y}^{\epsilon \cdot N}) \otimes \mathcal{H}^{\rho \cdot (\epsilon \cdot N - 1)}.$$

is nef, hence by the choice of ρ

$$(\rho - 1) \cdot \epsilon \cdot N < \rho \cdot (\epsilon \cdot N - 1)$$

or equivalently $\rho < \epsilon \cdot N$. Then

$$f_*(\mathcal{M}^N \otimes \omega_{X/Y}^{\epsilon \cdot N}) \otimes \mathcal{H}^{\epsilon^2 \cdot N^2}$$

is nef. This remains true if one replaces Y by any finite covering, and by [9], 2.2, one obtains that $f_*(\mathcal{M}^N \otimes \omega_{X/Y}^{\epsilon \cdot N})$ is nef. Applying 3.3 a second time, for the numbers (N', N_0) instead of (N, M_0) and for $\rho = 0$, one finds

CLAIM 3.4. For $\nu > 0$ and $\epsilon = e \cdot \nu$ or $\epsilon = e \cdot \nu - 1$ and for all $N' > 0$ the sheaf

$$f_*((\mathcal{M}^\nu \otimes \omega_{X/Y}^\epsilon)^{N'})$$

is nef.

In particular, choosing $N' = 1$ and $\epsilon = \nu e$ one obtains a).

For b) we consider the s -fold product $f^s : X^s \rightarrow Y$ for $s = r \cdot r(\nu)$. One has natural inclusions, splitting locally,

$$\mathcal{O}_Y = \det(f_*\mathcal{M})^{r(\nu)} \longrightarrow f_*^s \mathcal{P} = \bigotimes^s f_*\mathcal{M}$$

and

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X^s/Y}^{e,\nu}))^r \longrightarrow f_*^s(\mathcal{P}^\nu \otimes \omega_{X^s/Y}^{e,\nu}) = \bigotimes^s f_*(\mathcal{M}^\nu \otimes \omega_{X^s/Y}^{e,\nu}).$$

If Δ_1 and Δ_2 denote the corresponding zero-divisors on X^s then $\Delta_1 + \Delta_2$ does not contain any fibre of $V^s \rightarrow U$. Then

$$\mathcal{P}^{e,\nu} \otimes \omega_{X^s/Y}^{e,\nu \cdot (e-1)} = f^{s*} \det(f_*(\mathcal{M}^\nu \otimes \omega_{X^s/Y}^{e,\nu}))^{r \cdot (e-1)} \otimes \mathcal{O}_{X^s}((e-1) \cdot \Delta_2 + \nu \cdot \Delta_1),$$

and

$$\mathcal{P}^\gamma \otimes \omega_{X^s/Y}^{\gamma \cdot (e-1)} = (\mathcal{P} \otimes \omega_{X^s/Y}^{e-1})^{\gamma - \nu e} \otimes \mathcal{O}_X(\gamma \cdot f^{s*}H + (e-1) \cdot \Delta_2 + \nu \cdot \Delta_1).$$

By 3.4 the sheaf

$$f_*^s((\mathcal{P} \otimes \omega_{X^s/Y}^{e-1})^{\gamma - \nu \cdot e})^M = \bigotimes^s f_*((\mathcal{P} \otimes \omega_{X^s/Y}^{e-1})^{\gamma - \nu \cdot e})^M$$

is nef for all $M > 0$. 3.1 implies that

$$\mathcal{P} \otimes \omega_{X^s/Y}^{e-1} \otimes \omega_{X^s/Y} \left\{ \frac{-\gamma \cdot f^{s*}H + (e-1) \cdot \Delta_2 + \nu \cdot \Delta_1}{\gamma} \right\}$$

is nef. By (3.2.2) and (3.2.1)

$$e(((e-1) \cdot \Delta_2 + \nu \cdot \Delta_1)|_{F^s}) \leq e(\mathcal{P}|_{F^s}^{\nu \cdot e} \otimes \omega_{F^s}^{e,\nu \cdot (e-1)}) = e(\mathcal{M}|_F^{\nu \cdot e} \otimes \omega_F^{e,\nu \cdot (e-1)}) \leq \gamma$$

for all fibres F of $V \rightarrow U$. Hence the cokernel of

$$\omega_{X^s/Y} \left\{ \frac{-\gamma \cdot f^{s*}H + (e-1) \cdot \Delta_2 + \nu \cdot \Delta_1}{\gamma} \right\} \rightarrow \omega_{X^s/Y}(-f^{s*}H)$$

lies in $X^s \setminus V^s$, and thereby

$$f_*^s(\mathcal{P} \otimes \omega_{X^s/Y}^e) \otimes \mathcal{O}_Y(-H) = \left(\bigotimes^{r \cdot r(\nu)} f_*(\mathcal{M} \otimes \omega_{X/Y}^e) \right) \otimes \mathcal{O}_Y(-H)$$

is nef.

Part c) follows from part a) and Kollar's ampleness criterion (see [8], 4.34). Again we may assume that $\det(f_*\mathcal{M}) = \mathcal{O}_Y$. By part a) for all $\eta > 0$ the sheaf $\mathcal{E} = f_*(\mathcal{M}^\eta \otimes \omega_{X^s/Y}^{e,\eta})$ is nef. Choose $\nu > 0$ such that the multiplication map

$$\mu : S^\nu(\mathcal{E}^\eta) \rightarrow f_*(\mathcal{M}^{\nu \cdot \eta} \otimes \omega_{X^s/Y}^{e,\nu \cdot \eta})$$

is surjective over U . By [8], 4.34, $\det(\text{Im}(\mu))$ is ample, if the kernel \mathcal{K} of the multiplication map is of maximal variation. Let us recall the definition. For a point $y \in U$ choose a local trivialization of \mathcal{E} . Then $\mathcal{K}_y = \mathcal{K} \otimes \mathbb{C}(y)$ as a subvectorspace of $S^\nu(\mathbb{C}^{r(\eta)})$, defines a point $[\mathcal{K}_y]$ in the Grassmann variety

$$\text{Gr} = \text{Grass}(r(\nu \cdot \eta), S^\nu(\mathbb{C}^{r(\eta)})).$$

The group $G = \text{Sl}(r(\eta), \mathbb{C})$ acts on $S^\nu(\mathbb{C}^{r(\eta)})$, hence on Gr . Let G_y denote the orbit of $[\mathcal{K}_y]$. The kernel has maximal variation, if

$$\{z \in Y; G_z = G_y\}$$

is finite, as well as the stabilizer of $[\mathcal{K}_y]$.

The second condition holds true for η and ν sufficiently large. In fact, \mathcal{K}_y determines the fibre $f^{-1}(y)$ as a subvariety of

$$\mathbb{P}(H^0(f^{-1}(y), (\mathcal{M}^\eta \otimes \omega_{X/Y}^{e \cdot \eta})|_{f^{-1}(y)})),$$

and $[\mathcal{K}_y]$ is nothing but the point of the Hilbert scheme Hilb parameterizing subvarieties of this projective space. By [8], 7.2, the stabilizer of such a point is finite.

The assumption in c) implies that \mathcal{K} is not of maximal variation. Hence for all points z in a neighborhood U_y of y , the orbits G_z coincide. In different terms the images of $z \in U_y$ in Hilb all belong to the same G -orbit. Since M_h is a quotient of a subscheme of Hilb by the G -action, the morphism $\varphi : U \rightarrow M_h$ is constant, as claimed in c). □

4. Families of canonically polarized manifolds

For families with ω_F big and semi-ample the equivalence of birational and biregular isotriviality has been shown by Maehara in [6]. For families of canonically polarized manifolds, one just has to use, that the fibres are their own canonical model.

Or, to formulate the proof parallel to the one given below in the Kodaira dimension zero case, one could argue in the following way. Assume that U is a curve, and choose a semistable compactification $f : X \rightarrow Y$ of $V \rightarrow U$. By assumption X is birational to the trivial family $F' \times Y$ over Y , hence $f_*\omega_{X/Y}^\nu$ is a direct sum of copies of \mathcal{O}_Y , for all ν . Obviously, if \mathcal{M} is some power of $\omega_{X/Y}$ this implies that

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) \otimes \det(f_*\mathcal{M})^{-\nu r(\nu)} = \mathcal{O}_Y,$$

and by 3.2, c), one finds $V \rightarrow U$ to be biregularly isotrivial.

5. Families of manifolds of Kodaira dimension zero

Let U be a curve and $f : V \rightarrow U$ be a family of polarized manifolds F with $\omega_F^\delta = \mathcal{O}_F$. In order to prove 2.1 we may replace U by some finite cover, and we may choose a compactification $f : X \rightarrow Y$ satisfying the assumptions made in Theorem 3.2.

By assumption, $\lambda = f_*\omega_{X/Y}^\delta$ is an invertible sheaf, and the natural map $f^*\lambda \rightarrow \omega_{X/Y}^\delta$ is an isomorphism over V . Let E be the zero divisor of this map, i.e.

$$\omega_{X/Y}^\delta = f^*\lambda \otimes \mathcal{O}_X(E).$$

E is supported in $\Delta = X \setminus V$ and, since the fibres of f are reduced divisors, E can not contain a whole fibre. Hence for all $\mu > 0$ $f_*\mathcal{O}_X(\mu E) = \mathcal{O}_Y$ and $f_*\omega_{X/Y}^{\mu \cdot \delta} = \lambda^\mu$.

Let \mathcal{M}' be any polarization which is very ample and without higher cohomology on the fibres of $V \rightarrow U$. The sheaf

$$f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E)) = f_*(\mathcal{M}' \otimes (\lim_{\mu > 0} \mathcal{O}(\mu E)))$$

is coherent, hence locally free.

In fact, locally étale or locally analytic we can choose over a neighborhood \mathcal{U} of $s \in S = Y \setminus U$ a section $\sigma : \mathcal{U} \rightarrow X$ with image C , not meeting the support of E . If I denotes the ideal sheaf of C , for some $\rho \gg 0$

$$f_*(\mathcal{M}'|_{f^{-1}(\mathcal{U})} \otimes I^\rho) = 0.$$

Since direct images are torsion free, and since E is supported in fibres,

$$f_*((\mathcal{M}' \otimes \mathcal{O}_X(*E))|_{f^{-1}(\mathcal{U})} \otimes I^\rho) = 0.$$

Then $f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E))|_{\mathcal{U}}$ is a torsion free subsheaf of

$$f_*((\mathcal{M}' \otimes \mathcal{O}_X(*E))|_{f^{-1}(\mathcal{U})} \otimes \mathcal{O}_{f^{-1}(\mathcal{U})}/I^\rho) = f_*(\mathcal{M}'|_{f^{-1}(\mathcal{U})} \otimes \mathcal{O}_{f^{-1}(\mathcal{U})}/I^\rho),$$

hence coherent.

Let \mathcal{M} be the reflexive hull of the image of

$$f^* f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E)) \rightarrow \mathcal{M}' \otimes \mathcal{O}_X(*E).$$

\mathcal{M} is again coherent, and it must be contained in $\mathcal{M}' \otimes \mathcal{O}_X(\alpha \cdot E)$ for some α . Since it is reflexive, it is an invertible sheaf. By construction $\mathcal{M}|_V \simeq \mathcal{M}'|_V$ and

$$\begin{aligned} f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E)) &= f_* f^* f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E)) \subset \\ & f_* \mathcal{M} \subset f_*(\mathcal{M} \otimes \mathcal{O}_X(*E)) \subset f_*(\mathcal{M}' \otimes \mathcal{O}_X(*E)), \end{aligned}$$

hence all those sheaves coincide. We found an invertible sheaf \mathcal{M} satisfying the assumptions made in 3.2 with the additional condition

$$f_*(\mathcal{M} \otimes \omega_{X/Y}^e) = f_*(\mathcal{M} \otimes \mathcal{O}_X(\frac{e}{\delta} \cdot E) \otimes f^* \lambda^{\frac{e}{\delta}}) = f_* \mathcal{M} \otimes \lambda^{\frac{e}{\delta}},$$

for all multiples e of δ . For those e

$$\begin{aligned} (5.0.1) \quad \left(\bigotimes^r f_*(\mathcal{M} \otimes \omega_{X/Y}^e) \right) \otimes \det(f_* \mathcal{M})^{-1} &= \\ & \left(\bigotimes^r (f_* \mathcal{M}) \otimes \lambda^{\frac{e}{\delta}} \right) \otimes \det(f_* \mathcal{M})^{-1}. \end{aligned}$$

PROOF OF 2.1 FOR $\omega_F^\delta = \mathcal{O}_F$.

By assumption $f : X \rightarrow Y$ is birational over Y to the trivial family $pr_2 : F' \times Y \rightarrow Y$, hence

$$\lambda = f_* \omega_{X/Y}^\delta = \mathcal{O}_Y.$$

So the sheaf in (5.0.1) is

$$\left(\bigotimes^r (f_* \mathcal{M}) \right) \otimes \det(f_* \mathcal{M})^{-1}.$$

Since its determinant is of degree zero, it can not be ample. (5.0.1) and 3.2, b), imply that for no $\nu > 0$ the sheaf

$$\det(f_*(\mathcal{M}^\nu \otimes \omega_{X/Y}^{e \cdot \nu})) \otimes \det(f_* \mathcal{M})^{-\nu \cdot r(\nu)}$$

is ample. By 3.2, a), it is of non negative degree, and 3.2, c), implies that the family $V \rightarrow U$ is biregular isotrivial. \square

REMARK 5.1. Assume that $\delta = 1$, hence that $\omega_{V/U} = f^* \lambda|_U$. The argument used in the proof of 2.1 shows in this particular case that “ f non-isotrivial” implies that on the compactification Y of U

$$\deg(f_* \omega_{X/Y}) > 0.$$

Since the same holds true for all finite coverings of U , one obtains that the fibres of the period map from M_h to the period domain classifying the corresponding variations of Hodge structures can not contain a quasi-projective curve. Of course this is a well known consequence of the local Torelli Theorem for manifolds with a trivial canonical bundle.

6. Kodaira-Spencer maps

Recall first the following definition, replacing of nef and ample, on projective manifolds Y of higher dimension.

DEFINITION 6.1. Let \mathcal{F} be a torsion free coherent sheaf on a quasi-projective normal variety Y and let \mathcal{H} be an ample invertible sheaf.

- a) \mathcal{F} is generically generated if the natural morphism

$$H^0(Y, \mathcal{F}) \otimes \mathcal{O}_Y \longrightarrow \mathcal{F}$$

is surjective over some open dense subset U_0 of Y . If one wants to specify U_0 one says that \mathcal{F} is globally generated over U_0 .

- b) \mathcal{F} is weakly positive if there exists some dense open subset U_0 of Y with $\mathcal{F}|_{U_0}$ locally free, and if for all $\alpha > 0$ there exists some $\beta > 0$ such that

$$S^{\alpha \cdot \beta}(\mathcal{F}) \otimes \mathcal{H}^\beta$$

is globally generated over U_0 . We will also say that \mathcal{F} is weakly positive over U_0 , in this case.

- c) \mathcal{F} is big if there exists some open dense subset U_0 in Y and some $\mu > 0$ such that

$$S^\mu(\mathcal{F}) \otimes \mathcal{H}^{-1}$$

is weakly positive over U_0 . Underlining the role of U_0 we will also call \mathcal{F} ample with respect to U_0 .

Here, as in [8] and [10], we use the following convention: If \mathcal{F} is a coherent torsion free sheaf on a quasi-projective normal variety Y , we consider the largest open subscheme $i : Y_1 \rightarrow Y$ with $i^*\mathcal{F}$ locally free. For

$$\Phi = S^\mu, \quad \Phi = \bigotimes^{\mu} \quad \text{or} \quad \Phi = \det$$

we define

$$\Phi(\mathcal{F}) = i_*\Phi(i^*\mathcal{F}).$$

Again, $f : V \rightarrow U$ denotes a smooth family of manifolds over a quasi-projective manifold U , which is allowed to be of dimension larger than one. We choose non-singular projective compactifications Y of U and X of V , such that both $S = Y \setminus U$ and $\Delta = X \setminus V$ are normal crossing divisors and such that f extends to $f : X \rightarrow Y$. As usual η will denote a closed point in sufficient general position on U and X_η the fibre of f over η . We will write $T_{X_\eta}^i$ (or $T_{X/Y}^i(-\log \Delta) \dots$) for the i -th wedge product of T_{X_η} (or of $T_{X/Y}(-\log \Delta) = \Omega_{X/Y}^1(\log \Delta)^\vee \dots$).

Let T_η denote the restriction $T_U \otimes \mathbb{C}$ of the tangent sheaf of U to η . The Kodaira-Spencer map

$$T_\eta \longrightarrow H^1(X_\eta, T_{X_\eta})$$

gives rise to

$$\bigotimes^{\nu} T_\eta \longrightarrow \bigotimes^{\nu} H^1(X_\eta, T_{X_\eta}) \longrightarrow H^\nu(X_\eta, T_{X_\eta}^\nu).$$

The composite map factors through

$$S^\nu(T_\eta) \longrightarrow H^\nu(X_\eta, T_{X_\eta}^\nu).$$

One defines

$$\mu(f) = \text{Max}\{\nu \in \mathbb{N} - \{0\}; S^\nu(T_\eta) \longrightarrow H^\nu(X_\eta, T_{X_\eta}^\nu) \text{ is non zero}\}.$$

Of course, $\mu(f) \leq n = \dim(X_\eta)$. We do not know any criterion, implying that for $f : V \rightarrow U$ one has $\mu(f) = \dim(V) - \dim(U)$.

For example, if U is a curve and $f : V \rightarrow U$ a family of polarized manifolds, restricting the tautological sequence to $X_\eta = f^{-1}(\eta)$ one obtains an extension

$$0 \longrightarrow T_{X_\eta} \longrightarrow T_X|_{X_\eta} \longrightarrow \mathcal{O}_{X_\eta} \longrightarrow 0$$

and the induced class $\xi_\eta \in H^1(X_\eta, T_{X_\eta})$. Then $\mu(f) = \mu$ if and only if for η in general position, the wedge product $\wedge^\mu \xi_\eta \in H^\mu(X_\eta, T_{X_\eta}^\mu)$ is non-zero, whereas $\wedge^{\mu+1} \xi_\eta \in H^{\mu+1}(X_\eta, T_{X_\eta}^{\mu+1})$ is zero.

PROBLEM 6.2. Are there properties of X_η which imply that for all families $V \rightarrow U$ over a curve U , with general fibre $X_\eta = f^{-1}(\eta)$ the class

$$\wedge^n \xi_\eta \in H^n(X_\eta, \omega_{X_\eta}^{-1})$$

is non-zero?

Being optimistic, one could try in 6.2 the condition “ $\Omega_{X_\eta}^1$ ample”.

A slight extension of the main result of [10] says:

THEOREM 6.3. *Assume that for a general fibre X_η of $f : X \rightarrow Y$ either ω_{X_η} is ample, or $\omega_{X_\eta}^\delta = \mathcal{O}_{X_\eta}$, for some δ .*

- i. *Then for some $m > 0$ the sheaf $S^m(\Omega_Y^1(\log S))$ contains an invertible subsheaf \mathcal{M} of Kodaira dimension $\text{Var}(f)$.*
- ii. *If $\text{Var}(f) = \dim(U)$ the sheaf $S^{\mu(f)}(\Omega_Y^1(\log S))$ contains a big coherent subsheaf \mathcal{P} .*
- iii. *Let Z be a submanifold of Y such that $S_Z = S \cap Z$ remains a normal crossing divisor, and such that $W = X \times_Y Z$ is non-singular. For the induced family $h : W \rightarrow Z$ assume that $\mu(f) = \mu(h)$. Then, if $\text{Var}(f) = \dim(U)$, the restriction of the sheaf \mathcal{P} from part ii) to $S^{\mu(f)}(\Omega_Z^1(\log S_Z))$ is non trivial.*
- iv. *Assume in iii) that $h : W \rightarrow Z$ is a desingularization of the pullback of a family $h' : W' \rightarrow Z'$ under $\pi : Z \rightarrow Z'$, with Z' non-singular and with h' smooth over $Z' \setminus S_{Z'}$ for a normal crossing divisor $S_{Z'}$. Then then the restriction of the sheaf \mathcal{P} to $S^{\mu(f)}(\Omega_Z^1(\log S_Z))$ lies in $S^{\mu(f)}(\pi^*(\Omega_{Z'}^1(\log S_{Z'})))$.*

PROOF. Parts i) and ii) have been shown in [10], 1.4, for canonically polarized manifolds with $\mu(f)$ replaced by the fibre dimension n . We will just sketch the changes which allow to extend the arguments used in [10] to cover 6.3, ii), iii) and iv), for canonically polarized manifolds. Next we will try to convince the reader, that the same proof goes through for minimal models of Kodaira dimension zero.

As in [10] we drop the assumption that Y is projective. Leaving out a codimension two subscheme, we may assume that f is flat and that Δ is a relative normal crossing divisor. Then we have the tautological exact sequence

$$0 \longrightarrow f^* \Omega_Y^1(\log S) \longrightarrow \Omega_X^1(\log \Delta) \longrightarrow \Omega_{X/Y}^1(\log \Delta) \longrightarrow 0$$

and the wedge product sequences

$$(6.3.1) \quad 0 \longrightarrow f^* \Omega_Y^1(\log S) \otimes \Omega_{X/Y}^{p-1}(\log \Delta) \longrightarrow \text{gr}(\Omega_X^p(\log \Delta)) \longrightarrow \Omega_{X/Y}^p(\log \Delta) \longrightarrow 0,$$

where

$$\text{gr}(\Omega_X^p(\log \Delta)) = \Omega_X^p(\log \Delta) / f^* \Omega_Y^2(\log S) \otimes \Omega_{X/Y}^{p-2}(\log \Delta).$$

For the invertible sheaf $\mathcal{L} = \Omega_{X/Y}^n(\log \Delta)$ we consider the sheaves

$$F^{p,q} := R^q f_*(\Omega_{X/Y}^p(\log \Delta) \otimes \mathcal{L}^{-1})$$

together with the edge morphisms

$$\tau_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega_Y^1(\log S),$$

induced by the exact sequence (6.3.1), tensored with \mathcal{L}^{-1} . As explained in [10], Proof of 4.4 iii), over U the edge morphisms $\tau_{p,q}$ can also be obtained in the following way. Consider the exact sequence

$$0 \longrightarrow T_{V/U} \longrightarrow T_V \longrightarrow f^* T_U \longrightarrow 0,$$

and the induced wedge product sequences

$$0 \longrightarrow T_{V/U}^{n-p+1} \longrightarrow \tilde{T}_V^{n-p+1} \longrightarrow T_{V/U}^{n-p} \otimes f^* T_U \longrightarrow 0,$$

where \tilde{T}_V^{n-p+1} is a subsheaf of T_V^{n-p+1} . One finds edge morphisms

$$\tau_{p,q}^\vee : (R^q f_* T_{V/U}^{n-p}) \otimes T_U \longrightarrow R^{q+1} f_* T_{V/U}^{n-p+1}.$$

Restricted to η those are just the wedge product with the Kodaira-Spencer class. Moreover, tensoring with Ω_U^1 one gets back $\tau_{p,q}|_U$. Hence $\mu(f)$ is the smallest number m for which the composite

$$\tau^m : F^{n,0} = \mathcal{O}_Y \xrightarrow{\tau_{n,0}} F^{n-1,1} \otimes \Omega_U^1 \xrightarrow{\tau_{n-1,1}} F^{n-2,2} \otimes S^2(\Omega_U^1) \longrightarrow \dots \xrightarrow{\tau_{n-m+1,m-1}} F^{n-m,m} \otimes S^m(\Omega_U^1)$$

is non-zero. Next we used that (replacing Y by some covering) there is an ample invertible sheaf \mathcal{A} on Y such that the kernel \mathcal{K} of

$$\text{id}_{\mathcal{A}} \otimes \tau_{n-m,m} : \mathcal{A} \otimes F^{n-m,m} \longrightarrow \mathcal{A} \otimes F^{n-m-1,m+1} \otimes \Omega_Y^1(\log S)$$

is negative, or to be more precise, that its dual is weakly positive. This gives a non-trivial map

$$v : \mathcal{A} \otimes \mathcal{K}^\vee \longrightarrow S^m(\Omega_Y^1(\log S))$$

and we take for \mathcal{P} its image.

The number m was used in the proof of [10], 1.4 ii), hence there is no harm to replace the upper bound n , used there, by the more precise number $\mu(f)$ in 6.3, ii).

The sheaves $F^{p,q}$ are compatible with restriction to the subvariety Z . The assumption $\mu(f) = \mu(h)$ implies that the restriction

$$v|_Z : \mathcal{A}|_Z \otimes \mathcal{K}^\vee|_Z \longrightarrow S^m(\Omega_Z^1(\log S_Z))$$

is non-trivial. In fact, the kernel \mathcal{K}' of

$$\text{id}_{\mathcal{A}_Z} \otimes \tau_{n-m,m}^Z : \mathcal{A}|_Z \otimes F^{n-m,m}|_Z \longrightarrow \mathcal{A}|_Z \otimes F^{n-m-1,m+1}|_Z \otimes \Omega_Z^1(\log S_Z)$$

contains $\mathcal{K}|_Z$, and the diagram

$$\begin{array}{ccc} \mathcal{A}|_Z \otimes \mathcal{K}^\vee|_Z & \longrightarrow & S^m(\Omega_Y^1(\log S))|_Z \\ \uparrow & & \downarrow \\ \mathcal{A}|_Z \otimes \mathcal{K}'^\vee & \longrightarrow & S^m(\Omega_Z^1(\log S_Z)) \end{array}$$

is commutative. One obtains iii).

Since the sheaves $F^{p,q}$ and the maps $\tau_{p,q}$ are compatible with pullbacks, under the additional assumptions made in iv), the image of

$$\mathcal{A}|_Z \otimes \mathcal{K}'^\vee \longrightarrow S^m(\Omega_Z^1(\log S_Z))$$

lies in $S^m(\pi^*(\Omega_{Z'}^1(\log S_{Z'})))$ and the same holds true for the restriction of \mathcal{P} .

If one considers the proof of [10], 1.4, i) and ii), the assumption that the fibres are canonically polarized is used twice. First of all, since we apply in the proof of 4.4, iv), the Akizuki-Kodaira-Nakano vanishing theorem to the restriction of ω_F to a smooth multicanonical divisor B . If some power of ω_F is trivial the divisor B is empty, and there is nothing to show.

The second time is in the proof of [10], 4.8. We use the diagram (2.8.1) and the fact that the morphism $Z^\# \rightarrow Y^\#$ considered there is of maximal variation. The construction of (2.8.1) just uses the existence of the moduli scheme M_h , and it provides a morphism $Z^\# \rightarrow Y^\#$ induced by a generically finite morphism $Y^\# \rightarrow M_h$.

This construction works in particular for the moduli scheme of polarized manifolds F with $\omega_F^\delta = \mathcal{O}_F$, and 1.2 implies that the variation of the morphism $Z^\# \rightarrow Y^\#$ is again maximal.

The rest of the arguments, as given on page 311–313 of [10] remain unchanged, and one obtains 6.3, i), ii) and iii). \square

For families $f : X \rightarrow Y$ with ω_{X_η} semiample, and with $\mu(f) = n$ one can add to [10], 1.4, a statement similar to 6.3, iii) and iv). Since the later will not be used, we omit it.

THEOREM 6.4. *Assume ω_{X_η} is semiample, and $\text{Var}(f) = \dim(f)$.*

- i. *There exists a non-singular finite covering $\psi : Y' \rightarrow Y$ and a big coherent subsheaf \mathcal{P}' of $\psi^*S^m(\Omega_{Y'}^1(\log S))$, for some $m \leq \mu(f)$.*
- ii. *If $\mu(f) = n$, then one finds $m = n$ in i).*
- iii. *Let Z be a submanifold of Y such that $S_Z = S \cap Z$ remains a normal crossing divisor, and such that $W = X \times_Y Z$ is non-singular. For the induced family $h : W \rightarrow Z$ assume that $\mu(f) = \mu(h) = n$. Then one can choose the covering ψ such that $\psi^{-1}(Z)$ is non-singular, $\psi^{-1}(S_Z)$ a normal crossing divisor and such that the image of the sheaf \mathcal{P}' from part i) in $\psi^*S^n(\Omega_Z^1(\log S_Z))$ is non trivial.*

PROOF. We keep the notations from the sketch of the proof of 6.3. For part i) we replaced in [10], page 309 and 310, the sheaves $F^{p,q}$ (in fact a twist of those by some invertible sheaf on Y) by some quotient sheaves. But then $\mu(f)$ remains an upper bound for the number m , used there, and one obtains 6.3, i), as stated.

However, one has no control on the behavior of m under restriction to subvarieties. So for part ii) and iii) we have to recall the construction in more detail. To get the weak positivity of the kernels \mathcal{K} one has to replace (over some covering Y' of Y whose ramification divisor is in general position) $\mathcal{A} \otimes F^{n-m,m}$ by its image $\mathcal{A} \otimes \tilde{F}^{n-m,m}$ in some larger sheaf $E^{n-m,m}$. Here

$$\left(\bigoplus_{p+q=n} E^{p,q}, \bigoplus_{p+q=n} \theta_{p,q} \right)$$

is again a Higgs bundle, and $\theta_{p,q}$ is compatible with $\tau_{p,q}$.

As stated in the proof of [10], 4.4, iv) the kernel and cokernel of the map

$$\mathcal{A} \otimes F^{n-m,m} \rightarrow E^{n-m,m}$$

are direct images of the $n - m - 1$ -forms of a multicanonical divisor. If $n = m$ there are no such forms, and $\mathcal{A} \otimes F^{0,n}$ is a subsheaf of $E^{0,n}$. Hence $\tau^n \neq 0$ implies that the corresponding map for $\mathcal{A} \otimes \tilde{F}^{n-m,m}$ is non-zero. The compatibility with restrictions follows by the argument used in the proof of 6.3, iii). \square

7. Subvarieties of the moduli stack of polarized manifolds of Kodaira dimension zero

Theorem 6.3 has a number of geometric implication for manifolds U mapping to moduli stacks of polarized manifolds, i.e. for morphisms $\varphi : U \rightarrow M_h$ induced by a family $f : V \rightarrow U$. Those had been shown in [10] for the moduli stack of canonically polarized manifolds. The proves are all based on vanishing theorems for logarithmic differential forms, and they do not refer to the type of fibres of f , once 6.3, i) and ii), is established.

Using 1.2 we extended 6.3, i) and ii), to a larger class of families of polarized manifolds. Hence the geometric implications carry over to this larger class, i.e. to the moduli stack of polarized manifolds with $\omega_F^\delta = \mathcal{O}_F$, for some $\delta > 0$. For the readers convenience we recall the statements below.

For polarized manifolds with ω_F semiample, the lack of an affirmative answer to Problem 1.3 still forces us to assume that $\text{Var}(f) = \dim(U)$ to obtain similar results.

THEOREM 7.1 (see [10], 5.2, 5.3, 7.2, 6.4, and 6.7). *Let M_h be the moduli scheme of canonically polarized manifolds, or of polarized manifolds F with ω_F^δ trivial for some $\delta > 0$.*

I. *Assume that U satisfies one of the following conditions*

- a) *U has a non-singular projective compactification Y with $S = Y \setminus U$ a normal crossing divisor and with boundary $T_Y(-\log S)$ weakly positive.*
- b) *Let $H_1 + \dots + H_\ell$ be a reduced normal crossing divisor in \mathbb{P}^N , and $\ell < \frac{N}{2}$. For $0 \leq r \leq \ell$ define*

$$H = \bigcap_{j=r+1}^{\ell} H_j, \quad S_i = H_i|_H, \quad S = \sum_{i=1}^r S_i,$$

and assume $U = H \setminus S$.

- c) *$U = \mathbb{P}^N \setminus S$ for a reduced normal crossing divisor*

$$S = S_1 + \dots + S_\ell$$

in \mathbb{P}^N , with $\ell < N$.

Then a morphism $U \rightarrow M_h$, induced by a family, must be trivial.

II. For $Y = \mathbb{P}^{\nu_1} \times \cdots \times \mathbb{P}^{\nu_k}$ let

$$D^{(\nu_i)} = D_0^{(\nu_i)} + \cdots + D_{\nu_i}^{(\nu_i)}$$

be coordinate axes in \mathbb{P}^{ν_i} and

$$D = \sum_{i=1}^k D^{(\nu_i)}.$$

Assume that $S = S_1 + \cdots + S_\ell$ is a divisor, such that $D + S$ is a reduced normal crossing divisor, and $\ell < \dim(Y)$. Then there exists no morphism $\varphi : U = Y \setminus (D + S) \rightarrow M_h$ with

$$\dim(\varphi(U)) > \text{Max}\{\dim(Y) - \nu_i; i = 1, \dots, k\}.$$

III. Let U be a quasi-projective variety and let $\varphi : U \rightarrow M_h$ be a quasi-finite morphism, induced by a family. Then U can not be isomorphic to the product of more than $\mu(f)$ varieties of positive dimension.

8. Rigidity

Again, $f : V \rightarrow U$ denotes a smooth family of manifolds with $\omega_{V/U}$ f -semiample and with $\text{Var}(f) = \dim(U) > 0$. We say that f is rigid, if there exists no non-trivial deformation over a non-singular quasiprojective curve T .

Here a deformation of f over T , with $0 \in T$ a base point, is a smooth projective morphism

$$g : \mathcal{V} \rightarrow U \times T$$

for which there exists a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{\cong} & g^{-1}(U \times \{0\}) & \xrightarrow{\subset} & \mathcal{V} \\ f \downarrow & & \downarrow & & \downarrow g \\ U & \xrightarrow{\cong} & U \times \{0\} & \xrightarrow{\subset} & U \times T \end{array}.$$

If the fibres F of f are canonically polarized, or if some power of ω_F is trivial, this says that morphisms from U to the moduli stack do not deform.

PROPOSITION 8.1. Assume either that ω_{X_η} is ample, or that $\omega_{X_\eta}^\delta = \mathcal{O}_{X_\eta}$, for some δ . Assume that $\text{Var}(f) = \dim(U) > 0$. Let T be a non-singular quasi-projective curve. Let $g : \mathcal{V} \rightarrow U \times T$ be a deformation of f . If $\mu(f) = \mu(g)$, then $\text{Var}(g) = \dim(U)$.

PROOF. Suppose that $\text{Var}(g) > \dim(U)$. Then

$$\dim(U) + 1 = \dim U \times T \geq \text{Var}(g) > \dim(U),$$

hence, $\dim(U \times T) = \text{Var}(g)$. Let \bar{T} be a non-singular compactification of T , $S_{\bar{T}} = \bar{T} \setminus T$. Correspondingly we write $S_{Y \times \bar{T}}$ for the complement of $U \times T$ in $Y \times \bar{T}$.

By Theorem 6.3, ii, one finds a big coherent subsheaf \mathcal{P} of

$$S^{\mu(g)}(\Omega_{Y \times \bar{T}}^1(\log S_{Y \times \bar{T}})),$$

and by 6.3, iii) the image of \mathcal{P} in

$$S^{\mu(g)}(\Omega_{Y \times \{0\}}^1(\log S_{Y \times \{0\}})) = pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))|_{Y \times \{0\}}$$

is non-zero. Then, the image of \mathcal{P} in

$$pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))$$

is non-zero, and for a point $y \in Y$ in general position, the image of \mathcal{P} under

$$S^{\mu(g)}(\Omega_{Y \times \bar{T}}^1(\log S_{Y \times \bar{T}}))|_{\{y\} \times T} \longrightarrow pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))|_{\{y\} \times T}$$

is not zero. Note that any non-zero quotient of the coherent sheaf $\mathcal{P}|_{\{y\} \times T}$ for y in general position must be big. In fact, if \mathcal{P} is ample over some open dense subset W_0 of $Y \times \bar{T}$, one just has to make sure that $\{y\} \times T$ meets W_0 . Since $pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))|_{\{y\} \times T}$ is a direct sum of copies of $\mathcal{O}_{\{y\} \times T}$ this is not possible. \square

Using 6.4 instead of 6.3 one obtains a similar result for families with $\omega_{V/U}$ f semi-ample, whenever $\mu(f) = n$.

PROPOSITION 8.2. *Assume that ω_{X_η} is semiample, that*

$$\text{Var}(f) = \dim(U) > 0$$

and that

$$\mu(f) = \dim(X_\eta) = n.$$

Let T be a non-singular quasi-projective curve, and let $g : \mathcal{V} \rightarrow U \times T$ be a deformation of f . Then $\text{Var}(g) = \dim(U)$.

PROOF. If $\text{Var}(g) > \dim(U)$, again one finds $\dim(U \times T) = \text{Var}(g)$. Let us keep the notations from the proof of 8.1. By Theorem 6.4, ii, one finds a finite covering $\psi : Y' \rightarrow Y$ and a big coherent subsheaf \mathcal{P}' of

$$\psi^*(S^{\mu(g)}(\Omega_{Y \times \bar{T}}^1(\log S_{Y \times \bar{T}}))),$$

and by 6.4, iii) the image of \mathcal{P}' in

$$\psi^*(S^{\mu(g)}(\Omega_{Y \times \{0\}}^1(\log S_{Y \times \{0\}}))) = \psi^*pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))|_{Y \times \{0\}}$$

is non-zero. Then, for a point $y \in Y$, in general position, the image of \mathcal{P}' under

$$\psi^*(S^{\mu(g)}(\Omega_{Y \times \bar{T}}^1(\log S_{Y \times \bar{T}}))|_{\{y\} \times T}) \longrightarrow \psi^*(pr_1^*(S^{\mu(g)}(\Omega_Y^1(\log S)))|_{\{y\} \times T})$$

is not zero. Again, since the sheaf on the right hand side is trivial, one obtains a contradiction. \square

Let $h : \mathcal{X} \rightarrow Z$ be a polarized family of manifolds F with ω_F semiample and of maximal variation, over a non-singular quasi-projective manifold Z . Assume that \bar{Z} is a projective compactification of Z , such that $\bar{Z} \setminus Z$ is a normal crossing divisor. Assume in addition, that there is an open dense subscheme Z_0 such that for all subvarieties \mathcal{U} of Z meeting Z_0

$$\text{Var}(\mathcal{X} \times_Z \mathcal{U} \rightarrow \mathcal{U}) = \dim(\mathcal{U}).$$

Let Y be a non-singular projective curve and let $U \subset Y$ be open and dense. Let us write

$$\mathbf{H} = \mathbf{Hom}((Y, U), (\bar{Z}, Z))$$

for the scheme parameterizing non-trivial morphisms $\psi : Y \rightarrow \bar{Z}$ with $\psi(U) \subset Z$ and

$$\mathbf{H}_{Z_0} = \mathbf{Hom}((Y, U), (\bar{Z}, Z); Z_0) \subset \mathbf{H}$$

for those with $\psi(U) \cap Z_0 \neq \emptyset$. Based on the bounds obtained in [9] we have shown in [10] that \mathbf{H}_{Z_0} is of finite type.

COROLLARY 8.3.

- I. a. Let $\psi : U \rightarrow Z$ be a morphism and $f : V \rightarrow U$ the pull back family. Assume that $\psi(U) \cap Z_0 \neq \emptyset$ and that

$$\mu(f : V \rightarrow U) = \dim(F) = n$$

Then the point $[\psi : Y \rightarrow \bar{Z}]$ is isolated in \mathbf{H}_{Z_0} .

- b. Assume for all fibres F of $h^{-1}(Z_0) \rightarrow Z_0$ and for all $\xi \in H^1(F, T_F)$

$$0 \neq \wedge^n \xi \in H^n(F, \omega_F^{-1}).$$

Then \mathbf{H}_{Z_0} is a finite set of points.

- II. Assume that the fibres F of $h : \mathcal{X} \rightarrow Z$ are either canonically polarized, or of Kodaira dimension zero.

- a. Let $\psi : U \rightarrow Z$ be a morphism and $f : V \rightarrow U$ the pull back family. Assume that $\psi(U) \cap Z_0 \neq \emptyset$ and that

$$\mu(f : V \rightarrow U) = \mu(h : \mathcal{X} \rightarrow Z).$$

Then the point $[\psi : Y \rightarrow \bar{Z}]$ is isolated in \mathbf{H}_{Z_0} .

- b. Assume there exists a constant μ such that for all fibres F of $h^{-1}(Z_0) \rightarrow Z_0$ and for all $\xi \in H^1(F, T_F)$

$$0 \neq \wedge^\mu \xi \in H^\mu(F, T_F^\mu)$$

but

$$0 = \wedge^{\mu+1} \xi \in H^{\mu+1}(F, T_F^{\mu+1}).$$

Then \mathbf{H}_{Z_0} is a finite set of points.

PROOF. In both cases b) follows from a). For the latter assume that $[\psi]$ lies in a component of \mathbf{H} of dimension larger than zero. Let T be a curve in \mathbf{H} , containing the point $[\psi]$. Then one has a non-trivial deformation $\Psi : U \times T \rightarrow Z$ of ψ , hence a non-trivial deformation $g : \mathcal{V} \rightarrow U \times T$ of $f : V \rightarrow U$. By 8.1 in case II) or by 8.2 in case I)

$$\text{Var}(g) = \text{Var}(f) < \dim(U \times T) = \dim(U) + 1,$$

contradicting the assumption made on Z_0 and $\mathcal{X} \rightarrow Z$. □

Corollary 8.3, II), should imply certain finiteness results for curves in the moduli scheme M_h of canonically polarized manifolds, or the moduli scheme of minimal models of Kodaira dimension zero meeting an open subscheme W where the assumption corresponding to the one in 8.3, II), b), holds true. However, one would have to show, that morphisms φ which factor through the moduli stack, are parameterized by some coarse moduli scheme. Hopefully this can be done extending the methods used in [2] for moduli of curves to moduli of higher dimensional manifolds.

Here we will show a slightly weaker statement, which coincides with 1.4 for $\mu = n$.

COROLLARY 8.4. Let M_h be either the moduli scheme of canonically polarized manifolds or the moduli scheme of polarized manifolds F with $\omega_F^\delta = \mathcal{O}$ for some $\delta > 0$. Let $0 < \mu \leq n = \dim(F)$ be a constant such that for all (F, \mathcal{L}) , and for all $\xi \in H^1(F, T_F)$

$$0 = \wedge^{\mu+1} \xi \in H^{\mu+1}(F, T_F^{\mu+1}).$$

Then for a quasi-projective non-singular curve U there are only finitely many morphisms $\varphi : U \rightarrow M_h$ which are induced by a smooth family $f : V \rightarrow U$ with $\mu(f) = \mu$.

PROOF. Let us choose any projective compactification \bar{M}_h of M_h , and an invertible sheaf \mathcal{H} on \bar{M}_h which is ample with respect to M_h . As usual, Y will be a non-singular projective curve containing U . We write s for the number of points in $S = Y \setminus U$ and $g(Y)$ for the genus of Y .

To show that there are only finitely many components of the scheme

$$\mathbf{Hom}((Y, U), (M_h, \bar{M}_h))$$

which contain a morphism $\varphi : U \rightarrow M_h$ factoring through the moduli stack, one has to find an upper bound for $\varphi^*\mathcal{H}$. To this aim one may assume that \bar{M}_h is reduced. The proof for the boundedness follows the line of the proof of [10], 6.2.

Kollar and Seshadri constructed (see [8], 9.25) a finite covering of M_h which factors through the moduli stack.

Consider any finite morphism $\pi : Z \rightarrow M_h$ with this property. We choose a projective compactifications \bar{Z} of Z such that π extends to $\pi : \bar{Z} \rightarrow \bar{M}_h$. So $\pi^*\mathcal{H}$ is again ample with respect to $\pi^{-1}(M_h)$. Let M_0 be a non-singular subvariety of $\pi(Z) \cap M_h$ with $Z_0 = \pi^{-1}(M_0)$ non singular.

Recall that for a family of projective varieties we constructed in [10], 2.7, a good open subset of the base space. Applying this construction to the restriction of the universal family to Z_0 , we may assume furthermore, that Z_0 coincides with this subset.

By induction on the dimension of Z , we may assume that we have found an upper bound for $\varphi^*(\mathcal{H})$ whenever $\varphi(Y) \subset \pi(Z) \setminus M_0$. Hence it is sufficient to find such a bound under the assumptions that $\varphi(Y) \subset \pi(Z)$ and $\varphi(Y) \cap M_0 \neq \emptyset$. There exists a finite covering Y' of Y of degree $d \leq \deg(Z/\pi(Z))$, such that

$$Y' \xrightarrow{\sigma} Y \xrightarrow{\varphi} \pi(Z)$$

factors through $\varphi' : Y' \rightarrow Z$, and it is sufficient to bound the degree of $\sigma^*\varphi^*\mathcal{H}$. For simplicity, we assume that $\varphi : Y \rightarrow \pi(Z)$ factors through $\varphi' : Y \rightarrow Z$.

By [10], 2.6 and 2.7, blowing up Z with centers in $Z \setminus Z_0$ we may assume that Z is non singular, that there exists a certain invertible sheaf λ_ν on Z , and a constant $N_\nu > 0$, such that

$$\deg(\varphi'^*\lambda_\nu) \leq N_\nu \cdot \deg(\det(f_*\omega_{X/Y}^\nu)),$$

where, as usual, $f : X \rightarrow Y$ is an extension of $V \rightarrow U$ to a projective manifold X . By the explicit description of λ_ν in [10], 2.6, d), and by [10], 3.4, the sheaf λ_ν is ample with respect to Z_0 for some $\nu > 1$. Hence it is sufficient to give an upper bound for $\deg(\varphi'^*\lambda_\nu)$, or for $\deg(\det(f_*\omega_{X/Y}^\nu))$.

By [9] (see also [1] and [4]) there exists a constant e , depending only on the Hilbert polynomial h , with

$$\deg(\det(f_*\omega_{X/Y}^\nu)) \leq (n \cdot (2g(Y) - 2 + s) + s) \cdot \nu \cdot \text{rank}(f_*\omega_{X/Y}^\nu) \cdot e,$$

and we found the bound we were looking for.

It remains to show, that the points

$$[\varphi : Y \rightarrow \bar{M}_h] \in \mathbf{Hom}((Y, U), (M_h, \bar{M}_h))$$

which are induced by a family $f : X \rightarrow Y$ with $\mu(f) = \mu$, are discrete. If not, one finds a positive dimensional manifold T and a generically finite morphism to

$\mathbf{Hom}((Y, U), (M_h, \bar{M}_h))$ whose image contains a dense set of points where the corresponding morphism is induced by a family. Let us choose a smooth projective compactification \bar{T} with $S_{\bar{T}} = \bar{T} \setminus T$ a normal crossing divisor.

The induced morphism $Y \times T \rightarrow \bar{M}_h$ is not necessarily factoring through the moduli stack, but using again the Kollár Seshadri construction again, we find a generically finite morphism $\pi : Z \rightarrow Y \times \bar{T}$ which over $\pi^{-1}(U \times T)$ is induced by a smooth family. Assume that Z is non-singular and that $S_Z = Z \setminus U \times T$ is a normal crossing divisor. Write $p : Z \rightarrow \bar{T}$ for the induced morphism.

Applying 6.3, ii), one obtains a big coherent subsheaf

$$\mathcal{P} \subset S^\mu(\Omega_Z^1(\log S_Z)).$$

By part iii), its image \mathcal{P}' in $S^\mu(\Omega_{Z/T}^1(\log S_Z))$ is non zero, and iv) implies that for a dense set of points $t \in T$ the restriction $\mathcal{P}'|_{p^{-1}(t)}$ lies in

$$\pi^* S^\mu(\Omega_{Y \times \{t\}}^1(\log(S \times \{t\}))).$$

This is only possible, if \mathcal{P}' is a big subsheaf of

$$\pi^* S^\mu(\Omega_{Y \times \bar{T}}^1(\log(S \times \bar{T}))).$$

Restricting to $\pi^{-1}(\{y\} \times \bar{T})$, for general $y \in Y$ one obtains as in the proof of 8.2 a big subsheaf of a trivial sheaf, a contradiction. \square

Needless to say, Corollary 8.4 is sort of empty, as long as we do not know any answer to Problem 6.2.

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