On global existence of wave maps with critical regularity

Andrea Nahmod

ABSTRACT. We survey recent work on Wave maps from Minkowski space $\mathbb{R}^{1+n}$ into (compact) Riemannian manifolds. We focus on the results obtained and some of the methods from harmonic analysis and gauge theory used.

0. Introduction

This article arose from a lecture given by the author at the JDG Conference held at Harvard University, Cambridge Massachusetts in May 2002 and celebrating Karen Uhlenbeck's sixtieth birthday. The article is expository in nature and surveys recent joint work with A. Stefanov and K.Uhlenbeck on wave maps from Minkowski space $\mathbb{R}^{1+n}, n \geq 4$ into compact Riemannian manifolds. We mainly focus on the results obtained and some of the methods from harmonic analysis and gauge theory used. The results and techniques presented actually work on any constant curvature complete Riemannian manifold (e.g. Lie groups and their symmetric Riemannian spaces). It is also probable that they can be further extended to bounded geometry complete Riemannian manifolds, for example but we do not pursue the latter extension here. The paper is organized as follows. We first describe the wave map problem and some of the literature. We then present some of the relevant tools from harmonic analysis and their use in the study of a related nonlinear wave equation. Finally we briefly comment on the difficulties that arise when passing from 4 to 3 dimensions. The ideas and results on wave maps presented here by the author are in collaboration with A. Stefanov and K. Uhlenbeck.

Let $\mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{1+n}$ be Minkowski space with flat metric $\eta = (\eta_{\alpha\beta}) = \operatorname{diag}(-1, 1, \ldots, 1)$ Then $\eta^{\alpha\beta} = (\eta_{\alpha\beta})^{-1} = \eta_{\alpha\beta}$ in our case . By $\partial_\alpha, \partial^\alpha \quad \alpha = 0, 1, \ldots, n$ we denote the usual derivatives with respect to the Minkowski metric. Then $\partial^\alpha = \eta^{\alpha\beta}\partial_\beta$, $\Box := \partial^\alpha\partial_\alpha = \partial_t^2 - \Delta$ is the D'Lambert operator; and

$$\frac{1}{2}\langle\partial^\alpha u, \partial_\alpha u\rangle = \frac{1}{2}(|u_t|^2 - |\nabla u|^2)$$

is the 'Lagrangian density' of u.

Wave maps are the natural Minkowskian analogue of harmonic maps.

1991 *Mathematics Subject Classification.* Primary 35J10, Secondary 45B15, 42B35 .

The author was partially supported by NSF grant DMS 0202139.

For a map $\Phi : \mathbb{R}\times\mathbb{R}^n \to M$, where $(M, \mathfrak{g})$ is -for example- a compact Riemannian manifold, the critical points of the Euler-Lagrangian functional

$$\Phi \to \int_{\mathbb{R}^{n+1}} \langle \partial_\alpha \Phi, \partial^\alpha \Phi \rangle_{\mathfrak{g}} \, d\sigma = \int\int |\frac{\partial \Phi}{\partial t}|_{\mathfrak{g}}^2 - |\nabla_x \Phi|_{\mathfrak{g}}^2 dx dt$$

are the solutions to the *Wave Map equation.* This equation can be regarded as given through covariant derivatives and in coordinate free notation is given by:

$$\Phi^* \nabla_0 \frac{\partial \Phi}{\partial t} - \sum_{j=1}^{n} \Phi^* \nabla_j \frac{\partial \Phi}{\partial x^j} = 0. \tag{1}$$

where $\Phi^*\nabla$ are the covariant derivatives (corresponding to the pullback of the Levi-Civita connection on M via the map Φ). This is a nonlinear wave equation with a non-polynomial nonlinearity including derivatives.

For example, when $M = \mathbb{S}^{m-1} \hookrightarrow \mathbb{R}^m$, equation (1) looks like

$$\Box \Phi = -\frac{\partial^2 \Phi}{\partial t^2} + \Delta \Phi = -\Phi \left(|\nabla_x \Phi|^2 - |\frac{\partial \Phi}{\partial t}|^2 \right).$$

And in general a (classical) wave map is a (smooth) map $\Phi : \mathbb{R}\times\mathbb{R}^n \to M$ satisfying,

$$\Box \Phi = -\Phi (\partial_\alpha \Phi)^T \partial^\alpha \Phi \tag{WM}$$

(with, say, Φ constant outside the finite union of light cones). One key feature of wave maps is that of *energy conservation*:

$$E(\Phi(t)) := \frac{1}{2} ||D\Phi(t)||^2_{L^2(\mathbb{R}^n)} = \frac{1}{2} ||D\Phi(0)||^2_{L^2(\mathbb{R}^n)}.$$

The study of well-posedness of the Cauchy problem with initial data in the Sobolev spaces $H^s \times H^{s-1}(\mathbb{R}^n; TM)$, $s \geq 1$ seeks answers to the following questions.

Local in time Existence and Uniqueness: for what values of s does the initial value problem admits a unique local solution ?

Local well posedness: in addition to the above, does the solution depend continuously on the initial data ?

Global well-posedness: for what values of s does this solution extend for all time ?

Global regularity: does the solution corresponding to smooth initial data stays smooth for all times ?

Wave maps have a natural scale invariance of the form

$$\Phi(x, t) \to \Phi(\frac{x}{\lambda}, \frac{t}{\lambda}).$$

As a consequence, the Sobolev norms $||\Phi||_{\dot{H}^s}$ become dimensionless when $s = n/2$.

This number is referred to as the *critical -regularity- exponent* or alternatively $||\cdot||_{\dot{H}^{n/2}}$ as the *critical* or *scale invariant* norm. One expects well posedness for the Cauchy problem with data in a Sobolev space with exponent above the critical one (i.e. subcritical case).

In short, wave maps have a natural scaling from where the *critical index* for the Cauchy problem with data in the Sobolev spaces is $s_c = \dfrac{n}{2}$.

Classical energy estimates for equations of the form $\Box u = F(u, \partial u)$ - hence for wave maps- imply local well-posedness of the Cauchy problem in H^s, $s > \frac{n}{2}+1$. The special structure associated to the nonlinearity in the wave map system however, allows for improvement.

0.1. Some background.

• When $n = 1$ global existence and regularity of wave maps with smooth data into complete Riemannian manifolds was established by Gu Chao-Hao [**12**] and Ginibre-Velo [**10**]. The idea was to use *characteristic coordinates* in $\mathbb{R}^{1+1}$,

$$\eta = t + r, \qquad \xi = t - r$$

to rewrite the wave map system in the form

$$-u_{\eta\,\xi} = A(u)(\partial_\eta u, \partial_\xi u).$$

The latter form allowed them to use L^∞ estimates to obtain g.w.p. of finite energy solutions.

• Keel and Tao [**16**] studied the one dimensional Cauchy problem with data in $H^s \times H^{s-1}$ further and in particular established local well-posedness. for $s > 1/2$ and global well-posedness when the target is a sphere and $s > 3/4$.

• In higher dimensions there are several special existence type results; such as the global well-posedness for smooth Cauchy data close to a geodesic and more. (c.f. [**4**] [**5**] [**31**])

• When $n \geq 3$ Shatah [**27**]; Cazenave, Shatah, Tahvildar-Zadeh [**3**] showed that solutions to the Cauchy problem for wave maps may blow up in finite time. Singularities can form from large data even when data is smooth and rotationally symmetric. Targets could be quite general as well (e.g. in $n \geq 4$ could be convex manifolds).

• When $n = 2$ (energy critical case) the first results were for equivariant maps. Equivariant maps give rise to semilinear wave equations in $\mathbb{R}^{n+3}$ spatial dimensions with critical growth. The structure of the nonlinearity is determined by the geometry of the target manifold.

Christodoulou and Shatah [**6**] obtained regularity of equivariant maps to hyperbolic space. Christodoulou and Tahvildar-Zadeh [**7**] studied the regularity of spherically symmetric wave maps assuming a convexity condition in the target manifold. Shatah and Tahvildar-Zadeh [**30**] studied the regularity of equivariant wave maps into two-dimensional rotationally symmetric and geodesically convex Riemannian manifolds. More work in this area has been done by M. Grillakis [**11**] and more recently by M. Struwe [**34**] who relaxed the convexity assumption.

• When $n \geq 2$ Klainerman and Machedon [**17**] [**18**] and Klainerman and Selberg [**22**] obtained the 'almost optimal' local well posedness results for the Cauchy problem for wave maps. That is, for data $(\Phi, \partial_t\Phi) \restriction_{t=0} = (f, g) \in H^s \times H^{s-1}$ with $s > n/2$, they showed that the Wave Map system is locally well posed. The so called '*null form* structure of the nonlinearity played an important role in their work [**19**][**20**].

0.2. Two pivotal breakthroughs at critical level. The first one came from Tataru when $n \geq 5$ [**38**] and for $n = 2, 3\ldots$ afterward [**39**]. His results established that for $n \geq 2$ and Cauchy data $(\Phi, \partial_t\Phi)\restriction_{t=0} = (f, g) \in \dot{B}^{n/2}_{2,1} \times \dot{B}^{n/2-1}_{2,1}$ *sufficiently small* there is global well posedness, regularity and scattering for wave maps.

An important reason for studying well posedness in Besov spaces instead was that $\dot{B}^{n/2}_{2,1} \hookrightarrow L^\infty$ while $\dot{H}^{n/2}$ does not embed into L^∞. The latter is a big problem; for starters one cannot localize to small coordinate patches, and loses worse as well.

The second one came shortly afterward by T. Tao , once again first for $n \geq 5$ [**35**] and later for $n = 2, 3, \dots$ [**36**]. Tao studied the *global regularity for small data* problem, asking whether solutions to (WM) corresponding to smooth initial data, *small* in $\|\cdot\|_{s_c}$ stay smooth for all time. He proved the following.

For $G = \mathbb{S}^m$, $n \geq 2$ and Cauchy data $(\Phi, \partial_t \Phi)\restriction_{t=0} = (f, g) \in H^s \times H^{s-1}$ with $s > n/2$ and critical norm $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ *sufficiently small*, WM have global regularity. Furthermore, for s close to $n/2$ have global bounds

$$\| (\Phi, \partial_t \Phi) \|_{L^\infty_t (H^s_x \times H^{s-1}_x)} \lesssim \|(f, g)\|_{H^s_x \times H^{s-1}_x}.$$

0.3. Recent developments. For spatial dimensions $n \geq 5$, similar results to those of Tao were obtained by Klainerman and Rodnianski [**21**] for target manifolds admitting a parallelizable structure (e.g. general Lie groups) . Roughly at the same time and independently, Shatah and Struwe [**28**] on the one hand and on the other Nahmod, Stefanov and Uhlenbeck [**25**] established the following result.

Let M be a compact Lie group or Riemannian symmetric space (e.g. $\mathbb{S}^m$)
Main Theorem *Let $n \geq 4$, $\Phi : \mathbb{R} \times \mathbb{R}^n \to M$. Suppose the Cauchy data $(\Phi, \partial_t \Phi)\restriction_{t=0} = (f, g)$ has sufficiently small norm in $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$. Then there exists a <u>unique</u> <u>global</u> solution to the WM problem such*

$$\|((\Phi, \partial_t \Phi)\|_{L^\infty_t (\dot{H}^{n/2}_x \times \dot{H}^{n/2-1}_x)} \lesssim \|(f, g)\|_{\dot{H}^{n/2}_x \times \dot{H}^{n/2-1}_x}.$$

Moreover, there is global regularity; i.e. if in addition $(f, g) \in H^s \times H^{s-1}$ with $s > n/2$ then the solution $(\Phi, \partial_t \Phi)$ belongs to $H^s \times H^{s-1}$ for all time and satisfies global bounds

$$\| (\Phi, \partial_t \Phi) \|_{L^\infty_t (H^s_x \times H^{s-1}_x)} \lesssim \|(f, g)\|_{H^s_x \times H^{s-1}_x}.$$

Remarks

• Shatah and Struwe's result is more general. In their case the target is any complete Riemannian manifold with bounded geometry.

The restriction in [**25**] to compact manifolds stemmed from their use of a Nash embedding into an Euclidean space with bounded geometry.

The authors of [**25**] learned later from Shatah and Struwe's work that Matthias Günther ([**13**] and references therein) extended Nash's embedding theorem to hold for any complete Riemannian manifold with bounded geometry (bounded second fundamental form) -not necessarily compact-; showing that it too is isometrically embedded into an Euclidean space with bounded geometry. Using this embedding, the results in [**25**] would extend to any complete Riemannian manifold with <u>constant</u> curvature.

• On the other hand Fabrice Planchon has pointed out that certain multiplications theorems for Besov spaces and L^∞ are sufficient to include variable curvature in our proof in [**25**]. Thus *a posteriori* our results seem might also hold in the case of variable bounded curvature $R(\Phi)(x)$ as well.

• The method in **[25]** combines both delicate techniques from harmonic analysis with fairly standard global gauge theoretic geometric methods. Both **[25]** and **[28]** works use the same gauge change : the Hodge or Coulomb gauge. The analytic approach is significantly different as Shatah-Struwe base their results on Lorentz spaces and we use Besov spaces. Besov spaces are contained in Lorentz spaces -for appropriate indexes-. Lorentz spaces seem better behaved under coordinate transformations.

• One indeed has well posedness for the *gauged map* but there are no estimates available on *differences* for the *original wave map* itself at the critical level. One cannot obtain any continuous dependence of the map on the data in the coordinate setting. The problem stems in that *well posedness* is not a gauge invariant notion; it is not even necessarily true that uniqueness in one coordinate system implies uniqueness in another directly. Hence, in none of the works above is possible to obtain (strong) well posedness at the critical level for the wave map itself.

We now proceed to explain the ideas behind **[21]**.

1. Wave Maps for $n \geq 4$

1.1. Beginning of the Proof. Regard the wave map equation as an equation given through covariant derivatives.

$$\Phi : \mathbb{R} \times \mathbb{R}^n \to M \qquad d\Phi : \mathcal{T}(\mathbb{R} \times \mathbb{R}^n) \to \mathcal{T}M$$

where M is an arbitrary Riemannian manifold and $\mathcal{T}(\mathbb{R}\times\mathbb{R}^n) = (\mathbb{R}\times\mathbb{R}^n)\times(\mathbb{R}\oplus\mathbb{R}^n)$. If we let $\Phi^*\nabla$ be the pullback of the Levi-Civita connection on M to $\Phi^*\mathcal{T}M$ via the map Φ then in coordinate free notation, where we have set $t = x^0$

$$\Phi^*\nabla_0 \frac{\partial \Phi}{\partial t} - \sum_{j=1}^{n} \Phi^*\nabla_j \frac{\partial \Phi}{\partial x^j} = 0 \tag{1}$$

• The Levi-Civita connection on M is torsion free; i.e. if we set $t = x^0$,

$$\Phi^*\nabla_j \frac{\partial \Phi}{\partial x^k} = \Phi^*\nabla_k \frac{\partial \Phi}{\partial x^j} \qquad \text{for } j = 0, 1, \ldots, n\,,\ k = 1, \ldots, n. \tag{2}$$

We also have control on the curvature of $\Phi^*\nabla$ via the equation

$$[\Phi^*\nabla_j, \Phi^*\nabla_k] = R(\Phi)\Big(\frac{\partial \Phi}{\partial x^j}, \frac{\partial \Phi}{\partial x^k}\Big) \tag{3}$$

The *wave map system* (1)-(3) is overdetermined. We assume the map $\Phi : \mathbb{R}^n \cup \{\infty\} \to M$ is topologically trivial. Hence, $\Phi^*\mathcal{T}M$ is the trivial bundle $(\mathbb{R}^1 \times \mathbb{R}^n) \times \mathbb{R}^m$. By our choice of target M , we have $R(\Phi) \equiv R$ constant.

Next, under smallness assumptions on $\Phi \in L_t^\infty \dot{W}_x^{1,n/2}$, we obtain a unique choice of coordinates for $\Phi^*\mathcal{T}M$. This follows from the existence and uniqueness of a *gauge change* g under suitable hypothesis on the space-time curvature $F_A = dA + [A, A]$ and connection A. The following result is proved following the methods used by K. Uhlenbeck in **[40]**.

Theorem 1 (Existence of a good gauge) *Let $d + A$ be a smooth connection with compact structure groups G over $\mathbb{R}\times\mathbb{R}^n$ or $I\times\mathbb{R}^n$. Assume $A \sim 0$ at spatial infinity and let $F_A = dA + [A, A]$ be the space-time curvature. Then there exists a positive constant $\epsilon = \epsilon(n, G)$ such that if the mixed space-time Lebesgue norm*

$$\|F_A\|_{L_t^\infty L_x^{n/2}} < \epsilon\,,$$

then, there exits a unique smooth gauge change g, $g \sim I$ at spatial infinity, such that if $\tilde{A} = gAg^{-1} - dg\, g^{-1}$ we have,

1. $\|\tilde{A}\|_{L^\infty_t \dot{W}^{1,n/2}_x} \leq c(n,G)\|F_A\|_{L^\infty_t L_x{}^{n/2}}$

2. $\sum_{j=1}^n \frac{\partial}{\partial x^j}\tilde{A}_j = 0$

Corollary *The above remains true if $A \in L^\infty_t \dot{W}^{1,n/2}_x$ and $F_A \in L^\infty_t L^{n/2}_x$.*

We now apply this gauge change in the wave map system (1)-(3). If we denote by $b = d\Phi$ schematically we get :

$$b_j \to g b_j g^{-1} := \tilde{b}_j$$

$$g s^* \nabla_j g^{-1} \to \frac{\partial}{\partial x_j} + \frac{1}{2} g b_j g^{-1} - dg g^{-1} := \frac{\partial}{\partial x_j} + a_j := D_j$$

In the new gauge, we have that $\operatorname{div} a = 0$ and the *same* equations (1)-(3) albeit with D_j replacing $\Phi^*\nabla_j$ and $\tilde{b}_j$, b_j.

More precisely, let $b = d\Phi$. Then the equations themselves are written

$$D_0 \tilde{b}_0 - \sum_{j=1}^n D_j \tilde{b}_j = 0.$$

$$D_k \tilde{b}_j = D_j \tilde{b}_k\,, \quad k = 0, 1, \ldots, n\,,\ j = 1, 2, \ldots, n.$$

This is a non-linear first order hyperbolic system. Moreover we also have

$$da + [a,a] + R[\tilde{b}, \tilde{b}] \ = \ 0.$$

All in all we have

$$\begin{array}{ll}
\text{(a)} & \frac{\partial}{\partial t}\tilde{b}_0 - \sum \frac{\partial}{\partial x_j}\tilde{b}_j + (a\cdot \tilde{b})_{\text{space-time}} = 0 \\
\text{(b)} & d\tilde{b} + a \wedge \tilde{b} = 0 \\
\text{(c)} & da + [a,a] = R[\tilde{b}, \tilde{b}] \\
\text{(d)} & \sum_{j=1}^n \frac{\partial}{\partial x^j}\, a_j = 0
\end{array}$$

We refer to this as the *Gauged Wave Map* system.

This system however, is still not as nice to work with because of the presence of the $\frac{\partial a}{\partial t}$ terms. So we go one step further and convert it to a single equation using Hodge theory. From now on we abuse notation and call $\tilde{b}$ just b and let $b = d\phi + d^*\psi$. Then we can show **[25]** that the gauged wave map system can be rewritten as

$$\begin{array}{ll}
\text{(a)} & \Box\phi + (a,b) = 0 \\
\text{(b)} & \Box\psi + a \wedge b = 0 \\
\text{(c)} & b = d\phi + d^*\psi \\
\text{(d)} & da + [a,a] = R[b,b] \\
\text{(e)} & \sum_{j=1}^n \frac{\partial}{\partial x^j}\, a_j = 0.
\end{array}$$

The initial data on ϕ and ψ can be taken to be

$$\begin{array}{ll}
\phi(0,x) = 0\,, & \psi(0,x) = 0 \\
\frac{\partial\phi}{\partial t}(0,x) = b_0(0,x) & \frac{\partial\psi}{\partial t}_{0,j}(0,x) = b_j(0,x) \\
\frac{\partial}{\partial t}\psi_{j,k}(0,x) = 0 & j,\, k \neq 0
\end{array}$$

Since the gauged wave map system is still overdetermined we consider a subset of it. Incorporating (e) into (d) we obtain the following.

Theorem 2 *Under our assumptions on the target manifold M, a* subset *of the* gauged wave map equations *(a)- -(e) has a structure of a non-linear wave system of integral differential operators. Namely,*

$$\begin{array}{ll}(a) & \Box\phi + (a,b) = 0 \\ (b) & \Box\psi + a \wedge b = 0 \\ (c) & b = d\phi + d^*\psi \\ (d) & \Delta a_j + \sum_{j=1}^{n} \frac{\partial}{\partial x^k}[a_k, a_j] + \frac{\partial}{\partial x^k}[b_k, b_j] = 0,\ j = 0, 1, \dots n\end{array}$$

We refer to this system as the Modified Wave Map system - or simply MWM .

The existence and uniqueness of *wave maps* will follow from the next Theorems 3-5. In section 1.2 we explain how the global well posedness and higher regularity of the MWM, together with a stronger uniqueness result come together to give the *Main Theorem* above.

Theorem 3 (Well posedness of the MWM) *There exists $\varepsilon > 0$ such that whenever the initial data*

$$||(f,g)||_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} < \varepsilon,$$

the system above has a unique global solution $v = (\phi, \psi)$ which belongs both to

$$\bullet\, L^\infty(\mathbb{R}; \dot{H}_x^{n/2}) \cap L^2(\mathbb{R}; \dot{B}^1_{2n,2}) \qquad \textit{and}$$

$$\bullet\, W^{1,\infty}(\mathbb{R}; \dot{H}_x^{n/2-1}) \cap W^{1,2}(\mathbb{R}; \dot{B}^0_{2n,2}).$$

Moreover, there is stability; i.e.

$$\mathit{esssup}_t ||v_1 - v_2||_{\dot{H}^{n/2}} \lesssim ||(f_1, g_1) - (f_2, g_2)||_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}}$$

provided the r.h.s. is small enough.

Remark The theorem above gives the *existence* part of our Main Theorem on *wave maps.* The uniqueness of solutions to the MWM however is *solely* in the Besov spaces which is **not enough** to claim the solution to the MWM system came from a wave map. Thus an additional argument is needed. The following is a stronger uniqueness result which will indeed suffice to return to the wave map (ie. will give uniqueness of the *original* wave map).

Theorem 4 (Uniqueness) *Suppose $(v_1\,,\, a_1)$ and $(v_2\,,\, a_2)$ are two solutions to*

$$\begin{aligned}\Box v + B(a, dv) &= 0 \\ \Delta a + divB(a,a) + divB(dv,dv) &= 0.\end{aligned}$$

such that $dv_j = b_j$, for $j = 1, 2$ are small in $L_t^\infty L_x^n$. Suppose that $dv_j = b_j \in L_t^2 L_x^{2n}$ for $j = 1, 2$. Assume in addition that $a_1 = a_1(v_1) \in L_t^1 L_x^\infty$. Then $v_1 = v_2$.

The smallness of dv_j in $L_t^\infty L_x^n$ is the necessary condition to solve the 'gauged' equation. The proof follows a scheme devised by Shatah-Struwe to establish uniqueness. Essentially follows via *energy estimates* under minimal assumptions on the solution. If $w = v_1 - v_2$, then $\exists\ h \in L_t^1(\mathbb{R})$ such that

$$\frac{1}{2}\partial_t ||Dw(t)||^2_{L^2_x} = \int_{\mathbb{R}^n} \langle \Box w, w_t \rangle dx \lesssim h(t) ||Dw(t)||^2_{L^2_x}.$$

Gronwall's Lemma then implies $w \equiv 0$.

Finally, by differentiating the MWM system and observing that the resulting nonlinearity has the same bilinear structure -for which the necessary 'multiplication estimates' hold- the following regularity result follows.

Theorem 5 (Higher Regularity)*Suppose the initial data (f, g) to (MWM) is in $H^{n/2+1} \times H^{n/2}$ and has sufficiently small $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ norm. Then the solution v to the Cauchy problem (MWM) with initial data (f, g) can be continued in $H^{n/2+1} \times H^{n/2}$ globally in time. Furthermore, we have the global bounds*

$$\|v\|_{L_t^\infty(\mathbb{R};\dot{H}_x^{n/2+1})} \lesssim \|(f,g)\|_{\dot{H}_x^{n/2+1} \times \dot{H}_x^{n/2}}.$$

1.2. The Return to the Map. The well-posedness results on the modified wave map apply to a larger class of formal solutions (a, b) to the equation than those which come from wave maps. Our method of using the results on the modified wave map equation to show existence of wave maps is similar to the idea we used for non-linear Schrödinger and not very different from the technique used by Shatah-Struwe Roughly: regularize the data to the WM system. Then it has a *local smooth* solution. If *in addition* the $\dot{H}^{n/2} \times \dot{H}^{n/2-1}$ norm of the data is *sufficiently small* the local smooth solution is *global and smooth* and satisfies the a priori global estimates satisfied by the solution to the MWM system. This a priori estimate is now used to pass to the limit. The translation depends on the compactness of M (or certain bounds on the isometric Nash embedding of a non-compact M in an Euclidean space).

2. Basic Littlewood-Paley theory

We now introduce and develop the techniques and tools from harmonic analysis that enter in the proof of the well posedness result of the MWM system in $n \geq 4$.

2.1. Background of the classical theory.

Let $f(x)$ be a function on $\mathbb{R}^n$ and $\hat{f}(\xi)$ its Fourier transform. Consider $m(\xi)$ to be a non-negative radial bump function supported on the ball $|\xi| \leq 2$ and equal to 1 on the ball $|\xi| \leq 1$. Then for each integer k let $P_k(f)$ be the *Littlewood-Paley* projection operator onto *frequencies* $|\xi| \lesssim 2^k$. This is defined by

$$\widehat{P_k(f)}(\xi) := m(2^{-k}\xi)\hat{f}(\xi).$$

• $P_k \to 0$ as $k \to -\infty$ and $P_k \to I$ as $k \to \infty$ in any reasonable sense (e.g. L^2). The function $P_k(f)(x)$ is a (smoothed) average of f localized to *physical scales* $\lesssim 2^{-k}$. By the uncertainty principle one expects $P_k(f)$ to be essentially constant at scales much smaller than 2^{-k}. The operator Q_k is the projection onto the *frequency annulus* $|\xi| \sim 2^k$ given by the formula,

$$Q_k := P_k - P_{k-1}.$$

Hence $\psi(\xi) := m(\xi) - m(2\xi)$ is supported on the annulus $1/2 \leq |\xi| \leq 2$, for all $\xi \neq 0$, $\sum_{k\in\mathbb{Z}} \psi(2^{-k}\xi) \equiv 1$, and

$$\widehat{Q_k(f)}(\xi) = \psi(2^{-k}\xi)\hat{f}(\xi).$$

The Littlewood-Paley projections are bounded operators in all the Lebesgue spaces. In fact, Q_k is given by a convolution kernel whose L^p-norm equals $2^{(kn)(1-1/p)}$,

$1 \leq p \leq \infty$. In particular its L^1- norm is identically 1 for all $k \in \mathbb{Z}$. It is essentially constant on physical scales $<< 2^{-k}$ and it has mean zero at scales $\lesssim 2^{-(k-10)}$. In fact, on a ball in physical space of radius $O(2^{-k})$ the function $Q_k(f)$ is *smooth* at physical scales $<< 2^{-k}$ and all moments of $Q_k(f)$ vanish: $\partial_\xi^\alpha(\widehat{Q_k(f)})(0) = 0$.

By telescoping the series we have the *Littlewood-Paley decomposition*

$$f := \sum_{k \in \mathbb{Z}} Q_k(f) \quad \text{in the sense of } L^2$$

or for any locally integrable function with decay at infinity. We have thus written f as a superposition of functions $Q_k(f)$, each of which has frequency of magnitude $\sim 2^k$. *Lower values* of k represent *low frequency* components of f. *Higher values* represent *high frequency* components.

The Haar system on $\mathbb{R}$ given by $h_I(x) = 2^{k/2}h(2^k - m)$ as $I = [2^{-k}m, 2^{-k}(m+1))$, $k, m \in \mathbb{Z}$ and where $h(x) = 1$ for $0 \leq x < 1/2$; $h(x) = -1$ for $1/2 \leq x < 1$ and $h(x) = 0$ otherwise is a good 'model' to bear in mind. This is the Walsh analogue of the Littlewood-Paley decomposition.

To relate the Littlewood-Paley pieces $Q_k(f)$ back to the function f itself, suppose $f \in L^2$. By construction and Plancherel,

$$||f||_2 \sim \big(\sum_k ||Q_k(f)||_2^2\big)^{1/2} \sim ||\big(\sum_k |Q_k(f)(\cdot)|^2\big)^{1/2}||_2$$

The *function* $S(f)(x) := \left(\sum_k |Q_k(f)(x)|^2\right)^{1/2}$ is known as the Littlewood-Paley *Square Function.* In general, for any $1 < p < \infty$, on has the *Littlewood-Paley Inequality:*

$$||Sf||_p \sim ||f||_p$$

The proof relies on standard harmonic analysis and follows in a straightforward fashion from Calderón-Zygmund theory.

We thus have a nice characterization of the Lebesgue spaces in terms of very friendly building blocks. From the PDE viewpoint one of the advantages of using Littlewood-Paley theory lies in the simple equivalence,

$$||\nabla Q_k(f)||_p \sim 2^k ||Q_k(f)||_p, \qquad 1 \leq p \leq \infty$$

Roughly, ∇ is multiplication by $2\pi i\xi$ and $|\xi| \sim 2^k$ on the support of $Q_k(f)$. So -morally- one can decompose a derivative as a linear combination of its LP pieces, $2^k\, Q_k(f)$. We see the the effect of a derivative on a function f is to *accentuate* the *high frequencies* and *diminish* the *low frequencies.* A similar principle applies, of course, to other differentiation or pseudo-differential operators such as $(-\Delta)^{s/2}$.

Thus Littlewood-Paley is nicely adapted to dealing with spaces which combine L^p-type norms with derivatives: *Sobolev spaces, Besov spaces, Hölder spaces, etc.* For example, the Sobolev spaces $W^{s,p}$, consisting of those functions f (distributions) such that f and the first s derivatives of f are in L^p can be very simply characterized using Littlewood-Paley decompositions. Indeed,

$$||f||_{W^{s,p}} \sim ||f||_p + |||\nabla|^s f||_p \quad \text{where} \quad \mathcal{F}(|\nabla|^s f)(\xi) =: (2\pi|\xi|)^s \hat{f}(\xi).$$

Then,

$$||f||_{W^{s,p}} \sim ||f||_p + ||\big(\sum_k |2^{ks}Q_k(f)|^2\big)^{1/2}||_p$$

Once again we note that $|\nabla|^s$ accentuates the k^{th} frequency piece of f by a factor of 2^{ks}; ie. $|\nabla|^s Q_k(f) \sim 2^{ks}Q_k(f)$.

2.2. Products and Product Estimates.

Let f and g be two nice functions. By splitting them using Littlewood-Paley decompositions we analyze *bilinear* expressions such as the pointwise product $B(f,g)(x) = f(x)g(x)$. Since $\widehat{(f\,g)}(\xi) = \int \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,d\eta$ we have that

$$\text{supp }\widehat{(f\,g)} \subseteq \text{ supp }\widehat{f} + \text{ supp }\widehat{g}$$

We write,

$$\begin{aligned} fg &= \textstyle\sum_{k,j} Q_k(f)Q_j(g) = \sum_{k\geq j} Q_k(f)Q_j(g) + \sum_{k<j} Q_k(f)Q_j(g) \\ &= \textstyle\sum_{k\in\mathbb{Z},m\geq 0} Q_k(f)Q_{k-m}(g) + \sum_{j\in\mathbb{Z},m>0} Q_{j-m}(f)Q_j(g) \end{aligned}$$

Further splitting gives,

$$\begin{aligned} fg &= \textstyle\sum_l Q_l(fg) \\ &= \textstyle\sum_{l,k}\sum_{m\geq 0} Q_l(Q_k(f)Q_{k-m}(g)) + \sum_{l,j}\sum_{m>0} Q_l(Q_{j-m}(f)Q_j(g)). \end{aligned}$$

By inspecting the Fourier supports we find that :

$$\text{(1)} \qquad \text{supp}\,(\widehat{Q_k(f)Q_{k-m}}(g)) \subseteq \{|\xi| \lesssim 2^k\}.$$

Hence

$$Q_l(Q_k(f)Q_{k-m}(g)) = 0 \text{ unless } k \geq l$$

$$\text{(2)} \quad \text{supp}\,(\widehat{Q_k(f)Q_{k-m}}(g)) \cap \{|\xi| << 2^{k-m}\} = \emptyset \text{ if } m > 5.$$

Hence,

$$\begin{aligned} Q_l(Q_k(f)Q_{k-m}(g)) &= 0 \quad \text{unless :} \\ & l = k \text{ and } m > 5 \qquad \text{or} \\ & l < k \text{ and } 0 \leq m \leq 5 \end{aligned}$$

All in all we only have three types of sums :

$$\begin{aligned} fg &= \textstyle\sum_l\sum_{m\geq 5} Q_l(Q_l(f)Q_{l-m}(g)) + \sum_{m=0}^{5}\sum_l\sum_{k>l} Q_l(Q_k(f)Q_{k-m}(g)) \\ &+ \textstyle\sum_l\sum_{m\geq 5} Q_l(Q_{l-m}(f)Q_l(g)) + \sum_{m=0}^{5}\sum_l\sum_{j>l} Q_l(Q_{j-m}(f)Q_j(g)) \end{aligned}$$

This could be re-written as well as :

$$\begin{gathered} fg = \textstyle\sum_\ell P_{\ell+1}(f)P_{\ell+1}(g) - P_\ell(f)P_\ell(g) \\ = \textstyle\sum_\ell Q_\ell(f)P_\ell(g) + \sum_\ell P_\ell(f)Q_\ell(g) + \sum_\ell Q_\ell(f)Q_\ell(g) \\ \text{Paraproduct} \;+\; \text{Paraproduct} \;+\; \text{Diagonal} \end{gathered}$$

In various applications the *high-low* interactions are 'easily dealt with' ($\sim$ paraproducts). It is the *high-high* interactions what usually may account for energy cascade effects; they are subtler to analyze.

We finish this brief introduction to Littlewood-Paley theory with two well known results that can be derived as applications of the above.

The Div-Curl Lemma

Convergence of approximating solutions for partial differential equations is not clear when weak continuity is not available. The notion of compensated compactness was developed to overcome these difficulties for non-linear equations in elasticity and fluid flow by exploiting cancellation properties of certain nonlinear quantities, usually bilinear, which arise naturally in studying the existence of global solutions. In this context, the decomposition above gives a very simple proof of the div-curl lemma used in compensated compactness.

If f, g are L^2 functions fg is only in L^1 which is not enough for weak continuity arguments since passing to weak limits is a discontinuous operation. If one considers $\varphi \in C_0^\infty$, then $\varphi e^{in\cdot} = f_n \rightharpoonup 0$ and $\varphi e^{-in\cdot} = g_n \rightharpoonup 0$. But $f_n g_n = \varphi^2$ does not converge weakly to 0. The *renormalization of the product* however, given by $fg - \sum_\ell Q_\ell(f)Q_\ell(g))$ is smoother than fg , and has *more cancellation.* Thus it is in a much 'regular' space, namely in the Hardy space $\mathcal{H}^1 \subset L^1$ which allows for weak continuity arguments to ensure convergence of approximating solutions. The point is that when $\operatorname{div} f = 0$ and $\operatorname{curl} g = 0$ one can essentially replace fg by its renormalization; thus the div-curl lemma of Murat and Tartar follows ([**8**]).

The Leibniz Rule for Fractional Derivatives

Let $s \in \mathbb{R}$, $s > 0$. Then for any $1 < p, q, r < \infty$ such that $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$,

$$\||\nabla|^s (fg)\|_r \lesssim \||\nabla|^s f\|_p \|g\|_q + \|f\|_p \||\nabla|^s g\|_q$$

This estimate has played a fundamental role in many nonlinear estimates such as those arising in well-posedeness problems below the energy norm. It essentially follows from the work of R. Coifman and Y. Meyer [**9**] and it is closely related to the Kato-Ponce commutator estimate [**14**].

3. Function Spaces and Multiplication Estimates

3.1. Set up and Function Spaces. For the linear wave equation,

$$\Box u \;=\; 0 \qquad u(0,x) = f(x), \quad \partial_t u(0,x) = g(x)$$

in higher dimensions it is simple to find an explicit formula by solving the equation using the Fourier transform:

$$\phi(t,x) \sim \int e^{ix\cdot\xi}\big(\cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi)\big)d\,\xi$$

These formulas can be rewritten by setting $\phi = \phi^+ + \phi^-$ where

$$\phi(t,x)^\pm \sim \int e^{ix\cdot\xi} e^{\pm it|\xi|} \hat{f_\pm}(\xi) d\,\xi$$

where

$$\hat{f_\pm}(\xi) = \hat{f}(\xi) \pm i\frac{\hat{g}(\xi)}{|\xi|}.$$

In other words,

$$\phi^\pm = e^{\pm it|\nabla|}\frac{1}{2}(f \;\mp\; i|\nabla|^{-1}g)$$

where we have denoted by $|\nabla|$ the pseudodifferential operator whose symbol is $m(\xi) = |\xi|$

In other words this shows that the *space- time* Fourier transform of ϕ defined by

$$\tilde{\phi}(\tau,\xi) := \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{-i(t\tau + x\cdot\xi)} \phi(t,x)\,dx\,dt$$

is a *distribution* supported on the *light cone* $|\tau| = |\xi|$ in $\mathbb{R} \times \mathbb{R}^n$. More precisely,

$$\tilde{\phi}^{\pm}(\tau,\xi) \sim \delta(\tau \mp |\xi|)\hat{f}_{\pm}(\xi)$$

are supported on the positive cone $\tau = |\xi|$ and negative cone $\tau = -|\tau|$ respectively. These formulas say that ϕ may be viewed as the (adjoint to the) *restriction* of the Fourier transform to the light cone. One may then try to perform a similar analysis than the one carried out by E. Stein and P. Thomas to studied the restriction to the sphere of the Fourier transform. The cone however is a non-compact manifold with one degenerate direction along which the curvature vanishes. Still, there are $n-1$ non-zero principal curvatures at each point; and this is actually enough to obtain good decay estimates for $\|\phi(t,\cdot)\|_{L^\infty_x}$ (i.e. ' the dispersive inequality') by stationary phase methods. (**[33]** and references therein).

For the inhomogeneous problem for the wave equation with zero initial data,

$$\Box\psi = F, \quad \psi(0,x) = 0, \quad \partial_t\psi(0,x) = 0.$$

We have by the *Duhamel's principle* that

$$\psi(t,x) \sim \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin((t-s)|\xi|)}{|\xi|} \widehat{F}(s,\xi)\,d\xi\,ds$$

Definition *Let $1 \le q,r \le \infty$. The space-time mixed Lebesgue spaces $L^q_t L^r_x$ is the set of functions ϕ on $\mathbb{R} \times \mathbb{R}^n$ with*

$$\Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} |\phi(t,x)|^r dx\Big)^{q/r} dt\Big)^{1/q} := \|\phi\|_{L^q_t L^r_x} < \infty$$

Strichartz Estimates for the Wave Equation *Let $n \ge 2$ and let u be a solution to the problem*

$$\Box u(t,x) = F(t,x), \quad t > 0, \quad x \in \mathbb{R}^n$$

with initial data

$$u(0,x) = f(x), \qquad \partial_t u(0,x) = g(x)$$

Then we have the estimate

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

provided

(1) the norms are dimensionally balanced (ie. they scale properly)

$$\frac{1}{q} + \frac{n}{r} \;=\; \frac{n}{2} - s \;=\; \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2$$

(2) $\dfrac{1}{\tilde{q}} + \dfrac{1}{\tilde{q}'} = 1$ *and* $\dfrac{1}{\tilde{r}} + \dfrac{1}{\tilde{r}'}$

(3) The pairs (q,r) and $(\tilde{q},\tilde{r})$ are wave admissible pairs. In other words they both belong to the set

$$\mathcal{A} := \{(q,r)\,:\, 2 \le q,\, r \le \infty,\ \frac{1}{q} + \frac{n-1}{2r} \le \frac{n-1}{4}\} \setminus \{(2,\infty) \text{ when } n = 3\}$$

For future reference, we say that $(q, r) \in \mathcal{A}$ is <u>*sharp*</u> <u>*admissible*</u> if $\frac{1}{q} + \frac{n-1}{2r} = \frac{n-1}{4}$.

Let $\phi(t, x)$ be now a function on $\mathbb{R} \times \mathbb{R}^n$ and $\hat{\phi}(t, \xi)$ be its *spatial* Fourier transform.

Just as before, for each integer k let $P_k(\phi)$ be the usual Littlewood-Paley projection operator onto spatial frequencies $|\xi| \lesssim 2^k$ and let the operator Q_k be the projection onto the *spatial* frequency annulus $|\xi| \sim 2^k$. We denote by $\phi_k(t, x) := Q_k(\phi)(t, x)$.

Introduce $\mathcal{S}_k$ the *localized Strichartz space* at frequency 2^k, as the set of functions ϕ_k whose space-time norm is given by:

$$||\phi_k||_{\mathcal{S}_k} := \sup_{(q,r) \in \mathcal{A}} 2^{k(\frac{1}{q} + \frac{n}{r})}(||\phi_k||_{L_t^q L_x^r} + 2^{-k}||\partial_t \phi_k||_{L_t^q L_x^r}).$$

where $\mathcal{A}$ is, as before, the set of wave admissible Strichartz exponents.

The space $\mathcal{S}$ is now defined as an l^2 based Besov space, i.e.

$$\mathcal{S} = \oplus_{l^2} \mathcal{S}_k = \{f : ||f||_{\mathcal{S}} = \left(\sum_k ||f||^2_{\mathcal{S}_k} \right)^{1/2} \}$$

We also introduce the space $\mathcal{S}^{(-1)}$, which is defined so that

$$\phi \in \mathcal{S} \quad \text{if and only if} \quad \partial_x \phi \in \mathcal{S}^{(-1)}.$$

If we denote by $b = \partial_x \phi$ then,

$$||b_k||_{\mathcal{S}_k^{(-1)}} := \sup_{(q,r) \in \mathcal{A}} 2^{k(\frac{1}{q} + \frac{n}{r} - 1)}(||b_k||_{L_t^q L_x^r} + 2^{-k}||\partial_t b_k||_{L_t^q L_x^r}).$$

- The spaces $\mathcal{S}$ are $\dot{H}^{n/2}$- normalized while the $\mathcal{S}^{(-1)}$ are $\dot{H}^{n/2-1}$ -normalized:

$$\begin{aligned} ||\partial_x^{n/2} Q_k(f)||_{L_t^\infty L_x^2} &\sim 2^{n/2k}||Q_k(f)||_{L_t^\infty L_x^2} \\ &\leq ||Q_k(f)||_{\mathcal{S}_k} \end{aligned}$$

Hence by taking ℓ^2 norms both sides

$$\Big(\sum_{k \in \mathbb{Z}} 2^{2k(n/2)}||Q_k(f)||^2_{L_t^\infty L_x^2}\Big)^{1/2} \lesssim ||f||_{\mathcal{S}}.$$

In other words,

$$\mathcal{S} \hookrightarrow L_t^\infty \dot{H}_x^{n/2}.$$

Moreover for $n \geq 4$ one has for example the estimates:

$$\begin{aligned} ||Q_k(\phi)||_{L_t^\infty L_x^2} &\leq 2^{-\frac{n}{2}k}\, ||Q_k(\phi)||_{\mathcal{S}_k} \\ ||Q_k(\phi)||_{L_t^2 L_x^{\frac{2(n-1)}{(n-3)}}} &\leq 2^{k(\frac{n}{(n-1)} - \frac{(n+1)}{2})}\, ||Q_k(\phi)||_{\mathcal{S}_k} \\ ||Q_k(\phi)||_{L_t^2 L_x^\infty} &\leq 2^{-\frac{k}{2}}\, ||Q_k(\phi)||_{\mathcal{S}_k}; \end{aligned}$$

from where we notice that to control high frequencies one should use $L_t^q L_x^r$ with small r; while large r is instrumental to control the low frequencies.

The Strichartz estimates in this context now read,

Strichartz Estimates Revisted (T. Tao; Keel-Tao) *Let $k \in \mathbb{Z}$ and let $Q_k(\phi)(t,x)$ be any function on $\mathbb{R} \times \mathbb{R}^n$ with spatial Fourier support on the annulus $|\xi| \sim 2^k$.*

(SE)

$$\|Q_k(\phi)\|_{S_k} \lesssim \|Q_k(\phi)(0,\cdot)\|_{\dot{H}_x^{n/2}} + \|\partial_t Q_k(\phi)(0,\cdot)\|_{\dot{H}_x^{n/2-1}} + 2^{k(\frac{n}{2}-1)}\|\Box Q_k(\phi)\|_{L_t^1 L_x^2}.$$

3.2. Multiplication Estimates. Denote by $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{G}$ the following sets of pairs.

$$\mathcal{C} =: \{(q,p) \in \mathcal{A} : \frac{1}{q} + \frac{n}{p} \leq 1^-\}$$

$$\mathcal{D} =: \{(q,p) : \frac{1}{2q} + \frac{n-1}{4p} \leq (\frac{n-1}{4})^-\}$$

$$\mathcal{E} =: \{(q,p) : \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2};\ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{with } (q_1,p_1) \in \mathcal{A} \text{ and } (q_2,p_2) \in \mathcal{C}\}$$

where $\mathcal{A}$ is the set of all wave admissible pairs. Finally, let

$$\mathcal{A} \subset \mathcal{G} := \mathcal{D} \cap \mathcal{E}.$$

Definition *Let $\mathcal{S}_+^{(-1)}$ be the space of functions on $\mathbb{R} \times \mathbb{R}^n$ whose norm is given by*

$$\|\phi\|_{\mathcal{S}_+^{(-1)}} := \sum_{k \in \mathbb{Z}} \|Q_k(\phi)\|_{S_{k}{}_+^{(-1)}}$$

where

$$\|\Phi\|_{S_{k}{}_+^{(-1)}} := \sup_{(q,p)\in\mathcal{G}} 2^{k(\frac{1}{q}+\frac{n}{p}-1)}(\|\Phi\|_{L_t^q L_x^p} + 2^{-k}\|\partial_t \Phi\|_{L_t^q L_x^p})$$

Let us denote by $|\nabla|^{-1} := \nabla\Delta^{-1}$ the pseudo-differential operator of order -1; i.e.

$$\widehat{|\nabla|^{-1} f}(t,\xi) = \frac{1}{|\xi|}\hat{f}(t,\xi).$$

The principal technical result in our main theorem is the 'multiplication estimate' :

$$\||\nabla|^{-1}(f \cdot g)\|_{\mathcal{S}_+^{(-1)}} \lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}},$$

The space $\mathcal{S}_+^{(-1)}$ is a similar to $\mathcal{S}^{(-1)}$ but with a larger norm that controls a *larger collection* of space-time Lebesgue norms than those in $\mathcal{A}$. For example when $n \geq 4$ one has the embeddings,

$$\mathcal{S}_+^{(-1)} \hookrightarrow \mathcal{S}^{(-1)} \cap L_t^q \dot{B}_{p,2}^s \quad \text{where} \quad q,p \geq 2;\ s = \tfrac{1}{q} + \tfrac{n}{p} - 1$$

$$\mathcal{S}_+^{-1} \hookrightarrow \{f : \textstyle\sum_k \|Q_k(f)\|_{L_t^1 L_x^\infty} < \infty\},$$

which are *crucial in closing the estimates.* In fact in **[25]** we show the following.

Main Multiplication Estimate

$$|\nabla|^{-1} : \mathcal{S}^{(-1)} \times \mathcal{S}^{(-1)} \longrightarrow \mathcal{S}_{+}^{(-1)}$$

In other words, for f, $g \in \mathcal{S}^{(-1)}$ we have

$$\sum_{l\in\mathbb{Z}} \sup_{(\tilde{q},\tilde{p})\in\mathcal{G}} 2^{l(1/\tilde{q}+n/\tilde{p}-1)} \quad \|Q_l(|\nabla|^{-1}(f\cdot g))\|_{L_t^{\tilde{q}}L_x^{\tilde{p}}}$$
$$\lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}}$$

Perform Littlewood-Paley decomposition on f, g, and the product $f \cdot g$ as we discuss earlier. Denote $f_k := Q_k(f)$. By studying the supports of the L-P projections and symmetry considerations the result follows from the following two estimates.

- High-High into Low:

$$\sum_{l\in\mathbb{Z}} \sup_{(\tilde{q},\tilde{p})\in\mathcal{G}} 2^{-l}\, 2^{l(1/\tilde{q}+n/\tilde{p}-1)} \sum_{k>l} \quad \|Q_l((f_k\cdot g_k))\|_{L_t^{\tilde{q}}L_x^{\tilde{p}}}$$
$$\lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}}$$

- High-Low into High:

$$\sum_{l\in\mathbb{Z}} \sup_{(\tilde{q},\tilde{p})\in\mathcal{G}} 2^{-l}\, 2^{l(1/\tilde{q}+n/\tilde{p}-1)} \|Q_l\big(\sum_{m>10} f_l\cdot g_{l-m}\big)\|_{L_t^{\tilde{q}}L_x^{\tilde{p}}}$$
$$\lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}}.$$

The dimension restriction enters only to control the *high-high* term.

In order to shed some light into the above we will present an alternative route which though longer might be more enlightening as to the action of $|\nabla|^{-1}$ on products of 'solutions'. It also establishes a somewhat stronger multiplication result. We start with an auxiliary Lemma.

Lemma 6 *Let $n \geq 4$ and let f be a function on $\mathcal{S}^{(-1)}$. For any $q \geq 2$, and p' defined by $\frac{1}{q}+\frac{n}{p'}=1$ we have that*

$$(i) \qquad \big(\sum_{k\in\mathbb{Z}} \|Q_k(f)\|^2_{L_t^q L_x^{p'}}\big)^{1/2} \lesssim \|f\|_{\mathcal{S}^{(-1)}}$$

In addition, we also have

$$(ii) \qquad \|f\|_{L_t^\infty \dot{W}^{n/2-1,2}} \lesssim \big(\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}\|Q_k(f)\|^2_{L_t^\infty L_x^2}\big)^{1/2} \lesssim \|f\|_{\mathcal{S}^{(-1)}}$$

The same conclusions hold for $2^{-k}\|\partial_t Q_k(f)\|$ replacing $\|Q_k(f)\|$.

Proof. Let $f_k = Q_k(f)$. Clearly,

$$\|\partial_x^{n/2-1} f_k\|_{L_t^\infty L_x^2} \sim 2^{k(n/2-1)}\|f_k\|_{L_t^\infty L_x^2} \leq \|f_k\|_{\mathcal{S}_k^{(-1)}}.$$

Then (ii) follows by taking ℓ^2 norms both sides.

To prove (i) we proceed as follows. Given $q \geq 2$, let p' be defined by $\frac{1}{q}+\frac{n}{p'}=1$. In particular we have that $4 \leq n \leq p' \leq 2n$. Since $n \geq 4$ we can now choose $2 \leq r < p'$ such that (q,r) is sharp admissible and

$$\|f_k\|_{L^q_t L^{p'}_x} \lesssim 2^{k\gamma}\|f_k\|_{L^q_t L^r_x}$$

by the Sobolev embedding where γ is given by and $\frac{1}{p'} = \frac{1}{r} - \frac{\gamma}{n}$.

In particular then $0 < \gamma = \frac{n}{r} - \frac{n}{p'} = \frac{n}{r} + \frac{1}{q} - 1$. From where we can conclude that

$$\|f_k\|_{L^q_t L^{p'}_x} \lesssim \|f_k\|_{S^{(-1)}_k}$$

since by Sobolev embedding , (c.f. Lemma 2.7 in [**25**]) we have that

$$\sup_{(q,r)-\text{admissible}} 2^{k(\frac{1}{q}+\frac{n}{r}-1)}\|Q_k(f)\|_{L^q_t L^r_x} = \sup_{(q,r)-\text{sharp admissible}} 2^{k(\frac{1}{q}+\frac{n}{r}-1)}\|Q_k(f)\|_{L^q_t L^r_x}.$$

The desired conclusion follows by taking ℓ^2 norms both sides $\square$.

First Multiplication estimate *For $1 \le q < \infty$, let $\mathcal{L}^1_t \dot{B}^0_{\infty,q}$ be the Banach space of functions on $\mathbb{R} \times \mathbb{R}^n$ whose norm is given by*

$$\|f\|_{\mathcal{L}^1_t \dot{B}^0_{\infty,q}} := \Big(\sum_{k\in\mathbb{Z}} \|Q_k(f)\|^q_{L^1_t L^\infty_x}\Big)^{1/q}.$$

Then

$$|\nabla|^{-1} : \mathcal{S}^{(-1)} \times \mathcal{S}^{(-1)} \longrightarrow \mathcal{L}^1_t \dot{B}^0_{\infty,1}.$$

Proof. Let f and g be in $\mathcal{S}^{(-1)}$ and let $f_k = Q_k(f)$ and $g_j = Q_j(g)$ be their corresponding Littlewood-Paley projections. We write

$$\begin{aligned} |\nabla|^{-1}(f \cdot g) &= |\nabla|^{-1}\Big(\sum_{k,j\in\mathbb{Z}} f_k \cdot g_j\Big) \\ &= |\nabla|^{-1}\Big(\sum_{k,j\in\mathbb{Z}:k\ge j} f_k \cdot g_j\Big) + |\nabla|^{-1}\Big(\sum_{k,j\in\mathbb{Z}:k<j} f_k \cdot g_j\Big). \end{aligned}$$

By symmetry of the sums, it is enough to consider only one of them. The proof for the other is identical after exchanging k and j. Hence we need to estimate

$$\begin{aligned} &\sum_{l\in\mathbb{Z}} \|Q_l\big(|\nabla|^{-1}\big(\sum_{k,j\in\mathbb{Z}:k\ge j} f_k \cdot g_j\big)\big)\|_{L^1_t L^\infty_x} \\ &= \sum_{l\in\mathbb{Z}} \|Q_l\big(|\nabla|^{-1}\big(\sum_{k\in\mathbb{Z}}\sum_{m\ge 0} f_k \cdot g_{k-m}\big)\big)\|_{L^1_t L^\infty_x} \end{aligned}$$

Since supp $(\widehat{f_k \cdot g_{k-m}}) \subseteq \{\xi : |\xi| \le 2^k\}$ we have that $Q_l(f_k \cdot g_{k-m}) \equiv 0$ unless $k \ge l$.

Therefore we can make the last sum less than or equal to

$$\sum_{l\in\mathbb{Z}} 2^{-l} \sum_{k\ge l}\sum_{m\ge 0} \|Q_l(f_k \cdot g_{k-m})\|_{L^1_t L^\infty_x}.$$

On the other hand, we have that supp $(\widehat{f_k \cdot g_{k-m}}) \cap \{\xi : |\xi| << 2^{k-m}\} = \emptyset$ if $m > 5$

Hence, $Q_l(f_k \cdot g_{k-m}) \equiv 0$ unless $l = k$ and $m > 5$ or $m \le 5$ and $l < k$.

We must then have that the above sum is

$$\lesssim \sum_{0\leq m\leq 5}\sum_{l\in\mathbb{Z}}2^{-l}\sum_{k>l}\|Q_l(f_k\cdot g_{k-m})\|_{L^1_tL^\infty_x}+\sum_{m>5}\sum_{l\in\mathbb{Z}}2^{-l}\|Q_l(f_l\cdot g_{l-m})\|_{L^1_tL^\infty_x}.$$

We consider the first sum first.

$$\sum_{0\leq m\leq 5}\sum_{l\in\mathbb{Z}}2^{-l}\sum_{k>l}\|Q_l(f_k\cdot g_{k-m})\|_{L^1_tL^\infty_x}$$
$$\lesssim \sum_{0\leq m\leq 5}\sum_{l\in\mathbb{Z}}2^{-l}\sum_{k>l}(2^{nl})^{\frac{n-3}{n-1}}\|f_k\|_{L^2_tL^{2p}_x}\|g_{k-m}\|_{L^2_tL^{2p}_x},$$

by Young's inequality with $p=\frac{n-1}{n-3}$, $\frac{1}{p}+\frac{1}{p'}=1+\frac{1}{\infty}$ and Cauchy-Schwartz inequality.
The endpoint Strichartz estimates (2.1) now yield the bound

$$\sum_{0\leq m\leq 5}\sum_{l\in\mathbb{Z}}2^{-l}\sum_{k>l}(2^{nl})^{\frac{n-3}{n-1}}2^{2k(1+\frac{n}{(n-1)}-\frac{(n+1)}{2})}\|f_k\|_{\mathcal{S}_k^{(-1)}}\|g_{k-m}\|_{\mathcal{S}_k^{(-1)}}$$
$$\sim \sum_{0\leq m\leq 5}\sum_{k\in\mathbb{Z}}\sum_{l\leq k}2^{l\frac{w}{n-1}}2^{-k\frac{w}{n-1}}\|f_k\|_{\mathcal{S}_k^{(-1)}}\|g_{k-m}\|_{\mathcal{S}_k^{(-1)}}$$

where $w=(n-2)^2-3$ which is positive provided $n\geq 4$. Hence by summing first in l and then applying Cauchy-Schwartz to the sum in k we get that the above is $\lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}}$ as desired.

We proceed next with the second sum.

$$\sum_{m>5}\sum_{l\in\mathbb{Z}}2^{-l}\|Q_l(f_l\cdot g_{l-m})\|_{L^1_tL^\infty_x}$$
$$\lesssim \sum_{m>5}\sum_{l\in\mathbb{Z}}2^{-l}2^{nl/r}\|(f_l\cdot g_{l-m})\|_{L^1_tL^r_x}$$

by Young's inequality with $r=2n$. Now, by Hölder's inequality we can bound the last sum by

$$\sum_{m>5}\sum_{l\in\mathbb{Z}}2^{-l}2^{nl/r}\|f_l\|_{L^2_tL^{2r}_x}\|g_{l-m}\|_{L^2_tL^{2r}_x}.$$

Since the pair $(2,2r)$ is admissible we have by the Strichartz estimates that the above sum is up to a constant less than or equal to

$$\sum_{m>5}\sum_{l\in\mathbb{Z}}2^{-l}2^{nl/r}2^{2l(1/2-n/(2r))}2^{-m(1/2-n/(2r))}\|f_l\|_{\mathcal{S}_l^{(-1)}}\|g_{l-m}\|_{\mathcal{S}_l^{(-1)}}$$
$$\lesssim \sum_{m>5}2^{-m(1/2-n/(2r))}\sum_{l\in\mathbb{Z}}\|f_l\|_{\mathcal{S}_l^{(-1)}}\|g_{l-m}\|_{\mathcal{S}_l^{(-1)}}$$
$$\lesssim \sum_{m>5}2^{-m(1/2-n/(2r))}\Big(\sum_{l\in\mathbb{Z}}\|f_l\|^2_{\mathcal{S}_l^{(-1)}}\Big)^{1/2}\Big(\sum_{l\in\mathbb{Z}}\|g_{l-m}\|^2_{\mathcal{S}_l^{(-1)}}\Big)^{1/2}$$
$$\lesssim \|f\|_{\mathcal{S}^{(-1)}}\|g\|_{\mathcal{S}^{(-1)}}$$

since by our choice of r, $1/2-n/(2r)=1/4>0$ □.

Second Multiplication estimate. Let $q\geq 2$ and $p<2$ such that $\dfrac{1}{q}+\dfrac{n}{p}=\dfrac{n}{2}+1$. Then

$$|\nabla|^{-1}:\mathcal{S}^{(-1)}\times\mathcal{S}^{(-1)}\longrightarrow L^q_t\dot{B}^s_{\tilde{p},2}$$

for any $\tilde{p}\geq p$ and $s=\frac{1}{q}+\frac{n}{\tilde{p}}-1$.

In particular, we have that $|\nabla|^{-1}: \mathcal{S}^{(-1)} \times \mathcal{S}^{(-1)} \longrightarrow \mathcal{L}^2_t \dot{B}^s_{\tilde{p},2}$; that is

$$\Big(\sum_k 2^{2ks} \|Q_k(|\nabla|^{-1}(f\cdot g)\|^2_{L^2_t L^{\tilde{p}}_x}\Big)^{1/2} \lesssim \|f\|_{\mathcal{S}^{(-1)}} \|g\|_{\mathcal{S}^{(-1)}}$$

for any $2 \leq \tilde{p} < 2n$ and $s = \frac{n}{\tilde{p}} - \frac{1}{2}$.

Proof Let f and g be two function on $\mathcal{S}^{(-1)}$.

Let $q \geq 2$ and let p' and p be defined by

$$\frac{1}{q} + \frac{n}{p'} = 1 \quad \text{and} \quad \frac{1}{p} = \frac{1}{2} + \frac{1}{p'}.$$

Claim:

$$\||\nabla|^{-1}(f\cdot g)\|_{L^q_t \dot{W}^{n/2,p}} \lesssim \|f\|_{\mathcal{S}^{(-1)}} \|g\|_{\mathcal{S}^{(-1)}}.$$

Assuming the claim we note that since $p < 2$ and $\frac{1}{q} + \frac{n}{p} = 1 + \frac{n}{2}$

$$L^q_t \dot{W}^{n/2,p} \hookrightarrow L^q_t \dot{B}^{n/2}_{p,2} \hookrightarrow L^q_t \dot{B}^s_{\tilde{p},2}$$

provided $s = \frac{1}{q} + \frac{n}{\tilde{p}} - 1$ and $\tilde{p} \geq p$ as desired.

To prove the claim we first note that by Lemma 6 we have, in particular, the following two 'endpoint estimates'.

$$\|f\|_{L^\infty_t \dot{W}^{n/2-1,2}} \lesssim \|f\|_{\mathcal{S}^{(-1)}}. \tag{i}$$

$$\|f\|_{L^q_t L^{p'}_x} \lesssim \|f\|_{L^q_t \dot{B}^0_{p',2}} \lesssim \|f\|_{\mathcal{S}^{(-1)}}. \tag{ii}$$

since $p' \geq 4 > 2$.

If $n = 4$, the above two estimates suffice. For then,

$$\|f\cdot g\|_{L^q_t \dot{W}^{n/2-1,p}} \lesssim \|f\|_{\mathcal{S}^{(-1)}} \|g\|_{\mathcal{S}^{(-1)}} \text{ where } \frac{1}{p} = \frac{1}{2} + \frac{1}{p'}$$

since $p' \geq n$. In turn, this implies that

$$|\nabla|^{-1}: \mathcal{S}^{(-1)} \times \mathcal{S}^{(-1)} \longrightarrow L^q_t \dot{W}^{n/2,p}$$

as desired.

In the case $n \geq 5$ however, we need to prove additional estimates. We consider separately the cases when n is *even* first and then indicate the necessary modifications when n is *odd*.

More precisely, let $n \geq 5$ be even. Given q, p' as above and $1 \leq j \leq \frac{n}{2} - 2$ let $0 < \theta_j < 1$ be defined by $\theta_j = \frac{j}{(n/2-1)}$. Next, let q_j, $q_{n/2-1-j}$ and p'_j, $p'_{n/2-1-j}$ be any solutions to the following equations:

$$\frac{1}{q_j} + \frac{n}{p'_j} = 1 \qquad \text{and} \qquad \frac{1}{q_{n/2-1-j}} + \frac{n}{p'_{n/2-1-j}} = 1$$

$$\frac{1-\theta_j}{q_j} + \frac{\theta_j}{q_{n/2-1-j}} = \frac{1}{q}, \qquad \frac{1-\theta_j}{p'_j} + \frac{\theta_j}{p'_{n/2-1-j}} = \frac{1}{p'}$$

Next, let $\frac{1}{\tilde{q}_j} = \frac{1-\theta_j}{q_j}$ and $\frac{1}{\tilde{p}'_j} = \frac{1-\theta_j}{p'_j} + \frac{\theta_j}{2}$.

We claim that

$$(iii) \qquad \|\partial_x^j f\|_{L_t^{\tilde q_j} L^{\tilde p'_j}} \lesssim \|\partial_x^j f\|_{L_t^{\tilde q_j} \dot B^0_{\tilde p'_j,2}} \lesssim \|f\|_{\mathcal{S}^{(-1)}}.$$

Indeed, the first inequality follows from the embeddings between Sobolev and Besov spaces since $\tilde p'_j \geq 2$. For the second one we have that

$$\begin{aligned}
\|\partial_x^j f_k\|_{L_t^{\tilde q_j} L^{\tilde p'_j}} &\sim 2^{kj} \|f_k\|_{L_t^{\tilde q_j} L^{\tilde p'_j}} \\
&= 2^{k(\theta_j(n/2-1))} \|f_k\|_{L_t^{\tilde q_j} L^{\tilde p'_j}} \\
&= 2^{k(\frac{1}{\tilde q_j} + \frac{n}{\tilde p'_j} - 1)} \|f_k\|_{L_t^{\tilde q_j} L^{\tilde p'_j}} \lesssim \|f_k\|_{S_k^{(-1)}}
\end{aligned}$$

whence the second inequality follows by taking ℓ^2-norms.

Finally we put together (i), (ii) and (iii) to obtain that

$$\begin{aligned}
\|f \cdot g\|_{L_t^q \dot W^{n/2-1,p}} &= \|\partial_x^{n/2-1}(f \cdot g)\|_{L_t^q L_x^p} \\
&\lesssim \sum_{j=0}^{n/2-1} \|\partial_x^j f\|_{L_t^{\tilde q_j} L_x^{\tilde p'_j}} \|\partial_x^{n/2-1-j} g\|_{L_t^{\tilde q_{n/2-1-j}} L_x^{\tilde p'_{n/2-1-j}}} \\
&\lesssim \|f\|_{\mathcal{S}^{(-1)}} \|g\|_{\mathcal{S}^{(-1)}},
\end{aligned}$$

as desired.

We indicate now the technicalities needed when $n \geq 5$ is odd. Given q, p' as above and $0 \leq j \leq [\frac{n}{2}] - 1$ let $0 < \theta_j < 1$ to be defined in a moment. As before, let q_j, $q_{n/2-1-j}$ and p'_j, $p'_{n/2-1-j}$ be any solutions of the equations as above for θ_j and as before let $\frac{1}{\tilde q_j} = \frac{1-\theta_j}{q_j}$ and $\frac{1}{\tilde p'_j} = \frac{1-\theta_j}{p'_j} + \frac{\theta_j}{2}$. Now,

$$\begin{aligned}
\|f \cdot g\|_{L_t^q \dot W^{n/2-1,p}} &= \|\partial_x^{n/2-1}(f \cdot g)\|_{L_t^q L_x^p} \\
&\lesssim \sum_{j=0}^{[n/2]-1} \Big(\|\partial_x^{j+1/2} f\|_{L_t^{\tilde q_j} L_x^{\tilde p'_j}} \|\partial_x^{n/2-j-3/2} g\|_{L_t^{\tilde q_{n/2-1-j}} L_x^{\tilde p'_{n/2-1-j}}} + \\
&\qquad + \|\partial_x^j f\|_{L_t^{\tilde q_j} L_x^{\tilde p'_j}} \|\partial_x^{n/2-j-1} g\|_{L_t^{\tilde q_{n/2-1-j}} L_x^{\tilde p'_{n/2-1-j}}} \Big)
\end{aligned}$$

For the first term inside the big sum we take $\theta_j = \frac{j+1/2}{(n/2-1)}$; note that $\theta_{n/2-j-1} = 1 - \theta_j = \frac{n/2-j-3/2}{n/2-1}$. Then for $0 \leq j \leq [n/2]-1$ we have that

$$\|\partial_x^{j+1/2} f\|_{L_t^{\tilde q_j} L_x^{\tilde p'_j}} \lesssim \|\partial_x^{j+1/2} f\|_{L_t^{\tilde q_j} \dot B^0_{\tilde p'_j,2}} \lesssim \|f\|_{\mathcal{S}^{(-1)}};$$

while

$$\|\partial_x^{n/2-j-3/2} g\|_{L_t^{\tilde q_{n/2-1-j}} L_x^{\tilde p'_{n/2-1-j}}} \lesssim \|\partial_x^{n/2-j-3/2} g\|_{L_t^{\tilde q_{n/2-j-1}} \dot B^0_{\tilde p'_{n/2-j-1},2}} \lesssim \|g\|_{\mathcal{S}^{(-1)}}$$

For the second we take $\theta_j = \frac{j}{(n/2-1)}$ as in (iii), and the needed estimates follow just as in the even case.

All in all we have that

$$\|f \cdot g\|_{L^q_t \dot{W}^{n/2-1,p}} \lesssim \|f\|_{\mathcal{S}^{(-1)}} \|g\|_{\mathcal{S}^{(-1)}},$$

as desired $\square$.

4. The Modified Wave Map System

The general scheme to find a solution to a nonlinear wave equation

$$\begin{aligned} \square u &= \mathcal{N}(u) \\ u(\cdot, 0) &= f \\ \partial_t u(\cdot, 0) &= g, \end{aligned}$$

relies on Picard iteration. We denote by $u_{-1} \equiv 0$ and let u_0 be the solution to the homogeneous problem

$$\square u_0 = 0 \qquad u_0(\cdot, 0) = f, \quad \partial_t u_0(\cdot, 0) = g.$$

Subsequent iterates u_m, $m \geq 1$ are obtained by solving

$$\square u_m = \mathcal{N}(u_{m-1}) \qquad u_m(\cdot, 0) = f \quad \partial_t u_m(\cdot, 0) = g.$$

In other words, formally

$$u_m = u_0 + \square^{-1} \mathcal{N}(u_{m-1}), \quad m \geq 1$$

where by $\square^{-1}$ we really mean the Duhamel operator giving the solution to the inhomogeneous linear problem with zero data as described in section 3.1.

To find a solution then we need to identify two Banach spaces $\mathcal{X}$, the '*solution space*' and $\mathcal{Y}$ the '*nonlinearity space*' such that:

- Free solutions $\subset \mathcal{X}$ with norm controlled by that of the initial data
- $u_{m-1} \in \mathcal{X}$ implies that $u_m \in \mathcal{X}$ (for $u \in \mathcal{X}$, $\mathcal{N}(u) \in \mathcal{Y}$ and $\square^{-1} \mathcal{Y} \subset \mathcal{X}$)
- $\{u_m\}$ is bounded in $\mathcal{X}$
- $\{u_m\}$ is Cauchy in $\mathcal{X}$ (estimates on differences).

We will follow the above scheme to study the well-posedness of the MWM system. Since $n \geq 4$ the actual form of the bilinear nonlinearity on the right hand side of the MWM plays no role: estimates on products are sufficient. Thus for simplicity we denote by $B(a, b)$ any finite linear combination of functions $a \in \mathcal{S}_+^{(-1)}$ and $b \in \mathcal{S}^{(-1)}$ of the form $\sum_{\kappa, \ell} c_{\kappa \ell} a_\kappa b_\ell$ where $a_\kappa \in \mathcal{S}_+^{(-1)}$, $b_\ell \in \mathcal{S}^{(-1)}$ and $c_{\kappa \ell} \in \mathbb{C}$.

Thus consider the MWM system of coupled wave equations in R^{n+1}, $n \geq 4$.

$$\begin{aligned} \square v &= B(a, b) \sim a \cdot b \\ v(x, 0) &= f(x) \\ v_t(x, 0) &= g(x) \end{aligned}$$

where $v = (\varphi, \psi)$, $b = d\varphi + \, div_{(sp.t)}\, \psi$ and

$$da = [a, a] + [b, b], \quad i.e. \quad a = |\nabla|^{-1}[a, a] + |\nabla|^{-1}[b, b]$$

In our case the space $\mathcal{X}$ will be $\mathcal{S}$. The nonlinearity space $\mathcal{Y}$ is determined by the following result:

Theorem 7 (Main Nonlinear Estimate) *Let $a \in \mathcal{S}_+^{(-1)}$ and $b \in \mathcal{S}^{(-1)}$ then*

$$\Big(\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||Q_k(a\cdot b)||^2_{L^1_tL^2_x}\Big)^{1/2} \lesssim ||a||_{\mathcal{S}_+^{(-1)}}||b||_{\mathcal{S}^{(-1)}}$$

The proof starts by performing a Littlewood-Paley decomposition of a and b to obtain

$$\begin{aligned}
&\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||Q_k(a\cdot b)||^2_{L^1_tL^2_x} \lesssim \\
&\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||\sum_{m>5}\big(Q_k(Q_k(a)\cdot Q_{k-m}(b))\big)||^2_{L^1_tL^2_x} + \\
&\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||\sum_{m>5}\big(Q_k(Q_{k-m}(a)\cdot Q_k(b))\big)||^2_{L^1_tL^2_x} + \\
&\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||\sum_{k<l}\big(Q_k(Q_l(a)\cdot Q_l(b))\big)||^2_{L^1_tL^2_x}.
\end{aligned}$$

Now since a and b belong to different spaces we lose the 'symmetry' and need to consider all three cases separately. We show how the most delicate case of *low frequencies* in the curvature term a versus *high frequencies in b* proceeds:

$$\begin{aligned}
&\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||\sum_{m>5}\big(Q_k(Q_k(b)\cdot Q_{k-m}(a))\big)||^2_{L^1_tL^2_x} \\
&\lesssim \textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||Q_k(b)||^2_{L^\infty_tL^2_x}\big(\sum_{m>5}||Q_{k-m}(a)||_{L^1_tL^\infty_x}\big)^2 \\
&\lesssim ||a||^2_{\mathcal{S}_+^{(-1)}}\ \textstyle\sum_{k\in\mathbb{Z}}||Q_k(b)||^2_{\mathcal{S}_k^{(-1)}} \\
&\lesssim ||a||^2_{\mathcal{S}_+^{(-1)}}||b||^2_{\mathcal{S}^{(-1)}}
\end{aligned}$$

invoking the fact that

$$a \in \mathcal{S}_+^{(-1)} \hookrightarrow \{f : \sum_k ||Q_k(f)||_{L^1_tL^\infty_x} < \infty\}$$

Lemma 8 (A priori estimate) *Let $a \in \mathcal{S}_+^{(-1)}$ and $b \in \mathcal{S}^{(-1)}$. Then the solution to the MWM system with initial data $(f,g) \in \dot{H}^{n/2}\times\dot{H}^{n/2-1}$ satisfies*

$$||v||_{\mathcal{S}} \lesssim ||f||_{\dot{H}^{n/2}} + ||g||_{\dot{H}^{n/2-1}} + ||a||_{\mathcal{S}_+^{(-1)}}||b||_{\mathcal{S}^{(-1)}}$$

Proof Let us denote by $v_k = Q_k(v)$. By the Strichartz's estimates (SE) at the end of section 3.1 , we have that

$$||v_k||_{\mathcal{S}_k} \lesssim ||f_k||_{\dot{H}^{n/2}} + ||g_k||_{\dot{H}^{n/2-1}} + 2^{k(n/2-1)}||B(a,b)||_{L^1_tL^2_x}.$$

The nonlinear estimate above then gives that

$$\begin{aligned}
||v||_{\mathcal{S}} &= \big(\textstyle\sum_{k\in\mathbb{Z}}||v_k||^2_{\mathcal{S}_k}\big)^{1/2} \\
&\lesssim ||f||_{\dot{H}^{n/2}} + ||g||_{\dot{H}^{n/2-1}} + \big(\textstyle\sum_{k\in\mathbb{Z}} 2^{2k(n/2-1)}||B(a,b)||^2_{L^1_tL^2_x}\big)^{1/2} \\
&\lesssim ||f||_{\dot{H}^{n/2}} + ||g||_{\dot{H}^{n/2-1}} + ||a||_{\mathcal{S}_+^{(-1)}}||b||_{\mathcal{S}^{(-1)}}
\end{aligned}$$

as desired. □

We turn now to the proof of global well posedness of the MWM with small data. The *a priori* estimate we just proved together with the multiplication estimates allow us to control $||a||_{\mathcal{S}_+^{(-1)}}$ by $||b||_{\mathcal{S}^{(-1)}}$ and close the estimates as follows. We proceed proceeds by Picard's iteration relying on the smallness of the data.

Suppose $||(f,g)||_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} = \delta$ and let v_0 be the solution to

$$\Box v_0 = 0; \qquad v_0(0,\cdot) = f \quad \partial_t v_0(0,\cdot) = g.$$

By the Strichartz's estimates

$$||v_0||_{\mathcal{S}} \leq c_1 ||(f,g)||_{\dot{H}^{n/2}\times\dot{H}^{n/2-1}} = c_1\delta.$$

Now, $v_0 = (\varphi_0, \psi_0)$ produces $b_0 = d\,\varphi_0 + div_{(sp.t)}\,\psi_0$ with $||b_0||_{\mathcal{S}^{(-1)}} \leq c_2||v_0||_{\mathcal{S}} \leq c_3\delta$.

Next, the multiplication estimates allow one to perform a fixed point argument to produce a_0 from b_0 by solving

$$a_0 = |\nabla|^{-1}[a_0, a_0] + |\nabla|^{-1}[b_0, b_0].$$

Moreover, $||a_0||_{\mathcal{S}_+^{(-1)}} \leq c_4||b_0||^2_{\mathcal{S}^{(-1)}} \leq c_5\delta^2$

Let v_1 be the solution of

$$\Box v_1 = B(a_0, b_0) \qquad v_1(0,\cdot) = f \quad \partial_t v_1(0,\cdot) = g.$$

By the *a priori* estimate,

$$||v_1||_{\mathcal{S}} \leq c_0\big(\delta + ||a_0||_{\mathcal{S}_+^{(-1)}}||b_0||_{\mathcal{S}^{(-1)}}\big) \leq 2c_0\delta$$

provided δ is small enough.

We proceed next by induction to show :

• For any $j \geq 0$, $\quad ||b_j||_{\mathcal{S}} \leq 2c_2c_o\delta$ and $\quad ||a_j||_{\mathcal{S}} \leq c_5\delta^2$.

Hence if

$$\Box v_{j+1} = B(a_j, b_j) \qquad v_{j+1}(0,\cdot) = f \quad \partial_t v_{j+1}(0,\cdot) = g,$$

by the *a priori* estimates we have that

$$||v_{j+1}||_{\mathcal{S}} \leq 2c_0\delta,$$

provided $\delta > 0$ is small enough (indep. of j). Moreover we have estimates for the differences,

$$||b_{j+1} - b_j||_{\mathcal{S}^{(-1)}} \leq c_2||v_{j+1} - v_j||_{\mathcal{S}}$$

and

$$||a_{j+1} - a_j||_{\mathcal{S}_+^{(-1)}} \leq c\delta||b_{j+1} - b_j||_{\mathcal{S}^{(-1)}} \leq c\delta||v_{j+1} - v_j||_{\mathcal{S}}.$$

Whence all in all, by choosing δ small enough we have that

$$||v_{j+2} - v_{j+1}||_{\mathcal{S}} \leq \frac{1}{2}||v_{j+1} - v_j||_{\mathcal{S}}.$$

Hence v_j is Cauchy in $\mathcal{S}$, thus establishing existence and uniqueness. For the stability result one proceeds in the same fashion as in the proof of being Cauchy; thus concluding the proof of the theorem. $\Box$.

5. Remarks on wave maps in $\mathbb{R}^{1+3}$ with critical regularity

We now turn to the question of extending the results in the previous sections to spatial dimension $n = 3$. In [**36**] T. Tao established global regularity in $n = 2, 3$ when the target is any sphere, $\mathbb{S}^m$. His result has recently been extended by J. Krieger for $n = 3$ and when the target is the hyperbolic space $\mathbb{H}^2$ instead [**24**].

In this section we describe some ideas and estimates developed jointly with A. Stefanov and K. Uhlenbeck and contained in [**26**] to study the problem of establishing existence and uniqueness of solutions to the wave map equations from Minkowski space $\mathbb{R}^{3+1}$ into -say- a constant curvature complete Riemannian manifold.

The first point is that when $n = 3$ and unlike the case in higher dimensions the precise dependence of the curvature term a on b, the derivative of the solution, is needed. If one uses the Coulomb gauge as we did in higher dimensions the curvature term looks essentially like $|\nabla|^{-1}(b{\cdot}b)$ and the lack of any structure whatsoever makes it impossible to prove good a priori estimates to control the nonlinearity; *even* if the 'missing' Strichartz estimate, $L^2_t L^\infty_x$ in dimension 3 were true. If one considers the very simple 'model problem' $\Box u = |\nabla|^{-1} u \cdot du$ it is clear that the nonlinearity is dangerous at low-high interactions. More precisely for $m > 0$,

$$(|\nabla|^{-1}u)_{k-m} \cdot (du)_k \sim 2^m u_{k-m} u_k$$

which in low dimensions this may cause blow up. A similar situation occurs in low dimensions for the 'model equation' $\Box v = |\nabla|^{-1}(dv \cdot dv) \cdot dv$. Dangerous low-high interactions in the nonlinearity occur for example when $m > 0$ and $k - m << l << k$ in

$$(|\nabla|^{-1}((dv)_l \cdot (dv)_l))_{k-m} \cdot (dv)_k \sim 2^m 2^{2l} v_l v_l v_k.$$

However, in general it is possible to obtain 'extra structure' in which case there is some hope to rule out blow up. For example if the curvature term in nonlinearity of the latter model problem has a 'null form' $Q_{ij}(u, v) := \partial_i u \partial_j v - \partial_j u \partial_i v$ instead of just a product, then the curvature term at low frequencies gains a factor of $2^{(k-m-l)}$; that is now

$$(|\nabla|^{-1}(Q_{ij}(u_l, v_l)))_{k-m} \cdot (dw)_k \sim 2^k 2^l v_l v_l v_k.$$

The principle behind this is very simple. The Fourier transform of a solution to the free wave equation is a distribution supported on the light cone Λ. As we have seen in section 2 the support of the Fourier transform of the product of u_1 and u_2, two such solutions, lies in the algebraic sum of the support of $\hat{u}_1$ and $\hat{u}_2$. On the other hand the sum of two 'vectors' on Λ is close to the light cone if and only if they are collinear; e.g. if u_1 and u_2 are two waves (wave packets) traveling in the same direction. The 'null form' has a symbol that vanishes precisely when the two arguments are collinear and on Λ; i.e.

$$\widehat{Q_{ij}(u, v)}(\xi) = \int \langle \xi, \eta^\perp \rangle \hat{u}_1(\xi - \eta) \hat{u}_2(\eta)\, d\eta$$

In other words the 'null form' precisely helps with parallel interactions. This idea follows from the works of J. Bourgain, Klainerman-Machedon, Klainerman-Tataru, T. Wolff and T. Tao ([**2**], [**17**][**23**], [**41**][**37**]) who realized that a null form structure or some angular separation between the Fourier transform of the two factors in a product helps rule out parallel interactions; thus allowing for the range of possible $L^q_t L^r_x$ bilinear estimates to be enlarged considerably.

5.1. Gauge fixing and the equations for curvature. Following K. Uhlenbeck's ideas we now describe a gauge fixing for $n = 3$ that yields a better structure for the curvature term; as described above [**26**]. The following holds on an appropriate band in time-space. However, we ignore this issue in what follows and leave the localization in time out of the present discussion [**26**].

Given the (derivative) solution space is in $L^4_t L^4_x$, we assume that $b := du$ is sufficiently small in $L^4_t L^4_x$. Then the curvature $F = [b, b]$ term is small in $L^2_t L^2_x$. Thus the "good gauge theorem" above ([**25**] [**40**]) says that for any target we can fix a gauge ψ so that the *elliptic space-time* divergence of $a = (a_0, a_1, a_2, a_3)$ vanishes. In other words so that,

$$d^* a = \frac{\partial a_0}{\partial t} + \sum_{j=1}^{3} \frac{\partial a_j}{\partial x_j} = 0 = \sum_{j=0}^{3} \frac{\partial a_j}{\partial x_j}$$

by letting $t = x_0$. Moreover we have the bound

$$||a||_{H^1(\mathbb{R}\times\mathbb{R}^3)} \lesssim ||b||^2_{L^4_{x,t}}$$

Now, depending on whether the target is abelian (e.g. $\mathbb{S}^2$, $\mathbb{H}^2$) or non-abelian, we have respectively that

$$da = [b, b] \quad \text{or} \quad da = [a, a] + [b, b]$$

where d is in $\mathbb{R}^4$; e.g. d acting on 0-forms is the $\nabla = (\partial_t, \nabla_x)$. To get 'a null form' structure in the equation for a, we 'approximate' b by q as follows. We write $b = (b_0, b_{sp})$ and do a Hodge decomposition of the spatial part, $b_{sp} = d_{sp}q + d^*_{sp}r$ where now (d_{sp}, d^*_{sp}) are the exterior differentiation and its dual over $\mathbb{R}^3$. Then if let $r_0 := b_0 - \partial_t q$ and $R = (r_0, d^* r)$ we get that

$$b := (b_0, b_{sp}) \sim (\partial_t q + r_0, d_{sp}q + d^*_{sp}r) \sim dq + R$$

Revisiting then the equations for a,

$$\begin{aligned} d^* a \quad &= 0 \\ da \quad &= [b, b] \quad \text{or} \quad da = [a, a] + [b, b]] \end{aligned}$$

we find that the second equation now becomes

$$da = dq \wedge dq \;+\; 2[dq, R] \;+\; [R, R] \quad (\text{ in the abelian case) or}$$

$$da = dq \wedge dq \;+\; 2[dq, R] \;+\; [R, R] + [a, a] \quad (\text{ in the non-abelian case})$$

where now the first term on the r.h.s is elliptic in $3+1$ and is the only term that has a 'null form' structure. The last terms despite not having any special structure, do possess better regularity properties and hence are overall somewhat better behaved. For example, the term $[R, R]$ behaves like the nonlinearity in the $4 + 1$-dimensions wave map equation.

If we denote the bilinear forms above by $B(\alpha, \gamma) := \sum c_{ij}\alpha_i\gamma_j$,with $c_{ij} \in \mathbb{C}$ and

$$Q_{ij}(u, v) := \partial_i u \partial_j v - \partial_j u \partial_i v,$$

Note that $Q_{ii} = 0$ and that $i, j = 0$ signify as usual, the time derivatives.
Then we can schematically write the second equations for the curvature term as

$$da = Q_{i,j}(q, q) + B(R, dq) + B(R, R) + B(a, a) \tag{3}$$

Heuristically, at least in the estimates, $R \sim |\nabla|^{-1}(a, b)$ and hence R will satisfy the same *a priori* estimates $|\nabla|^{-1}(b, b)$ satisfies.

Thus from now on we abuse notation and simply write the equations for a as

$$(4)\qquad \begin{aligned} d^*a &= 0 \\ da &= Q_{ij}(u,u) + B(du,a) + B(a,a) \end{aligned}$$

where u is the solution and $du = b$.

5.2. Null forms and a priori curvature estimates. First, a technical lemma establishing the boundedness of products of Riesz transforms in the mixed Lebesgue spaces $L^q_t L^r_x$. This is of course well known in the case of ordinary Lebesgue spaces (i.e. $q = r$), but seems to be missing from the literature in this general context.

Lemma 9[**26**]*Let* $1 \leq q, r \leq \infty$ *and let*

$$||\phi||_{\mathcal{L}^q_t L^r_x} := \left(\sum_k ||\phi_k||^2_{L^q_t L^r_x}\right)^{1/2}.$$

Then each of the products of Riesz transforms in $T_i = \partial_i \nabla_{x,t} |\Delta_{xt}|^{-1}$, $i = 0,1,2,3$ *is a bounded mapping from* $\mathcal{L}^q_t L^r_x$ *to itself.*

A proof of this lemma can be found in [**26**]. Recently, Stefanov and Torres studied Calderón-Zygmund operators in mixed Lebesgue space-time norms [**32**] thus extending this boundedness result to a large class of operators.

Remark: In order to include the endpoints $q = \infty$ or $r = \infty$, we consider the action of the operators only on spatially frequency localized pieces. We remark that even in the diagonal case $q = r$, one does not have in general $R_j = \partial_j |\nabla|^{-1} : L^\infty \to L^\infty$.

The so called null form structure is given by the bilinear form

$$Q_{ij}(u,v) = \partial_i u \partial_j v - \partial_j u \partial_i v,$$

for $i, j = 0, \ldots 3$. The operator can be written as a Fourier multiplier operator as follows

$$Q_{ij}(u,v)(x) = \int (\xi_i \eta_j - \xi_j \eta_i) \hat{u}(\xi) \hat{v}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

Note that ξ_0 and η_0 signify the time components of the corresponding vectors.

The special structure of the symbol is exploited in the following manner.

Lemma 10 *For all* $1 \leq q, r, q_1, q_2, r_1, r_2 \leq \infty$, *with*

$$\begin{aligned} &1/q_1 + 1/q_2 = 1/q\ , \quad 1/r_1 + 1/r_2 = 1/r \\ &||\nabla_{xt}|\Delta_{xt}|^{-1} Q_{ij}(u,v)||_{L^q_t L^r_x} \\ \lesssim\ & \min(||\partial u||_{L^{q_1}_t L^{r_1}_x} ||v||_{L^{q_2}_t L^{r_2}_x}, ||u||_{L^{q_1}_t L^{r_1}_x} ||\partial v||_{L^{q_2}_t L^{r_2}_x}).||\nabla_{xt}|\Delta_{xt}|^{-1} Q_{ij}(u,v)||_{L^q_t L^r_x} \\ \lesssim\ & \min(||\partial u||_{L^{q_1}_t L^{r_1}_x} ||v||_{L^{q_2}_t L^{r_2}_x}, ||u||_{L^{q_1}_t L^{r_1}_x} ||\partial v||_{L^{q_2}_t L^{r_2}_x}). \end{aligned}$$

Proof Write $\xi_i \eta_j - \xi_j \eta_i = (\xi_i + \eta_i)\eta_j - (\xi_j + \eta_j)\eta_i$. Thus

$$\frac{\nabla_{xt}}{\Delta_{xt}} Q_{ij}(u,v) = c \frac{\nabla_{x,t} \partial_i}{\Delta_{xt}} (u \partial_j v) - c \frac{\nabla_{xt} \partial_j}{\Delta_{xt}} (u \partial_i v).$$

The result follows from Lemma 9 and the Hölder inequality.

Remark: Note that if the 'missing' Strichartz estimate $L_t^2 L_x^\infty$ in dimension 3 *were* true then the Lemma above would yield the needed $\mathcal{L}_t^1 \dot{B}^0_{\infty,1}$ estimate for the delicate portion of the curvature term. Just as in the higher dimensional case such an estimate would then be enough to prove existence and uniqueness of wave maps into constant curvature complete Riemannian manifolds in $\mathbb{R}^{1+3}$.

Because of the failure of such Strichartz estimate; the largest range of *a priori* estimates in the mixed-Lebesgue norm that one can establish for the curvature term are contained in the following two Lemmata. First we need to modify the definition of the set $\mathcal{E}$ according to the exclusion of the 'missing' Strichartz pair in dimension 3. Let $\kappa > 0$ be small and fixed. Define the set of exponents $\mathcal{E}$ to be the convex hull of the points,

$$(0,0)\,,\ (0, 5/6-\kappa) \quad \text{and} \quad \{(q,r) : 1-\kappa > 1/q > 1/4+\kappa \ \ 1/q + 1/r \le 1\}\,.$$

In particular, $\mathcal{A} \subset \mathcal{E}$.

Lemma 11 [**26**] *For every* $(q,r) \in \mathcal{E}$ *we have*

$$\sum_k 2^{k(1/q+3/r-1)} \|(d^*_{xt}\Delta^{-1}_{xt} Q_{ij}(u,v))_k\|_{L^q_t L^r_x} \le C_\kappa \|u\|_{\mathcal{S}} \|v\|_{\mathcal{S}}.$$

Lemma 12 [**26**] *Suppose that* $u \in \mathcal{S}$ *with* $\|u\|_{\mathcal{S}}$ *sufficiently small. Then the solution* a *of the problem*

$$d^* a = 0$$
$$da + B(a,a) + B(a,du) = Q_{ij}(u,u),$$

satisfies the a priori estimate

$$\sum_k 2^{k(1/q+3/r-1)} \|a_k\|_{L^q_t L^r_x} \lesssim \|u\|^2_{\mathcal{S}}$$

for every $(q,r) \in \mathcal{E}$. *In particular, if we denote by*

$$\|a\|_{\mathcal{S}'} := \sup_{(q,r)\in\mathcal{E}} \sum_k 2^{k(1/q+3/r-1)} \ \|a_k\|_{L^q_t L^r_x}$$

we have that

$$\begin{aligned}
\sum_k \sum_{l>k} 2^{k(1/q+3/r-1)} \||\nabla|^{-1}(a_l du_l)_k\|_{L^q_t L^r_x} &\lesssim \|a\|_{\mathcal{S}'} \|u\|_{\mathcal{S}},\\
\sum_{m>0} \sum_k 2^{k(1/q+3/r-1)} \||\nabla|^{-1}(a_{k-m} du_k)\|_{L^q_t L^r_x} &\lesssim \|a\|_{\mathcal{S}'} \|u\|_{\mathcal{S}},\\
\sum_k \sum_{l>k} 2^{k(1/q+3/r-1)} \||\nabla|^{-1}(a_l b_l)_k\|_{L^q_t L^r_x} &\lesssim \|a\|_{\mathcal{S}'} \|b\|_{\mathcal{S}'},\\
\sum_k 2^{k(1/q+3/r-1)} \||\nabla|^{-1}(a_{k-m} b_k)\|_{L^q_t L^r_x} &\lesssim \|a\|_{\mathcal{S}'} \|b\|_{\mathcal{S}'}.
\end{aligned}$$

The proof of the previous two lemmata can be found in [**26**].

The estimates in Lemma 12 however are not sufficient to find good a priori estimates for the nonlinearity $\mathcal{N}(u) := a \cdot du$, where $da = Q_{i,j}(q,q) + B(R,dq) + B(R,R) + B(a,a)$ as above. In other words it is not possible any longer to place the solution of $\Box u = a \cdot du$ in the $\mathcal{S}$ space and close the estimates in $L^1_t \dot{H}^{1/2}_x$ for example. The majority of nonlinear terms are indeed controllable mostly in the mixed-Lebesgue norms. But not all of them. The main obstacle is the contribution of the low-high interactions term $a_{<k} du_k$ in the non-linearity. As it is clear from its form, the derivative lands on the high frequency part u_k, which makes the control of that term problematic in the mixed Lebesgue spaces (mainly due to the "missing" Strichartz estimate at $L^2_t L^\infty_x$).

If one ignores the quadratic correction term $B(a,a)$ in the second equation for the curvature term a in Lemma 10, and applies the space-time divergence operator d^*_{xt}, one gets,

$$\Delta_{xt} a = d^* da = d^*_{xt} Q_{ij}(u,u).$$

Thus, for the purposes of estimates in this case, the non-linearity (essentially) looks like

$$(|\nabla_{tx}|^{-1} Q_{ij}(u,u))_{<k} du_k.$$

At this point roughly what one wishes is to find (local in time) Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ replacing $\mathcal{S}$ and $L^1_t \dot{H}^{1/2}_x$ respectively such that :

1. $\mathcal{X} \subset L^\infty_t \dot{H}^{3/2}_x \qquad L^\infty_t \dot{H}^{5/2}_x \subset \mathcal{X} \subset \mathcal{S}$
2. Free solutions with data in $\dot{H}^{3/2} \times \dot{H}^{1/2}$ are in $\mathcal{X}$.
3. For u a solution of $\Box u = F$, $u(x,0) = f$, $u_t(x,0) = g$.

$$||u||_{\mathcal{X}} \lesssim ||(f,g)||_{\dot{H}^{3/2} \times \dot{H}^{1/2}} + ||F||_{\mathcal{Y}}$$

4. $||\mathcal{N}(u)||_{\mathcal{Y}} \lesssim ||u||^3_{\mathcal{X}}$. In particular, the following estimates hold
   - (high-high interactions in the null form)

$$||(|\nabla_{tx}|^{-1} Q_{ij}(u_l, v_l))_{<k} dw_k||_{\mathcal{Y}} \lesssim ||u_l||_{\mathcal{X}} ||v_l||_{\mathcal{X}} ||w_k||_{\mathcal{X}}$$

   - (high-low interactions in the null form) For $m > 0$ and $l < k - m$,

$$|||\nabla_{tx}|^{-1} Q_{ij}(u_l, v_{k-m}) dw_k||_{\mathcal{Y}} \lesssim 2^{-(k-m-l)/2} ||u_l||_{\mathcal{X}} ||v_{k-m}||_{\mathcal{X}} ||w_k||_{\mathcal{X}}.$$

Remarks. (i) It is important to note that the a priori estimates needed to control the problematic term in the nonlinearity are really trilinear –and not bilinear–. That is they come from considering truly quatrilinear forms and where also the precise dependence of the curvature term a on b, the derivative of the solution, is needed. T. Tao [**36**] and J. Krieger [**24**] had to deal with similar type of trilinear estimates .

(ii) The scheme outline in this section provides a path to establish global existence, uniqueness and regularity of wave maps with small data in $\dot{H}^{3/2}(\mathbb{R}^3) \times \dot{H}^{1/2}(\mathbb{R}^3)$. Once a solution space $\mathcal{X}$ and a nonlinearity space $\mathcal{Y}$ are found so that (1)–(4) hold, the theorem giving the existence of a local in time gauge alluded above and in **[26]** produces a local in time solution. A global regularity result then implies this solution turns out to be a global solution, which is the weak limit in $L_t^\infty(\dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$ of smooth solutions. Uniqueness holds in the sense that if $\tilde{u} \in L_t^\infty \dot{H}_x^{3/2} \cap \dot{W}^{1,4}$ is a 'small' solution to the wave map problem, then there exists a gauge transformation such that $\tilde{u} = u$ where $u \in \mathcal{X}$ is the local solution constructed by the special choice of gauge **[26]**.

(iii) Finally we note that a possible approach to this problem is to use for $\mathcal{X}$ and $\mathcal{Y}$ the corresponding solution and nonlinear spaces T. Tao introduced in **[36]**. This approach would of course mean one is assuming most of Tao's difficult and delicate paper at this point. It is possible however that in dimensions 3 one could find alternative spaces $\mathcal{X}$ and $\mathcal{Y}$ that would allow one to establish (1)–(4) above more directly. This is part of subject under investigation in **[26]**. [1]

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A. Nahmod, Department of Mathematics and Statistics, Lederle GRT, University of Massachusetts, Amherst, MA 01003-4515

E-mail address: nahmod@math.umass.edu