Geometric results in classical minimal surface theory

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1. Introduction.

Although classical minimal surface theory dates back to the work of Euler [**23**] and Lagrange [**52**] in the 18th century, most advances in the global theory have been obtained in the past twenty-five years. The primary goal of this survey is to present the recent theorems in minimal surface theory together with a sufficient amount of background information so that these theorems can be understood, appreciated and applied. The presentation here of both the old and the new results in this classical subject is more from a geometric rather than an analytic point of view. The interested reader can also find more detailed history and further results in the following surveys, reports and popular science articles [**4, 18, 19, 37, 38, 39, 40, 46, 63, 65, 87**].

In the next two sections we introduce the concept of minimal surface from several different equivalent points of view and include many of the basic definitions, notation and results. In Section 4 we give a description of eight well-known classical examples of minimal surfaces. These examples motivate many of the theoretical results in later sections and so, the reader should make an effort to understand their geometry and special properties before proceeding. Section 5 introduces the important notion of stable or locally-least-area minimal surface and includes some of the basic theorems on stable minimal surfaces. Part of the importance of stable minimal surfaces is that they are an essential tool for studying many of the difficult global problems in the classical theory.

Section 6 deals with the questions of existence and of regularity of solutions to the classical Plateau problem; this problem asks whether a simple closed curve in $\mathbb{R}^3$ is the boundary of a least-area surface. This section includes several different formulations of this least-area problem, including a short discussion of the barrier construction of Meeks and Yau [**83**] which we need in some later discussions.

The remainder of the survey deals with advances made in the past decade. Section 7 explains the solution of the generalized Nitsche Conjecture given by Collin [**15**] and the recent theorem of Meeks and Rosenberg [**74**] on the uniqueness of the helicoid. Together, the theorem of Collin and the theorem of Meeks and Rosenberg give a satisfactory theory for describing all properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ in terms of meromorphic data on their conformal compactifications; these conformal compactifications are closed Riemann surfaces. This analytic representation of finite topology examples leads to real analytic structures on the associated moduli spaces of examples of a fixed topology, to a description of the asymptotic behavior of the examples and, in certain cases, to the classification of all examples of a fixed topological type.

In Section 8 we enter the realm of active research on the local geometry of embedded minimal surfaces in Riemannian three-manifolds near points of large Gaussian curvature, under the additional hypothesis of having a fixed bound on the genus of the surface in a small neighborhood of such a point of large curvature. This description follows the pioneering work of Colding and Minicozzi [**9, 10, 11, 12, 13**] and subsequent geometric extensions by Meeks [**61**], Meeks and Rosenberg [**74**] and Meeks, Perez and Ros [**70, 66, 67, 69, 68**].

Section 9 presents all known topological obstructions for properly minimally embedding a noncompact orientable surface into $\mathbb{R}^3$. Here we include a discussion of the classical results of Schoen [**100**] and Lopez-Ros [**56**], which together with Collin's theorem [**15**] and the Meeks-Rosenberg theorem [**74**], give a complete classification of all properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ which have either genus-zero or two ends. Recently Meeks, Perez and Ros [**69**] have shown that a properly embedded minimal surface in $\mathbb{R}^3$ of finite genus g and a finite number of ends has a bound on the number of its ends that only depends on g. These topological obstructions and classification theorems represent the most important theoretical results that one might hope to obtain in this subject.

Much of present day research in classical minimal surface theory is focussed on describing the asymptotic geometry of properly embedded minimal surfaces with finite genus and infinitely generated fundamental group; this means that we are considering surfaces of finite genus with an infinite number of ends (see Section 3 for a description of the space of ends of a noncompact surface). In this regard, we include in Section 9 a discussion of the fundamental result of Collin, Kusner, Meeks and Rosenberg [**16**] on the structure of the space of ends of a properly embedded minimal surface with an infinite number of ends. This structure theorem, together with related topological obstructions, plays a fundamental role in restricting the geometry and topology of properly embedded minimal surfaces in $\mathbb{R}^3$ which have infinite topology. This structure theorem is essential in proving that a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and an infinite number of ends must have exactly two limit ends [**67**].

In Section 10 we describe the recent proof by Frohman and Meeks [**29**] of the "Topological Classification Theorem for Minimal Surfaces". This theorem gives a complete cook-book type description of how a minimal surface is embedded in $\mathbb{R}^3$ in terms of calculable algebraic-topological invariants; it gives necessary and sufficient conditions for two different properly embedded minimal surfaces to be properly ambiently isotopic in $\mathbb{R}^3$.

For many of the global questions we would like to answer for some class of minimal surfaces in $\mathbb{R}^3$, it is essential to know the underlying conformal structure. For example, near the end of the proof of the uniqueness of the helicoid in [**74**] (see Sections 7 and 8), one needs to know that certain properly embedded simply-connected minimal surfaces are conformally the complex plane $\mathbb{C}$. This result on conformal structure depends on a general theorem in [**16**] that asserts that any component of the intersection of a properly immersed minimal surface with a closed halfspace in $\mathbb{R}^3$ is parabolic, in the sense that bounded harmonic functions on the component are determined by their boundary values. In Section 11 we prove this theorem and discuss the related question of recurrence for Brownian motion in minimal surfaces.

In Section 12 we describe the theory of periodic minimal surfaces in $\mathbb{R}^3$ as developed by Meeks and Rosenberg [**75**], [**77**], especially in the case of properly embedded surfaces with quotient having finite topology. Here we also discuss the uniqueness of some of the classical periodic minimal surfaces described in Section 4.

In Section 13 we briefly leave the classical setting in order to describe several surprising and deep results that are likely to have important applications. This section covers the theoretical results that Meeks and Rosenberg [**72, 73**] have obtained for properly embedded minimal surfaces in a Riemannian product $M \times \mathbb{R}$, where M is a compact Riemannian surface. The applications of these results that we have in mind pertain to the case where M is the two-sphere S^2 of constant Gaussian curvature 1. These applications are related to the existence and classification of harmonic maps of Riemann surfaces to S^2. Also, we hope to apply these theoretical results to better understand constant mean curvature surfaces in $\mathbb{R}^3$ and minimal and constant mean curvature surfaces in the three-sphere S^3.

In Section 14 we present a brief discussion of 16 fundamental conjectures of the author and others. Since these conjectures are discussed in detail there, we just briefly list them by name here: Convex Curve Conjecture, 4π Conjecture, Finite Topology Conjecture, Properness Conjecture, Liouville Conjecture, Removal Singularities Conjecture, Isometry Conjecture, Genus-Zero Conjecture, Geometric Flux Conjecture, Scherk Uniqueness Conjecture, Uniqueness of Limit Tangent Cone Conjecture, Graph Connectedness Conjecture, Quadratic Area Growth Conjecture, One-ended Conjecture, Infinite Topology Conjecture, Singular Curve Conjecture and the Uniqueness of the $\mathcal{A}$-family Conjecture.

2. The definition of minimal surface.

By a *surface* in $\mathbb{R}^3$ we mean a subset Σ of $\mathbb{R}^3$ that is locally parametrized by the open unit disk D in $\mathbb{R}^2$. In other words, for each $p \in \Sigma$, there is a neighborhood $U_p \subset \Sigma$ together with a map $f\colon D \to \mathbb{R}^3$ which is an injective smooth immersion with $f(D) = U_p$. If Σ is a smooth abstract Riemannian surface, then we say that $f\colon \Sigma \to \mathbb{R}^3$ is a *smooth embedding* or an *isometric injective immersion* if f is an injective immersion and the flat Riemannian metric in $\mathbb{R}^3$ induces the Riemannian metric on the surface Σ. We say that a surface $\Sigma \subset \mathbb{R}^3$ is *complete* if it is a complete metric space with respect to the natural distance function obtained from taking the infimum of the lengths of curves which join pairs of points. By the Hopf-Rinow theorem, Σ is complete if and only if every geodesic segment on Σ can be continued indefinitely. If Σ is allowed to have boundary, then we take the same definition for completeness, except now the Hopf-Rinow theorem states that a geodesic segment can be continued indefinitely or continued until it arrives at the boundary of Σ. At times it is convenient to consider a surface $\Sigma \subset \mathbb{R}^3$ as an embedding under inclusion: $f\colon \Sigma \to \mathbb{R}^3$ with respect to its underlying induced Riemannian structure. It is a standard fact for two-dimensional Riemannian surfaces that the Riemannian metric of a disk $F \subset \Sigma$ multiplied by some positive function is a new metric on F for which F is isometric to the unit disk D in $\mathbb{R}^2$. It follows that when Σ is orientable (Σ has a well-defined unit normal vector field called an *orientation*), then it has a system of local coordinates $\phi_\alpha\colon D \to U_\alpha \subset \Sigma$ such ϕ_α is conformal or angle preserving; such coordinates are called *isothermal* or *conformal* coordinates. With these elementary concepts in mind, we now define the concept of minimal surface by way of a list of equivalent properties.

THEOREM 2.1. *$f\colon \Sigma \to \mathbb{R}^3$ is an oriented* **minimal** *surface if Σ is oriented and any of the following equivalent properties hold:*

1. *Σ has zero mean curvature;*
2. *Small disks in Σ have least-area relative to their boundaries;*
3. *Small disks in Σ have least-energy relative to their boundaries;*
4. *Small disks in Σ are equal to the unique idealized soap film surfaces with the same boundary;*
5. *The coordinate functions f_1, f_2, f_3 of f are harmonic functions;*
6. *The Gauss or unit normal map $G\colon \Sigma \to S^2$ is conformal(we mean here that the derivative map is angle preserving wherever it is nonzero) and its stereographic projection $g\colon M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function.*

PROOF. I have just a few comments to make on the equivalences in the above Theorem. If $G\colon \Sigma \to S^2$ is the unit normal map, then the tangent space $T_p\Sigma$ of Σ at $p \in \Sigma$ is parallel in $\mathbb{R}^3$ to the tangent space $T_{G(p)}S^2$ to S^2 at the point $G(p) \in S^2$, and so after identifying these spaces under translation, the derivative map can be thought of as a linear map $S_p\colon T_p\Sigma \to T_p\Sigma$ called the *shape operator*. S_p is a symmetric linear transformation whose orthogonal eigenvectors are called the *principal directions* of Σ and the corresponding eigenvalues are called the *principal curvatures* of Σ. The *mean curvature* function H of Σ is the pointwise trace of the shape operator or, equivalently, $H(p)$ is equal to the sum of the principal curvatures at p.

In reference to Statement 2 of Theorem 2.1, one has the following more general result:

THEOREM 2.2. *(First Variation of Area Formula). If Σ is a compact, not necessarily minimal, surface with unit normal vector field N, and $\Sigma(t)$, $-\varepsilon < t < \varepsilon$, is a smooth deformation of Σ with $\partial\Sigma(t) = \partial\Sigma$, then the first derivative of area of this variation at $t = 0$, can be calculated as:*

$$A'(0) = \frac{dA}{dt}\Big|_{t=0} = \int_\Sigma H < N,\ V > dA,$$

where H is the mean curvature function on Σ and V is the variational vector field of $\Sigma(t)$ at $t = 0$.

It follows from the first variation of area formula that a compact minimal surface is a critical point to the area functional. The fact that Statements 1 and 2 in Theorem 2.1 are equivalent follows from this interpretation of the first variation of area formula, together with the fact that critical points of area are always local minima in small neighborhoods of every point, which can be derived for instance from the minimizing area property of minimal graphs.

The fact that Statement 2 is equivalent to Statement 3 follows from the standard inequality between energy and area. The energy we refer to here is the Dirichlet energy $\int_{\tilde\Sigma} |\nabla F|^2\, d\tilde A$ of a parametrization $F\colon \tilde\Sigma \to \Sigma \subset \mathbb{R}^3$ of the surface Σ by a Riemannian surface $\tilde\Sigma$, where $d\tilde A$ is the area form of $\tilde\Sigma$. If dA denotes the area form for Σ and $F^*(dA)$ denotes the pull-back form, then this inequality states that pointwise $|\nabla F|^2\, d\tilde A \geq 2F^*(dA)$ with equality if and only if the map is conformal. Since the inclusion mapping f is conformal (f is an isometry), the energy form of f is just twice its area form. Thus, Σ locally minimizes area precisely when considered as a map, it locally minimizes its energy.

Statements 3 and 4 are equivalent since idealized soap films are just surfaces which, by surface tension, locally minimize their energy. Surface tension on a minimal surface creates a static force which separates at each point on the surface into two forces that act oppositely and orthogonally to the surface along orthogonal directions on the surface (principal directions) and these forces are proportional pointwise to the principal curvatures at the point.

Statements 1 and 5 are equivalent by the easy to derive formula: If $f\colon M \to \mathbb{R}^3$ is an immersed oriented surface, then

$$\Delta f = (\Delta f_1, \Delta f_2, \Delta f_3) = H \cdot N,$$

where Δ is the Laplace operator on Σ, H is the mean curvature function and N is the unit normal field $N\colon \Sigma \to S^2 \subset \mathbb{R}^3$ of Σ.

Statements 1 and 6 are equivalent since the derivative of the Gauss map at a point $p \in \Sigma$ can be identified with the shape operator $S_p\colon T_p\Sigma \to T_p\Sigma = T_{G(p)}S^2$ which is a symmetric transformation with trace equal to H. If one takes the orientation of $T_{G(p)}S^2$ to be given by the orientation of S^2 coming from stereographic projection, which is opposite to the orientation on $T_{G(p)}S^2$ induced by parallel translation to $T_p\Sigma$, we see that the derivative map $G'_p\colon T_p\Sigma \to T_{G(p)}S^2$ is angle preserving wherever the derivative is not zero. This completes our proof of Theorem 2.1.

$\square$

For later purposes note that the Gaussian curvature $K(p)$ of a point p on a minimal surface Σ is nonpositive and equal to the determinant of $S_p\colon T_p\Sigma \to T_p\Sigma$ which is equal to the product of the principal curvatures at p. Thus, on a compact minimal surface, it follows that the total Gaussian curvature of Σ is equal to

$$C(\Sigma) = \int_\Sigma K dA = -\text{Area}(G\colon \Sigma \to S^2),$$

where the area is counted with multiplicity.

3. Basic definitions and results.

An important analytic result is the classical Weierstrass representation of a minimal surface. Basically it gives a cook-book type recipe for analytically defining a minimal surface $f\colon \Sigma \to \mathbb{R}^3$. The approach we take is a variant of the Weierstrass representation given by Osserman in [**90**].

THEOREM 3.1. *(Weierstrass Representation) Suppose Σ is a Riemann surface and $f\colon \Sigma \to \mathbb{R}^3$ is a conformal harmonic map (i.e., f is a branched minimal surface) with $f(p_0) = (0,0,0)$. Let $\eta = dx_3 + i dx_3^*$ be the holomorphic one-form where x_3 is the third coordinate of Σ and x_3^* is the locally defined harmonic conjugate function of x_3. Let $g\colon \Sigma \to \mathbb{C} \cup \{\infty\}$ be the meromorphic Gauss map for Σ. Then:*

$$f(p) = \text{Re} \int_{p_0}^{p} (\frac{1}{2}(\frac{1}{g} - g)\eta, \frac{i}{2}(\frac{1}{g} + g)\eta, \eta).$$

Conversely, if η is a nonzero holomorphic one-form and $g\colon \Sigma \to \mathbb{C} \cup \{\infty\}$ is a nonconstant meromorphic function on a Riemann surface Σ such that the function $f\colon \Sigma \to \mathbb{R}^3$ given by the above formula is well-defined (the holomorphic one-forms in the formula have no real periods on Σ), then f is a conformal branched minimal immersion of Σ into $\mathbb{R}^3$ whose stereographically projected Gauss map is the meromorphic function g.

We are interested in understanding the space of complete embedded minimal surfaces in $\mathbb{R}^3$. All known examples of such surfaces satisfy the stronger hypothesis given in the next definition. Recall that a surface M has *more than one end* if it contains a smooth compact subdomain such that the complement of the interior of this domain in M has more than one noncompact component.

Definition 3.1. An immersion $f\colon M \to \mathbb{R}^3$ is *proper* if for every compact ball B, $f^{-1}(B)$ is compact in M. Let $\mathcal{P}$ denote the space of all properly embedded connected minimal surfaces in $\mathbb{R}^3$ and let $\mathcal{M} \subset \mathcal{P}$ be the subspace of examples with more than one end. The topology on $\mathcal{P}$ is the topology of smooth convergence on compact subsets of $\mathbb{R}^3$.

Definition 3.2. A Riemannian manifold M with nonempty boundary is *parabolic* if every bounded harmonic function on M is determined by its boundary values.

Definition 3.3. Given a Riemannian manifold M with nonempty boundary and a point $p \in \mathrm{Int}(M)$, one can define the *harmonic* or *hitting* measure μ_p of an interval $I \subset \partial M$ as the probability that a Brownian path, beginning at p, exits the boundary ∂M somewhere on the interval I.

Instead of defining the harmonic measure μ_p of $I \subset \partial M$ in terms of probability and Brownian motion, one can also define it as follows. Consider a compact exhaustion $I \subset \partial M_1 \subset M_1 \subset M_2 \subset \ldots$ of M. Let $h_n\colon M_n \to [0,1]$ be the bounded harmonic function with boundary values 1 on $\mathrm{Int}(I)$ and 0 on the interior of $\partial M_n - I$. Since h_n is an increasing sequence of harmonic functions on M bounded by the constant function 1, h_n has a unique limit harmonic function h. In this case $\mu_p(I) = h(p)$.

The following useful Proposition is an elementary consequence of the definition of *harmonic* or *hitting* measure.

PROPOSITION 3.2. *Suppose M is a Riemannian manifold with nonempty boundary. The following are equivalent:*

1. *M is parabolic;*
2. *Bounded harmonic functions on M are determined by their boundary values;*
3. *For some $p \in Int(M)$, the measure μ_p is full on ∂M, i.e., $\int_{\partial M} \mu_p = 1$;*
4. *Given any $p \in Int(M)$ and any bounded harmonic function $f\colon M \to \mathbb{R}$, then $f(p) = \int_{\partial M} f(x)\mu_p$;*
5. *There exists a proper positive superharmonic function on M.*

The property of a Riemannian manifold with boundary being parabolic is closely related to the following notion of recurrence for Brownian motion. See [**32**] for an excellent survey of recurrence and Brownian motion on Riemannian manifolds.

Definition 3.4. A Riemannian manifold M (without boundary) is *recurrent* or *recurrent for Brownian motion* if and only if for any given point $p \in M$, almost all continuous paths $\alpha\colon [0,\infty) \to M$ with $\alpha(0) = p$ are dense in M.

It is well known [**32**] that $\mathbb{R}^n$ is recurrent for Brownian motion if and only if $n \leq 2$.

The following Lemma makes clear the relationship between the concepts of recurrence and of parabolicity.

LEMMA 3.3. *A connected Riemannian manifold M without boundary is recurrent if and only if for any nonempty open set $O \subsetneqq M$, $M - O$ is parabolic.*

PROOF. Suppose M is recurrent and $O \subsetneqq M$ is a nonempty open subset. Let C be a component of $M - O$ and $p \in \text{Int}(C)$. Since almost all Brownian paths beginning at p are dense in M, almost all Brownian paths beginning at p must enter O. But, in order to enter O, such a path must cross ∂C which means that the hitting measure μ_p on ∂C is full and so Proposition 3.2 implies that C is parabolic.

We now prove the converse statement. Suppose that for any nonempty open set $O \subsetneqq M$, $M - O$ is parabolic. Let $p, q \in M$ and let $\alpha \subset M$ be a Brownian path starting at p. For any open ball $B(q, \varepsilon)$ centered at q, $M - B(q, \varepsilon)$ is parabolic and so, with probability 1, the path α will enter the closed ball $\overline{B}(q, \varepsilon)$. Since ε is arbitrary, with probability 1 the closure of α in M is all of M.

□

Recent work of Meeks and Rosenberg [**74, 76**] proves:

THEOREM 3.4. *An $M \in \mathcal{P}$ of finite topology is conformally diffeomorphic to a finitely punctured compact Riemann surface. In particular, such a minimal surface is recurrent for Brownian motion.*

In order to understand generalizations of the above theorem, we now discuss the topological notion of "ends" of a surface.

Definition 3.5. Suppose M is a noncompact connected manifold. The *space of ends* of M, denoted by $\mathcal{E}(M)$, is the set of equivalence classes of proper arcs $\alpha \colon [0, \infty) \to M$ where α_1 is equivalent to α_2 if for every smooth compact subdomain $C \subset M$, α_1 and α_2 intersect the same component of $M - \text{Int}(C)$ in a noncompact set.

A basis for the topology of $\mathcal{E}(M)$ is defined as follows. For each compact set $C \subset M$, define the basis open set $B(C) \subset \mathcal{E}(M)$ to be those equivalence classes of proper arcs in M which have representatives contained in $M - C$.

It can be shown that $\mathcal{E}(M)$ is a totally disconnected compact Hausdorff space that embeds as a subspace of the unit interval (this is not difficult to prove once one knows that it is true but I do not have a reference for it). Conversely, every compact totally disconnected subset C of the unit interval corresponds to the space of ends of a noncompact surface; namely, consider C to lie on an Jordan curve in S^2, then $S^2 - C$ has C as its space of ends. It is also interesting to note that two connected genus-zero surfaces are homeomorphic if and only if their spaces of ends are homeomorphic.

Definition 3.6. An end $e \in \mathcal{E}(M)$ is a *simple* end of M if it is an isolated point in $\mathcal{E}(M)$. Note that in dimension two that e is simple if and only if there exists a representative proper arc $\alpha \in e$ and a proper subdomain $W \subset M$ containing α such that W is homeomorphic to $S^1 \times [0, \infty)$ or to $S^1 \times [0, \infty)$ connected sum with an infinite number of tori where the n^{th} connected sum occurs at the point $(1, n) \in S^1 \times [0, \infty)$. In the first case we refer to e as an *annular* end and in the second case we refer to e as a *simple end of infinite genus.* We say that the domain W *represents* the end e.

Definition 3.7. An end $e \in \mathcal{E}(M)$ is a *limit* end of $\mathcal{E}(M)$ if it is not a simple end. In other words, e is a limit end if it is a limit point of $\mathcal{E}(M)$. As in the previous definition, a limit end has *genus-zero* if it can be represented by a proper domain $W \subset M$ with compact boundary and the genus of W is zero. If a limit end e does not have genus-zero, then we say that e has *infinite genus*; in this case every proper subdomain with compact boundary representing e has infinite genus.

Definition 3.8. *The limit tangent plane at infinity* of a properly embedded minimal surface M in $\mathbb{R}^3$ with more than one end is the plane passing through the origin whose normal vector equals the normal vector of some end of a noncompact properly embedded minimal surface $\Sigma \subset (\mathbb{R}^3 - M)$ with compact boundary and finite total curvature (see Theorem 7.1); see [**6**] for further details on the existence and uniqueness of the limit tangent plane at infinity when $M \in \mathcal{M}$. We say that the limit tangent plane at infinity is *horizontal* if it is the x_1x_2-plane.

THEOREM 3.5. [**30**] *[Ordering Theorem] Suppose $M \in \mathcal{M}$ has horizontal limit tangent plane at infinity. Then the ends of M can be linearly ordered geometrically by their relative heights over the x_1x_2-plane. Furthermore, this ordering is a topological ordering in the following sense. If M is properly isotopic to a properly embedded minimal surface M' with horizontal limit tangent plane at infinity, then the associated ordering of the ends of M' either agrees with or is opposite to the ordering coming from M.*

Definition 3.9. Consider the ordering on the ends $\mathcal{E}(M)$ given by the above theorem. The end $e_T \in \mathcal{E}(M)$ is the *top* end of M if it is the unique end in $\mathcal{E}(M)$ that is maximal in the ordering. The top end e_T exists since $\mathcal{E}(M)$ is compact. The end $e_B \in \mathcal{E}(M)$ is the *bottom* end of M if it is the unique minimal element in the ordering of the ends. An end of M that is neither the top nor the bottom end of M is called a *middle* end of M.

4. Eight classical examples of minimal surfaces.

1. **Plane:** $\mathbf{P} = x_2x_3$-plane. Weierstrass data: $\Sigma = \mathbb{C},\ \eta = dz,\ g(z) = 1$.

   A twisted plane would be a helicoid which is ruled by straight lines which rotate around the axis of the helicoid.
2. **Helicoid:** $\mathbf{H} = \{(t\ \cos(s), t\ \sin(s), s) \mid t, s \in \mathbb{R}\}$.
   Weierstrass data: $\Sigma = \mathbb{C},\ \eta = idz,\ g(z) = e^z$.

   By taking the conjugate surface to the helicoid H, using the conjugate harmonic coordinate functions, we obtain an image surface which is a surface of revolution called a catenoid.
3. **Catenoid:** $\mathbf{C} = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = \cosh^2(x_3)\}$.
   Weierstrass data: $\Sigma = \mathbb{C} - \{0\},\ \eta = \frac{1}{z}dz,\ g(z) = z$.
   Weierstrass data on the universal cover of $\mathbf{C}$: $\Sigma = \mathbb{C},\ \eta = dz,\ g(z) = e^z$.

   In 1835, Scherk [**98**] defined five new minimal surfaces. Two of these examples, 4 and 5 below, have had an important influence on the theory of minimal surfaces. Scherk's singly-periodic minimal surface $\mathbf{S}_\theta$ defined below is asymptotic, away from the x_3-axis, to two planes P_1, P_2 containing the x_3-axis and which make an angle of 2θ with each other. In particular, under homothetic shrinkings, $\frac{1}{t}\mathbf{S}_\theta \to P_1 \cup P_2$ as $t \to \infty$. For t small, $\frac{1}{t}\mathbf{S}_\theta$ has the appearance of an embedded minimal surface which approximates $P_1 \cup P_2$ with the self-intersection curve $(P_1 \cap P_2) \subset (P_1 \cup P_2)$ desingularized by adding small one-handles along it. The example $\mathbf{S}_{\frac{\pi}{4}}$ can be defined implicitly by the formula $\sin(z) = \sinh(x)\sinh(y)$.

Scherk's doubly-periodic minimal surface $\hat{\mathbf{S}}_\theta$ defined below, which is the conjugate surface to $\mathbf{S}_\theta$, is asymptotic, away from the x_1x_2-plane, to families of equally spaced vertical parallel halfplanes in the respective halfspaces, with the halfplanes in $\{x_3 > 0\}$ and the halfplanes in $\{x_3 < 0\}$ making an angle of 2θ with each other. These doubly-periodic minimal surfaces are invariant under translation by a rombus lattice in $\mathbb{R}^2 \times \{0\}$. The example $\hat{\mathbf{S}}_{\frac{\pi}{4}}$ can be defined implicitly by the formula $e^z \cos y = \cos x$.

4. **Scherk's Singly-Periodic Surfaces:** $\mathbf{S}_\theta$, $0 < \theta \leq \frac{\pi}{4}$.
Weierstrass data: $\Sigma = \mathbb{C} \cup \{\infty\} - \{\pm e^{\pm i\theta}\}$, $\eta = \frac{zidz}{\Pi(z \pm e^{\pm i\theta})}$, $g(z) = z$.

5. **Scherk's Doubly-Periodic Surfaces:** $\hat{\mathbf{S}}_\theta$, $0 < \theta \leq \frac{\pi}{4}$.
Weierstrass data: $\Sigma = \mathbb{C} \cup \{\infty\} - \{\pm e^{\pm i\theta}\}$, $\eta = \frac{zdz}{\Pi(z \pm e^{\pm i\theta})}$, $g(z) = z$.

After Scherk's discovery, the next important examples were found by Riemann [**95**] who classified all of the minimal surfaces in $\mathbb{R}^3$ which are foliated by single circles and lines in horizontal planes. He wrote down a one-parameter family $\mathcal{R}_t$ of these examples defined for $t \in (0, \infty)$. The $\mathcal{R}_t$ converge to catenoids as $t \to 0$ and to helicoids as $t \to \infty$ (when appropriately normalized). Up to scaling by a homothety, $\mathcal{R}_t$ intersects the horizontal planes at integer heights in lines parallel to the x_1-axis and intersects other horizontal planes in circles symmetric with respect to reflection in the x_2x_3-plane. Each $\mathcal{R}_t$ has two limit ends with planar horizontal middle ends.

We will use the Weierstrass data to define the surfaces $\mathcal{R}_t$, up to a possible rotation by $\frac{\pi}{2}$ around the x_3-axis. Let $\mathbb{T}_t$ be the rectangular elliptic curve $\mathbb{T}_t = \mathbb{C}/\Lambda_t, \Lambda_t = \{m + tni \mid m, n \in \mathbb{Z}\}$. Let $\mathcal{P}_t$ be the meromorphic function on $\mathbb{T}_t$ with a double zero at 0 and a double pole at $\frac{1+ti}{2}$ and with value $\mathcal{P}_t(\frac{1+ti}{4}) = i$. Let $\bar{\Sigma}_t = \mathbb{T}_t - (Z_t \cup P_t)$, where Z_t, P_t are the zeros and poles of $\mathcal{P}_t$. Consider the infinite cyclic cover $\pi \colon \mathbb{C}/ti\mathbb{Z} \to \mathbb{C}/\Lambda_t$ and let $\Sigma_t = \pi^{-1}(\bar{\Sigma}_t)$ and $\tilde{\mathcal{P}}_t = \mathcal{P}_t \circ \pi$. Then the Riemann example $\mathcal{R}_t$ has the following Weierstrass data, when considered to be a periodic minimal surface in $\frac{\mathbb{R}^3}{<V>}$, where V is some vector in the x_1x_3-plane.

6. **Riemann Minimal Examples $\mathcal{R}_t$:**
Weierstrass data: $\Sigma = \Sigma_t$, $\eta = dz$, $g_t(z) = \tilde{\mathcal{P}}_t(z)$.

In 1982 Costa [**17**] wrote down, in terms of elliptic functions on the square torus $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$, for $\Sigma = \mathbb{T} - \{0, \frac{1}{2}, \frac{1+i}{2}\}$, a conformal harmonic immersion $f \colon \Sigma \to \mathbb{R}^3$. Costa proved that $f(\Sigma)$ was an embedded surface outside of a ball in $\mathbb{R}^3$. Later Hoffman and Meeks [**44**] proved that the Costa surface was embedded and constructed for every positive integer k a related properly embedded minimal surface Σ_k in $\mathbb{R}^3$ of genus k with three ends, where Σ_1 is Costa's surface. We call this sequence of minimal surfaces:

7. **Costa-Hoffman-Meeks Examples:** Weierstrass data: for some $\lambda > 0$, $\Sigma_k = \{(z,w) \in \mathbb{C}^2 \mid w^{k+1} = z^k(z^2-1)\} - \{(1,0),(1,0)\}, \eta = \frac{dz}{z^2-1}, g = \frac{\lambda}{w}$.

A couple of years after the discovery of the Costa-Hoffman-Meeks examples, Callahan, Hoffman and Meeks produced by computer graphics techniques many other new examples of finite total curvature. As a limit of one family of these finite topology examples, they wrote down the Weierstrass data for a sequence of very symmetric properly embedded minimal surfaces $M(n)$ which are invariant under vertical translation by $v = (0,0,2)$, had $2n-1$ vertical planes of symmetry containing the x_3-axis and making equal angles, had planar middle ends at integer heights with $2n-1$ horizontal lines meeting the x_3-axis at such heights, horizontal planes of symmetry at heights of the form $k + \frac{1}{2}, k \in \mathbb{Z}$, and such that horizontal planes at non-integer heights intersected $M(n)$ in simple closed curves. These examples were the first properly embedded minimal surfaces with an infinite number of ends and infinite genus.

We refer the interested reader to **[5]** for a beautiful full page colored computer graphics rendered photo of the surface $M(1)$. Also, in **[5]**, one can find pictures of the surface $M(2)$ and one of the Riemann examples. Unfortunately, these examples do not have a simple Weierstrass representation. The computer graphics pictures of these surfaces are obtained as numerical solutions to associated period problems on Riemann surfaces modelled on certain infinite cyclic branched covers of rectangular elliptic curves.

8. **Callahan-Hoffman-Meeks Examples:** $M(n), n \in \mathbb{N}$.

5. Stable minimal surfaces.

By definition, a minimal surface is locally a surface of least-area where by "local" we mean small disks on the surface. If instead we use "local" to mean in a small neighborhood of the entire surface, then we say that the minimal surface is stable. More precisely we have the following definition.

Definition 5.1. A *stable* minimal surface Σ in $\mathbb{R}^3$ is a surface such that every smooth compact subdomain $\bar{\Sigma}$ is stable in the following sense: if $\bar{\Sigma}(t)$ is a smooth family of minimal surfaces with $\partial\bar{\Sigma}(t) = \partial\bar{\Sigma}$ and $\bar{\Sigma}(0) = \Sigma$, then the second derivative of the area function $A(t)$ of the family $\bar{\Sigma}(t)$ is nonnegative at $t = 0$. We will say that Σ has *finite index* if outside of a compact subset it is stable.

Given a smooth variation $\bar{\Sigma}(t)$ of a compact minimal surface $\bar{\Sigma}$ with $\bar{\Sigma}(0) = \bar{\Sigma}$ and $\partial\bar{\Sigma}(t) = \partial\bar{\Sigma}$, one can express for t small the surfaces $\bar{\Sigma}(t)$ as normal graphs over $\bar{\Sigma}$ and so one obtains a normal variational vector field V on $\bar{\Sigma}$ which is zero on $\partial\bar{\Sigma}$. Assume that $\bar{\Sigma}$ is orientable with unit normal field N. Then $V = fN$ where $f\colon \bar{\Sigma} \to \mathbb{R}$ is a smooth function with zero boundary values. Conversely, if $f\colon \bar{\Sigma} \to \mathbb{R}$ is a smooth function with zero boundary values, then for small t one can find normal graphs $\bar{\Sigma}(t)$ which are the graphs $p + tf(p)N(p)$ over $\bar{\Sigma}$ with variational vector field fN. An elementary calculation gives the following second variational formula **[89]**.

THEOREM 5.1. *(Second Variation of Area Formula) Suppose Σ is a compact oriented minimal surface and $f\colon \Sigma \to \mathbb{R}$ is a smooth function with zero boundary values. Let $\Sigma(t)$ be a variation of Σ with variational vector field fN and let $A(t)$*

be the area of $\Sigma(t)$. *Then*

$$A''(0) = -\int_{\Sigma} f(\Delta f - 2Kf)dA,$$

where K *is the Gaussian curvature function on* Σ *and* Δ *is the Laplace operator on* Σ.

Definition 5.2. If Σ is a minimal surface, then $f\colon \Sigma \to \mathbb{R}$ is a *Jacobi function* if $\Delta f - 2Kf = 0$.

Jacobi functions f on Σ arise from normal variations $\Sigma(t)$, not necessarily with the same boundary, where the $\Sigma(t)$ are minimal surfaces with $\Sigma(0) = \Sigma$ and with variational vector field fN on Σ.

Using standard elliptic theory, it is easy to prove that an open oriented minimal surface Σ is stable if and only if it has a positive Jacobi function. Since the universal covering space of an orientable stable minimal surface is stable (it has a positive Jacobi function by composing), for many theoretical questions concerning a stable minimal Σ, we may assume Σ is simply-connected.

Suppose Σ is a minimal surface and $D \subset \Sigma$ is a geodesic disk of radius R on Σ centered at p which is stable. A short calculation (see below) by way of the second variation of area formula, using the function $f(r,\theta) = \frac{(R-t)}{R}$ defined in polar geodesic coordinates (t,θ) on D, gives a proof of the following beautiful formula of Colding-Minicozzi [**14**] for estimating the area of D.

THEOREM 5.2. *If* $D \subset \Sigma$ *is a stable minimal disk of geodesic radius* r_0 *on a minimal surface* $\Sigma \subset \mathbb{R}^3$, *then*

$$\pi r_0^2 \leq \text{Area}(D) \leq \frac{4}{3}\pi r_0^2.$$

PROOF. We now give the proof of the above formula, following the calculation in [**14**]. This calculation is excerpted from [**60**].

Since D has nonpositive Gaussian curvature, the area of D is at least as great as the comparison Euclidean disk of radius r_0, which implies $\pi r_0^2 \leq \text{Area}(D)$. Consider a test function $f(r,\theta) = \eta(r)$ on the disk $D = D(r_0)$ that is a function of the radial coordinate r and which vanishes on ∂D. By the second variation of area formula, Green's formula and the coarea formula, we obtain:

(1)
$$0 \leq \int_D -f\Delta f + 2Kf^2 = \int_D |\nabla f|^2 + 2\int_D Kf^2 = \int_0^{r_0} (\eta'(s))^2 l(s) + 2\int_0^{r_0} \left(\int_{r=s} K\right)\eta^2(s),$$

where K is the Gaussian curvature function on $D(s)$ of radius s and $l(s)$ is the length of $\partial D(s)$.

Let $K(s) = \int_{D(s)} K$. Then, by the first variation of arc length and the Gauss-Bonnet formula, we obtain:

$$l'(s) = \int_{\partial D(s)} \kappa_g = 2\pi - K(s) \Rightarrow K(s) = 2\pi - l'(s). \tag{2}$$

Since $K'(s) = \int_{r=s} K$, substituting in (1) yields:

$$0 \leq \int_0^{r_0} (\eta'(s))^2 l(s) + 2\int_0^{r_0} K'(s)\eta^2(s). \tag{3}$$

Integrating (3) by parts and then substituting the value of $K(s)$ given in (2) yields:
(4)

$$0 \le \int_0^{r_0} (\eta'(s))^2 l(s) - 2\int_0^{r_0} K(s)(\eta^2(s))' = \int_0^{r_0} (\eta'(s))^2 l(s) - 2\int_0^{r_0} (2\pi - l'(s))(\eta^2(s))'.$$

Now let $\eta(s) = 1 - \frac{s}{r_0}$ and so $\eta'(s) = \frac{-1}{r_0}$ and $(\eta^2(s))' = \frac{-2}{r_0}(1 - \frac{s}{r_0})$. Substituting these functions in (4) and then rearranging gives the following inequality:

$$(5) \qquad -\frac{1}{r_0^2}\int_0^{r_0} l(s) + \frac{4}{r_0}\int_0^{r_0} l'(s)(1 - \frac{s}{r_0}) \le \frac{8\pi}{r_0}\int_0^{r_0} (1 - \frac{s}{r_0}) = 4\pi.$$

Integration of (5) by parts followed by an application of the coarea formula yields:

$$\frac{-1}{r_0^2}\int_0^{r_0} l(s) + \frac{4}{r_0^2}\int_0^{r_0} l(s) = \frac{3}{r_0^2}\int_0^{r_0} l(s) = \frac{3}{r_0^2}\text{Area}(D) \le 4\pi \Rightarrow \text{Area}(D) \le \frac{4}{3}\pi r_0^2.$$

□

We now apply the above area estimate for stable minimal disks to give a short proof of the famous classical result of do Carmo and Peng [**20**] and of Fischer-Colbrie and Schoen [**26**] which states:

THEOREM 5.3. *The plane is the only complete stable orientable minimally immersed surface in* $\mathbb{R}^3$.

PROOF. If Σ is a complete orientable stable minimal surface in $\mathbb{R}^3$, then the universal covering space of Σ composed with the inclusion of Σ in $\mathbb{R}^3$ is also a complete minimally immersed stable minimal surface in $\mathbb{R}^3$. Since Σ is a plane if and only if its universal cover is a plane, we may assume that Σ is simply-connected. Since the Gaussian curvature of Σ is nonpositive, Hadamard's theorem implies that, after picking a base point $p_0 \in \Sigma$, we obtain global geodesic polar coordinates (t, θ) on Σ centered at p_0. In these coordinates let $D(R)$ denote the disk of radius R centered at p_0.

Let $A(R)$ be the area of $D(R)$ and note that $A(R)$ is a smooth function of R. The first derivative of $A(R)$ is equal to

$$A'(R) = \text{Length}(\partial D(R)).$$

Also it is easy to see by the first variation of arc length that

$$A''(R) = \int_{\partial D(R)} \kappa_g,$$

where κ_g is the geodesic curvature of $\partial D(R)$. By the Gauss-Bonnet formula, we obtain

$$A''(R) = 2\pi - \int_{D(R)} K dA,$$

and so $A''(R)$ is monotonically increasing as a function of R. Since $A''(R)$ is monotonically increasing and $A(D(R)) \le \frac{4}{3}\pi R^2$, then $A''(R) \le \frac{8}{3}\pi$ and so $-\int_{D(R)} K dA$ is less than $\frac{2}{3}\pi$. Thus, Σ has total absolute Gaussian curvature which is finite and at most $\frac{2}{3}\pi$. At this point one obtains a contradiction in any of several different ways. One way is to appeal to a theorem of Osserman (Theorem 7.1) that states that the total curvature of a complete orientable nonplanar minimal surface is an integer multiple of -4π. Since the absolute total curvature of Σ is at most $\frac{2}{3}\pi$, its total curvature must be zero and we conclude Σ is a plane.

□

REMARK 5.4. Let $D(2R_0)$ be a stable minimal geodesic disk of radius $2R_0$ and $D(R_0)$ be the subdisk of radius R_0. The calculations and estimates used in the proof of Theorem 5.3 easily yield an upper bound of $\frac{8}{3}\pi$ for the total absolute curvature of $D(R_0)$. Since $A''(R) \leq 2\pi + \frac{8}{3}\pi = \frac{14}{3}\pi$ in the range $0 \leq R \leq R_0$, one obtains the estimate of $\frac{14}{3}\pi R$ for the Length$(\partial D(R)) = A'(R)$ for $0 \leq R \leq R_0$. We will use these estimates in the proof of Theorem 5.6. Theorem 5.3 is also an immediate consequence of Theorem 5.6 below for which we will give a self-contained proof that does not appeal to Osserman's Theorem.

Modifications of the arguments in the previous two Theorems (using different cut-off functions, etc. [**60**]) show that a related Theorem holds for complete orientable minimal surfaces of finite index; this result is the following theorem of Fischer-Colbrie [**25**].

THEOREM 5.5. *If Σ is a complete orientable minimal surface with compact boundary and finite index, then Σ has finite topology and finite total curvature.*

An important consequence of Theorem 5.3, using a blow-up argument, is that orientable minimally immersed stable surfaces with boundary in $\mathbb{R}^3$ have curvature estimates up to their boundary of the form given in the next Theorem. These curvature estimates by Schoen play an important role in numerous applications.

THEOREM 5.6. [**99**] *There exists a constant $c > 0$ such that for any stable orientable minimally immersed surface Σ in $\mathbb{R}^3$ and p a point in Σ of intrinsic distance $\mathbf{d}(p)$ from the boundary of Σ, then the absolute Gaussian curvature of Σ at p is less than $\frac{c}{(\mathbf{d}(p))^2}$.*

The above theorem by way of the same blow-up argument implies a similar estimate for stable minimal surfaces in a Riemannian three-manifold N^3 with injectivity radius bounded from below and which is uniformly locally quasi-isometric to Euclidean space; in particular, one obtains a similar curvature estimate for any compact Riemannian three-manifold M, where the constant c depends on M. We now give a sketch of the construction of the aforementioned blow-up argument, which gives a different proof for Theorem 5.6 from the argument given by Schoen [**99**].

PROOF. Suppose the desired curvature estimate were to fail. By taking universal covering spaces, we may assume that the stable minimal surfaces we are considering are simply-connected. We may assume that there exists a sequence of points $p(n) \in \Sigma(n)$ in the interior of stable orientable simply-connected minimal surfaces $\Sigma(n)$ such that the absolute Gaussian curvature at $p(n)$ is at least $\frac{n}{(d(p(n),\partial\Sigma(n))^2}$. Let $D(p(n))$ be the geodesic disk in $\Sigma(n)$ centered at $p(n)$ of radius $d(p(n), \partial\Sigma(n))$. Let $q(n) \in D(p(n))$ be a point in $D(p(n))$ where the function $\mathbf{d}^2|K|\colon D \to [0,\infty)$ has a maximum value; here $\mathbf{d}$ is the distance function to the boundary of D and $|K|$ is the absolute value of the Gaussian curvature (for simplicity we omit the dependence of d, D and $|K|$ on n). Let $D(n) \subset D(p(n))$ be the geodesic disk of radius $\frac{\mathbf{d}(q(n))}{2}$ centered at $q(n)$. Let $\tilde{D}(n)$ be the disk obtained by first translating $D(n)$ so that $q(n)$ is moved to the origin in $\mathbb{R}^3$ and then homothetically expanding the translated disk by the scaling factor $\sqrt{|K(q(n))|}$. The normalized disks $\tilde{D}(n)$ have Gaussian curvature -1 at the origin, Gaussian curvature bounded from below by -4, and the radii $r(n)$ of the $\tilde{D}(n)$ go to infinity as $n \to \infty$. A standard compactness result (see for example [**74**] or [**93**]) shows that a subsequence of the $\tilde{D}(n)$ converges smoothly as subsets to a complete simply-connected immersed minimal surface $D(\infty)$ passing through the origin of bounded Gaussian curvature and with no boundary. It is straightforward to show that the limit of stable minimal surfaces is stable and so $D(\infty)$ is stable. By Theorem 5.3, $D(\infty)$ is a plane but by construction $D(\infty)$ has Gaussian curvature -1 at the origin since each of the $\tilde{D}(n)$ have this property. This contradiction proves the desired curvature estimate of Schoen.

For the sake of completeness we give a self-contained modification of the end of the proof of Theorem 5.6 that does not depend on the stated compactness result or on the statement of Theorem 5.3. A slight modification of these same arguments can be used to give a simple complete proof (see [**60**] or [**107**]) of Osserman's Theorem, which is Theorem 7.1 and was used in our proof of Theorem 5.3, without appealing to the theorem of Huber (see the paragraph following the statement of Theorem 7.1).

A standard compactness argument shows the following: There exists an $\varepsilon > 0$ such that for any minimal disk E in $\mathbb{R}$ of geodesic radius at least 1 and center p with $|K(p)| = 1$ and $|K|\colon E \to [0,4]$, the image $G(E) \subset S^2$ of the Gauss map contains a geodesic cap centered at $G(p)$ of radius ε. An important application and immediate consequence of this result on the size of the Gauss map, together with a slight variation of the blow-up argument in the previous paragraph, is the following curvature estimate: For all $\eta > 0$, there exists a $\delta > 0$ such that if the total absolute curvature of a minimal disk D centered at p of geodesic radius at least 1 is less than δ, then $|K(p)| < \eta$. (See [**60**] for simple proofs of these and other related results.) We will now make use of both of these elementary results in order to complete the proof of Schoen's curvature estimate.

Recall that $r(n)$ is the radius of the disk $\tilde{D}(n)$ and the $r(n) \to \infty$ as $n \to \infty$. For each $r, 0 < r < r(n)$, let $\tilde{D}(r,n)$ be the geodesic subdisk of radius r and that each $D(n)$ is contained in a stable minimal disk of radius $2r(n)$. From the Remark 5.4, $\tilde{D}(r,n)$ has absolute total curvature at most $\frac{8}{3}\pi$ and the length of $\partial\tilde{D}(r,n)$ is less than $\frac{14}{3}\pi r$. Since $r(n) \to \infty$ as $n \to \infty$ and the total absolute curvature of each $\tilde{D}(n)$ is at most $\frac{8}{3}\pi$, there exist positive integers $k(n)$ such that $2^{k(n)+2} < r(n)$ and such that the total curvatures $C(n)$ of the annuli $\tilde{A}(n)$ bounded by $\partial\tilde{D}(2^{k(n)}, n)$ and $\partial\tilde{D}(2^{k(n)+2}, n)$ satisfy $C(n) \to 0$ as $n \to \infty$.

Now consider the new geodesic disks $\hat{D}(n)$ obtained by homothetically scaling $\tilde{D}(n)$ by the factor $2^{-k(n)}$ and let $\hat{A}(n) \subset \hat{D}(n)$ be the correspondingly scaled annuli. Let $\partial\hat{D}(2,n) \subset \hat{D}(n)$ be the circle of geodesic radius 2 in $\hat{D}(n)$. Since for any point of $\partial\hat{D}(2,n)$, the geodesic disk of radius 1 in $\hat{D}(n)$ centered at such a point is contained in $\hat{A}(n)$ and so has total absolute curvature approaching zero as $n \to \infty$. Our previous curvature estimate implies that the Gaussian curvature of the $\hat{D}(n)$ uniformly approach zero along $\partial\hat{D}(2,n)$ as $n \to \infty$. Since the length of $\partial\hat{D}(2,n)$ is less than $\frac{28}{3}\pi$, it follows that as $n \to \infty$, the length of $G(\partial\hat{D}(2,n))$ in S^2 approaches zero, where G is the Gauss map of $\hat{D}(2,n)$.

By our previous discussion, there exists an $\varepsilon > 0$ such that the Gaussian image $G(\tilde{D}(1,n))$ contains a spherical cap of radius ε centered at the value of G at the center of $\tilde{D}(1,n) \subset \tilde{D}(n)$. It follows that $G(\hat{D}(2,n))$ contains the same spherical cap. Since the Gauss map of $\hat{D}(2,n)$ is an open mapping, $G(\hat{D}(2,n))$ contains a fixed size spherical cap and Length $(G(\partial\hat{D}(2,n))) \to 0$ as $n \to \infty$, for n large the image by G of $\hat{D}(2,n)$ must have area approaching the area of S^2 which is 4π. But this contradicts the fact that the total curvature of $\hat{D}(2,n)$ is at most $\frac{8}{3}\pi$. This contradiction proves the desired curvature estimate.

$\square$

6. The Plateau problem and the Meeks-Yau barrier construction.

The classical Plateau problem is the following:

Classical Plateau problem: Suppose Γ is a smooth embedded simple closed curve in $\mathbb{R}^3$. Does there exist a smooth map $f\colon D \to \mathbb{R}^3$ where D is the unit disk such that $f|\partial D$ is a parametrization of Γ and f has least-area with respect to all such mappings?

The answer to the above question is yes. One particularly natural solution to this problem was given by Douglas [**21**] who gave a solution $f\colon D \to \mathbb{R}^3$ which minimizes the energy over all harmonic maps (harmonic coordinate functions) whose boundaries monotonically parametrize the curve Γ. Later Morrey [**85**] solved the similar Plateau problem in Riemannian manifolds which satisfy a technical condition called homogeneously regular.

Some years after Douglas solved the classical Plateau problem, geometers became interested in solving the following related least-area question where Γ is a finite collection of pairwise disjoint smooth simple closed curves in $\mathbb{R}^3$.

Area-minimizing Plateau Problem:
Does Γ bound a smooth least-area surface Σ with $\partial\Sigma = \Gamma$?

The answer to this problem can be found in [**24**] and is yes, up to a question of boundary regularity. In other words, there exists an open surface Σ which is smooth and embedded such that the homological boundary of Σ is Γ and any other rectifiable two-chain which has homological boundary Γ with $\mathbb{Z}_2$-coeficients has area at least as big as Σ. If Σ has finitely generated fundamental group, then a neighborhood of $\partial\Sigma$ is orientable. It follows from the next Theorem and boundary regularity [**36**] that such a finite topology Σ attaches to its boundary in a smooth way.

THEOREM 6.1. *(Hardt-Simon* [**35**]*)*

1. Γ *is the boundary of a smooth immersed orientable surface of least-area;*
2. *Every such least-area surface is embedded with finite topology;*
3. *There are a finite number of such solutions.*

In general, the classical Douglas solution of least-area is not embedded. However, by an important result of Osserman [**91**], it has no interior branch points. In certain cases it can be shown that this least-area disk is an immersion along its boundary as well. One basic open problem in the classical theory is to prove that the Douglas solution has no boundary branch points. When Γ is analytic, then this result is a theorem of Gulliver and Lesley [**34**]. When Γ is extremal (lies on the boundary of its convex hull), then one has the following regularity theorem for the Douglas solution.

THEOREM 6.2. *(Meeks-Yau* [**82**]*) If* Γ *is extremal, then every Douglas solution to the classical Plateau problem is a smooth injective immersion. In particular, every solution is a smooth embedded disk.*

Meeks and Yau also proved, using the Morrey solution, the regularity of the classical Plateau in the following setting.

THEOREM 6.3. *(*[**83**]*) Suppose* Γ *is a smooth simple closed curve on the boundary of a homogeneously regular Riemannian three-manifold* N^3 *such that the boundary of* N^3 *has nonnegative mean curvature. If* Γ *is homotopically trivial in* N^3*, then there exists a Morrey disk* $f\colon D \to N^3$ *of least-energy and every such disk is a smooth embedding.*

In the above situation the nonnegative mean curvature condition makes the boundary into a good barrier for solving Plateau problems in N^3, including the previous possibly nonorientable and the Hardt-Simon solutions. By using a minimal surface as a barrier against itself, one can often prove the existence of least-area minimal surfaces in the complement of a given minimal surface. An important case of this barrier argument is the theorem in [**78**] which states that if Σ_1, Σ_2 are two properly immersed minimal surfaces in $\mathbb{R}^3$ which are disjoint, then Σ_1 and Σ_2 are contained in closed halfspaces of $\mathbb{R}^3$. In this case one proves that there is a properly embedded least-area surface Σ which separates Σ_1 and Σ_2; Σ is a plane by Theorem 5.3. Later Hoffman and Meeks [**47**] proved that a properly immersed minimal surface contained in a closed halfspace of $\mathbb{R}^3$ is a plane. Therefore, the original Σ_1, Σ_2 we were considering must be planes. This result, called the Strong Halfspace Theorem, has many important applications. In a different direction, if Γ is an extremal simple closed curve in $\mathbb{R}^3$ which does not bound a unique compact branched minimal surface, then Meeks and Yau [**83**] prove that Γ is the boundary of two stable embedded minimal disks; here one uses the union of two minimal surfaces bounding Γ as a barrier to solving the classical Plateau problem in certain mean convex three-manifolds that lie in the convex hull of Γ. This result by Meeks and Yau, together with a disk uniqueness theorem of Nitsche [**88**], has the following corollary. In the proof of Nitsche's theorem, Nitsche assumes the boundary curve is analytic because of the consideration of boundary branch points. When Γ is extremal there are never boundary branch points as shown in [**83**].

THEOREM 6.4. [**83**] *If* Γ *is a smooth extremal curve with total curvature at most* 4π*, then* Γ *is the boundary of a unique compact branched minimal surface and this surface is a smooth embedded minimal disk of least-area.*

7. Minimal surfaces of finite topology.

The deepest results in classical minimal surface theory concern the geometry of properly embedded minimal surfaces in $\mathbb{R}^3$ with finite topology. An important subcollection of these surfaces are the examples which have finite total Gaussian curvature. In this regard, one has the following classical theorem of Osserman [**90**]. (See [**7**], [**60**], and [**107**], for the n-dimensional version of Osserman's theorem.) It follows from this Theorem that such minimal surfaces are defined analytically in terms of meromorphic data on a closed Riemann surface.

THEOREM 7.1. *Suppose* M *is a complete oriented minimal surface in* $\mathbb{R}^3$ *with finite total Gaussian curvature* $C(M) = \int_M K dA$*. Then:*

1. *$C(M)$ is a integer multiple of -4π;*
2. *M has finite conformal type, which means M is conformally diffeomorphic to a compact Riemann surface $\bar{M}$ punctured in a finite number of points;*
3. *The meromorphic Gauss map $g\colon M \to C \cup \{\infty\}$ extends to a meromorphic function on $\bar{M}$;*
4. *The holomorphic one-form $\eta = dx_3 + idx_3^*$ on M given in the Weierstrass representation extends to meromorphic one-form on $\bar{M}$.*

The proof of the above theorem is straightforward if one assumes the result of Huber [**49**] that a complete Riemannian surface M of nonpositive curvature and finite total curvature is conformally $\bar{M} - \{p_1, \cdots, p_n\}$ where $\bar{M}$ is a compact Riemann surface. In this case the meromorphic Gauss map $g\colon \bar{M} - \{p_1, \cdots, p_n\} \to \mathbb{C} \cup \{\infty\}$ has finite area $-C(M)$ counted with multiplicity. Picard's theorem implies g extends analytically across the punctures to a meromorphic function $\bar{g}\colon \bar{M} \to \mathbb{C} \cup \{\infty\}$ of integer degree k. Thus, $C(M) = -4\pi k$, since the area of the unit sphere S^2 is 4π. The theorem then follows rather easily from these observations. We refer the reader to [**60**] for a simple short proof of Osserman's Theorem, which does not assume Huber's or Picard's theorem.

Until 1982 there were only three known examples of properly embedded minimal surfaces of finite topology: they are the plane, helicoid and catenoid. This situation changed radically after this date with the discovery of many new examples of positive genus with finite total curvature. From the pioneering work by Ros [**96**] (also see [**92, 93, 94**]), we now understand reasonably well the structure of the moduli spaces of examples with some bound on the genus and number of ends. In particular, we know that these moduli spaces are real semi-analytic varieties and we understand something about the degeneration of sequences of examples which diverge to points in the boundary of the spaces.

In Section 4 we explained how the Cost-Hoffman-Meeks minimal surfaces $\Sigma(g)$ of genus $g, g \geq 1$, with three ends could be constructed by using the classical Weierstrass representation. Hoffman-Meeks (unpublished) proved that these surfaces could each be deformed analytically through embedded minimal surfaces $\Sigma(g,t)$ of finite total curvature whose middle end is a graph with logarithmic growth t. By the maximum principle these surfaces are embedded in the parameter t, beginning at $t = 0$, until the logarithmic growth of the middle end is equal to the logarithmic growth of one of the other two ends. In [**39**] Hoffman and Karcher proved that for $g \geq 2$, the logarithmic growth of the middle end of $\Sigma(g,t)$ is always less than the logarithmic growth of the other two ends and so the $\Sigma(g,t)$ are embedded for all t when $g \geq 2$.

Using computer graphics techniques, Calahan, Hoffman and Meeks constructed many other properly embedded minimal surfaces of finite total curvature with more than three ends. Within a short time, it became clear that there were probably examples with an arbitrarily large number of ends. While they could not give a rigorous proof of the existence of such surfaces, they were able to give a proof of the existence of properly embedded minimal surfaces which are limits of the expected finite topology examples with more and more ends. These are the Callahan-Hoffman-Meeks [**5**] examples $M(n)$ discussed in Section 4. The $M(n)$ are periodic with infinite genus and an infinite number of planar-type ends; in other words, their middle ends are asymptotic to planes. Since these surfaces are periodic, they have two limit ends which are the top and bottom ends of the surfaces.

For a few years computer graphics ruled the existence part of the theory as geometers constructed more and more intricate examples of properly embedded minimal surfaces of finite positive genus and at least three ends. All of these constructions supported the conjecture of Hoffman and Meeks (see Conjecture 3 in Section 14) that when an example Σ has at least two ends, then $e(\Sigma) \leq g(\Sigma) + 2$, where $e(\Sigma)$ is the number of ends of Σ and $g(\Sigma)$ is the genus of Σ.

In the past decade there has been much success in creating new theoretical methods for constructing properly embedded minimal surfaces which are not obtained by the Weierstrass representation. Shortly after Hoffman and Meeks gave a proof of the existence of the $\Sigma(g)$ examples [**41**] using the Weierstass representation, they gave an abstract minimax construction of these examples [**42**]. In [**45**] Meeks and Hoffman proved that, when properly normalized, the surfaces $\Sigma(g)$ of Costa-Hoffman-Meeks converge as $g \to \infty$ to the union of a vertical catenoid $\mathbf{C}$ and the x_1x_2-plane P and that, on the scale of the maximum curvature along the forming intersection curve $\mathbf{C} \cap P$, the surfaces converge to Scherk's one-periodic minimal surface $\mathbf{S}_{\frac{\pi}{4}}$ described in Section 4. These results motivated the general question of whether two transversely intersecting minimal surfaces could be desingularized by sewing in a "curve" of "Scherk" surfaces. This proposed desingularization became known as the procedure of "minimal" surgery. The theoretical procedure of minimal surgery was given a rigorous basis by the work of Kapouleas [**51**]. Kapouleas was able to prove that if $C_1, C_2, \ldots, C_n$ are a finite collection of catenoids with axes the x_3-axis and of varying logarithmic growth, then the union of these catenoids can be approximated by properly embedded minimal surfaces with $2n$ catenoid type ends and large genus and which approximate scaled down Scherk surfaces near the intersection curves as the genus approaches infinity. Actually one can also take C_1 to be the x_1x_2-plane, and one then obtains examples with $2n-1$ ends and large genus.

More recently Weber and Wolf [**106**] have combined the Weierstrass representation with new conformal methods to produce properly immersed minimal surfaces of every possible odd $e \geq 3$ and genus g which satisfy the Hoffman-Meeks inequality. These surfaces are almost certainly embedded but a rigorous proof of embeddedness seems difficult. In a different direction, which uses an algebraic-geometric type implicit function theorem, Traizet [**104**] has been able to verify the existence of many properly embedded minimal surfaces which satisfy the Hoffman-Meeks inequality but his methods fall short of proving the existence part of it holds in general. Traizet obtains many families of examples of varying dimensions depending on the genus and the number of ends under consideration.

During the 1980's, there were a number of partial results on what became known as the generalized Nitsche Conjecture. Nitsche's original conjecture [**86**] was that if Σ is a complete minimal surface which is the union of simple closed curves in parallel planes, then Σ is a catenoid. The generalized Nitsche conjecture states that if Σ is a properly embedded minimal surface of finite topology $\mathbb{R}^3$ with more than one end, then Σ has finite total Gaussian curvature $C(\Sigma) = \int_\Sigma K dA$. Partial results on the generalized Nitsche conjecture were obtained by Hoffman and Meeks [**43**] and by Meeks and Rosenberg [**76**]. This conjecture was finally proven by Collin in 1997.

THEOREM 7.2. [**15**] *If $\Sigma \in \mathcal{M}$, then each annular end of Σ is asymptotic to the end of a plane or catenoid. In particular, if Σ also has finite topology, then* *by the formula of Jorge and Meeks* [**50**], *Σ has total absolute curvature $C(\Sigma) = -4\pi(g(\Sigma) + e(\Sigma) - 1)$, where $g(\Sigma)$ is the genus of Σ and $e(\Sigma)$ is the number of ends of Σ.*

Finally, in the case where $\Sigma \in \mathcal{P}$ has one end which is annular and Σ is not a plane, Meeks and Rosenberg [**74**] proved that Σ is asymptotic to a helicoid. Their work is based on recent important pioneering work of Colding and Minicozzi which we will discuss in the next section. Putting together the results of Collin [**15**], of Meeks and Rosenberg [**74**], we have the following theorem:

THEOREM 7.3. *Suppose $\Sigma \in \mathcal{P}$ has finite topology and Σ is not a plane. Then:*

1. *Σ is conformally a compact Riemann surface $\bar{\Sigma}$ punctured in a finite number of points;*
2. *The embedding of Σ into $\mathbb{R}^3$ via the Weierstrass representation can be obtained in terms of meromorphic data on $\bar{\Sigma}$;*
3. *The moduli space of examples in $\mathcal{P}$ with the same topology as Σ is an semi-analytic variety;*
4. *If Σ has more than one end, then each end of Σ is asymptotic to the end of a plane or catenoid;*
5. *If Σ has one end, then Σ is asymptotic to a helicoid.*

In certain cases it turns out that knowing that a $\Sigma \in \mathcal{P}$ of finite topology can be described as in Theorem 7.3, implies that Σ must have a particular topology. The first result of this type was proven by Jorge and Meeks [**50**] where they showed that the sphere S^2 punctured in 3, 4 or 5 points can not be properly minimally embedded in $\mathbb{R}^3$ with finite total curvature. Next Schoen [**100**] proved that a $\Sigma \in \mathcal{M}$ of finite total curvature and two ends is a catenoid. Then Lopez and Ros [**56**] generalized the Jorge-Meeks obstruction by proving that every $\Sigma \in \mathcal{M}$ with finite topology and genus-zero is a catenoid. Next, Meeks and Rosenberg [**74**] proved that if $\Sigma \in \mathcal{P}$ is simply-connected, then Σ is a plane or a helicoid. Recently, Meeks, Perez and Ros [**69**] have shown that there is an upper bound on the number of ends of $\Sigma \in \mathcal{M}$ with finite topology and fixed genus. Putting these results together, Theorem 7.3 implies:

THEOREM 7.4. *If $\Sigma \in \mathcal{P}$ has finite topology, then:*

1. *If Σ has genus-zero, then Σ is a plane, a helicoid or a catenoid;*
2. *If Σ has two ends, then Σ is a catenoid;*
3. *For every genus g, there exists an integer $e(g)$ such that if Σ has genus g, then the number of ends of Σ is at most $e(g)$.*

8. The local theory of properly embedded minimal surfaces.

In a recent series of papers, Colding and Minicozzi [**9, 10, 11, 12, 13**], have attempted to describe the basic structure of compact embedded minimal surfaces M of fixed genus which are contained in the unit ball B and which have their boundary on the boundary of B. The most important case of their structure theorem is when M is a disk which passes through the origin where its Gaussian curvature is large. In this case Colding and Minicozzi prove that M has the appearance, in a smaller ball $B(\varepsilon)$ centered at the origin, of a multisheeted graph with many sheets and an axis similar to the axis of a helicoid. They then use this local picture to prove the following beautiful compactness theorem:

THEOREM 8.1. *If $M(n) \subset B, n \in \mathbb{N}$, is a sequence of properly embedded minimal disks in the interior of B with $\partial M(n) \subset \partial B$ and the curvature of the family $M(n)$ is unbounded at the origin, then a subsequence of the $M(n)$ converges to a minimal lamination $\mathcal{L}$ by minimal disks of the interior of B and which is a foliation in a neighborhood of the origin. Furthermore, the convergence of this subsequence is smooth except along a connected Lipschitz curve $S(\mathcal{L})$ passing through the origin.*

Meeks and Rosenberg [**74**] then applied this local structure theorem to prove that the plane and the helicoid are the only properly embedded simply-connected minimal surfaces in $\mathbb{R}^3$. Meeks [**61**] recently applied this uniqueness of the helicoid theorem to prove that the singular curve $S(\mathcal{L})$ in the above theorem is of class $C^{1,1}$ and is orthogonal to the leaves of $\mathcal{L}$. Of class $C^{1,1}$ means that $S(\mathcal{L})$ is of class $C^1$and the unit tangent vector field to $S(\mathcal{L})$ extends to an ambient Lipschitz vector field.

These results of Colding and Minicozzi, of Meeks and of Meeks and Rosenberg involve a geometric analysis of the local geometry of embedded minimal surfaces $M(n)$ at a sequence of points of large normalized curvature, a concept that we now define.

Definition 8.1. A *sequence of points of large normalized curvature* is a sequence $p(n) \in M(n) \subset B$ such that:

1. $\lambda(n) := \sqrt{|K_{M(n)}|(p(n))}$ tends to ∞ as $n \to \infty$;
2. $B(p(n), \frac{\tau(n)}{\lambda(n)}) \subset B$ for each n for some positive $\tau(n)$, where $\tau(n) \to \infty$ as $n \to \infty$;
3. There exists $c > 0$ such that $|K_{M(n)}| \leq c\lambda(n)^2$ in $M(n) \cap B(p(n), \frac{\tau(n)}{\lambda(n)})$.

The definition of points of large normalized curvature is made so that the minimal surfaces $M(n) \cap B(p(n), \frac{\tau(n)}{\lambda(n)})$ translated by $-p(n)$ and then scaled homothetically by the factor $\lambda(n)$, are embedded minimal surfaces with Gaussian curvature -1 at the origin, properly embedded in balls of radii $\tau(n) \to \infty$ and in these balls the surfaces have uniformly bounded Gaussian curvature. It follows from [**74**] that a subsequence of these related normalized minimal surfaces converges with multiplicity one to a connected properly embedded minimal surface Σ in $\mathbb{R}^3$ with bounded nonzero Gaussian curvature.

Suppose $M(n)$ is a sequence of properly embedded minimal disks in B which converges to $\mathcal{L}$ defined above with singular curve $S(\mathcal{L})$. For any point $q \in S(\mathcal{L})$ there exist a sequence of points $q(n) \in M(n)$ of large normalized curvature that converge to q. It follows from the results in [**74**] that a subsequence of the surfaces $\tilde{M}(n)$, obtained by translating $M(n)$ by $-q(n)$ and then homothetically expanding this surface by the factor $\sqrt{|K(q(n))|}$, converges with multiplicity one to a surface $M(\infty)$, called a *normalized blow-up* of the $M(n)$, which is a properly embedded minimal surface in $\mathbb{R}^3$ of bounded nonzero absolute curvature. Suppose now the $M(n)$ are not necessarily disks. It follows from the convex hull property for minimal surfaces that the limit $M(\infty)$ has genus less than or equal to any upper bound of the genus of the $M(n)$ and also that $M(\infty)$ has at most as many generators in its fundamental group as the $M(n)$ have. Thus, when the surfaces $M(n)$ are simply-connected, $M(\infty)$ is simply-connected, and then by [**74**], $M(\infty)$ is a helicoid. Hence, in a small neighborhood of a point of $M(n)$ of very large normalized curvature, $M(n)$ has the appearance of a homothetically shrunk helicoid with a large number of sheets.

These results lead to basic curvature estimates for embedded minimal surfaces. These curvature estimates have proven useful in describing the local geometry of a sequence of properly embedded minimal surfaces $M(n)$ of fixed genus, but not necessary with a bound on the number of generators of their fundamental groups, near points of large normalized curvature. The desired description can be obtained from the special case where the genus of $M(n)$ is zero but $M(n)$ is not necessarily simply-connected. If at such a point p of large normalized curvature the components of $M(n)$ in a small ball centered at p are simply-connected, then the previous analysis implies that in some smaller neighborhood of p, $M(n)$ has the appearance of a homothetically shrunk helicoid with many sheets. It is convenient to make the following definition.

Definition 8.2. A sequence $M(n) \subset B$ is **not** *uniformly-locally-simply-connected* if at some point x in the interior of B there exists a sequence $\varepsilon(n) \to 0$ such that for some large $k(n)$, $M(n + k(n)) \cap B(x, \varepsilon(n))$ contains at least one component which is not simply-connected.

It follows that if the $M(n)$ are not uniformly-locally-simply-connected in B, then there exists a point x in the interior of B and a subsequence $n(k)$ with associated points $p(n(k)) \in M(n(k))$ which converge to x and which are points of large normalized curvature. In particular such a sequence of minimal surfaces has a normalized blow-up. Under the assumption that the $M(n)$ have fixed finite genus, there is a conjecture as to what are the possible normalized blow-ups of such a sequence. Since in this case the normalized blow-up is a properly embedded minimal surface in $\mathbb{R}^3$ of finite genus and bounded curvature, the next more general conjecture explains what the possible normalized blow-ups should be. Note that all of these surfaces actually occur as normalized blow-ups.

CONJECTURE 8.1. *(Meeks, Perez, Ros) If $M \in \mathcal{P}$ has finite genus and is not a plane, then:*

1. *M has bounded curvature;*
2. *If M has one end, then M is asymptotic to a helicoid;*
3. *If M has a finite number of ends greater than one, then M has finite total curvature;*
4. *If M has an infinite number of ends, then M has two limit ends, each of which is asymptotic as $x_3 \to \infty$ to translated limit ends of one of the classical Riemann minimal examples (see Section 4 and also* [**70**] *for a description of these beautiful singly-periodic minimal surfaces of genus-zero which are foliated by circles and lines in horizontal planes);*
5. *If M has genus-zero, then M is a helicoid, a catenoid, or a Riemann minimal example.*

In the case M has finite topology, the conjecture holds for M by Theorem 7.4. If M has an infinite number of ends, then the results in [**16**] show that M can have at most two limit ends (see Section 2 for definitions and Section 9 for results). Meeks, Perez and Ros have shown that a finite genus M cannot have one limit end. An important partial result on the above conjecture is the next Theorem.

THEOREM 8.2. [**66, 67**] *If M is a properly embedded minimal surface in $\mathbb{R}^3$ with finite genus and an infinite number of ends, then:*

1. *M has bounded curvature;*

2. *M has two limit ends.*

If $\Sigma(n)$ is a sequence of properly embedded minimal surfaces in a Riemannian three-manifold N^3 which has a lower bound on its injectivity radius and a bound on its sectional curvature, then it makes sense to talk about a sequence of points $p(n) \in \Sigma(n)$ of large normalized curvature, and one obtains by the arguments in [**74**] a normalized blow-up $\Sigma(\infty)$ as a limit of some subsequence of the $\Sigma(n)$ around $p(n), \Sigma(\infty)$ being a properly embedded minimal surface in $\mathbb{R}^3$ with bounded Gaussian curvature. As the limit $\Sigma(\infty)$ is nonflat, it can be shown that the convergence $\Sigma(n) \to \Sigma(\infty)$ has multiplicity one. Our previous discussion implies that local bounds on the genus or the numbers of generators of the fundamental group in a local neighborhood of $p(n)$ on $\Sigma(n)$ give the same bounds on genus and number of generators of the fundamental group of $\Sigma(\infty)$.

THEOREM 8.3. ([**66, 67**]) *If a sequence of properly embedded minimal surfaces $\Sigma(n) \subset N^3$ has uniformally-locally-bounded-genus and has a normalized blow-up $\tilde{M} \subset \mathbb{R}^3$, then the sequence $\Sigma(n)$ has another normalized blow-up M satisfying:*

1. *M is a helicoid;*
2. *M has a finite number of ends greater than one and M has finite total curvature;*
3. *M has two limit ends and genus-zero.*

9. Minimal surfaces of infinite topology.

Before about 1980, there were only three known classical examples in $\mathcal{P}$ which had finite topology; these surfaces are the plane, the helicoid and the catenoid. The remainder of the classical examples were periodic and not simply-connected and so had infinitely generated fundamental groups. Except for the Riemann minimal examples, all of the remaining known examples in $\mathcal{P}$ had infinite genus and one end. Most of these examples were doubly or triply-periodic and so, by the following Theorem 9.1, they had infinite genus and one end.

THEOREM 9.1. [**6**] *If $\Sigma \in \mathcal{P}$ is not a plane and it is doubly-periodic or triply-periodic, then Σ has infinite genus and one end.*

Later it was shown by Frohman and Meeks [**31**] that given two surfaces in $\mathcal{P}$ with one end and the same genus, there is a diffeomorphism of $\mathbb{R}^3$ which takes one to the other. Thus, for example, one of Scherk's singly-periodic and one of Scherk's doubly-periodic differ by a diffeomorphism of $\mathbb{R}^3$, even though their geometric appearance is completely different.

In [**5**] Callahan, Hoffman and Meeks constructed many new singly-periodic examples in $\mathcal{P}$ with two limit ends and infinite genus; these examples are described in Section 4 and have middle ends which are annular and are asymptotic to horizontal planes. Thus, these new infinite ended examples all have end representatives contained in horizontal slabs and these representatives have asymptotic area growth of πR^2. It turns out that some related properties hold for any example in $\mathcal{P}$ with two limit ends, which we now explain.

Definition 9.1. A surface $M \subset \mathbb{R}^3$ has *quadratic area growth* if there exists a positive constant c such that for all large positive R, the area of M inside the ball $B(R)$ of radius R centered at the origin is less than cR^2.

By the proof of the ordering theorem [**30**], a middle end of a properly embedded minimal surface M with horizontal limit tangent plane at infinity can be represented by a proper subdomain $W \subset M$ with compact boundary such that W "lies between two catenoids." This means that W is contained in a neighborhood S of the x_1x_2-plane, S being topologically a slab, whose width grows logarithmically with the distance from the origin.

One of the fundamental results in the global theory of properly embedded minimal surfaces is that the middle ends of an $M \in \mathcal{M}$ are never limit ends. This is shown by first proving that if W is a properly immersed minimal surface with compact boundary and contained between two catenoids, then W has quadratic area growth [**16**]. By the monotonicity formula for area [**101**], every subend of a representative of a limit end has limiting area growth at least πR^2. Hence, a limit end never has a representative with quadratic area growth. Thus:

THEOREM 9.2. [**16**] *If $M \in \mathcal{M}$, then a limit end of M must be a top or bottom end of M. In particular, M can have at most two limit ends. Furthermore, each middle end has limiting area growth which is approximately like a positive integer times πR^2; the* parity *of a middle end is the parity of this integer.*

In fact, a theorem from [**16**] states that when $\Sigma \in \mathcal{P}$ has two limit ends and horizontal limit tangent plane at infinity, then there exists a proper family $\{P_n \mid n \in \mathbb{Z}\}$ of horizontal planes ordered by their relative heights, each of which intersects Σ transversely in a compact family of curves. Furthermore, the slab determined by P_n and P_{n+1} intersects Σ in a proper subdomain which represents the n^{th} middle end of Σ.

In [**106**] Weber and Wolf consider a method which proves the existence of a sequence $M(n) \subset \mathbb{R}^3$ of properly immersed minimal surfaces of odd genus n with $n+2$ horizontal planar ends. Computer graphics pictures of these surfaces for n relatively large indicate that they are all embedded and it is believed that, when properly normalized, sequences of these surfaces apparently converge to the properly embedded periodic minimal surfaces $M(1)$ of Callahan, Hoffman and Meeks in Section 4. Assuming that these surfaces are embedded, a slightly different normalization by a vertical translation should yield as a limit a properly embedded minimal surface with a bottom catenoid end, middle planar ends and a top limit end. Such a limit minimal surface would then have infinite genus and one limit end. One should compare the probable existence of this infinite genus minimal surface with one limit end to Theorem 8.2 in Section 14, which implies that a one limit end example must have infinite genus.

10. The topological classification theorem for minimal surfaces.

In 1992, Meeks and Yau [**84**] proved that properly embedded minimal surfaces of finite topology in $\mathbb{R}^3$ are unknotted in the sense that any two such homeomorphic surfaces are properly ambiently isotopic. Later Frohman [**28**] proved that any two triply periodic minimal surfaces are properly ambiently isotopic. Recall that a *handlebody* is a three-manifold with boundary which is homeomorphic to the closed regular neighborhood of a connected properly embedded one-dimensional CW-complex in $\mathbb{R}^3$. A surface in a three-manifold is called a *Heegaard surface* if it separates the three-manifold into two closed complements which are handlebodies. More recently Frohman and Meeks [**31**] proved that a properly embedded minimal surface in $\mathbb{R}^3$ with one end is a Heegaard surface in $\mathbb{R}^3$ and that Heegaard surfaces of

$\mathbb{R}^3$ with the same genus are unknotted; hence, properly embedded minimal surfaces in $\mathbb{R}^3$ with one end are unknotted even when the genus is infinite. These topological uniqueness theorems of Meeks and Yau, Frohman, and Frohman and Meeks are special cases of the following general classification theorem which was conjectured in **[31]** and which represents the final solution to the topological classification problem. The space of ends of a properly embedded minimal surface in $\mathbb{R}^3$ has a natural linear ordering which is determined up to reversal by Theorem 3.5 and the middle ends in this ordering have a parity (even or odd) according to Theorem 9.2.

THEOREM 10.1. **[29]** *(Topological Classification Theorem for Minimal Surfaces) Two properly embedded minimal surfaces in $\mathbb{R}^3$ are properly ambiently isotopic if and only if there exists a homeomorphism between the surfaces that preserves the ordering of their ends and preserves the parity of their middle ends.*

The constructive nature of the proof of the Topological Classification Theorem provides an explicit description of the topological embedding of any properly embedded minimal surface in terms of the ordering of the ends, the parity of the middle ends, the genus of each end - zero or infinite - and the genus of the surface. This topological description depends on several major advances in the classical theory of minimal surfaces. First, associated to any properly embedded minimal surface M with more than one end is a unique plane passing through the origin called the limit tangent plane at infinity of M (see Definition 3.8). Furthermore, the ends of M are geometrically ordered over its limit tangent plane at infinity and this ordering is a topological property of the ambient isotopy class of M by Theorem 3.5. Second, the proof of the classification theorem depends on the nonexistence of middle limit ends for properly embedded minimal surfaces given in Theorem 9.2. Third, the proof relies heavily on a topological description of the complements of M in $\mathbb{R}^3$; this topological description of the complements was carried out by Frohman and Meeks **[31]** when M has one end and by Freedman **[27]** in the general case.

Here is an outline of the proof of the classification theorem. The first step is to construct a proper family $\mathcal{F}$ of topologically parallel standardly embedded planes in $\mathbb{R}^3$ such that the closed slabs and halfspaces determined by $\mathcal{F}$ each contains exactly one end of M and each plane in $\mathcal{F}$ intersects M transversely in a simple closed curve. The next step is to reduce the global classification problem to a tractable topological-combinatorial classification problem for "Heegaard" decompositions of closed slabs or half spaces in $\mathbb{R}^3$.

Recently Meeks and Rosenberg have been able to generalize some of the above arguments to prove the following unknotted theorem:

THEOREM 10.2. **[73]** *(Unknotted Theorem) Suppose S^2 is a two sphere endowed with a Riemannian metric with no stable simple closed geodesics. Then:*

1. *If Σ is a noncompact properly embedded minimal surface in $S^2 \times \mathbb{R}$, then Σ is a Heegaard surface for $S^2 \times \mathbb{R}$;*
2. *Every Heegaard surface for $S^2 \times \mathbb{R}$ has two ends and if Σ is a connected orientable surface with two ends, then Σ embedds in $S^2 \times \mathbb{R}$ as a Heegaard surface;*
3. *Heegaard surfaces of $S^2 \times \mathbb{R}$ are unknotted in the sense that if two such surfaces are diffeomorphic, then there exists an orientation preserving diffeomorphism of $S^2 \times \mathbb{R}$ which takes one surface to the other.*

11. The conformal structure of properly embedded minimal surfaces.

By Theorem 7.3, if $\Sigma \in \mathcal{P}$ has finite topology, then it is conformally a compact Riemann surface $\bar{\Sigma}$ punctured in a finite number of points and the Weierstrass representation for Σ can be expressed in terms of meromorphic data on $\bar{\Sigma}$. In the case that Σ has more than one end and finite topology, the conformal structure of being $\bar{\Sigma}$ punctured in a finite number of points was first proven in [**76**]. The case when Σ has one end appears in [**74**]. The proof in [**74**] that a simply connected Σ is conformally $\mathbb{C}$ proceeds by first proving that there exists a plane that intersects Σ transversely in a single proper arc. The result on the conformal structure of Σ, then follows from the next theorem [**71**].

THEOREM 11.1. *If Σ is a properly immersed minimal surface of finite topology and one end which intersects some plane transversely in a finite number of immersed (possibly noncompact) curves, then Σ is conformally a compact Riemann surface punctured in one point.*

The main tool for proving the above theorem is Fatou's Lemma on the almost everywhere radial limits of a bounded harmonic function on the open disk, which is applied in conjunction with the next general theorem.

THEOREM 11.2. [**16**] *If Σ is a properly immersed minimal surface with boundary (possibly empty) in $\mathbb{R}^3$, then every component of the intersection of a closed halfspace with Σ is a parabolic Riemannian surface with boundary.*

The proof of the above basic result introduces an important new definition to the subject; we will give the proof here of the special case when Σ has no boundary.

Definition 11.1. A function $f\colon \Omega \to \mathbb{R}$ defined on a domain $\Omega \subset \mathbb{R}^3$ is called a *universal superharmonic* function if its restriction to any minimal surface Σ in Ω is superharmonic, i.e., $\Delta(f|\Sigma) \leq 0$.

Examples of universal superharmonic functions on all of $\mathbb{R}^3$ include coordinate functions such as x_1 or the function $-x_1^2$. In the proof of the quadratic area growth property of middle ends, one uses the universal superharmonic function $\ln(\sqrt{x_1^2+x_2^2}) - x_3\tan^{-1}(x_3) + \frac{1}{2}\ln(1+x_3^2)$ on a certain region of $\mathbb{R}^3$.

Recall by Proposition 3.1 that a Riemannian surface M with boundary is parabolic if and only if there exists a proper positive superharmonic function on M. We now use this defining property of parabolicity and the universal superharmonic function $\ln(\sqrt{x_1^2+x_2^2}) - \frac{1}{2}x_3^2$ on the complement of the cylinder $\{(x_1,x_2,x_3) \mid x_1^2+x_2^2 < 1\}$ to prove Theorem 11.2 for a properly immersed minimal surface M without boundary; the proof of the case when M has boundary is a small modification of the proof of the empty boundary case.

PROOF. We will show that $M(+) = \{(x_1,x_2,x_3) \in M \mid x_3 \geq 0\}$ is parabolic. Assume that $M(+)$ is connected; the general case can be obtained by proving each component of $M(+)$ is parabolic. For each positive integer n define $M(n) = \{(x_1,x_2,x_3) \in M \mid 0 \leq x_3 \leq n\}$ and let $M(n,*) = \{(x_1,x_2,x_3) \in M(n) \mid 1 \leq x_1^2+x_2^2\}$. Let h_n be the restriction of the universal superharmonic function $\ln(\sqrt{x_1^2+x_2^2}) - \frac{1}{2}x_3^2$ to $M(n,*)$ and note that $h_n\colon M(n,*) \to \mathbb{R}$ is proper and bounded from below. (This function is superharmonic on $M(n,*)$ by the easy to calculate estimate [**16**], $\Delta\ln(x) \leq \frac{|\nabla x_3|^2}{r^2}$ where $r = \sqrt{x_1^2+x_2^2}$ and Δ is the surface

Laplacian). Hence, $M(n,*)$ is parabolic. Since $M(n)$ is the union of a compact surface with $M(n,*)$, then $M(n)$ is also parabolic.

For n large choose a $p \in M(n)$ such that $x_3(p) = 1$. Let $\partial(n)$ denote the part of $\partial M(n)$ at x_3-height n and let $\partial(0)$ denote the part of $\partial M(n)$ at height zero. Since $M(n)$ is parabolic and $x_3|M(n)$ is a bounded harmonic function on $M(n)$, for the hitting measure $\mu_p(n)$ on $\partial M(n)$, we have by Proposition 3.2,

$$1 = x_3(p) = \int\limits_{\partial M(n)} x_3(x)\mu_p(n) = n \int\limits_{\partial(n)} \mu_p(n).$$

Hence, $\int\limits_{\partial(n)} \mu_p(n) = \frac{1}{n}$ and since $M(n)$ is parabolic, $\int\limits_{\partial(0)} \mu_p(n) = 1 - \frac{1}{n}$. Hence, for all positive integers n,

$$\int\limits_{\partial M(+)} \mu_p = \int\limits_{\partial(0)} \mu_p \geq \int\limits_{\partial(0)} \mu_p(n) = 1 - \frac{1}{n}.$$

Therefore, $\int\limits_{\partial M(+)} \mu_p = 1$, which proves that $M(+)$ is parabolic.

□

COROLLARY 11.3. *If M is a properly immersed minimal surface in $\mathbb{R}^3$ and M intersects some plane in a compact set, then M is recurrent for Brownian motion. In particular, by the discussion after Theorem 9.2, if $M \in \mathcal{P}$ has two limit ends, then M is parabolic.*

Theorem 8.2 and the above Corollary imply the next Theorem.

THEOREM 11.4. *Every $\Sigma \in \mathcal{P}$ of finite genus is conformally a closed Riemann surface $\bar{\Sigma}$ punctured in a closed countable set of points W and when this set of points is infinite, then W has exactly two limit points on $\bar{\Sigma}$. In particular, Σ is recurrent for Brownian motion.*

For some related results see [**55**].

12. Periodic minimal surfaces.

In [**75**] Meeks and Rosenberg developed the classical theory of properly embedded doubly-periodic minimal surfaces in $\mathbb{R}^3$. Theoretical questions concerning the geometry of properly embedded doubly-periodic minimal surfaces are usually most easily approached by studying the quotient surface Σ in $\mathbb{T} \times \mathbb{R}$ where $\mathbb{T}$ is a flat two dimensional torus. One of the main theorems in [**75**] is that such a Σ has total Gaussian curvature $c(\Sigma) = 2\pi\chi(\Sigma)$ where $\chi(\Sigma)$ is the Euler characteristic of Σ; thus, Σ has finite total curvature if it has finite topology. This finite total curvature property for finite topology Σ leads to strong restrictions on the geometry and the topology of such surfaces and forces each annular end of Σ to be asymptotic to the end of a flat annulus in $\mathbb{T} \times \mathbb{R}$.

Later Meeks [**65**] generalized these results by proving that any properly embedded minimal surface Σ in $\mathbb{T} \times \mathbb{R}$ has a finite number of ends and that if the genus of Σ is finite, then Σ has finite topology and linear area growth. Since any complete Riemannian surface Σ with at most linear area growth and nonpositive curvature has total curvature $c(\Sigma) = 2\pi\chi(\Sigma)$, Meeks' theorem gave a new proof of the total curvature formula of Meeks and Rosenberg. More importantly, Meeks' theorems identified properly embedded minimal surfaces in $\mathbb{T} \times \mathbb{R}$ of finite total curvature with those surfaces of finite genus. In particular, if Σ has genus-zero, then Σ has finite total curvature. This result, together with some other constraints finite total curvature planar domains in $\mathbb{T} \times \mathbb{R}$ satisfy [**75**], was then applied by Lazard-Holly and Meeks [**53**] to prove the deep result:

THEOREM 12.1. *A genus-zero properly embedded minimal $\Sigma \subset \mathbb{T} \times \mathbb{R}$ is the quotient of one of the classical doubly-periodic examples defined by Scherk* [**98**] *in 1835. (See Section 4).*

Recently, Meeks and Wolf [**81**] have been able to prove the following related uniqueness theorem for the conjugate surfaces to the minimal surfaces in Theorem 12.1. The proof of these theorems are similar in approach but quite different in their details. They are hopeful that they will be able to prove the same result without the symmetry assumption.

THEOREM 12.2. *If M is a connected minimal surface with area less than $2\pi R^2$ in balls of radius R and the symmetry group of M is infinite, then M is a singly-periodic Scherk surface $\mathbf{S}_\theta$ described in Section 4, M is a catenoid or M is a plane.*

Another important uniqueness theorem for periodic minimal surfaces is the uniqueness of the Riemann minimal examples first defined by Riemann in [**95**]. See Section 4 for a description of these beautiful singly-periodic minimal surfaces of genus-zero which are foliated by circles and lines in horizontal planes. The following uniqueness of the Riemann examples was proved by Meeks, Perez and Ros in [**70**]. They are actively working on proving this theorem without the hypothesis of infinite symmetry group. See Conjecture 8.1 and Section 8 for some related discussion on the Riemann examples.

THEOREM 12.3. *The plane, catenoid, helicoid and Riemann minimal examples are the only properly embedded minimal surfaces in $\mathbb{R}^3$ of genus zero with infinite symmetry group.*

More generally, Meeks and Rosenberg [**77**] prove the following theorem for singly-periodic minimal surfaces whose quotient surfaces have finite topology.

THEOREM 12.4. *A properly embedded minimal surface Σ in a nonsimply connected complete flat three-manifold N^3 has finite total curvature if and only if it has finite topology. If $N^3 = \mathbb{R}^3/S_\theta$, where S_θ is a screw motion symmetry with angle $\theta, 0 \leq \theta \leq \pi$, with axis being the x_3-axis and $\Sigma \subset N^3$ is a properly embedded minimal surface with finite topology, then Σ has finite conformal type and can be defined analytically in terms of meromorphic data on its conformal compactification. Furthermore, each annular end of Σ is asymptotic to a horizontal plane in N^3, a vertical half-plane in N^3 or an end of a helicoid in N^3.*

As we already remarked, Meeks' theorem [**65**] states that a properly embedded minimal surface in $\mathbb{T} \times \mathbb{R}$ always has a finite number of ends. On the other hand, there exist properly embedded minimal surfaces of finite genus in $\mathbb{R}^2 \times S^1$ and $\mathbb{R}^3/S_\pi$ with an infinite number of ends (see [**58**]). For example, a singly-periodic quotient $\bar{\mathbf{S}}_{\frac{\pi}{4}} \subset \mathbb{R}^2 \times S^1$ of a doubly-periodic Scherk surface can have an infinite number of ends. The following theorem shows that these flat three-manifolds with infinite cyclic fundamental groups are special cases.

THEOREM 12.5. [**58**] *If $\Sigma \subset \mathbb{R}^3/S_\theta, \theta \neq 0$ or π, is a properly embedded minimal surface with more than one end, then Σ has at most quadratic area growth and has a finite number of ends. If $\Sigma \subset \mathbb{R}^3/S_\theta, \theta \neq 0$ or π, has finite genus, then Σ has at most quadratic area growth, finite topology and finite total curvature.*

The author refers the interested reader to [**65**] for a detailed survey of the classical theory of periodic minimal surfaces and to [**64**] for the more specialized theory of triply-periodic minimal surfaces.

13. Minimal surfaces in $M \times \mathbb{R}$.

In Section 12 we discussed briefly some of the theoretical results of Meeks [**65**] and of Meeks and Rosenberg [**75**] concerning minimal surfaces in $\mathbb{T} \times \mathbb{R}$ where $\mathbb{T}$ is a flat two-dimensional torus; these surfaces are just the quotients of doubly-periodic minimal surfaces in $\mathbb{R}^3$. When $\Sigma \subset \mathbb{T} \times \mathbb{R}$ is a properly embedded minimal surface, then, by the theorem of Meeks [**65**], Σ has a finite number of ends and if Σ also has finite genus, then it has bounded curvature, linear area growth, total curvature $2\pi\chi(\Sigma)$ and finite index with respect to the stability operator. In [**72**] and [**73**] Meeks and Rosenberg generalized many of their results for $\mathbb{T} \times \mathbb{R}$ to the case $M \times \mathbb{R}$ where M is a compact Riemannian surface. In particular, they prove that if M is endowed with a metric of nonpositive curvature, then the just described result of Meeks for a properly embedded minimal Σ in $M \times \mathbb{R}$ holds (see the Finiteness of Ends Theorem and the Bounded Curvature Theorem at the end of this Section). The four main theorems in these papers - The Linear Area Growth Theorem, the Stability Theorem, the Finiteness of Ends Theorem and the Bounded Curvature Theorem - all represent surprisingly strong theoretical results which will likely have an impact on research in other areas of classical surface theory.

In part because of the possible applications of these results to the study of constant mean curvature surfaces in $\mathbb{R}^3$ and S^3, we will briefly go over their statements and some of the ideas behind these proofs. In all of these theorems M denotes a compact Riemannian surface.

Given a properly immersed minimal surface Σ in $M \times \mathbb{R}$, we define the *flux of* Σ to be the flux of the gradient ∇h across $\Sigma \cap (M \times \{0\})$ where $h \colon \Sigma \to \mathbb{R}$ is the harmonic height function $h(p,t) = t$. Since h is a proper harmonic function, the flux of Σ is the flux of ∇h across any level set of h, not just the level set at height zero. The invariance of the flux of Σ plays a crucial role in the proofs of many of these theorems, including the following.

THEOREM 13.1. [**73**] *(Linear Area Growth Theorem) If Σ is a properly embedded noncompact minimal surface in $M \times \mathbb{R}$ of bounded curvature, then Σ has a finite number of ends and linear area growth, in the sense that $c_1 t \leq \mathrm{Area}(\Sigma \cap (M \times [-t,t])) \leq c_2 t$ where $c_1 > 0$ depends only on the injectivity radius of M and c_2 depends only on the geometry of M, a lower bound of the flux of Σ and an upper bound on the absolute Gaussian curvature of Σ.*

Every sequence of properly embedded minimal surfaces in a three-dimensional Riemannian manifold, which intersect a compact domain and satisfy uniform local area and local curvature estimates, has a subsequence that converges to another properly embedded minimal surface with local area and local curvature estimates (see for example [**74**]). A simple consequence of this compactness result and Theorem 13.1 is that every noncompact properly embedded minimal surface in $M \times \mathbb{R}$

with bounded curvature is quasiperiodic in the following sense. A properly embedded surface Σ in a Riemannian three-manifold M is *quasiperiodic* if there exists a discrete infinite closed subset $S = \{T_n \mid n \in \mathbb{N}\}$ of the isometry group of M such that $T_n(\Sigma)$ converges on compact subsets of M to a properly embedded surface.

COROLLARY 13.2. *If Σ is a properly embedded noncompact minimal surface of bounded curvature in $M \times \mathbb{R}$, then Σ is quasiperiodic. In fact, any sequence of vertical translations of Σ in $M \times \mathbb{R}$ contains a convergent subsequence to another properly embedded minimal surface with the same bound on its curvature.*

By the curvature estimates of Schoen [**99**], every properly embedded stable minimal surface in $M \times \mathbb{R}$ has bounded curvature. Therefore, every properly embedded noncompact stable minimal surface in $M \times \mathbb{R}$ is quasiperiodic. This quasiperiodicity property is essential in proving the next theorem.

THEOREM 13.3. [**72**] *(Stability Theorem) Suppose that Σ is a connected properly embedded stable orientable minimal surface in $M \times \mathbb{R}$. Then Σ is one of the surfaces described in (1)-(4) below:*

1. *Σ is compact and $\Sigma = M \times \{t\}$ for some $t \in \mathbb{R}$;*
2. *Σ is of the form $\gamma \times \mathbb{R}$ where γ is a simple closed stable geodesic in M;*
3. *Σ is periodic and has a quotient $\bar{\Sigma}$ in $M \times S^1(r)$ where r is the circumference of the circle. In this case, for every $p \in M, \{p\} \times S^1(r)$ intersects $\bar{\Sigma}$ transversely in a single point and the orbit of the natural action of $S^1(r)$ on $M \times S^1(r)$ gives rise to a product minimal foliation of $M \times S^1(r)$. In particular, $\bar{\Sigma}$ is homeomorphic to M and is area minimizing in its integer homology class;*
4. *Σ is a graph over an open connected subdomain of M bounded by a finite number of stable geodesics with each end of Σ asymptotic to the end of one of the flat vertical annuli described in (2);*
5. *The moduli space of examples described in (3) in the case M is orientable is naturally parametrized by $P(H_1(M)) \times \mathbb{R}^+$ where $P(H_1(M))$ consists of the primitive (non-multiple) elements in the first homology group of M. Given an example $\bar{\Sigma} \subset M \times S^1(r)$ we obtain the corresponding element $([\bar{\Sigma}] \cap [M \times \{*\}], r) \in P(H_1(M)) \times \mathbb{R}^*$, where $\cap$ is the intersection pairing of the associated homology classes in $M \times S^1(r)$ and $*$ is a base point on $S^1(r)$.*

Theorem 13.1 states, among other things, that a properly embedded minimal surface of bounded curvature in $M \times \mathbb{R}$ must have a finite number of ends. The next theorem demonstrates that the bounded curvature hypothesis on the surface can be dropped and one still obtains the finite number of ends conclusion; Lemmas and Assertions used in the proof of Theorem 13.3 play a fundamental role in proving this more general result.

THEOREM 13.4. [**73**] *(Finiteness of Ends Theorem) If Σ is a properly embedded minimal surface in $M \times \mathbb{R}$, then Σ has a finite number of ends.*

We next focus our attention on the case when the properly embedded minimal surface Σ in $M \times \mathbb{R}$ has finite genus. By Theorem 13.4, such a surface Σ has a finite number of ends and so each end of Σ is an annulus and Σ has finite topology. Meeks and Rosenberg then use this finite topology property of Σ to prove that Σ has bounded curvature and so, by Theorem 13.1, Σ has linear area growth. The proof that Σ has bounded curvature is difficult and uses some of the recent results of Meeks, Perez and Ros [**66**] on the local structure of properly embedded minimal surfaces in three manifolds with bounded genus in a neighborhood of a point of large curvature; these results depend on recent curvature estimates of Colding and Minicozzi [**11, 12, 13**], and results of Meeks [**61**] and of Meeks and Rosenberg in [**74**]. See Section 8 for a more detailed discussion of these topics. With some further geometric analysis, Meeks and Rosenberg obtain the following theorem which significantly generalizes their earlier stated results.

THEOREM 13.5. [**73**] *(Bounded Curvature Theorem) Suppose Σ is a properly embedded minimal surface of finite genus in $M \times \mathbb{R}$. Then:*

1. *Σ has finite topology, finite conformal type, bounded curvature, and linear area growth;*
2. *If M has nonpositive curvature, then Σ has finite index with respect to the stability operator and Σ has total curvature $2\pi\chi(\Sigma)$;*
3. *If M has nonpositive curvature and M is not a torus, then each end of Σ is asymptotic to $\gamma \times \mathbb{R}$ where γ is a stable simple closed geodesic in M.*

In [**73**] Meeks and Rosenberg discuss several general methods for constructing minimal surfaces of finite topology in $M \times \mathbb{R}$, in particular the minimal graphs described below.

THEOREM 13.6. *If M is an orientable Riemannian surface of genus at least one and M is not a torus endowed with a metric which admits a foliation by closed geodesics, then there exists an infinite number of non-isotopic domains in M bounded by a finite number of stable simple closed geodesics and proper minimal graphs in $M \times \mathbb{R}$ over these domains.*

In the case M is a two sphere S^2 endowed with a metric of constant positive curvature, Meeks and Rosenberg write down a two-parameter family $\mathcal{A}$ of properly embedded minimal annuli in $S^2 \times \mathbb{R}$ that are closely related to the infinite-ended periodic Riemann examples of genus-zero in $\mathbb{R}^3$. The surfaces in $\mathcal{A}$ coincide with the properly embedded minimal annuli in $S^2 \times \mathbb{R}$ foliated by circles, one in each $S^2 \times \{t\}$. This family is defined in [**73**] in terms of meromorphic functions on rectangular elliptic curves and is closely related to a family of "tori" of constant mean curvature in $\mathbb{R}^3$ defined by Abresh [**1**].

14. Sixteen of my favorite conjectures.

In this section the author will present sixteen fundamental conjectures in the classical theory of minimal surfaces. For the most part these conjectures are motivated by the author's own research and are not widely known except to classical minimal surface specialists. Hopefully, the presentation of these problems and suggestions for a plan of attack on solving them will speed up their solution and stimulate further interest in this beautiful subject. In the statement of each conjecture the author has included a suggested expected time frame for a solution; only time will tell how accurate this time frame is. The author has listed in the statement of each conjecture the principal researchers to whom the conjecture might be attributed. These conjectures are listed approximately in order according to the author's interest in them or by his personal feeling of their general importance or deepness.

Most of these problems and many others appear in [**59**] along with further discussion; also, see the author's 1978 book [**62**] for a much longer list of conjectures in the subject, some of whose solutions we have discussed in this survey. In the following discussion we again let $\mathcal{P}$ denote the space of all properly embedded connected minimal surfaces in $\mathbb{R}^3$ and let $\mathcal{M} \subset \mathcal{P}$ denote the subspace of examples with more than one end.

CONJECTURE 1. *[Convex Curve Conjecture (Meeks) Time Frame = 30 years] Two convex Jordan curves in parallel planes cannot bound a compact minimal surface of positive genus.*

There are some partial results on the Convex Curve Conjecture under the assumption of some symmetry on the curves (see [**79, 97, 100**]). Also, the results in [**79, 80**] indicate that the Convex Curve Conjecture probably holds in the more general case where the two convex planar curves do not necessarily lie in parallel planes but rather lie on the boundary of their convex hull, in this case the planar Jordan curves are called extremal. Recent results by Ekholm, White and Wienholtz [**22**] show that every compact orientable minimal surface that arises as a counterexample to the convex curve conjecture is embedded and that for a fixed pair of extremal convex planar curves there is a bound on the genus of such a minimal surface.

The next conjecture is motivated in part by the case where Γ is extremal (see Theorem 6.4), where it is known even in the more general case where the minimal surface is allowed to be nonorientable.

CONJECTURE 2. *[4π-Conjecture (Meeks-Yau, Nitsche) Time Frame = 20 years] If Γ is a simple closed curve in $\mathbb{R}^3$ with total curvature at most 4π, then Γ bounds a unique orientable branched minimal surface and this unique minimal surface is an embedded disk.*

There exists a conjecture by Ekholm, White and Wienholtz [**22**] that generalizes Conjecture 2, removing the orientability assumption on the minimal surface spanning Γ (these authors conjecture that besides the unique minimal disk given by Nitsche's Theorem, only one or two Möbius strips can occur).

CONJECTURE 3. *[Finite Topology Conjecture (Hoffman and Meeks) Time Frame = 100 years] A noncompact orientable surface M of finite topology with genus g and k ends, $k \neq 2$, occurs in $\mathcal{P}$ if and only if $k \leq g + 2$.*

See [**39, 44, 104, 106**] and the discussion in Section 7 for partial existence results which seem to indicate that the existence implication in Finite Topology Conjecture holds when $k > 2$. There is experimental computer evidence that every orientable surface with finite genus and one end properly minimally embedds in $\mathbb{R}^3$ as a minimal surface of finite type (see [**2, 3, 74**]); also see the later Conjecture 14. Theorem 7.4 shows that for each positive genus g, there exists an upper bound $e(g)$ on the number of ends of an $M \in \mathcal{M}$ with finite topology and genus g. Results of Collin [**15**] and Schoen [**100**] imply that the only examples in $\mathcal{M}$ with finite topology and two ends are catenoids. Results of Collin [**15**] and Lopez-Ros [**56**] imply that if M has finite topology, genus zero and at least two ends, then M is a catenoid.

CONJECTURE 4. *[Liouville Conjecture (Meeks, Sullivan) Time Frame = 50 years] If $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a positive harmonic function, then h is constant.*

The above conjecture is closely related to work in [**16, 71**]. For example, from the discussion in Section 11, we know that if $M \in \mathcal{P}$ has finite genus or two limit ends, then M is recurrent for Brownian motion which implies M satisfies the Liouville Conjecture.

CONJECTURE 5. *[Properness Conjecture (Calabi, Meeks) Time Frame = 100 years] If $f\colon M \to \mathbb{R}^3$ is a complete injective minimal immersion, then $M \in \mathcal{P}$.*

The author has an outline for a possible proof of properness in the finite topology case. The author in conjunction with Perez and Ros have conjectured [**66**] that if a complete embedded minimal $M \subset \mathbb{R}^3$ has finite genus, then M has bounded Gaussian curvature (also see, [**74**]). It follows from work in [**66, 67, 74**] that if such an M has locally bounded Gaussian curvature in $\mathbb{R}^3$ and finite genus, then M is properly embedded.

Gulliver and Lawson [**33**] proved that if Σ is a stable orientable minimal surface with compact boundary that is properly embedded in the punctured unit ball in $\mathbb{R}^3$, then its closure is an embedded surface. If Σ is not stable, then the corresponding result is not known. Recent results in [**12, 67**] indicate that a more general result might hold.

CONJECTURE 6. *[Isolated Singularities Conjecture (Gulliver-Lawson) Time Frame = 8 years] There does not exist a properly embedded minimal surface in the punctured ball $B - \{(0,0,0)\}$ whose closure is not a surface at $(0,0,0)$.*

CONJECTURE 7. *[Isometry Conjecture (Meeks) Time Frame = 20 years] If $M \in \mathcal{P}$, then intrinsic isometries of M extend to ambient isometries of $\mathbb{R}^3$. Furthermore, if M is not simply-connected, then it is "minimally rigid" in the sense that any isometric minimal immersion of M into $\mathbb{R}^3$ is congruent to M.*

This Isometry Conjecture is known if $M \in \mathcal{P}$ has more than one end (see [**8**]). Results of Meeks and Rosenberg [**74**] and [**77**] imply that the isometry conjecture can only fail if M has one end and infinite genus. It is also known to hold for doubly-periodic minimal surfaces [**75**]. One way to prove the conjecture would be to prove that if $M \in \mathcal{P}$ has one end and infinite genus, then there exists a plane in $\mathbb{R}^3$ that intersects M in a set that contains a simple closed curve.

CONJECTURE 8. *[Genus-zero Conjecture (Meeks-Perez-Ros) Time Frame = 2 years] If $M \in \mathcal{P}$ has genus-zero, then M is a plane, a catenoid, a helicoid or a Riemann example. In particular, M is foliated by lines and circles in parallel planes.*

The above conjecture is known if M has genus-zero and finite topology by results in [**15, 56, 74**]. By the main theorem in [**16**], if M has infinite topology, then it must have one or two limit ends. Theorem 8.2 states that if M has finite genus and infinite topology, then M must have two limit ends; in fact, the main goal of [**67**] is to prove this result. If M has finite genus and two limit ends, then the curvature estimates in [**66**] show that M is quasiperiodic and the results in [**70**] imply the conjecture if M is actually periodic. See Section 8 for related discussions and an explanation of the importance of Conjecture 8.

CONJECTURE 9. *[Geometric Flux Conjecture (Meeks-Rosenberg) Time Frame = 80 years] Suppose $M \in \mathcal{P}$ and $h\colon M \to \mathbb{R}$ is a nonconstant coordinate function on M. Consider the set I of integral curves of ∇h. Then there exists a countable set $C \subset I$ such that for any integral curve $\alpha \in I - C$, $h|\alpha\colon \alpha \to \mathbb{R}$ is a diffeomorphism. Here we consider $\alpha\colon \mathbb{R} \to M$ to be, after a choice of $p \in \alpha$, a curve $\alpha(t)$ with $\alpha(0) = p$ and $\alpha'(t) = \nabla h(\alpha(t))$.*

One could weaken the hypothesis in the above conjecture that "except for a countable number of integral curves, h restricted to an integral curve of ∇h is a diffeomorphism with $\mathbb{R}$" to the hypothesis that "for almost-all integral curves of ∇h, h restricted to an integral curve is a diffeomorphism with $\mathbb{R}$". This weaker version of the Flux Conjecture is, for technical reasons, more likely to be solved with a suggested time frame of only two years for its solution. This weaker conjecture, via Stokes theorem, has as a consequence the recent Algebraic Flux Lemma [**59**] by Meeks. The Algebraic Flux Lemma and Theorem 11.2 imply that for any coordinate function $h\colon M \to \mathbb{R}$ on an a properly immersed minimal surface in $\mathbb{R}^3$, the flux of ∇h across a level set of h is independent of the level set. The author feels that this flux result may have important theoretical consequences.

CONJECTURE 10. *[Scherk Uniqueness Conjecture (Meeks) Time Frame = 5 years] If M is a properly immersed minimal surface in $\mathbb{R}^3$ and, in balls $B(R)$ of radius R, Area$(M \cap B(R)) < 2\pi R^2$, then M is a Scherk singly-periodic minimal surface, a catenoid or a plane.*

A related conjecture on the uniqueness of Scherk's doubly-periodic minimal surfaces was recently solved by Lazard-Holly and Meeks [**53**]; they proved that if $M \in \mathcal{P}$ is doubly-periodic and the quotient surface has genus-zero, then M is one of Scherk's doubly-periodic minimal surfaces. The basic approach used in [**53**] can be adapted to prove the above conjecture under the assumption that the surface is periodic; this result is a recent theorem of Mike Wolf and the author (see Theorem 12.2). In fact their result gives a proof of the above conjecture in the case where M has an infinite symmetry group. Their approach for solving the general conjecture is first to prove the following conjecture on the uniqueness of the limit tangent cone of M, from which it follows by unpublished work of Meeks and Ros that M has two Alexandrov-type planes of symmetry. From these planes of symmetry one can describe the Weierstrass representation of M, which hopefully would be useful in completing the proof of the conjecture. Much of the interest in the previous conjecture arises from the role that Scherk surfaces play in desingularizing two intersecting minimal surfaces (see Kapouleas [**51**]).

CONJECTURE 11. *[Unique Limit Tangent Cone Conjecture (Meeks) Time Frame = 4 years] If $M \in \mathcal{P}$ is not a plane and has quadratic area growth, then $\lim_{t\to\infty} \frac{1}{t}M$ exists and is a cone. Furthermore, if M has area less than $2\pi R^2$ in balls of radius R, then the limit tangent cone is the union of two planes or one plane of multiplicity two passing through the origin.*

CONJECTURE 12. *[Connected Graph Conjecture (Meeks) Time Frame = 8 years] A minimal graph in $\mathbb{R}^{n+1}$ with zero boundary values over a proper, possibly disconnected, domain in $\mathbb{R}^n$ can have at most two nonplanar components. If the graph also has sublinear growth, then such a graph with no planar components is connected.*

The above conjecture was made by Meeks a number of years ago. The first important partial result came out of work by Meeks and Rosenberg on the uniqueness of the helicoid (see [**74**]). They proved, under the additional hypothesis of gradient estimates, that such a graph can only have a finite number of nonplanar components. Spruck [**102**] has given some related results and Li and Wang [**54**] have recently proven finiteness of the number of nonflat components without assuming gradient bounds.

CONJECTURE 13. *[Quadratic Area Growth Conjecture (Meeks) Time Frame = 9 years] $M \in \mathcal{P}$ has quadratic area growth if and only if there exist a double cone C (of the form $x_3^2 = \lambda(x_1^2 + x_2^2)$ and possibly rotated) that intersects M in a compact set.*

It follows from computations in [**16**] that $M \in \mathcal{P}$ has quadratic area growth if M intersects the union of the negative end of a vertical catenoid and a positive vertical cone in a compact set. If the conclusion of the previous unique limit tangent cone conjecture holds for an $M \in \mathcal{P}$ with quadratic area growth, then for such an M there exists a double cone that intersects M in a compact set. Hence, the validity of the unique limit tangent cone conjecture would give one of the implications in the quadratic area growth conjecture.

CONJECTURE 14. *[One-ended Conjecture (Meeks and Rosenberg) Time Frame = 20 years] For every nonnegative integer g, there exists a unique nonplanar $M \in \mathcal{P}$ with genus g and one end.*

There are some partial results on the above conjecture. Meeks and Rosenberg have shown that every example $M \in \mathcal{P}$ of finite genus and one end has a special analytic representation which makes M into a "surface of finite type" (see [**74**]). In the case of genus-zero, Meeks and Rosenberg proved that the plane and the helicoid are the only genus-zero examples. Based on an earlier computational proof of the existence of a helicoid with a handle by Hoffman, Karcher and Wei [**40**], Hoffman, Weber and Wolf [**48**] have given a rigorous mathematical proof of the existence of the genus-one helicoid. Work in progress by Martin and Weber [**57**] indicates that this genus-one helicoid is unique. Other computational results in [**2, 3, 103**] indicate the conjecture is true for genus 2, 3, 4 and 5. We believe that a recent theorem of Traizet and Weber [**105**] will eventually lead to a proof of the existence part of the above conjecture.

The Finite Topology Conjecture of Hoffman and Meeks and the One-ended Conjecture of Meeks and Rosenberg together propose the precise topological conditions under which a noncompact orientable surface of finite topology would properly minimally embed in $\mathbb{R}^3$. What about the case where the noncompact orientable surface M has infinite topology; i.e., either M has infinite genus or M has an infinite number of ends? By Theorem 9.2, such an M can have at most one or two limit ends. Theorem 8.2 states that such an M cannot have one limit end and finite genus. The following conjecture is nothing more than the claim that these restrictions are the only ones.

CONJECTURE 15. *[Infinite Topology Conjecture (Meeks) Time Frame = 50 years] A noncompact orientable surface of infinite topology occurs in $\mathcal{P}$ if and only if it has at most one or two limit ends and when it has one limit end, then it also has infinite genus.*

Meeks and Rosenberg [**74**] and Meeks, Perez and Ros [**68**] have obtained some partial results on the following conjecture (See Section 13). It is closely related to Conjecture 8.

CONJECTURE 16. *[Uniqueness of the $\mathcal{A}$-family Conjecture (Meeks, Rosenberg) Time Frame = 2 years.] Let $S^2 \subset \mathbb{R}^3$ be the unit sphere centered at the origin. A properly embedded minimal annulus in $S^2 \times \mathbb{R}$ is in the family $\mathcal{A}$ described in Section 13. In particular, every such minimal annulus is foliated by circles in level set spheres.*

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