Local rigidity for cocycles

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ABSTRACT. In this paper we study perturbations of constant cocycles for actions of higher rank semi-simple algebraic groups and their lattices. Roughly speaking, for ergodic actions, Zimmer's cocycle superrigidity theorems implies that the perturbed cocycle is measurably conjugate to a constant cocycle modulo a compact valued cocycle. The main point of this article is to see that a cocycle which is a continuous perturbation of a constant cocycle is actually continuously conjugate back to the original constant cocycle modulo a cocycle that is continuous and "small".

We give some applications to perturbations of standard actions of higher rank semisimple Lie groups and their lattices. Some of the results proven here are used in our proof of local rigidity for affine and quasi-affine actions of these groups.

We also improve and extend the statements and proofs of Zimmer's cocycle superrigidity.

1. Introduction

Let G be a connected semisimple Lie group with no compact factors and all simple factors of real rank at least two. Further assume G is simply connected as a Lie group or simply connected as an algebraic group. The latter means that $G = \mathbb{G}(\mathbb{R})$ where $\mathbb{G}$ is a simply connected semisimple $\mathbb{R}$-algebraic group. Let $\Gamma < G$ be a lattice and L be the k points of an algebraic k-group where k is a local field of characteristic zero. Roughly speaking, Zimmer's cocycle superrigidity theorems imply that any cocycle into L over an ergodic action of G or Γ is measurably conjugate to a constant cocycle, modulo some compact noise. (See below for a precise formulation.) This theorem has many consequences for the dynamics of smooth actions of these groups. Even stronger results would follow if one could produce a continuous or smooth conjugacy. The main purpose of this paper is to prove that a perturbation of a constant cocycle is conjugate back to the constant cocycle via a small (and often continuous) conjugacy, modulo "small" noise. We also prove stronger and more general versions of the cocycle superrigidity theorems than had previously been known. In particular, we do not need to pass to a finite ergodic extension of the action and we obtain more general statements when k is non-Archimedean.

First author partially supported by NSF grants DMS-9902411 and DMS-0226121. Second author partially supported by NSF grant DMS-9800607. The authors would also like to thank the FIM at ETHZ for hospitality and support.

Throughout we work with a more general group G. We let I be a finite index set and for each $i \in I$, we let k_i be a local field of characteristic zero and $\mathbb{G}_i$ be a connected simply connected semisimple algebraic k_i-group. We first define groups G_i, and then let $G = \prod_{i \in I} G_i$. If k_i is non-Archimedean, $G_i = \mathbb{G}_i(k_i)$ the k_i-points of $\mathbb{G}_i$. If k_i is Archimedean, then G_i is either $\mathbb{G}_i(k_i)$ or its topological universal cover. (This makes sense, since when $\mathbb{G}_i$ is simply connected and k_i is Archimedean, $\mathbb{G}_i(k_i)$ is topologically connected.) Throughout the introduction, we assume that the k_i-rank of any simple factor of any $\mathbb{G}_i$ is at least two.

We first state a version of our main result for G actions and cocycles.

THEOREM 1.1. *Let G be as above, $L = \mathbb{L}(k)$ where $\mathbb{L}$ is an algebraic k-group and k is a local field of characteristic zero and let $\pi_0 : G \to L$ be a continuous homomorphism. Let (S, μ) be a standard probability measure space, ρ a measure preserving action of G on S, and $\alpha_{\pi_0} : G \times S \to L$ be the constant cocycle over the action ρ given by $\alpha_{\pi_0}(g, x) = \pi_0(g)$. Assume $\alpha : G \times S \to L$ is a Borel cocycle over the action ρ such that α is L^∞ close to α_{π_0}. Then there exists a measurable map $\phi : S \to L$, and a cocycle $z : G \times S \to Z$ where $Z = Z_L(\pi_0(G))$, the centralizer in L of $\pi_0(G)$, such that*

1. *we have $\alpha(g, x) = \phi(gx)^{-1}\pi_0(g)z(g, x)\phi(x)$;*
2. *$\phi : S \to L$ is small in L^∞;*
3. *the cocycle z is L^∞ close to the trivial cocycle*
4. *the cocycle z is measurably conjugate to a cocycle taking values in a compact subgroup C of Z where C is contained in a small neighborhood of the identity.*

Furthermore if S is a locally compact topological space, μ is a Borel measure on S with $\operatorname{supp}(\mu) = S$ *and α and ρ are continuous then both ϕ and z can be chosen to be continuous.*

Remark: If k is Archimedean, point (4) implies that z is measurably conjugate to the trivial cocycle.

Before stating the analogous theorem for Γ actions and cocycles, we need to recall a consequence of the superrigidity theorems [**M1, M2, M3**]. We will use the notation introduced here in the statements below. If G is as above and $\Gamma < G$ is a lattice, we call a homomorphism $\pi : \Gamma \to L$ *superrigid* if it almost extends to a homomorphism of G. This means that there is a continuous homomorphism $\pi^E : G \to L$ and a homomorphism $\pi^K : \Gamma \to L$ with bounded image such that $\pi(\gamma) = \pi^E(\gamma)\pi^K(\gamma)$ and $\pi^E(\Gamma)$ commutes with $\pi^K(\Gamma)$. The superrigidity theorems imply that any continuous homomorphism of Γ into an algebraic group is superrigid. This can be deduced easily from Lemma VII.5.1 and Theorems VII.5.15 and VII.6.16 of [**M3**].

THEOREM 1.2. *Let Γ be as above, $L = \mathbb{L}(k)$ be as in Theorem 1.1 and $\pi_0 : \Gamma \to L$ be a continuous homomorphism. Let (S, μ) be a standard probability measure space, ρ be a measure preserving action of Γ on S, and let $\alpha_{\pi_0} : \Gamma \times S \to L$ be the constant cocycle over the action ρ given by $\alpha_{\pi_0}(\gamma, x) = \pi_0(\gamma)$. Assume $\alpha : \Gamma \times S \to L$ is a Borel cocycle over the action ρ such that α is L^∞ close to α_{π_0}. Then there exists a measurable map $\phi : S \to L$, and a cocycle $z : \Gamma \times S \to Z$ where $Z = Z_L(\pi_0^E(G))$ such that*

1. *we have $\alpha(\gamma, x) = \phi(\gamma x)^{-1}\pi_0^E(\gamma)z(\gamma, x)\phi(x)$;*
2. *$\phi : S \to L$ is small in L^∞;*
3. *the cocycle z is L^∞ close to the constant cocycle defined by π_0^K*

4. *z is measurably conjugate to a cocycle taking values in a compact subgroup C of Z where C is contained in a small neighborhood of $\pi_0^K(\Gamma)$.*

Furthermore if S is a locally compact topological space, μ is a Borel measure on S with $\mathrm{supp}(\mu) = S$ *and α and ρ are continuous then both ϕ and z can be chosen to be continuous.*

Remark: If k is Archimedean, point (4) implies that z is measurably conjugate to a cocycle taking values in the closure of $\pi_0^K(\Gamma)$.

To prove Theorems 1.1 and 1.2, we prove a very general result about perturbations of cocycles over measure preserving actions of groups with property T. The result shows that any perturbation of a cocycle taking values in a compact group also takes values in a compact group, see Theorem 5.4.

Use of (an extension and modification of) Zimmer's cocycle superrigidity theorems is a key step in the proof of Theorem's 1.1 and 1.2. The cocycle superrigidity theorems are generalizations of the second author's superrigidity theorems. Our strongest results require an integrability condition on the cocycles considered.

DEFINITION 1.3. *Let D be a locally compact group, (S,μ) a standard probability measure space on which D acts preserving μ and L be a normed topological group. We call a cocycle $\alpha : D\times S\to L$ over the D action* D-integrable *if for any compact subset $M \subset D$, the function $Q_{M,\alpha}(x) = \mathrm{supp}_{m\in M} \ln^+ \|\alpha(m,x)\|$ is in $L^1(S)$.*

Any continuous cocycle over a continuous action on a compact topological space is automatically D-integrable. We remark that a cocycle over a cyclic group action is D-integrable if and only if $\ln^+ \|(\alpha(\pm1,x)\|$ is in $L^1(S)$.

THEOREM 1.4. *Let G, S, μ, L be as in Theorem 1.1. Assume G acts ergodically on S preserving μ. Let $\alpha : G\times S\to L$ be a G-integrable Borel cocycle. Then α is cohomologous to a cocycle β where $\beta(g,x) = \pi(g)c(g,x)$. Here $\pi : G\to L$ is a continuous homomorphism and $c : G\times S\to C$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

THEOREM 1.5. *Let G, Γ, S, L and μ be as Theorem 1.2. Assume Γ acts ergodically on S preserving μ. Assume $\alpha : \Gamma\times S\to L$ is a Γ-integrable, Borel cocycle. Then α is cohomologous to a cocycle β where $\beta(\gamma,x) = \pi(\gamma)c(\gamma,x)$. Here $\pi : G\to L$ is a continuous homomorphism of G and $c : \Gamma\times X\to C$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

The principal improvements over earlier results is that we do not need to pass to a finite ergodic extension of the action and that we consider the case where k is a non-Archimedean fields of characteristic 0. This builds on work of the second author, Zimmer, Stuck, Lewis, Lifschitz, Venkataramana and others [**L, Li, M1, M2, M3, Z2, Z1, Z4, Stu, V**]. In the case where S is a single point, Theorem 1.5 is equivalent to the fact that all homomorphisms of Γ to algebraic groups are superrigid. Theorem 1.4 is equivalent to the same fact when applied to $S = G/\Gamma$.

Remark: When k is non-Archimedean, it is not always the case that the algebraic hull of the cocycle is reductive unlike the case $k = \mathbb{R}$ treated in [**Z4**].

Remark: We also prove a result showing uniqueness of the homomorphism π occurring in Theorems 1.4 and 1.5. See subsection 3.8 for details.

Remark: Most of the results here should be true for k_i and k of positive characteristic as well, though additional arguments, similar to those in [**V, Li**] are apparently required. Some partial results in this direction are in [**Li**].

The main applications of our results on perturbations of constant cocycles are to studying perturbations of affine actions of G and Γ.

DEFINITION 1.6. **1** *Let A and D be topological groups, and $B < A$ a closed subgroup. Let $\rho : D\times A/B\to A/B$ be a continuous action. We call ρ* affine, *if, for every $d\in D$ there is a continuous automorphism L_d of A and an element $t_d\in A$ such that $\rho(d)[a] = [t_d\cdot(L_d(a))]$.*

2 *Let A and B be as above. Let C and D be two commuting groups of affine diffeomorphisms of A/B, with C compact. We call the action of D on $C\backslash A/B$ a* generalized affine action.

3 *Let A, B, D and ρ be as in* 1 *above. Let M be a compact Riemannian manifold and $\iota : D\times A/B\to \mathrm{Isom}(M)$ a C^1 cocycle. We call the resulting skew product D action on $A/B\times M$ a* quasi-affine action. *If C and D are as in* 2, *and $\alpha : D\times C\backslash A/B\to \mathrm{Isom}(M)$ is a C^1 cocycle, then we call the resulting skew product D action on $C\backslash A/B\times M$ a* generalized quasi-affine action.

Our notion of *generalized affine action* is from [**F**]. The main application of our results on local rigidity of constant cocycles is as part of our work on local rigidity of volume preserving quasi-affine actions of G and Γ on compact manifolds. We believe that volume preserving generalized quasi-affine actions on compact manifolds are locally rigid as well. As evidence for this, we have the following local entropy rigidity result. For any measure preserving action ρ of D, we denote by $h_\rho(d)$ the entropy of $\rho(d)$. Let $\mathbb{H}$ be an algebraic group defined over $\mathbb{R}$. We will refer to the connected component of the identity in $\mathbb{H}(\mathbb{R})$ as a connected real algebraic group.

COROLLARY 1.7. *Let H be a connected real algebraic group, $\Lambda < H$ a cocompact lattice and $K < H$ a compact subgroup. Let $D = G$ or Γ be as above and let ρ be a C^2 generalized affine action of D on $K\backslash H/\Lambda$. Let ρ' be any C^2 action sufficiently C^1 close to ρ. Then $h_\rho(d) = h_{\rho'}(d)$ for all $d\in D$.*

This result generalizes the one in [**QZ**]. Given the description of generalized standard affine actions below, the proof in [**QZ**] actually applies. We will prove Corollary 1.7 as a corollary of (part of) the proof of Theorems 1.1 and 1.2.

We note here that our techniques prove local rigidity results for perturbations of more general cocycles over actions of G and Γ than those in Theorems 1.1 and 1.2. We can prove an analogous theorem for perturbations of cocycles that are products of compact valued cocycles with constant cocycles. More generally, the original cocycle and the perturbed cocycle need not be cocycles over the same action, but only over actions that are "close". For example if S is a topological space, then the actions being C^0 close is sufficient. (Since constant cocycles are cocycles over any action, one need only consider a single action in the formulations of Theorems 1.1 and Theorem 1.2.) The proof of Corollary 1.7 then implies a local entropy rigidity result for generalized quasi-affine actions of G and Γ. The interested reader is welcome to adjust the proofs below to cover these situations, but for the sake of clarity we have restricted to the generality that we need for our next set of applications.

We now state a theorem which is used in our work on local rigidity of quasi-affine actions [**FM1, FM2**]. This theorem shows that any perturbation of any quasi-affine action is continuously semi-conjugate back to the original action, at least "along hyperbolic directions".

Let H be a connected real algebraic group and $\Lambda < H$ a discrete cocompact subgroup and let D be either G or Γ. Let ρ be a quasi-affine action of D on $H/\Lambda\times M$ which lifts to $H\times M$. By the discussion in section 6 there is a unique subgroup Z in H which is the maximal subgroup of H such that the derivative of ρ on Z cosets is an isometry for an appropriate choice of metric on H/Λ. The description given there shows that the lift of ρ to $H\times M$ descends to an action $\bar{\rho}$ on $Z\backslash H$. For example, if $G < H$ acts on H/Λ by left translations, then $Z = Z_H(G)$.

THEOREM 1.8. *Let $H/\Lambda\times M, \rho, D, Z$ and $\bar{\rho}$ be as in the preceding paragraph. Given any action ρ' sufficiently C^1 close to ρ, there is a continuous $D\times\Lambda$ equivariant map $f : (H\times M, \rho')\to(Z\backslash H, \bar{\rho})$, and f is C^0 close to the natural projection map.*

For actions by left translations this follows from Theorems 1.1 and 1.2. To prove Theorem 1.8 as stated here, we need a stronger result which is Theorem 5.1 in section 5. Theorem 1.8 holds more generally for any skew product action of D on H/Λ which is affine on H/Λ and given by a cocycle $\iota : D\times H/\Lambda\to \mathrm{Diff}^1_\omega(M)$ where ω is a volume form on M and M is compact. The version stated here is what is needed in [**FM1**]. We note that, by Theorems 6.4 and 6.5 below any quasi-affine D action on $H/\Lambda\times M$ lifts to $H\times M$ on a finite index subgroup $D' < D$.

Theorem 1.1, Theorem 1.2 and their applications hold in a wider setting than the groups G and Γ discussed above. The proof uses only that the cocycles we are considering satisfy the conclusion of the cocycle superrigidity theorems and that the group G has "few" representations. For example, for $Sp(1,n)$, F_4^{-20} and their lattices, our techniques can be combined with the results of [**CZ**] to obtain local rigidity theorems for certain perturbations of certain cocycles of these groups. If variants of Theorems 1.4 and 1.5 hold for $Sp(1,n)$ and F_4^{-20} and their lattices, then Theorems 1.1, 1.2 and 5.1 hold for these groups as well.

In section 2 we collect various standard definitions used throughout the paper. Section 3 concerns superrigidity for cocycles. Section 4 proves that certain orbits in representation varieties are closed. Section 5 contains the proof of our main results. The final section of the paper contains the proofs of Corollary 1.7 and Theorem 1.8. This section also contains a detailed description of all affine actions of G and Γ as above.

2. Preliminaries

We now collect various definitions that will be used in the course of the paper.

2.1. Algebraic groups. In this paper the words "algebraic group" mean a linear algebraic group defined over a local field k in the sense of [**B2**]. Unless otherwise noted, throughout this paper k will be a local field of characteristic zero. For background on algebraic groups particularly relevant to what follows, we refer the reader to [**M3**, I.1-2].

2.2. Cocycles and ergodic theory. Given a group D, a space X and an action $\rho : D\times X\to X$, we define a *cocycle over the action* as follows. Let L be a group, the cocycle is a map $\alpha : D\times X\to L$ such that $\alpha(g_1g_2, x) = \alpha(g_1, g_2x)\alpha(g_2, x)$ for all $g_1, g_2\in D$ and all $x\in X$. The regularity of the cocycle is the regularity of the map α. If the cocycle is measurable, we only insist on the equation holding almost everywhere in X. Note that the cocycle equation is exactly what is necessary to define a skew product action of D on $X \times L$ or more generally an action of D on $X \times Y$ by $d(x,y) = (dx, \alpha(d,x)y)$ where Y is any space with an L action.

We say two cocycles α and β are *cohomologous* if there is a map $\phi : X \to L$ such that $\alpha(d,x) = \phi(dx)^{-1}\beta(d,x)\phi(x)$. Again we can define the cohomology relation in any category, depending on how much regularity we seek or can obtain on ϕ. A cocycle is called *constant* if it does not depend on x, i.e. $\alpha_\pi(d,x) = \pi(d)$ for all $x \in X$ and $d \in D$. One can easily check from the cocycle equation that this forces the map π to be a homomorphism $\pi : D \to L$. When α is cohomologous to a constant cocycle α_π we will often say that α is cohomologous to the homomorphism π. The cocycle superrigidity theorems imply that many cocycles are cohomologous to constant cocycles, at least in the measurable category.

A measurable cocycle $\alpha : D \times S \to L$ is called *strict* if it is defined for all points in $D \times S$ and the cocycle equation holds everywhere instead of almost everywhere. For a dictionary translating facts about strict cocycles on homogeneous D-spaces to facts about homomorphisms of subgroups of D, see [**Z2**, Section 4.2].

An action of a group D on a topological space X is called *tame* if the quotient space $D \backslash X$ is T_0, i.e. if for any two points in $D \backslash X$, there is an open set around one of them not containing the other.

Given a locally compact group D and a discrete subgroup $\Gamma < D$, there is a particularly important strict cocycle $\beta_X : D \times D/\Gamma \to \Gamma$. We define this by choosing a fundamental domain X for the Γ action on D. By this we mean that there is a unique representation $d = \omega(d)\tau(d)$ where $\omega(d)$ is in X and $\tau(d)$ is in Γ. Identifying D/Γ with $X \subset D$, we define $\beta_X(d,x) = \tau(dx)^{-1}$. This cocycle is of particular interest when $\Gamma < D$ is a lattice. We call β_X the *strict cocycle corresponding to the fundamental domain X.*

Let D be a compactly generated group, with compact generating set K. Let A be a metrizable, locally compact group and fix a distance function $d : A \times A \to \mathbb{R}$. Given two measurable cocycles $\alpha, \beta : D \times S \to A$ into a locally compact group A, we can define a measurable function on S by $d(\alpha(d,x), \beta(d,x))$. We say that α and β are *L^∞ close* if there exists a small $\varepsilon > 0$ such that $\|d(\alpha(k,x), \beta(k,x))\|_\infty < \varepsilon$ for any $k \in K$.

Let $\mathbb{L}$ be an algebraic k-group and $L = \mathbb{L}(k)$. Let a group D act ergodically on a measure space S and let $\alpha : D \times S \to L$ be a cocycle. There is a unique (up to conjugacy), minimal algebraic subgroup $\mathbb{H}$ in $\mathbb{L}$ such that α is cohomologous to a cocycle taking values in $H = \mathbb{H}(k)$. The group H is referred to as the *algebraic hull* for the cocycle. This is a generalization the Zariski closure of a subgroup of an algebraic group. For more details, see chapter 9 of [**Z2**].

We recall that given any group D acting on a compact metric space X preserving a Borel measure μ, there is an *ergodic decomposition* of μ. That is, there are Borel measures μ_i on X, where each μ_i is an invariant ergodic measure for the action of D, and the measure μ is obtained as an integral of the μ_i over a specific measure $\tilde{\mu}$ on the space of measures on X. Furthermore, the measures μ_i are mutually singular.

2.3. The space of actions. In the introduction, some statements are made about actions being C^k close. Let D be a locally compact topological group and X a smooth manifold. Since an action is a map $D \to \mathrm{Diff}^k(X)$ we can topologize the space of actions by taking the compact open topology on $\mathrm{Hom}(D, \mathrm{Diff}^k(X))$. Two actions are C^k close if they are close with respect to this topology. If D is compactly generated with compact generating set K, this means that ρ and ρ' are C^k close if and only if $\rho(d)\circ\rho'(d)^{-1}$ is in a small neighborhood of the identity in $\mathrm{Diff}^k(X)$ for all $d\in K$. Given a manifold or a space X equipped with an action ρ, we often write (X,ρ) to denote the space with the action. Similarly a map written $(X,\rho)\rightarrow(X',\rho')$ is a map of D-spaces or a D equivariant map.

3. Superrigidity for Cocycles

In this section we prove Theorems 1.4 and 1.5 as well as some related results. Our integrability condition allows us to use Oseledec' Multiplicative Ergodic Theorem to obtain our general result. Some partial results below do not require the integrability condition. Theorem 1.5 is deduced from Theorem 1.4. The proof of Theorem 1.4 requires that one first argue the case where L is semi-simple and then use the result in that case to prove the more general result.

Theorems 1.4 and 1.5 imply a general result on the algebraic hull of the cocycles considered. In fact, at least for G cocycles, this result is a step in the proof of Theorem 1.4, see Theorem 3.10. It is proved in [**M3**] that for any field k and any homomorphism $\pi : \Gamma\rightarrow\mathbb{L}(k)$, the Zariski closure of $\pi(\Gamma)$ is semisimple. This is equivalent to saying that the algebraic hull of the cocycle $\pi\circ\beta : G\times G/\Gamma\rightarrow L$ is semisimple. In [**Z4**], it is shown that if $k=\mathbb{R}$, any G-integrable cocycle $\alpha : G\times X\rightarrow L$ has algebraic hull reductive with compact center. If k is non-Archimedean, it is no longer the case that the algebraic hull is reductive. The following example shows that our results on the algebraic hull are sharp.

EXAMPLE 3.1. We let J be a finite index set and for each $j\in J$, we let k_j be a local field of characteristic zero and $\mathbb{H}_j$ be a connected simply connected semisimple algebraic k_j-group. We let $H_j = \mathbb{H}_j(k_j)$ the k_j-points of $\mathbb{H}_j$ and $H=\prod_{j\in J}H_j$. We further assume that there is an irreducible lattice $\Lambda < H$. For many examples where irreducible Λ exist, we refer the reader to [**M3**, IX.1.7]. Let $\pi : G\rightarrow H$ be a homomorphism and assume that $Z_H(\pi(G))$ contains a non-trivial unipotent subgroup $U < H_l$ for some $l\in J$ where k_l is non-Archimedean. (We leave the easy construction of explicit examples to the reader.) Let $K < U$ be a Zariski dense compact subgroup and consider the G action on $K\backslash H/\Lambda$ and H/Λ. Choosing a measurable trivialization of the K-bundle $H/\Lambda\rightarrow K\backslash H/\Lambda$ defines a cocycle α : $G\times K\backslash H/\Lambda\rightarrow K$, which we view as $\alpha : G\times K\backslash H/\Lambda\rightarrow U$ via the inclusion of $K<U$. Standard arguments using Mautner's Lemma show that the G actions on H/Λ and $K\backslash H/\Lambda$ are ergodic. A simple argument using the fact that the Mackey range of the cocycle α is H/Λ and ergodicity of the G action on H/Λ shows that U is the algebraic hull of α. See [**Z2**, 4.2.24] for definitions and discussion of the Mackey range.

The reader should note the following

1. the above construction yields the same results when applied to the restriction of the actions and cocycles to any lattice $\Gamma < G$;
2. the construction gives non-trivial examples even when $G=\mathbb{G}(\mathbb{R})$;
3. one can take products of cocycles constructed as above with constant cocycles to obtain cocycles whose algebraic hull is neither unipotent nor reductive;
4. the argument above works for more general subgroups $Z < Z_H(\pi(G))\cap H_l$ where $K < Z$ is a Zariski dense compact subgroup. One can construct

examples where $Z = F \ltimes U$ is a Levi decomposition and the F action on U is non-trivial.

Let $\mathbb{L}$ be an algebraic group over k_l and $L = \mathbb{L}(k_l)$ and $D = G$ or Γ. The above outline constructs cocycles $\alpha : D \times S \to L$ of the form $\alpha = \pi \cdot c$ where $\pi : G \to L$ is a continuous homomorphism and $c : D \times S \to C$ is a cocycle taking values in a compact group $C < Z_L(\pi(G))$. We can construct α with algebraic hull L for any $\mathbb{L}$ provided we choose π so that $\pi(G)$ commutes with the unipotent radical of $\mathbb{L}$.

We now briefly indicate the plan of this section. Subsection 3.1 fixes notation for all of section 3 and contains some technical lemmas used throughout. In subsection 3.2 we prove a key technical result which shows that certain cocycles are cohomologous to constant cocycles. Subsection 3.3 applies the results of subsection 3.2 to prove a variant of Theorem 1.4 where the algebraic hull of the cocycle is assumed to be semisimple. Subsection 3.4 proves some conditional results on G-integrable cocycles, again using the results from subsection 3.2. We show how to use property T to control cocycles into amenable and reductive groups in subsection 3.5 and then prove Theorem 1.4 in subsection 3.6. Theorem 1.5 is also proven in subsection 3.6 modulo some facts concerning G-integrability of certain induced cocycles. These facts are then proven in subsection 3.7. Subsection 3.8 concerns the uniqueness of the homomorphism π in Theorems 1.4 and 1.5. These results are used in subsection 3.9 to prove some results on cocycles with constrained projections. The result on cocycles with constrained projections is required to prove Theorem 5.1 which is used in the proof of Theorem 1.8.

3.1. Notations and reductions. In this subsection, we fix notations and definitions for all of section 3. We also prove some technical lemmas that are used throughout this section.

The group G will be as specified in the introduction, but we both weaken the rank assumption and make some preliminary reductions. Let $\mathcal{S}$ be the union of primes of $\mathbb{Z}$ and $\{\infty\}$ and let $\mathbb{Q}_p$ be the p-adic completion of $\mathbb{Q}$, where as usual, $\mathbb{Q}_\infty = \mathbb{R}$. By application of restriction of scalars, we can assume that each $k_i = \mathbb{Q}_{p_\alpha}$, where the p_i are distinct elements of the set $\mathcal{S}$. As before, for the Archimedean factor, we can replace $\mathbb{G}_i(\mathbb{R})$ by it's topological universal cover. Actually this can be done or not done for each simple factor independently, though we simplify exposition by ignoring this nuance. Instead of assuming that each simple factor of $\mathbb{G}_i(k_i)$ has k_i-rank at least two, we let $r_i = k_i\text{-rank}(\mathbb{G}_i(k_i))$ and define the *rank of* G as $\sum_{i\in I} r_i$ and assume that the rank of G is at least two and that G has no non-trivial compact factors (or, equivalently, that every simple factor of $\mathbb{G}_i$ has k_i-rank at least one).

We specify a certain compact homogeneous G space, often called a *boundary* for G. Let $\mathbb{P}_i < \mathbb{G}_i$ be a minimal parabolic subgroup. we define P_i to be $\mathbb{P}_i(k_i)$ if $G_i = \mathbb{G}_i(k_i)$. If G_i is the topological universal cover of $\mathbb{G}_i(k_i)$, we define P_i to be the pre-image of $\mathbb{P}_i(k_i)$ under the covering map from G_i to $\mathbb{G}_i(k_i)$. We let $P = \prod_{i\in I} P_i$ and the homogeneous space we consider is $P\backslash G$. We note that the G action on $P\backslash G$ factors through the projection to $\prod_{i\in I} \mathbb{G}_i(k_i)$ and the space $P\backslash G$ can be identified with $\prod_{i\in I} \mathbb{P}_i(k_i)\backslash\mathbb{G}_i(k_i)$ which can be identified as a variety with $\prod_{i\in I}(\mathbb{P}_i\backslash\mathbb{G}_i)(k_i)$.

We fix (S, μ) to be a standard probability measure space. Also $\mathbb{L}$ will denote an algebraic k-group and $L = \mathbb{L}(k)$. We denote by $\mathbb{L}^0$ the connected component of

$\mathbb{L}$ and let $L^0 = \mathbb{L}^0(k)$. As above, we apply restriction of scalars and assume that $k = \mathbb{Q}_p$ for some $p \in \mathcal{S}$.

By a simple factor of G, we mean a subgroup $F < G$ which is either $\mathbb{F}(k_i)$ or its topological universal cover, where $\mathbb{F}$ is almost simple. We note that under our hypothesis, G is the direct product of all of its simple factors. We say a simple factor F_i has rank one if the k_i rank of $\mathbb{F}_i$ is one. If F_i is a simple factor of G then there is a group $F_i^c < G$ such that $G = F_i \times F_i^c$. We call F_i^c the complement of F_i.

DEFINITION 3.2. *Let (S, μ) be a finite measure space. Given a group G acting ergodically on S preserving μ, we call the action* weakly irreducible *if for any rank one simple factor $F < G$, the complement F^c acts ergodically on S.*

If no simple factor of G has rank 1, this is equivalent to the ergodicity of the G action. This is weaker than the standard definition of irreducibility where it is assumed that all simple factors act ergodically [**Z2**]. The definition of an irreducible action is motivated by properties of irreducible lattices. We call a lattice $\Gamma < G$ *weakly irreducible* if the projection of Γ to any rank 1 factor of G is dense. Standard arguments using the generalized Mautner phenomenon, see [**M3**, II.3.3], show that a lattice is weakly irreducible if and only if the action of G on G/Γ is weakly irreducible.

We will use the following elementary lemmas repeatedly. The first is obvious.

LEMMA 3.3. *Let A be a group and let $\alpha : D \times S \to A$ and $\beta : D \times S \to A$ be cocycles over the action of a group D on a set S. Assume that $\beta(D \times X)$ is contained in a subgroup $B < A$ and let $Z = Z_A(B)$. Let $\eta : D \times S \to Z$ be a map. If $\alpha(d,x) = \beta(d,x)\eta(d,x)$, then η is a cocycle over the D action.*

We let $\tau_i : G \to \mathbb{G}_i(k_i)$ be the natural projection.

LEMMA 3.4. *Given a non-trivial continuous homomorphism $\pi : G \to L$ there is $i \in I$ such that $k = k_i$ and a k-rational homomorphism $\pi_i : \mathbb{G}_i \to \mathbb{L}$ such that $\pi = \pi_i \circ \tau_i$. From this we can deduce:*

1. *the Zariski closure of $\pi(G)$ is semisimple and connected and;*
2. *if $\mathbb{L}' \to \mathbb{L}$ is a k-isogeny, then π lifts to a continuous homomorphism $\pi' : G \to \mathbb{L}'(k)$*

PROOF. We first give the proof where all G_i are k_i-points of algebraic k_i-groups and then describe the modifications necessary when G_i is the topological universal of such a group.

Let the projection from G to G_i be τ_i. Since $k = \mathbb{Q}_p$, by [**M3**, I.2.6] any continuous homomorphism of any G_i into L is the restriction of rational map from $\mathbb{G}_i$ to $\mathbb{L}$. This implies there is an i and a rational homomorphism $\bar{\pi} : \mathbb{G}_i \to \mathbb{L}$ such that π is the restriction of $\tau_i \circ \bar{\pi}$.

Since $\mathbb{G}_i$ is connected and semisimple and the characteristic of k is zero, it follows that the Zariski closure of $\bar{\pi}(\mathbb{G}_i)$ is connected and semisimple. If $\mathbb{L}' \to \mathbb{L}$ is an isogeny, then $\bar{\pi}$ lifts to a map $\bar{\pi}' : \mathbb{G} \to \mathbb{L}'$ since $\mathbb{G}_i$ is simply connected.

Now assume that G_i is the topological universal cover of $\mathbb{G}_i(\mathbb{R})$. If $k \neq \mathbb{R}$ then any continuous homomorphism from G_i to L is trivial, so we are done by the discussion above. If $k = \mathbb{R}$ then π factors through a continuous homomorphism $\bar{\pi} : G_i \to L$. The image of $\bar{\pi}$ is a closed subgroup of L and so is the real points of a real algebraic subgroup. This implies that $\bar{\pi}$ factors through the covering map $G_i \to \mathbb{G}_i(\mathbb{R})$. The conclusions of the lemma now follow as before. $\square$

3.2. α-invariant maps into algebraic varieties. Given two G-spaces S and Y, an L space R and a cocycle $\alpha : G{\times}S{\to}L$, we call a map $f : Y{\times}S{\to}R$ *α-invariant* if $f(gy, gs) = \alpha(g, s)f(y, s)$ for all g and almost every (y, s). Note that this definition differs slightly from the one in [**Z2**], where this map would be called $\tilde{\alpha}$-invariant where $\tilde{\alpha}$ is the pullback of α to $G{\times}Y{\times}S$.

The following theorem will play a key role in all proofs in this section. The assumption on the rank of G is only used to be able to apply this theorem.

THEOREM 3.5. *Assume G acts weakly irreducibly on S preserving μ. Let M be the k points of an algebraic variety $\mathbb{M}$ defined over k on which $\mathbb{L}$ acts k rationally. Assume that $\alpha : G{\times}S{\to}L$ is a Borel cocycle whose algebraic hull is L and that there exists a measurable α-invariant map $\phi : P\backslash G{\times}S{\to}M$ such that the essential image of ϕ is not contained in the set of L-fixed points of M. Then there is a normal k-subgroup $\mathbb{H} < \mathbb{L}$ of positive codimension such that:*

1. *$p_H{\circ}\alpha$ is cohomologous to a continuous homomorphism $\pi_H : G{\to}L/H$, where $p_H : L{\to}L/H$ and $H = \mathbb{H}(k)$;*
2. *$\mathbb{L}/\mathbb{H}$ is semisimple and connected.*

PROOF. Let $\phi_s(x) = \phi(x, s)$ where $x{\in}P\backslash G$ and $s{\in}X$. First one shows that either ϕ_s is rational for almost every s or ϕ_s is constant for almost every s. By rational we mean that there is $i{\in}I$ such that $k = k_i$, the map ϕ factors through the projection $p_i : P\backslash G{\to}\mathbb{P}_i(k_i)\backslash\mathbb{G}_i(k_i)$ which means that $\phi = p_i{\circ}\bar{\phi}$ where that $\bar{\phi}$ is a k rational map $\mathbb{P}_i\backslash\mathbb{G}_i{\to}\mathbb{M}$. Rationality of ϕ was shown by Zimmer in [**Z1**] for irreducible actions with each $G_i = \mathbb{G}_i(k_i)$ using an adaptation of an argument due to the second author [**M1, M2**]. The proof goes through almost verbatim for weakly irreducible actions, as well as for the case where one G_i is the universal cover of $\mathbb{G}_i(\mathbb{R})$. See also pages 104-5 of [**Z2**] or [**Fu3**] for accessible presentations of special cases. Our definition of weak irreducibility is motivated by the ergodicity needed at this step of the proof. We now assume that ϕ_s is rational and proceed in this case, the case of ϕ_s constant is discussed at the end of the proof.

Secondly, one sees that the map $\Phi : S{\to}\mathrm{Rat}(P\backslash G, M)$ defined by $\Phi(s) = \phi_s$ takes values in a single orbit. This follows from tameness of the $G{\times}L$ action on $\mathrm{Rat}(P\backslash G, M)$ and the ergodicity of the G action on S, see the "Proof of Step 3" on pages 105-6 and also Proposition 3.3.2 of [**Z2**].

One now picks a rational map ψ in this orbit and defines a map $l : S{\to}L$ such that $\phi_s = l(s)\psi$. Letting $\beta(g, s) = l(gs)^{-1}\alpha(g, s)l(s)$ we have that $\beta(g, s)\psi(x) = \psi(gx)$. Let H denote the point-wise stabilizer of $\psi(P\backslash G)$ in M. Since $M = \mathbb{M}(k)$ and $\mathbb{L}$ acts rationally on $\mathbb{M}$, $H = \mathbb{H}(k)$ where $\mathbb{H} < \mathbb{L}$ is an algebraic subgroup defined over k. Since $\beta(g, s)\psi(x) = \psi(gx)$, the Zariski closure of $\psi(P\backslash G)$ is invariant under $\beta(G{\times}S)$ and since the algebraic hull of β is L, the Zariski closure of $\psi(P\backslash G)$ is L-invariant. Therefore H is normal in L, and $\mathbb{H}$ is normal in $\mathbb{L}$. Fixing (almost any) s, and writing $\beta_s(g) = \beta(g, s)$, we have that $\beta_s(g_1g_2)\psi(x) = \psi(g_1g_2x) = \beta_s(g_1)\psi(g_2x) = \beta_s(g_1)\beta_s(g_2)\psi(x)$. Therefore $\beta_s(g_1g_2)\beta(g_2)^{-1}\beta(g_1)^{-1}$ fixes $\psi(P\backslash G)$ pointwise. It follows that $p_H{\circ}\beta_s : G{\to}L/H$ is a homomorphism. That $\pi = p_H{\circ}\beta_s$ is continuous follows from a result of Mackey, see [**Z2**, B.3]. The remaining conclusions of the theorem follow from Lemma 3.4.

If ϕ is constant for almost every $s{\in}S$, we have an α-invariant map $\Phi : S{\to}M$. The image of this map is contained in a single H orbit since the L action on M is tame and the G action on S is ergodic. Since the L action on M is defined by an algebraic action of $\mathbb{L}$ on $\mathbb{M}$, the stabilizer of this orbit is $H = \mathbb{H}(k)$ where $\mathbb{H} < \mathbb{L}$ is an algebraic subgroup. This means that we have an α invariant map $\phi : S \to L/H$. By [**Z2**, Lemma 5.2.11], this implies that the cocycle α is equivalent to one taking values in H, which contradicts our assumption on the algebraic hull of the cocycle unless $H = L$ in which case we contradict our assumption that the essential image of ϕ is not contained in the set of L-fixed points. □

The reader should note that essentially the same result can be proven by considering equivariant measurable maps from $G \times X$ to vector spaces, as in [**M3**, VII.1-4]. The argument there requires some modification since, in the language of that text, one needs to consider maps that are not strictly effective.

3.3. Algebraic hull semisimple. We now prove Theorems 1.4 and Theorem 1.5 in the case where the algebraic hull of the cocycle is semisimple.

THEOREM 3.6. *Let G act weakly irreducibly on S preserving μ and let $\alpha : G \times S \to L$ be a Borel cocycle with algebraic hull L. Further assume that $\mathbb{L}$ is semisimple. Then α is cohomologous to a cocycle $\beta = \pi \cdot c$. Here $\pi : G \to L$ is a continuous homomorphism and $c : G \times S \to C$ is a cocycle taking values in a compact group C centralizing $\pi(G)$.*

THEOREM 3.7. *Let $\Gamma < G$ be a weakly irreducible lattice. Assume Γ acts ergodically on S preserving μ. Let $\alpha : \Gamma \times S \to L$ be a Borel cocycle with algebraic hull L. Further assume $\mathbb{L}$ is semisimple. Then α is cohomologous to a cocycle β where $\beta(\gamma, x) = \pi(\gamma)c(\gamma, x)$. Here $\pi : G \to L$ is a continuous homomorphism and $c : \Gamma \times S \to C$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

To reduce Theorem 3.6 to Theorem 3.5 we need to find a k variety M on which H acts k-rationally and an α-invariant map $f : P\backslash G \times S \to M$. To produce α invariant maps, one uses the following modification of a lemma of Furstenberg from [**Fu2**] which can be deduced from Propositions 4.3.2, 4.3.4 and 4.3.9 of [**Z2**]. The lemma holds under more general circumstances than those needed here. For the lemma, G can be a locally compact, σ-compact group, P a closed amenable subgroup and L any topological group.

LEMMA 3.8. *Assume G acts on S preserving μ. Let $\alpha : G \times S \to L$ be a Borel cocycle. Let B be any compact metrizable space on which L acts continuously and $\mathcal{P}(B)$ the space of Borel regular probability measures on B. Then there is an α-invariant map $f : P\backslash G \times S \to \mathcal{P}(B)$.*

We note here that we give a proof using only amenability of P, without reference to the notion of an amenable actions, though we do rely on ideas of Zimmer's to construct a convex compact space on which P acts affinely and continuously.

PROOF. Let B be any compact L-space. Via α we can define a skew product action of G on $S \times B$. We consider the diagonal G action on $G \times S \times B$ given by the right G action on G and the skew product action on $S \times B$, which we note commutes with the left G action on G. Let μ_G be Haar measure on G and $\mathcal{M}(G \times S \times B)$ be the space of regular Borel measures on $G \times S \times B$ which are invariant under the diagonal action and project to $\mu_G \times \mu$ on $G \times S$. We want to topologize $\mathcal{M}(G \times S \times B)$ so the left G action is continuous and the space is a compact convex affine G-space. Using disintegration of measures, we can identify $\mathcal{M}(G \times S \times B)$ with $F(G \times S, \mathcal{P}(B))$ the space of measurable maps from $G \times S$ to $\mathcal{P}(B)$. Let $C(B)$ be the Banach space of continuous functions on B. We identify $F(G \times S, \mathcal{P}(B))$ as a subset of

$L^\infty(G\times S, C(B)^*)$ and give $L^\infty(G\times S, C(B)^*)$ the weak topology coming from the identification $L^\infty(G\times S, C(B)^*) = L^1(G\times S, C(Y))^*$. In this topology the action of G is continuous and $F(G\times S, \mathcal{P}(B))$ is the unit ball in $L^\infty(G\times S, C(B)^*)$. See [**Z2**, Section 4.3] for more discussion of this and related constructions. It follows that there is a fixed point $\mu^P \in \mathcal{M}(G\times S\times B)$ for the left P action. By applying disintegration of measures, this is left P-invariant, α-invariant map $\tilde{f} : G\times S \to \mathcal{P}(B)$ or equivalently an α-invariant map $f : P\backslash G\times S \to \mathcal{P}(B)$ □

We will also need the following lemma essentially due to Furstenberg.

LEMMA 3.9. *Let $J < GL(V)$, where $V = k^n$ and k is a local field of characteristic zero. Let J act on the projective space $P(V)$ preserving a measure μ. Then either J is projectively compact (i.e. the image of J in $PGL(V)$ is compact) or there is a proper subspace $W < V$ with $\mu(W) > 0$.*

For a proof, we refer the reader to [**Z2**, Lemma 3.2.2] or the original article of Furstenberg [**Fu1**] in the case where $k = \mathbb{R}$.

PROOF OF THEOREM 3.6. We call a representation of an algebraic group *almost faithful* if the kernel of the representation is finite. We choose an almost faithful irreducible k-rational representation σ of $\mathbb{L}$ on V such that the restriction of σ to any $\mathbb{L}^0$ invariant subspace is almost faithful, where as usual $\mathbb{L}^0$ denotes the connected component. (This can be done by inducing an almost faithful irreducible $\mathbb{L}^0$ representation.) Let $B = \mathbb{P}(V)$ be the corresponding projective space.

Since $P < G$ is an amenable subgroup Lemma 3.8 provides an α equivariant map $f : P\backslash G\times X \to \mathcal{M}(B)$. In fact this map takes values in a single H orbit $\mathcal{O}$ in $\mathcal{P}(B)$. This is deduced from ergodicity of the G action on S and the tameness of the action of L on $\mathcal{P}(B)$ which is [**Z2**, Corollary 3.2.12].

Let J be the stabilizer of a point μ for the L action on $\mathcal{O}$. We prove that either J is compact or J is contained in an algebraic subgroup of positive codimension in $\mathbb{L}$. If $\mathbb{L}$ is connected, this is Proposition 3.2.15 of [**Z2**]. By Lemma 3.9, if J is not projectively compact, then there is a proper subspace $W < V$ such that $\mu([W]) > 0$. Since $\mathbb{L}$ is semisimple and the representation on V is almost faithful, the map from $\mathbb{L}$ to $PGL(V)$ has finite kernel and only compact subgroups of L are projectively compact. Assuming J is non-compact, we choose W of minimal dimension among subspaces with positive μ measure. Since the measure of W is positive, $J{\cdot}W$ must be a finite union of disjoint subspaces $\cup_{l=1}^n W_l$, and we let $\mathbb{F}$ be the stabilizer of the J orbit $J{\cdot}W$. Let $F = \mathbb{F}(k)$. The stabilizer J^W in J of W is of finite index in J and, by minimality of W and Lemma 3.9, acts on $P(W)$ via a homomorphism to a compact subgroup of $PGL(W)$. If $\dim(\mathbb{F}) = \dim(\mathbb{L})$, then the connected component of $\mathbb{L}$ preserves $\cup_{l=1}^n W_l$ and by connectedness preserves W. Since $J^W < L$ and acts compactly on W, we then have that $J^W \cap \mathbb{L}^0(k)$ is projectively compact. But, since we have that $\mathbb{L}^0$ is semisimple and the representation of $\mathbb{L}^0$ on W is almost faithful, the map from $\mathbb{L}^0$ to $PGL(W)$ has finite kernel. This implies that $J^W \cap \mathbb{L}^0(k)$ is compact. The group $J^W \cap \mathbb{L}^0(k)$ has finite index in J^W which has finite index in J, so in this case, J is compact. Therefore either J is compact or $\mathbb{F}$ is of positive codimension in $\mathbb{L}$.

If J is compact, then Lemma 5.2.10 of [**Z2**] applies and shows that the cocycle α is cohomologous to one with bounded image. If $J < \mathbb{F}$ an algebraic subgroup of positive codimension, then we compose $\phi : P\backslash G\times X \to \mathcal{O}$ with the projection $p : \mathcal{O} \to L/F \subset (\mathbb{L}/\mathbb{F})(k)$. We note that the set of L fixed points in L/F is empty so we can apply Theorem 3.5. This theorem produces a normal k-subgroup of positive codimension $\mathbb{H} < \mathbb{L}$, such that the projection of α to $\mathbb{L}(k)/\mathbb{H}(k)$ is cohomologous to a continuous homomorphism π of G. That theorem also implies that $\mathbb{L}/\mathbb{H}$ is semisimple and connected. Since $\mathbb{L}$ is semisimple, there is a connected normal subgroup $\mathbb{H}^c < \mathbb{L}$ such that the map $\mathbb{H}^c \to \mathbb{H}/\mathbb{L}$ is an isogeny. Now $\mathbb{L} = \mathbb{H} \cdot \mathbb{H}^c$ where $\mathbb{H} \cap \mathbb{H}^c$ is finite. Because $[\mathbb{H}, \mathbb{H}^c] \subset \mathbb{H} \cap \mathbb{H}^c$ which is finite and $\mathbb{H}^c$ is connected, $\mathbb{H}$ and $\mathbb{H}^c$ commute. By Lemma 3.4 we can lift π to a homomorphism $\pi' : G \to \mathbb{H}^c$. It then follows that α is cohomologous to $\pi' \cdot \alpha'$ where α' takes values in $\mathbb{H}(k)$ and is a cocycle by Lemma 3.3. One can now replace α by α' and complete the proof of the theorem by induction on the dimension of L. □

PROOF OF THEOREM 3.7. This is proved by inducing actions and cocycles, exactly as in [**Z2**, Theorem 9.4.14]. We briefly outline the argument. Given a Γ action on S and a Γ cocycle $\alpha : \Gamma \times S \to L$, we consider the induced G action on $(G \times S)/\Gamma$ and a cocycle $\tilde{\alpha} : G \times G/\Gamma \times S \to L$. We define $\tilde{\alpha}$ by taking a fundamental domain X for Γ in G and the strict cocycle $\beta_X : G \times G/\Gamma \to \Gamma$ corresponding to X and letting $\tilde{\alpha}(g, [g_0, x]) = \alpha(\beta(g, [g_0]), x)$. It is straightforward to verify that weak irreducibility of Γ and ergodicity of the Γ action imply that the induced G action is weakly irreducible. One then shows that the algebraic hull of $\tilde{\alpha}$ is L and applies Theorem 3.6 to $\tilde{\alpha}$. Straightforward manipulation allows one to deduce the desired conclusions concerning α. □

3.4. Conditional results using G-integrability of α. In this section we prove a conditional result concerning the algebraic hull of G-integrable cocycles. The assumption of G-integrability is only used here.

Before stating our result, we fix some notation and assumptions. We assume that G has property T and that G acts weakly irreducible on (S, μ). Let $\alpha : G \times S \to L$ be a G-integrable Borel cocycle and assume that L is the algebraic hull of the cocycle. We can write $\mathbb{L} = \mathbb{F} \ltimes \mathbb{U}$ where $\mathbb{F}$ and $\mathbb{U}$ are k-subgroups, $\mathbb{U}$ is the unipotent radical of $\mathbb{L}$ and $\mathbb{F}$ is reductive. Let $p_F : \mathbb{L} \to \mathbb{F}$ and be the natural projection. We assume that the cocycle $p_F \circ \alpha$ is cohomologous to a cocycle of the form $\pi \cdot c$ where $\pi : G \to F$ is a continuous homomorphism and c is a cocycle taking values in a compact subgroup $C < Z_F(\pi(G))$. We note that π can be viewed as defining a homomorphism of G into L, and we let α' be the cocycle cohomologous to α that projects to $\pi \cdot c$.

THEOREM 3.10. *Under the hypotheses discussed in the preceding paragraph, U commutes with $\pi(G)$.*

We prove the theorem by contradiction. The general scheme is as follows. If U does not commute with $\pi(G)$ there exists a k-rational action of $\mathbb{L}$ on a variety $\mathbb{M}$ and an α-invariant map ϕ into $\mathbb{M}(k)$ such that the pointwise stabilizer $H = \mathbb{H}(k)$ of the image does not contain all of U. Applying Theorem 3.5 we obtain a contradiction, since number 2 of that theorem implies that $\mathbb{L}/\mathbb{H}$ is semisimple and this implies that $\mathbb{U} < \mathbb{H}$.

We will construct a measurable map ϕ that satisfies the hypotheses of Theorem 3.5 by using Oseledec' multiplicative ergodic theorem. We will give an argument that is close to the one in [**M3**, Section V.3-4], but also refer the reader to [**Z4**] for a somewhat different approach.

Let $\mathfrak{l}$ be the Lie algebra of L. Let $Gr_j(\mathfrak{l})$ be the Grassmann variety of j planes in $\mathfrak{l}$. We have an action of L on $\mathfrak{l}$ by the adjoint representation which also defines an action of L on $Gr_j(\mathfrak{l})$.

We look at the representation $\mathrm{Ad}_{\mathfrak{l}} \circ \pi$.

THEOREM 3.11. *Assume $\pi(G)$ does not commute with U. Then there is an integer $0 < m < \dim(\mathfrak{l})$ and an α-equivariant measurable map $\phi : P\backslash G \times S \to Gr_m(\mathfrak{l})$ such that the pointwise stabilizer of the image does not contain all of U.*

Before proving Theorem 3.11, we show how that result implies Theorem 3.10.

PROOF OF THEOREM 3.10. We now apply Theorem 3.5 to the map ϕ from Theorem 3.11. This is possible since the stabilizer of the essential image does not contain U and so the essential image is not contained in L fixed points. If H is the stabilizer of the essential image of ϕ this implies that $\mathbb{L}/\mathbb{H}$ is semisimple and therefore that $U < H$. But this implies that U is contained in the stabilizer of the essential image of ϕ', a contradiction. □

Before proving Theorem 3.11 we recall several facts from [**M3**]. The following, which is [**M3**, I.4.6.2], is a simple corollary of the Poincaré recurrence theorem.

LEMMA 3.12. *Let A be an automorphism of (S, μ) and f a non-negative measurable function on X. Then for almost all $x \in X$*

$$\liminf_{m\to\infty} \frac{1}{m} f(A^m(x)) = 0$$

and

$$\liminf_{m\to-\infty} \frac{-1}{m} f(A^m(x)) = 0.$$

Let A be an ergodic automorphism of (S, μ) and W be a k vector space. Let $u : \mathbb{Z} \times S \to GL(W)$ be a $\mathbb{Z}$-integrable cocycle over the $\mathbb{Z}$-action generated by A. If we define

$$\chi_+(u, x, w) = \lim_{m\to\infty} \frac{1}{m} \ln \|u(A^m, x)w)\|$$

and

$$\chi_-(u, x, w) = \lim_{m\to-\infty} \frac{1}{m} \ln \|u(A^m, x)w)\|$$

it follows from Oseledec multiplicative ergodic theorem that both $\chi_+(u, x, w)$ and $\chi_-(u, xw)$ exist for almost all $x \in X$ and all $w \in W$. Furthermore, that theorem shows that there exists a finite set J and real numbers $\chi_j(u)$ and maps $\omega_j(u, x) : S \to Gr_{l(j)}(W)$ such that

1. for almost all $x \in S$, the space W is the direct sum $\oplus_J \omega_j(u, x)$;
2. for almost all $x \in S$ the sequence $\{\frac{1}{m} \ln \|u(m, x)\|/\|w\|\}_{m\in\mathbb{N}^+}$ converges to $\chi_j(u)$ uniformly in $w \in \omega_j(u, x) - \{0\}$.

Furthermore

$$\{0\} \cup \{w \in \mathfrak{h} - \{0\} | \chi_+(u, x, w) \leq a\} = \oplus_{\chi_j \leq a} \omega_j(u, x)$$

and

$$\{0\} \cup \{w \in \mathfrak{h} - \{0\} | \chi_-(u, x, w) \geq a\} = \oplus_{\chi_j \geq a} \omega_j(u, x).$$

All of this is contained in [**M3**, V.2.1]. We refer to $\omega_j(u, x)$ as the *characteristic subspace* for u with *characteristic number* χ_j.

We call a cocycle $v : G{\times}X{\to}H$ *G-quasi-integrable* if v is cohomologous to a G-integrable cocycle u. If v is a $\mathbb{Z}$-quasi-integrable cocycle then we have $v(g,x) = \psi(gx)u(g,x)\psi(x)^{-1}$ for some $\mathbb{Z}$-integrable cocycle u. We then define $\omega_j(v,x) = \psi(x)\omega_j(u,x)$. It is easy to verify, using Lemma 3.12, that if v is in fact $\mathbb{Z}$-integrable, our two definitions of $\omega_j(v,x)$ agree and so $\omega_j(v,x)$ is well-defined and independent of the choice of u. Though $\omega_j(v,x)$ does not satisfy the dynamical condition 2 above, it can be shown to satisfy weaker dynamical conditions, see Definition 2.4 and Remark 2.4 in [**Z4**].

In the proof below we will need some functorial properties of characteristic subspaces. For $\mathbb{Z}$-integrable cocycles these are [**M3**, V.2.3] and it follows easily from the definition that they hold for $\mathbb{Z}$-quasi-integrable cocycles as well. Let $u : \mathbb{Z}{\times}S{\to}GL(W)$ be a $\mathbb{Z}$-quasi-integrable cocycle and let $Q < W$ be a subspace such that $u(m,x)Q = Q$. Let V be the quotient W/Q and $p : W{\to}V$ the projection. We have two additional cocycles $u^Q(m,x) = u(m,x)|_Q$ and $u^V(m,x) = p{\circ}u(m,x)$ both of which can easily be seen to be quasi-integrable. Then for any characteristic subspace $\omega_l(u^V,x)$ (respectively $\omega_l(u^Q,x)$) there is a characteristic subspace $\omega_{l(j)}(u,x)$ such that $\omega_l(u^V,x) = p(\omega_{l(j)}(u,x))$ (respectively $\omega_l(u^Q,x) = \omega_{l(j)}(u,x){\cap}Q$).

For certain cocycles it is easy to compute characteristic subspaces and numbers. Let $\sigma : \mathbb{Z}{\to}GL(W)$ be a homomorphism, let $M = \sigma(1)$ and let $c : \mathbb{Z}{\times}S{\to}GL(W)$ be a cocycle taking values in a compact subgroup of $GL(W)$. We let $u(m,x) = \sigma(m)c(m,x)$. We let $\Omega(M)$ be the set of all eigenvalues of M, $W_\lambda(M)$ the eigenspace corresponding to $\lambda{\in}\Omega(M)$ and $W_d(M) = [\oplus_{\ln|\lambda|=d}W_\lambda(M)]_k$. We also let $W_+(M) = \oplus_{d>0}W_d(M)$ and $W_{\leq 0}(M) = \oplus_{d\leq 0}W_d(M)$. We will call d a *characteristic number* of M and $W_d(M)$ a *characteristic subspace* of M. Then the characteristic numbers of u are the characteristic numbers of M and the characteristic subspaces for u are the characteristic numbers for M. Furthermore the space $\oplus_{\chi_j\leq 0}\omega_j(u,x) = W_{\leq 0}(M)$ and $\oplus_{\chi_j>0}\omega_j(u,x) = W_+(M)$

Proof of Theorem 3.11. As the proof is very involved, we divide it into several steps. The basic idea is to choose an element t of G and use Oseledec theorem to construct characteristic maps from $S{\to}Gr_m(\mathfrak{l})$ for α and each $g^{-1}tg$. This gives an α-invariant map $\phi : G{\times}S{\to}Gr_m(\mathfrak{l})$, which we show descends to an α-invariant map $\phi : P\backslash G{\times}S{\to}Gr_m(\mathfrak{l})$. We then pass to characteristic subspaces for the cocycle α' which is cohomologous to α and where $p_F{\circ}\alpha' = \pi{\cdot}c'$. We use the functoriality of characteristic subspaces and the form of α' to compute the characteristic subspaces quite explicitly. Finally using the assumption that $\pi(G)$ does not commute with U, we show that the stabilizer of the essential image does not contain U.

Step One: Choosing t.

We call a subgroup diagonalizable if it can be conjugated to a subgroup of the group of diagonal matrices. Recall that a subgroup $\mathbb{S}_i < \mathbb{G}_i$ is called a *maximal torus* if it is maximal diagonalizable subgroup of $\mathbb{G}_i$. We fix a torus $\mathbb{S}_i$ in each $\mathbb{G}_i$. We let $\mathbb{T}_i < \mathbb{S}_i$ be the *maximal split torus*, i.e. the maximal subgroup of $\mathbb{S}_i$ that is diagonalizable over k_i. We let $X(\mathbb{T}_i)$ be the group of k_i characters of $\mathbb{T}_i$, and $T_i = \mathbb{T}_i(k_i)$

Lemma 3.13. *There exists an element $t{\in}\prod_{i\in I}T_i$ such that*

1. *the group generated by t is not contained in any proper normal subgroup of G*
2. *for any $\chi{\in}X(\mathbb{T}_i)$ where $\chi(\tau_i(t))$ has modulus one it follows that $\chi(\tau_i(t)) = 1$.*

PROOF. To satisfy 1, it suffices to choose t such that it projects to an element which generates an infinite subgroup in each simple factor of each $\mathbb{G}_i$.

For $k_i = \mathbb{R}$ it suffices to assume that $\chi(\tau_i(t))$ is positive for every $\chi \in X(\mathbb{T}_i)$. If k_i is non-Archimedean, we identify T_i with $(k_i^*)^{l(i)}$ where $l(i)$ is the k_i-rank of $\mathbb{G}_i$. We choose π a uniformizer of k_i. We assume that the projection of $\tau_i(t)$ to each copy of k_i^* in $(k_{\alpha_0}^*)^{i(\alpha_0)}$ is the product of a unit of k_i^* with a non-zero power of π. □

Remark: Let π be finite dimensional representation of G on a vector space V. It follows from our choice of t that if $\pi(t)$ has all eigenvalues of modulus one then π is trivial.

Step two: Oseledec theorem and characteristic maps.

We will construct the map $\phi : G \times S \to Gr_k(\mathfrak{h})$ by applying Oseledec theorem to certain cocycles over the action of $g^{-1}tg$ on S.

Since G acts ergodically on S, it follows from the Mautner phenomenon that t and therefore $g^{-1}tg$ does as well [**M3**, II.3.3]. We define a map $\phi' : G \times S \to Gr_k(\mathfrak{l})$. The element $g^{-1}tg$ generates a $\mathbb{Z}$ action on S. We define a cocycle $u_g : \mathbb{Z} \times S \to GL(\mathfrak{l})$ over this $\mathbb{Z}$ action by $u_g(m,x) = \mathrm{Ad}_{\mathfrak{l}}\, \alpha(g^{-1}t^m g, x)$ and apply Oseledec theorem to each cocycle u_g. Since different choices of $g \in G$ define cohomologous cocycles over conjugate actions, it follows that the characteristic numbers $\chi_j(u_g)$ do not depend on g nor do the dimensions $l(j)$ of the subspaces $\omega_j(u_g, x)$. We $\chi_j = \chi_j(u_g)$ and $\omega_j(g,x) = \omega(u_g, x)$. We now have maps $\omega_j : G \times S \to Gr_{l(j)}(\mathfrak{l})$. If we let G on $G \times X$ by $h(g,x) = (gh^{-1}, hx)$, the map $\omega_j : G \times X \to H$ is α-invariant. To show this one uses the cocycle identity to see that

$$\alpha(hg^{-1}s^m gh^{-1}, hx) = \alpha(h, g^{-1}s^m gx)\alpha(g^{-1}s^m g, x)\alpha(h,x)^{-1}$$

and notes that

$$\liminf_{m\to\infty} \frac{1}{m} \ln^+ \|\alpha(h, g^{-1}s^m gx)\| = 0$$

for almost every x by Lemma 3.12.

Step 3: P-invariance of characteristic maps.

We now show that there is a minimal parabolic P such that the map $\omega_{\leq\chi}(g,x) = \oplus_{\chi_j \leq \chi}\omega_i(g,x)$ descends to a map from $P\backslash G \times S \to Gr_k(\mathfrak{h})$. This follows as in [**M3**, Theorem V.3.3]. If we let P be the set of elements in G such that the set $M = \{s^m p s^{-m} | m \in \mathbb{N}^+\}$ is relatively compact in G then P is a minimal parabolic in G as discussed in [**M3**, VI.4.8 and Lemma II.3.1(b)]. One then can compute that

$$\alpha((pg)^{-1}s^m pg, x) = \alpha((pg)^{-1}s^m ps^{-m} gg^{-1}s^m g, x)$$
$$= \alpha((pg)^{-1}, s^m pgx)\alpha(s^m ps^{-m}, s^m gx)\alpha(g, g^{-1}s^m gx)\alpha(g^{-1}s^m g, x).$$

Because α is G-integrable and M is precompact, it then follows that $Q_{M,\alpha}(s^m x) = \mathrm{supp}_{m \in \mathbb{N}^+} \ln^+ \|\alpha(s^m ps^{-m}, y)\| < \infty$ for almost all $y \in X$ and therefore that

$$\liminf_{m\to\infty} \frac{1}{m} \ln^+ \|\alpha(s^m ps^{-m}, s^m gx)\| = 0$$

almost everywhere by Lemma 3.12. Similarly both $\ln^+ \|\alpha((pg)^{-1}, y)\| < \infty$ and $\ln^+ \|\alpha(g,y)\| < \infty$ for almost all $y \in X$ and therefore

$$\liminf_{m\to\infty} \frac{1}{m} \ln^+ \|\alpha((pg)^{-1}, s^m pgx)\| = 0$$

and

$$\liminf_{m\to\infty}\frac{1}{m}\ln^+\|\alpha(g,g^{-1}s^mgx)\|=0$$

almost everywhere by Lemma 3.12. It follows that $\chi^+(pg,x,w)\leq\chi^+(g,x,w)$ and the reverse inequality follows by replacing p by p^{-1}. See [**M3**, V.3.3] for more detailed computations for certain choices of cocycle.

Let $\phi' = \oplus_{\chi_j\leq 0}\omega_i(g,x) : P\backslash G\times S\to Gr_k(\mathfrak{h})$.

Step 4: Modifying the map.

Let α' be the cocycle cohomologous to α which projects via p_F to $\pi{\cdot}c$, and let ψ be the map implementing the cohomology between α and α'. We will let $\phi([g],x)=\psi(x)\phi'([g],x)$. Since α' is G-quasi-integrable, it follows from the definitions that $\phi([g],x)$ is $\oplus_{\chi_j\leq 0}\omega'_j([g],x)$ where $\omega'_j([g],x)=\psi(x)\omega_j([g],x)$ are the characteristic subspaces for the cocycle $u'(m,x)=\alpha'(g^{-1}t^mg,x)$ over the $\mathbb{Z}$ action on S generated by $g^{-1}tg$.

Step 5: Application of functoriality.

Since L is an algebraic group with unipotent radical U and Levi complement F, we have that $\mathfrak{l}=\mathfrak{f}\oplus\mathfrak{u}$. Now $\mathfrak{f}$ and $\mathfrak{u}$ are invariant under $\mathrm{Ad}_{\mathfrak{l}}$ and so are invariant under the cocycle $\alpha' : G\times S\to GL(\mathfrak{l})$. By the functoriality of characteristic subspaces discussed above, we have that $\phi([g],x)\cap\mathfrak{f}$ is $\oplus_{\chi_j\leq 0}\omega^{\mathfrak{f}}_j([g],x)$ where $\omega^{\mathfrak{f}}_j([g],x)$ are the characteristic subspaces for the cocycle $u'^{\mathfrak{f}}_g(m,x)=\alpha'(g^{-1}t^mg,x)|_{\mathfrak{f}}$. Since $\mathrm{Ad}_{\mathfrak{l}}|_{\mathfrak{f}}$ factors through the map $p_F : L\to F$, and $p_F\circ\alpha'(g,x)=\pi(g)c(g,x)$ where c is a cocycle taking values in a compact group, it follows that $\phi([g],x)\cap\mathfrak{f}=\mathrm{Ad}_{\mathfrak{h}}(\pi(g)^{-1})W_{\leq 0}(\mathrm{Ad}_{\mathfrak{f}}\circ\pi(t))$.

The intersection of $\phi([g],x)$ with $\mathfrak{u}$ is more complicated to describe. To do this, we let $\mathfrak{u}_0=\mathfrak{u}$ and $\mathfrak{u}_i=[\mathfrak{u},\mathfrak{u}_{i-1}]$. Since $\mathfrak{u}$ is unipotent, there is a number k such that $\mathfrak{u}_k$ is the center of $\mathfrak{u}$ and $\mathfrak{u}_l=0$ for all $l>k$. Furthermore, $\mathfrak{u}_{i+1}$ is an ideal in $\mathfrak{u}_i$ and we have a sequence of quotients $\mathfrak{u}_i/\mathfrak{u}_{i+1}$. A key fact for what follows is that $\mathrm{Ad}(u)(\mathfrak{u}_i)\subset\mathfrak{u}_{i+1}$ for any $u\in U$. Since $\mathfrak{u}_i$ is $\mathrm{Ad}_{\mathfrak{l}}$ invariant, it follows that the cocycle $\mathrm{Ad}_{\mathfrak{l}}\circ\alpha'$ leaves each $\mathfrak{u}_i$ invariant. Let $p_i : \mathfrak{u}_i\to\mathfrak{u}_{i+1}$ be the projection. It follows that

$$p_i((\mathrm{Ad}_{\mathfrak{l}}\circ\alpha'(g,x)|_{\mathfrak{u}_i})v)=p_i(\mathrm{Ad}_{\mathfrak{l}}|_{\mathfrak{u}_i}(\pi(g)c(g,x))v)\tag{1}$$

for any $v\in\mathfrak{u}_i$.

Since the representation of G on $\mathfrak{h}$ is defined by $\mathrm{Ad}_{\mathfrak{h}}\circ\pi$, and each $\mathfrak{u}_i$ is $\mathrm{Ad}_{\mathfrak{h}}$ invariant, the G action leaves each $\mathfrak{u}_i$ invariant . Therefore,since G is semisimple, there are G invariant subspaces $\mathfrak{v}_i$ in $\mathfrak{u}_i$ such that $\mathfrak{u}_i=\mathfrak{v}_i\oplus\mathfrak{u}_{i+1}$. Since c takes values in a compact subgroup $C<F$, we can assume that $\mathfrak{v}_i$ is also c invariant. We identify $\mathfrak{v}_i$ and $\mathfrak{u}_i/\mathfrak{u}_{i+1}$ as G modules in the following paragraphs.

Now for any $v\in\mathfrak{v}_i$, we can rewrite equation 1 to

$$p_i(\mathrm{Ad}_{\mathfrak{l}}\circ\alpha'(g,x)|_{\mathfrak{u}_i})v)=\mathrm{Ad}_{\mathfrak{l}}((\pi(g)c(g,x)))v).\tag{2}$$

It follows from the definition of $\phi([g],x)$ in terms of characteristic subspaces, the functoriality of characteristic subspaces and equation 2 above, that $p_i(\phi([g],x)\cap\mathfrak{u}_i)=p_i(\mathrm{Ad}_{\mathfrak{l}}(\pi(g)^{-1})W_{\leq 0}((\mathrm{Ad}_{\mathfrak{f}}\circ\pi)|_{\mathfrak{v}_i}(t)))$.

Step 6: U does not stabilize the essential image of ϕ.

Since $\pi(G)$ does not commute with U by assumption, the representation of G on $\mathfrak{v}_i$ is non-trivial for some i. We fix one such i for what follows. Since G is semisimple $\mathrm{Ad}_{\mathfrak{l}}\circ\pi(G)|_{\mathfrak{v}_i}<SL(\mathfrak{v}_i)$. By our choice of t and i this implies that the decomposition

$$\mathfrak{v}_i=\big(\mathfrak{v}_i\cap W_+((\mathrm{Ad}_{\mathfrak{l}}\circ\pi)_{\mathfrak{v}_i}(t))\big)\oplus\big(\mathfrak{v}_i\cap W_{\leq 0}((\mathrm{Ad}_{\mathfrak{l}}\circ\pi)_{\mathfrak{v}_i}(t))\big)$$

is non-trivial. We let

$$V_+ = \mathfrak{v}_i \cap W_+((\mathrm{Ad}_{\mathfrak{l}} \circ \pi)|_{\mathfrak{v}_i}(t))$$

and

$$V_- = \mathfrak{v}_i \cap W_{\leq 0}((\mathrm{Ad}_{\mathfrak{l}} \circ \pi)|_{\mathfrak{v}_i}(t)).$$

For any choice of $g \in G$ we write V_+^g for $(\mathrm{Ad}_{\mathfrak{l}} \circ \pi(g)^{-1})V_+$ and V_-^g for $(\mathrm{Ad}_{\mathfrak{l}} \circ \pi(g)^{-1})V_-$. Note that $\mathfrak{v}_i = V_+^g \oplus V_-^g$ is non-trivial for all choices of g. Letting $\bar{V}_+^g$ (resp. $\bar{V}_-^g$) be the projection of $p_i(V_+^g)$ (resp. $p_i(V_-^g)$) to and $\bar{\phi}([g],x) = p_i(\phi([g],x) \cap \mathfrak{u}_i)$ we have that $\bar{V}_-^g = \bar{\phi}([g],x)$ so $\bar{V}_+^g \cap \bar{\phi}([g],x) = 0$.

Let $\mathfrak{t}$ be the Lie algebra of $\pi(T)$ and write ${}^g\mathfrak{t}$ for $\mathrm{Ad}_{\mathfrak{l}}(\pi(g)^{-1}(\mathfrak{t})$. Then for almost every $([g],x)$ we have ${}^g\mathfrak{t} \subset \phi'([g],x) \cap \mathfrak{f}$. Since $\mathrm{Ad}_{\mathfrak{h}}(\pi(g)^{-1}\pi(t)\pi(g))V_+^g = V_+^g$ we have $V_+^g \subset [{}^g\mathfrak{t}, V_+^g]$. We let U_g be the collection of elements of U of the form $\mathrm{Exp}(V_+^g)$ where $\mathrm{Exp} : \mathfrak{u} \to U$ is the Lie group exponential map. That $V_+^g \subset [{}^g\mathfrak{t}, V_+^g]$ implies that, for some $u_0 \in U_g$ the projection of $(\mathrm{Ad}_{\mathfrak{l}}(u_0)(\phi([g],x) \cap \mathfrak{f})) \cap \mathfrak{u}_i$ contains non-zero vectors in $\bar{V}_+^g$ and $p_i((\mathrm{Ad}_{\mathfrak{l}}(u_0)(\phi([g],x) \cap \mathfrak{f})) \cap \mathfrak{u}_i)$ is not contained in $\bar{\phi}([g],x) = \bar{V}_-^g$. This implies that, for $g = g_0$ fixed, the subgroup of U generated by u_0 does not stabilize the essential image of $\phi([g_0],x)$. We want the same conclusion for a set of g of positive measure. We note that the spaces $(\phi([g],x) \cap \mathfrak{f}) = \mathrm{Ad}_{\mathfrak{l}}(\pi(g^{-1}))W_{\leq 0}((\mathrm{Ad}_{\mathfrak{l}} \circ \pi)_{\mathfrak{f}}(t))$ and $\bar{V}_-^g$ depend continuously on g, and that the action of u_0 via $\mathrm{Ad}_{\mathfrak{l}}$ is continuous. So there exist a small $\epsilon > 0$ such that

$$p_i((\mathrm{Ad}_{\mathfrak{l}}(u_0)(\phi([g_0],x) \cap \mathfrak{f})) \cap \mathfrak{u}_i) \not\subseteq \bar{V}_-^{g_0}$$

implies

$$p_i((\mathrm{Ad}_{\mathfrak{l}}(u_0)(\phi([g],x) \cap \mathfrak{f})) \cap \mathfrak{u}_i) \not\subseteq \bar{V}_-^{g}$$

for all g in $B(g_0,\epsilon)$. This immediately implies that $\phi([B(g_0,\epsilon)] \times S)$ is not stabilized by u_0 which suffices to see that u_0 is not in the stabilizer of the essential image of ϕ. □

3.5. Property T and cocycles into amenable and reductive groups. In this section we note some results for cocycles for groups with property T of Kazhdan .

THEOREM 3.14. *Let D be a group with property T of Kazhdan and A be an amenable group. Assume D acts ergodically on S preserving μ. Let $\alpha : D \times S \to A$ be a cocycle. Then α is cohomologous to a cocycle taking values in a compact subgroup of A.*

This is [**Z2**, Theorem 9.1.1]. Our first application of this result is to cocycles with reductive target. We will also require the following algebraic lemma.

LEMMA 3.15. *Let $\mathbb{L}$ be a reductive group and $p_Z : \mathbb{L} \to \mathbb{L}/Z(\mathbb{L}^0)$ the natural projection. Let $\mathbb{F} < [\mathbb{L}^0, \mathbb{L}^0]$ be a connected subgroup. Let $g \in \mathbb{L}$, and assume that $[p_Z(g), p_Z(\mathbb{F})]$ is trivial. Then $[g,\mathbb{F}]$ is also trivial.*

PROOF. The commutator $[g,\mathbb{F}]$ is contained in $Z(\mathbb{L}^0)$ by assumption. Since $[\mathbb{L}^0, \mathbb{L}^0]$ is normal in $\mathbb{L}$ it follows that $[g,\mathbb{F}] < [\mathbb{L}^0, \mathbb{L}^0]$. Therefore $[g,\mathbb{F}]$ is a connected subgroup of the finite group $Z(\mathbb{L}^0) \cap [\mathbb{L}^0, \mathbb{L}^0]$. □

THEOREM 3.16. *Let G act weakly irreducibly on S preserving μ. Let $\alpha : G \times S \to L$ be a cocycle with algebraic hull L. Assume in addition that G has property T of Kazhdan and that $\mathbb{L}$ is reductive. Then α is cohomologous to a cocycle $\beta = \pi \cdot c$. Here $\pi : G \to L$ is a continuous homomorphism and $c : G \times S \to C$ is a cocycle taking values in a compact group C centralizing $\pi(G)$.*

PROOF. Since $\mathbb{L}$ is reductive, the connected component $\mathbb{L}^0$ is reductive. This implies that $\mathbb{L}^0 = [\mathbb{L}^0, \mathbb{L}^0]Z(\mathbb{L}^0)$ where $[\mathbb{L}^0, \mathbb{L}^0]$ is semisimple, $Z(\mathbb{L}^0)$ is the center of $\mathbb{L}^0$ and $[\mathbb{L}^0, \mathbb{L}^0] \cap Z(\mathbb{L}^0)$ is finite. Since $Z(\mathbb{L}^0)$ is characteristic in $\mathbb{L}^0$, it is normal in $\mathbb{L}$ and the quotient $\mathbb{J} = \mathbb{L}/Z(\mathbb{L}^0)$ is semisimple. We let $p_J : \mathbb{L} \to \mathbb{J}$ and look at the cocycle $p_J \circ \alpha$. By Theorem 3.6, we have that $p_J \circ \alpha$ is cohomologous to a $\pi' \cdot c'$ where $\pi' : G \to J$ is a continuous homomorphism and c' is a cocycle taking values in a compact subgroup C' of J which commutes with $\pi'(G)$. By Lemma 3.4, the Zariski closure $\mathbb{J}_E$ of $\pi'(G)$ is semisimple and connected. Letting $\mathbb{J}_K$ be the Zariski closure of $c'(G \times S)$, it is clear that $\mathbb{J}$ is the almost direct product $\mathbb{J}_E \times \mathbb{J}_K$. The map $[\mathbb{L}, \mathbb{L}] \to \mathbb{J}$ is an isogeny, so again by Lemma 3.4, we can lift π' to a homomorphism π from G to $[\mathbb{L}, \mathbb{L}] < \mathbb{L}$. Then the Zariski closure of $\pi(G)$ is a connected semisimple subgroup of $\mathbb{L}$ whose projection to $\mathbb{J}$ is $\mathbb{J}_E$. Let $\mathbb{L}_K < \mathbb{L}$ be the pre-image of $\mathbb{J}_K$. Then by Lemma 3.15, the Zariski closure of $\pi(G)$ commutes with $\mathbb{L}_K$. We now see that α is cohomologous to $\pi \circ \alpha'$ where α' takes values in a subgroup H of $\mathbb{L}_K(k) < L$ and is a cocycle by Lemma 3.3. In fact H is the pre-image in $\mathbb{L}_K(k)$ of C' and so is a compact extension of $Z(\mathbb{L}^0)(k)$ and is therefore an amenable group. Since G has property T of Kazhdan, it follows from 3.14 that α' is cohomologous to a cocycle taking values in a compact group. □

Remark: In Theorem 3.16 G can be replaced by Γ. This can be proven in two ways. It follows from Theorem 3.16 exactly as Theorem 3.7 follows from Theorem 3.6. Or, using the Borel density theorem and Theorem 3.7, one can modify the proof of Theorem 3.16 to prove the same result for Γ.

3.6. Proofs of Theorems 1.4 and 1.5. We note here that Theorem 1.4 holds under weaker hypotheses. Namely, it holds for G-integrable cocycles over weakly irreducible actions for G as in Theorem 3.16. However, the existence of ergodic decompositions of measures makes the formulation of Theorem 1.4 more useful in applications.

PROOF OF THEOREM 1.4. We first verify that we can use Theorems 3.10 and 3.16 under the assumptions of Theorem 1.4.

For a proof that G has property T, we refer the reader to [**M1**, III.5] for the case where all factors of G are algebraic. When we replace some factors by their topological universal covers, the resulting G is a central extension of a T group, and the extension does not split with respect to any non-trivial subgroup of the center. It follows that G has T by an argument due to Serre presented in [**HV**, Proposition.3.d.17]. The G action is weakly irreducible since it is ergodic and G contains no rank one simple factors.

Let $L = J \ltimes U$ be a Levi decomposition as above, and $p_J : L \to J$ be the natural projection. Combining Theorem 3.10 with Theorem 3.16, we have the following:

1. a cocycle β cohomologous to α;
2. a homomorphism $\pi : G \to J < L$ such that $\pi(G)$ commutes with U
3. a cocycle $c : G \times S \to C$ where $C < Z_J(\pi(G))$ is compact
4. $p_J \circ \beta(g, x) = \pi(g)c(g, x)$

We can define a map $\tilde{\alpha}(g, x) = \pi(g)^{-1}\beta(g, x)$. That $\tilde{\alpha}$ is a cocycle follows by Lemma 3.3 since U commutes with $\pi(G)$. It suffices to show that $\tilde{\alpha}$ is cohomologous to a cocycle taking values in a compact group. But $p_J \circ \tilde{\alpha} = c$ so $\tilde{\alpha}$ takes values in $K \ltimes U$, an amenable group. Since G has property T of Kazhdan, we are done by 3.14. □

PROOF OF THEOREM 1.5. We prove Theorem 1.5 from Theorem 1.4 by inducing cocycles and actions exactly as in the proof of Theorem 3.7. It is easy to see that the induced action is ergodic. The only additional difficulty is verifying that the induced cocycle is G-integrable. If Γ is cocompact in G, the argument is straightforward. Since, G/Γ is compact, the strict cocycle $\beta : G{\times}G/\Gamma{\to}\Gamma$ can be defined using a precompact fundamental domain. This forces $\beta(g,S)$ to be finite for any $g{\in}G$. In this case it is easy to verify that $\tilde{\alpha}$ is G-integrable.

For Γ non-uniform, we need to prove that the induced cocycle is G-integrable. Since these arguments take us rather far afield, we relegate the proof to the next subsection, see Proposition 3.17. □

3.7. Integrability of Cocycles. When Γ is non-uniform, we need to be careful to verify that the induced cocycle is G-integrable before we can apply Theorem 1.4 to the induced action and cocycle. For G algebraic, the solution is the close to the content of Lewis' note [**L**]. Since Lewis' note continues to circulate in draft form and not appear and since our integrability assumption for the Γ cocycle is weaker than his and our groups G and Γ are more general than his, we give an argument here.

PROPOSITION 3.17. *Let G be as in the introduction and $\Gamma < G$ a lattice. Assume Γ acts on a standard measure space (S,μ) preserving μ. Let $\alpha : \Gamma{\times}S{\to}L$ be a Γ-integrable Borel cocycle. Then there is a choice of fundamental domain $X{\subset}G$ for Γ such that the induced cocycle $\tilde{\alpha} : G{\times}(G{\times}S)/\Gamma{\to}L$ is G-integrable.*

As noted above, the proposition is only non-trivial when Γ is non-uniform. For now we restrict our attention to the case where $G = \prod_i \mathbb{G}_i(k_i)$ and consider the case where a factor of G is the topological universal cover of $\mathbb{G}_i(\mathbb{R})$ only at the end of the proof. We will work with a fundamental domain X that is a contained in a finite union of (generalized) Siegel domains.

Recall that the choice of a fundamental domain X allows us to write any element of G as $\omega(g)\tau(g)$ where $\omega(g){\in}X$ and $\tau(g){\in}\Gamma$. We fix an embedding of each $\mathbb{G}_i(k_i)$ in $GL(n,k_i)$ and use this to define a norm. We then take the supremum norm over factors to define a norm on G. In the case where $\mathbb{G}_i(\mathbb{R})$ is replaced by it's topological universal cover, we define a norm as in [**F**], section 7.2.

PROPOSITION 3.18. *There exists a fundamental domain X for Γ in G such that:*

1. $\int_X \ln^+ \|x\| d\mu_G(x) < \infty$;
2. *for any compact set $M{\subset}G$, there is a constant C_M such that*
$$\sup_{g\in M} \|w(gx)\| {\leq} C_M \|x\|$$
for all $x{\in}X$.

Remark: Proposition 3.18 is true with no assumption on the rank of $\mathbb{G}_\alpha$. For uniform lattices, the proposition is trivial. It can easily be reduced to the case where Γ is non-uniform and irreducible. For irreducible Γ there are two cases, one where the rank of G is one and the other where the rank of G is at least two. Though we do not use the first case, the fundamental domain constructed by Garland and Ragunathan for such Γ and G can easily be seen to satisfy the proposition [**GR**]. In the second case, by the second authors arithmeticity theorems, Γ is arithmetic in G. We give a proof for the case $G = \mathbb{G}(\mathbb{R})$ and Γ a non-uniform irreducible

arithmetic lattice, in which case Γ is commensurable to $\mathbb{G}(\mathbb{Z})$. The case of non-uniform irreducible arithmetic Γ in more general G is analogous though the notation becomes more involved. We note that for irreducible Γ, Proposition 3.17 is true as long as $G \neq SL_2(\mathbb{R})$. This requires different estimates from the ones given below.
Remark: The careful reader may have noted that Proposition 3.18 follows from [**M3**, VIII.1.2]. Though there may exist a fundamental domain for which that proposition is correct, the proof indicated there, using a fundamental domain X contained in a finite union of Siegel sets is not. More precisely, for a fundamental domain X of this type, part a of the conclusions there is true as stated, but part b is true only if X is contained in a single Siegel domain. All applications of [**M3**, VIII.1.2.b] both in that text and in the articles [**F**, **L**] can be replaced by the estimate in part 2 of Proposition 3.18.

Proof of proposition 3.18. Recall that we are proving the proposition in the case where $G = \mathbb{G}(\mathbb{R})$ and Γ commensurable to $\mathbb{G}(\mathbb{Z})$. Let T be a maximal $\mathbb{Q}$ split torus in G. Let F be a root system for G with respect to A, and let Δ be a set of simple roots. Let $A = \{a \in A | \alpha(a) \leq 1 \forall \alpha \in \Delta\}$ be a Weyl chamber. By standard reduction theory, we can assume there is a finite set $F \in \mathbb{G}(\mathbb{Q})$ and a bounded set $L \subset G$ such that $X \subset LAF$. The first conclusion of the proposition is standard, and proofs can be found in [**B1**] or [**PlRa**]. The proof of the second conclusion depends on standard facts from reduction theory. Our principle reference for these facts is [**B1**], particularly Section 14.4. Though the discussion there is restricted to real groups, analogous statements are known for more general G as above.

Let $l = \dim(A)$. As in [**B1**, Section 14], we can find a finite collection $\rho_1, \cdots, \rho_l$ of $\mathbb{Q}$ representations from $\mathbb{G}$ into $GL(W_i)$ and vectors $w_i \in W_i(\mathbb{Z})$ such that

$$\mathbb{P} = \{g \in \mathbb{G} | \rho_i(g) w_i = \chi_i(g) w_i\}$$

where $\chi_i : \mathbb{P} \rightarrow \mathbb{R}^*_{\geq 0}$ restricted to the split torus $\mathbb{A}$ is the highest weight of ρ_i and $\chi_i|_{\mathbb{A}} = d_i \alpha_i$ for some simple root α_i and an integer $d > 0$. This implies that for any other weight $\chi \neq \chi_i$ of ρ_i we have $|\chi(s)| > |\alpha_i(s)|^{-d} |\chi_i(s)|$ where $d > 0$ is an integer.

Given two real valued functions f and g on x, we write $f \prec g$ if $f(x) \leq C g(x)$ and $f \asymp g$ if $f \prec g$ and $g \prec f$.

Given $x \in X$, we write $x = laf$. We take a compact set $M \subset G$ and $x \in LA_1F$. We write $gx = glaf = l'a'f'\gamma$ for any $g \in M$ where $\gamma = \tau(gx)^{-1}$. It follows that $l'a'f' = \omega(gx)$. Since L is compact and F is finite, to prove the proposition it suffices to show that $\|a\| \asymp \|a'\|$. Since χ_i for $1 \leq i \leq l$ form a basis for $X(T) \bigotimes \mathbb{R}$, it is enough to show that $|\chi_i(a)| \asymp |\chi_i(a')|$ for all i.

We apply gx to $f^{-1}w_i$ for each ρ_i. Then $\|(glaf)(f^{-1}w_i)\| = \|glaw_i\| \asymp |\chi_i(a)|$. We also have $\|(glaf)(f^{-1}w_i)\| = \|(l'a'f'\gamma)(f^{-1}w_i)\|$. If $f'\gamma f^{-1}w_i$ is proportional to w_i we have $\|(l'a'f'\gamma)(f^{-1}w_i)\| \asymp |\chi_i(a')|$ and $\|(l'a'f'\gamma)(f^{-1}w_i)\| \succ |\alpha_i(a')|^{-d} |\chi_i(a')|$ otherwise. Therefore $|\chi_i(a')| \prec |\chi_i(a)|$. Replacing $f^{-1}w_i$ with $(f'\gamma)^{-1}w_i$ and arguing in the same manner yields $|\chi_i(a)| \prec |\chi_i(a')|$. Therefore we have $|\chi_i(a)| \asymp |\chi_i(a')|$ which suffices to prove the proposition. (We note that the constants implicit in the signs $\asymp$ and $\prec$ used here depend on the compact set M.) $\square$

Corollary 3.19. *There exists a fundamental domain X for Γ in G such that for any compact set $M \subset G$ there is a constant C_M such that*

$$\ln^+ \|\beta(g,x)\| \leq C_1(M) \ln^+ \|x\| + C_2(M)$$

for all x in X.

PROOF. We write $gx = \omega(gx)\tau(gx)$ and recall that $\beta(g,x) = \tau(gx)^{-1}$ is the strict cocycle $\beta : G{\times}G/\Gamma{\to}\Gamma$ corresponding to X. This implies that $\beta(g,x) = x^{-1}g^{-1}\omega(gx)$. Therefore

$$\ln^+ \|\beta(g,x)\| \leq \ln^+ \|x^{-1}\| + \ln^+ \|g^{-1}\| + \ln^+ \|\omega(gx)\|.$$

Since G is algebraic there exist positive constants c and d such that $\|x^{-1}\| < c\|x\|^d$. Combined with Proposition 3.18 this implies that the previous equation can be rewritten

$$\ln^+ \|\beta(g,x)\| \leq dC_M \ln^+ \|x\| + \ln^+(c) + \ln^+ \|g\|.$$

Letting $C_1(M) = dC_M$ and $C_2(M) = \mathrm{supp}_{g\in M} \ln^+ \|g\| + \ln^+(c)$, we are done. □

PROOF OF PROPOSITION 3.17. We have assumed that the cocycle α is Γ-integrable. If we let K be a finite generating set for Γ, and K^j the set of all words in S of length less than j, then Γ-integrability implies that $\int_S \ln^+ \|\alpha(\gamma,x)\| \leq Cj$ almost everywhere for some constant C (not depending on j) and all $\gamma \in K^j$. Recall that the cocycle $\tilde{\alpha}$ is defined by $\tilde{\alpha}(g,[g_0,s]) = \alpha(\beta(g,[g_0]),s)$. To show that $\tilde{\alpha}$ is quasi-integrable, it therefore suffice to show that the word length of $\beta(g,[g_0])$ is an L^1 function on G/Γ. We choose a fundamental domain to define β as in Proposition 3.18. For such a domain it follows that $\ln^+ \|x\|$ is in $L^1(X)$. In what follows, we identify G/Γ with the fundamental domain X and consider $\beta : G{\times}X{\to}\Gamma$ written as $\beta(g,x)$. To finish the argument, one then uses a theorem of Lubotzky, Mozes and Ragunathan. Define a distance function on G by choosing a right G and left K invariant Riemannian metric on G. Then Lubotzky, Mozes, and Ragunathan show that the word length metric on Γ is bilipschitz equivalent to the induced metric as a subset of G [**LMR**]. This result, combined with a simple computation [**F**, Proof of Proposition 7.9], shows that the word length of $\beta(g,x)$ is bounded by a multiple of $\ln^+ \|\beta(g,x)\|$ plus a constant. One then applies Corollary 3.19 to see that $\ln^+ \|\beta(g,x)\| < C_1(M)\ln^+ \|x\| + C_2(M)$ for any $g \in M$ where $M \subset G$ is pre-compact.

Letting $M \subset G$ be any pre-compact set, writing $\|\gamma\|_\Gamma$ for the word length of γ and collecting inequalities, we have:

$$\int_{S\times X} Q_{M,\tilde{\alpha}}(s,x) = \int_{S\times X} \mathrm{supp}_{g\in M}\, \tilde{\alpha}(g,s,x)$$

$$\leq C \int_X \mathrm{supp}_{g\in M} \|\beta(g,x)\|_\Gamma$$

$$\leq C' \int_X \mathrm{supp}_{g\in M} \ln^+ \|\beta(g,x)\| + B$$

$$\leq C_1(M)C' \int_X \ln^+ \|x\| + B + C_2(M).$$

This shows that $\tilde{\alpha}$ is G-integrable whenever G is an algebraic group.

When G or a simple factor of G is not algebraic, we need to extend the reduction theory arguments to fundamental domains for G/Γ in this setting. This is done at the end of section 7.2. of [**F**]. A key step in the argument there is showing that $\Gamma \cap Z(G) < Z(G)$ is a subgroup of finite index. This allows one to choose a fundamental domain for Γ in G which is contained in a finite union of connected components of pre-images of Siegel sets. In this context there is a choice of the norm on G, since G is not a linear group. For the choice made in [**F**], the extension of the results in [**LMR**] is obvious. We note that some statements in [**F**] inherit the inaccuracy of [**M3**, VIII.1.2]. All of these inaccuracies can be fixed easily using Proposition 3.18 above. □

3.8. Uniqueness of the superrigidity homomorphism. In this section, we show that the homomorphism π appearing in the formulation of Theorems 1.4 and 1.5 is unique up to conjugacy. In fact, we prove a more general fact that requires no assumption on the rank of G. In this subsection, G will be as in subsection 3.1, but with no assumption on the rank of G, but still assuming G has no compact factors. As usual, $\Gamma < G$ is a lattice.

THEOREM 3.20. *Let $D = G$ or Γ. Assume D acts on S preserving μ. For $j = 1, 2$, let $\pi_j : G \to L$ be continuous homomorphism, let $Z_k = Z_L(\pi_j(G))$ and let $c_j : D \times S \to C_j$ be a cocycle over the D action taking values in a compact subgroup $C_j < Z_j$. Let $\alpha_j : D \times S \to L$ be the cocycle over the D action defined by $\alpha_j(d, x) = \pi_j(d)c_j(d, x)$. Then if α_1 is cohomologous to α_2, the homomorphism π_1 and π_2 are conjugate.*

Before proving the theorem, we recall some terminology and notation. For any element g of $GL_n(\mathbb{R})$, there is a unique decomposition of $g = us = su$ where u is unipotent and s is semisimple. Further, we have a unique decomposition $s = cp = pc$ where all eigenvalues of p are positive and all eigenvalues of c have modulus one. We refer to p as the *polar part* of g and denote it by $\mathrm{pol}(g)$. For any subset $\Omega \subset GL_n(\mathbb{R})$ we define

$$P(\Omega) = \{\,\mathrm{pol}(h) : h \in \Omega\}.$$

In general $P(\Omega)$ is not a subset of Ω, but if Ω is a semisimple subgroup without compact factors, then the Zariski closure of $P(\Omega)$ is Ω.

For non-Archimedean fields k, the situation is more complicated. We fix a uniformizer π for the field k and define a *polar element* of $GL_n(k)$ to be an element p all of whose eigenvalues are powers of π. We call an element c in $GL_n(k)$ *compact* if c generates a bounded subgroup in $GL_n(k)$. Each element of $GL_n(k)$ can be written uniquely as su where u is unipotent and s is semisimple. We call an element g *quasi-polar* if $g = su$ as above and s can be written as $s = cp = pc$ where p is polar and c is compact. Once π is fixed, this decomposition is unique. We note that if s is semisimple then $s^{n!}$ is quasi-polar. As above for h quasi-polar, we denote the polar part by $\mathrm{pol}(h)$. For a subset Ω in $GL_n(k)$, we define

$$P(\Omega) = \{\mathrm{pol}(h) : h \in \Omega \text{ and } h \text{ quasi-polar}\}.$$

As before, in general $P(\Omega)$ is not a subset of Ω, but if Ω is a semisimple subgroup without compact factors, then the Zariski closure of $P(\Omega)$ is Ω.

Recall from subsection 3.4 that if M is a linear transformation and we let $\Omega(M)$ be the set of all eigenvalues of M, we call the numbers $d = |\lambda|$ for $\lambda \in \Omega(M)$ characteristic numbers of M. If $W_\lambda(M)$ is the eigenspace corresponding to $\lambda \in \Omega(M)$, we let $W_d(M) = [\oplus_{\ln|\lambda| = d} W_\lambda(M)]_k$ be the characteristic subspace of M with characteristic number d.

Remark: The key fact about polar elements is that a polar element is completely determined by it's characteristic numbers and subspaces. We note that under any rational homomorphism $\pi : \mathbb{G}(k) \to GL_n(k)$ the image of a polar element is a polar element. In fact, for g quasi-polar, $\mathrm{pol}(\pi(g)) = \pi(\mathrm{pol}(g))$. This implies that we can define polar and quasi-polar elements of a linear algebraic group $\mathbb{G}$ and that the definition is independent of the realization $\mathbb{G}$. However, the set of polar elements of $\mathbb{G}$ does depend on the choice of uniformizer for k.

For each $i \in I$ we fix an almost faithful representation of $\mathbb{G}_i(k_i)$ in $GL_n(k)$. We will call an element g of G polar if $\tau_i(g)$ is polar whenever it is non-trivial. We call a subgroup $F < G$ *Zariski dense* if $\tau_i(F)$ is Zariski dense in $\mathbb{G}_i$ for each i.

LEMMA 3.21. *There exists a finite collection of quasi-polar element $g_1, \dots, g_l \in \Gamma$ such that the group F generated by pol$(g_1), \dots,$pol(g_l) is Zariski dense in G.*

PROOF. This is similar to [**MQ**, Lemma 4.5]. Let Z be the Zariski closure of $< \mathrm{pol}(\gamma) | \gamma \in \Gamma >$. The proof follows from the fact Z is invariant under conjugation by elements of Γ and so also by elements of G by the Borel-Wang density theorem [**M3**, Theorem II.4.4]. Hence $G \cap Z$ is a normal subgroup of G. That it is all of G follows from results of Mostow and Prasad-Ragunathan [**M, PR**] which show that there is a maximal split torus $A < G$ such that $A \cap \Gamma$ is a lattice in A. This implies that $A \cap \Gamma$ projects to a non-compact subgroup of semisimple elements of each simple factor of G.

Since $< \mathrm{pol}(\gamma) | \gamma \in \Gamma >$ is Zariski dense in G and algebraic groups satisfy an ascending chain condition, it follows that there is a finite collection $\gamma_1, \dots, \gamma_l$ such that $< \gamma_1, \dots, \gamma_l >$ is Zariski dense in G.

Though the results in [**M, PR**] are only stated for the case of real algebraic G, the interested reader may generalize the proof of [**PR**] to the more general G considered in the statement of our theorems. □

PROOF OF THEOREM 3.20. If $D = \Gamma$, we apply Lemma 3.21 to obtain the group F and $g_1, \dots, g_l$. If $D = G$, there exists a Zariski dense finitely generated subgroup F generated by polar elements $g_1, \dots, g_l$. In either case, we fix F and $g_1, \dots, g_l$ for the remainder of the proof.

We define associated actions of D on $S \times \mathfrak{l}$ by $\rho_j(d)(x, v) = (dx, \mathrm{Ad}_{\mathfrak{l}} \circ \alpha_j(d, x))$. Oseledec' multiplicative ergodic theorem implies that there are characteristic exponents and subspaces for any $\mathbb{Z}$ action defined by powers of an element d in D. Let $\psi : S \to L$ be the measurable function such that $\alpha_1(d, x) = \psi(dx)^{-1} \alpha_2(d, x) \psi(x)$. It follows easily from Lemma 3.12 that for any $d \in D$ the characteristic numbers for $\rho_2(d)$ and $\rho_1(d)$ are equal. In fact, that lemma shows that if λ is a characteristic number for $\rho_1(d)$ with characteristic subspace ${}^1W^d_\lambda(x)$ then $(\psi(x)^{-1})\,{}^1W^d_\lambda(x)$ is a characteristic subspace for $\rho_2(d)$ with characteristic number λ. Since each α_j is the product of a constant cocycle and a compact valued cocycle, it follows from the discussion immediately proceeding the proof of Theorem 3.11 that the characteristic numbers and subspaces for $\rho_i(d)$ are just the characteristic numbers and subspaces for the linear representation $\mathrm{Ad}_{\mathfrak{a}} \circ \pi_j$. This implies that ${}^jW^d_\lambda(x) = W_\lambda(\pi_j(d))$. Combining these two facts, we see that $\psi(x) W_\lambda(\pi_1(d)) = W_\lambda(\pi_2(d))$ for every d, every λ, and almost every x.

We let $\{{}^j\lambda^m_k\}$ be the characteristic numbers of $\pi_j(g_m)$ for each $1 \le m \le l$. Then by the argument above, we see that ${}^1\lambda^m_k = {}^2\lambda^m_k$ and also that $\psi(x) W_{{}^1\lambda^m_k}(\pi_1(g_m)) = W_{{}^2\lambda^m_k}(\pi_2(g_m))$ for almost every x and $1 \le m \le l$. Fixing x for which the equation holds for all m, and letting $l = \psi(x)$, the definition of the polar part of an element implies that $\mathrm{pol}(\pi_1(g_m)) = l^{-1}\,\mathrm{pol}(\pi_2(g_m)l$. This implies that $(\pi_1(\mathrm{pol}(g_m))) = l^{-1}(\pi_2(\mathrm{pol}(g_m))l$. Since the group F generated by $\mathrm{pol}(g_1), \dots, \mathrm{pol}(g_l)$ is Zariski dense in G and since Lemma 3.4 implies each π_j factors through a rational homomorphism of some $\mathbb{G}_i$ this implies that $\pi_1 = l^{-1}\pi_2 l$. □

Remark: If one of the cocycles α_j is simply a homomorphism, it is possible to give a simpler proof based on the Borel density theorem.

3.9. Cocycles with prescribed projections. In this subsection we state and prove some variants on the cocycle superrigidity theorems. Though these variants hold in many settings, we only state variants of Theorems 1.4 and 1.5. Therefore, throughout this subsection G will be as in the introduction, i.e. with the assumption that G has no rank 1 simple factors. The variants stated below are needed for our applications to local rigidity of affine action and are used in the proof of Theorem 5.1, which in turn is used to prove Theorem 1.8.

Throughout this subsection $\mathbb{A}$ and $\mathbb{H}$ will be algebraic k-subgroups of $\mathbb{L}$ such that $\mathbb{L} = \mathbb{A} \ltimes \mathbb{H}$. We further assume that $\mathbb{A}$ is a connected semisimple k-group. We will denote the k points $\mathbb{H}(k) = H$ and $\mathbb{A}(k) = A$. We fix homomorphisms $\pi_A^G : G \to A$ and $\pi_A^\Gamma : \Gamma \to A$ with Zariski dense image. We let $p_A : L \to A$ denote the standard projection.

THEOREM 3.22. *Assume G acts ergodically on S preserving μ. Let $\alpha : G \times S \to L$ be a G-integrable Borel cocycle such that $p_A \circ \alpha = \pi_A^G$. Further assume that L is the algebraic hull of the cocycle. Then there is a measurable map $\phi : S \to H$ such that $\beta = \phi(gx)^{-1}\alpha(g,x)\phi(x) = \pi(g)c(g,x)$ where $\pi : G \to L$ is a continuous homomorphism and $c : G \times S \to C$ is a cocycle taking values in a compact subgroup $C < Z_L(\pi(\Gamma))$. The fact that $\phi(S) \subset H$ implies that $p_A \circ \beta = p_A^\Gamma$.*

THEOREM 3.23. *Assume Γ acts ergodically on S preserving μ. Let $\alpha : \Gamma \times S \to L$ be a Γ-integrable, Borel cocycle such that $p_A \circ \alpha = \pi_A^\Gamma$. Further assume L is the algebraic hull of the cocycle. Then there is a measurable map $\phi : S \to H$ such that $\beta = \phi(\gamma x)^{-1}\alpha(\gamma,x)\phi(x) = \pi(\gamma)c(\gamma,x)$ where $\pi : G \to L$ is a continuous homomorphism and $c : \Gamma \times S \to C$ is a cocycle taking values in a compact subgroup $C < Z_L(\pi(\Gamma))$. The fact that $\phi(S) \subset H$ implies that $p_A \circ \beta = p_A^\Gamma$.*

These variants are proven from Theorems 1.4 and 1.5 using Theorem 3.20, the following lemma and some facts about the structure of algebraic groups.

LEMMA 3.24. *Let D, R, F and A be groups. Assume $A \times F$ acts on R by a (possibly trivial) homomorphism into $Aut(R)$. We will write an element $g \in (A \times F) \ltimes R$ as (g_A, g_F, g_R) where $g_A \in A, g_F \in F$ and $g_R \in R$. Let $\pi_A : D \to A$ be a homomorphism and p_A the projection of from $(A \times F) \ltimes R$ to A. Let D act on a set X. Let $\alpha : D \times X \to (A \times F) \ltimes R$ be a cocycle over the D action on X. Assume:*

1. *$p_A \circ \alpha(d,x) = \pi_A(d)$;*
2. *Let $C < R$ be a subgroup such that A and F normalize C;*
3. *α is cohomologous to a cocycle β taking values in $(A \times F) \ltimes C$.*

Assume $\lambda(x) = (\lambda_A(x), \lambda_F(x), \lambda_R(x))$ is the function such that

$$\lambda(dx)^{-1}\alpha(d,x)\lambda(x) = \beta(d,x)$$

Let $\lambda'(x) = (1_A, \lambda_F(x), \lambda_R(x))$. Then

$$\lambda'(gx)^{-1}\alpha(g,x)\lambda'(x) = (\pi(d), \beta_F(d,x), \beta'_R(d,x)$$

where $\beta'_R(d,x) = {}^{\lambda_A(dx)}\beta_R(d,x)$ and $\beta'_R(D \times X) \subset C$.

PROOF. This is checked directly by multiplication. We write the multiplication in $(A \times F) \ltimes R$ as $(g_A, g_F, g_R)(h_A, h_F, h_R) = (g_A h_A, g_F h_F, g_R{}^{(g_A, g_F)}(h_R))$ where

${}^{(g_A,g_F)}(h_R)$ is the image of h_R under the automorphism of R given by (g_A, g_F). Then $\lambda'(dx)^{-1}\alpha(d,x)\lambda'(x) = \lambda_A(dx)\beta(d,x)\lambda_A(x)^{-1}$. Then

$$\begin{aligned}(\lambda_A(dx), 1_F, 1_R)&(\beta_A(d,x), \beta_F(d,x), \beta_R(d,x))(\lambda_A(x)^{-1}, 1_F, 1_R)\\ &= (\lambda_A(dx)\beta_A(d,x)\lambda_A(x)^{-1}, \beta_F(d,x), {}^{\lambda_A(dx)}\beta_R(d,x)).\end{aligned}$$

Defining $\beta'(d,x) = {}^{\lambda_A(dx)}\beta_R(d,x)$, since $\beta_R(D\times X)\subset C$ and A normalizes C, we have $\beta'(D\times X)\subset C$. □

PROOF OF THEOREM 3.22. We can write $\mathbb{H} = \mathbb{F}\ltimes\mathbb{U}$ where $\mathbb{F}$ is semisimple and $\mathbb{U}$ is solvable. We first show that we can change $\mathbb{A}$ so that it commutes with $\mathbb{F}$. If not, then since $\mathbb{A}$ is connected and semisimple $\mathbb{A}$ acts on $\mathbb{F}$ by inner automorphisms and therefore $\mathbb{A}$ is an almost direct product $\mathbb{A}_1\mathbb{A}_2$ where $\mathbb{A}_1$ commutes with $\mathbb{F}$ and $\mathbb{A}_2$ is (virtually) a subgroup of $\mathbb{F}$. Let $\Delta^{-1}(\mathbb{A}_2)$ be the antidiagonal embedding of $\mathbb{A}_2$ in $\mathbb{A}\ltimes\mathbb{F}$. We replace $\mathbb{A}$ by $\mathbb{A}' = \mathbb{A}_1\Delta^{-1}(\mathbb{A}_2)$. A simple computation shows that $\mathbb{F}$ and $\mathbb{A}'$ commute in the semidirect product $\mathbb{A}'\ltimes\mathbb{H}$. In what follows we replace $\mathbb{A}$ by $\mathbb{A}'$.

If we apply Theorem 1.4 to α we see that α is cohomologous to a cocycle β where β is of the form $\pi\cdot c$ for a homomorphism $\pi : G\to L$ and a cocycle $c : G\times S\to K$ where $K < L$ is compact. We let $C = K\cap\mathbb{U}(k)$. It follows from Theorem 3.20 that $p_A\circ\beta$ is cohomologous to a homomorphism conjugate to π_A^G and that A is the algebraic hull of $p_A\circ\alpha$. Since the algebraic hull of α contains the Zariski closure of C, it follows from Theorem 3.10 that $\mathbb{A}$ and C commute.

We now apply Lemma 3.24 with $A = A$, $F = \mathbb{F}(k)$, $R = \mathbb{U}(k)$ and $C = C$. □

PROOF OF THEOREM 3.23. In the case when π_A^Γ is the restriction to Γ of a continuous homomorphism of G, the proof above applies verbatim. Though Theorem 3.10 is only stated for G actions and cocycles, the analogous statement for Γ actions and cocycles is easily proven by inducing cocycles and actions.

When $k = \mathbb{R}$ and π_A^Γ does not extend, the argument is even simpler. We let $\mathbb{H} = \mathbb{F}\ltimes\mathbb{U}$ where $\mathbb{F}$ is reductive and $\mathbb{U}$ is unipotent. As above we modify $\mathbb{A}$ so that $\mathbb{A}$ commutes with $\mathbb{F}$. This is possible with $\mathbb{F}$ reductive since $\mathbb{A}\ltimes\mathbb{F}$ is reductive and therefore $\mathbb{A}$ commutes with the torus in $\mathbb{A}\ltimes\mathbb{F}$ which contains the torus in $\mathbb{F}$. We let $F = \mathbb{F}(\mathbb{R})$ and $U = \mathbb{U}(\mathbb{R})$. It follows from Theorem 1.5 and the fact that U contains no compact subgroups that $\beta(g,x)$ takes values in $A\times F$. The theorem now follows from Lemma 3.24 applied to $A, F, R = U$ and $C = 1_U$.

The case of k non-Archimedean and π_A^Γ not extendable is the most complicated and the only place where we need the full strength of both Theorem 3.20 and Lemma 3.24. Let $\mathbb{H} = \mathbb{F}\ltimes\mathbb{U}$ and $\mathbb{L} = (\mathbb{A}\times\mathbb{F})\ltimes\mathbb{U}$ as in the proceeding paragraph. Then $L = (A\times F)\ltimes U$ where $U = \mathbb{U}(k)$ and $F = \mathbb{F}(k)$. We can write $\mathbb{A}$ and A as almost direct products $\mathbb{A} = \mathbb{A}_1\mathbb{A}_2$ and $A = A_1A_2$ where $A_1 = \mathbb{A}_1(k), A_2 = \mathbb{A}_2(k)$ and $p_{A_1}\circ\pi_A$ extends to a continuous homomorphism of G and $p_{A_2}\circ\pi_A$ has bounded image. By Theorem 3.10 applied as above, $\mathbb{A}_1$ commutes with $\mathbb{U}$.

Note that $\mathbb{U}(k)$ is the injective limit of it's compact subgroups. This follows from the same fact for the additive group $\mathbb{Q}_p$, where the compact subgroups are subgroups with bounded denominators. Apply Theorem 1.5 to α to obtain a cocycle β cohomologous to α, where $\beta(\gamma,x) = \pi(\gamma)c(\gamma,x)$ where π is a continuous homomorphism of G and c is a cocycle taking values in a compact subgroup $C < Z_L(\pi(G))$. Write $\beta(\gamma,x) = (\beta_A(\gamma,x), \beta_F(\gamma,x), \beta_U(\gamma,x))$ using coordinates as in Lemma 3.24. Let C_1 be the smallest compact subgroup of U containing $\beta_U(\Gamma\times S)$. Note that A_1 commutes with C_1.

Let $p_A \circ \beta = \beta_A$. This is cohomologous to $(p_A \circ \pi) \cdot (p_A \circ c)$. By Theorem 3.20, $p_A \circ \pi$ is conjugate to π_A^E by an element $a \in A$. Let β' be the conjugate of β by a. Writing $\beta'(\gamma, x) = (\beta'_A(\gamma, x), \beta'_F(\gamma, x), \beta'_U(\gamma, x))$ as above, it is clear that $\beta'_F = \beta_F$ and that $\beta'_U(\Gamma \times X) \subset C_2 = {}^a C_1$ and that C_2 is a compact subgroup of U. Note that A_1 commutes with C_2. It also follows that $\beta'_A(\gamma, x) = \pi_A^E(\gamma) c_A(\gamma, x)$ where $c_A(\Gamma \times X) \subset C_A$ where $C_A < A_2$ is a compact subgroup and $\pi_A^E(\Gamma)$ is contained in A_1. We note that C_A acts on U by automorphisms, and let K be the set of all images of C_2 under the action of C_A. It is clear that K is compact and is the union of subgroups of U. Since U is the projective limit of it's compact subgroups, there is a compact subgroup $C < U$ with $K \subset C$. We now apply Lemma 3.24 with $A = A, F = F, R = U$ and $C = C$. □

4. Orbits in the space of representations

In this section we prove an independent result that is used in the proof of our results on local rigidity of constant cocycles. This result appears to be known, but we include a proof for completeness. In this section D will be any finitely presented group and $H = \mathbb{H}(k)$ will be the k points of a algebraic k-group $\mathbb{H}$ where k is a local field of characteristic 0. We will fix a realization $\mathbb{H} < GL(W)$. We will call a homomorphism ρ from a group D to $\mathbb{H}$ completely reducible if the representation on W given by $\rho(D) < \mathbb{H} < GL(W)$ is completely reducible. We let $\mathrm{Hom}(D, \mathbb{H})$ be the space of homomorphisms of D into $\mathbb{H}$ which has a natural structure as an algebraic subvariety of $\mathbb{H}^m$ where m is the number of generators of D. We note that the structure of $\mathrm{Hom}(D, \mathbb{H})$ as a variety is independent of the presentation of D and that $\mathrm{Hom}(D, H)$ is the set of k-points of the variety. Note that $\mathbb{H}$ (resp. H) acts on the space $\mathrm{Hom}(D, \mathbb{H})$ (resp. $\mathrm{Hom}(D, H)$) by conjugation.

THEOREM 4.1. *Let D and H be as above. Let $\pi : D \to H$ be any completely reducible homomorphism. Then*

1. *the $\mathbb{H}$ orbit of π in* $\mathrm{Hom}(D, \mathbb{H})$ *is Zariski closed and*
2. *the H orbit of π in* $\mathrm{Hom}(D, H)$ *is Hausdorff closed.*

Point (2) of Theorem 4.1 follows from point (1) and a result of Bernstein and Zelevinsky [**BZ**]. The result of Bernstein and Zelevinsky says that, given an action of a k group G on a k variety V, the k points of any orbit are locally closed in the Hausdorff topology. An examination of the proof shows that for Zariski closed orbits, the orbit is also Hausdorff closed. For an accessible proof in characteristic zero see [**AB**, proof of Theorem 6.1]. Let K be the algebraic closure of k and consider $\mathbb{H} < GL(W)$ where $W = K^n$. We prove part 1 of Theorem 4.1 from the following result.

THEOREM 4.2. *Let D be as above and let $\pi : D \to GL(W)$ be any completely reducible representation. Then the $GL(W)$ orbit of π in* $\mathrm{Hom}(D, GL(W))$ *is Zariski closed.*

PROOF OF THEOREM 4.2. We let $K[D]$ be the group ring of D. The representation π defines a representation $\tilde{\pi}$ of $K[D]$. This representation factors through a finite dimensional quotient $A = K[D]/\ker(\tilde{\pi})$ and since π is completely reducible A is a semisimple algebra. (See for example [**La**, XVII.6], particularly Theorem 6.1.)

If $\tilde{\pi}'$ is in the closure of the $GL(W)$ orbit of π then $\tilde{\pi}'$ also vanishes on $\ker(\tilde{\pi})$ and so $\tilde{\pi}'$ is also a representation of A. We recall that two representations of a semisimple algebra A are conjugate if and only if they have the same character, see for example [**La**, Theorem XVII.3.8]. This implies that the $GL(W)$ orbit of π is closed. $\square$

The proof of Theorem 4.1 is modelled on the proof that conjugacy classes of semisimple elements in algebraic groups are closed.

Proof of (1) in Theorem 4.1. The space $\mathrm{Hom}(D, \mathbb{H})$ is an algebraic variety over k. Assume that the $\mathbb{H}$ orbit of π is not closed. Then the orbit closure $\mathrm{Zar}(\mathbb{H}\cdot\pi)$ is the union of $\mathbb{H}\cdot\pi$ with a collection of subvarieties of strictly smaller dimension.

Given any homomorphism $\sigma : D \to G$ where G is any group, the orbit of σ in $\mathrm{Hom}(D, G)$ is naturally identified with $G/Z_G(\sigma(D))$. So given a homomorphism $\bar{\pi} \in \mathrm{Zar}(\mathbb{H}\cdot\pi)\backslash(\mathbb{H}\cdot\pi)$, we have that $\dim(Z_{\mathbb{H}}(\bar{\pi}(D))) > \dim(Z_{\mathbb{H}}(\pi(D)))$. We now show that $\dim(Z_{\mathbb{H}}(\bar{\pi}(D))) = \dim(Z_{\mathbb{H}}(\pi(D)))$ for any completely reducible homomorphisms π and $\bar{\pi}$ with $\bar{\pi} \in \mathrm{Zar}(\mathbb{H}\cdot\pi)$.

Let $\mathfrak{h}$ be the Lie algebra of $\mathbb{H}$ and $\mathrm{Ad}_{\mathfrak{h}}$ the adjoint representation of $\mathbb{H}$ on $\mathfrak{h}$. Let $\mathfrak{z}(\pi)$ (resp. $\mathfrak{z}(\bar{\pi})$) be the $\mathrm{Ad}_{\mathfrak{h}} \circ \pi(D)$ (resp. $\mathrm{Ad}_{\mathfrak{h}} \circ \bar{\pi}(D)$) invariant vectors in $\mathfrak{h}$. Since the characteristic of K is zero, we have that $\mathfrak{z}(\pi)$ is the Lie algebra of $Z_{\mathbb{H}}(\pi(D))$ and $\mathfrak{z}(\bar{\pi})$ is the Lie algebra of $Z_{\mathbb{H}}(\bar{\pi}(D))$ (see [**B2**], II.7). By construction $\mathrm{Ad}_{\mathfrak{h}} \circ \bar{\pi}$ is in the closure of the $\mathbb{H}$ orbit of $\mathrm{Ad}_{\mathfrak{h}} \circ \pi$ in $\mathrm{Hom}(D, GL(\mathfrak{h}))$. By Theorem 4.2 $\mathrm{Ad}_{\mathfrak{h}} \circ \bar{\pi}$ is conjugate to $\mathrm{Ad}_{\mathfrak{h}} \circ \pi$ by an element of $GL(\mathfrak{h})$ and therefore $\dim(\mathfrak{z}(\pi)) = \dim(\mathfrak{z}(\bar{\pi}))$. This implies that the $\mathbb{H}$ orbit of π is closed in the Zariski topology on $\mathrm{Hom}(D, \mathbb{H})$. $\square$

Theorem 4.1 would suffice to prove Theorems 1.1 and 1.2. However, to prove Theorem 5.1, we need the following.

Corollary 4.3. *Let $\mathbb{L} = \mathbb{A} \ltimes \mathbb{H}$ where all groups are k-algebraic. Let D be a finitely generated group. If $H = \mathbb{H}(k)$ and $L = \mathbb{L}(k)$, then H orbits of completely reducible homomorphisms in $\mathrm{Hom}(D, L)$ are Hausdorff closed.*

Proof. It follows from part 1 of Theorem 4.1 that $\mathbb{L}$ orbits in $\mathrm{Hom}(D, \mathbb{L})$ are Zariski closed. We let U be an $\mathbb{L}$ orbit in $\mathrm{Hom}(D, \mathbb{L})$. Then for any $u \in U$, then $\mathbb{L}u = \cup_{a \in \mathbb{A}} a\mathbb{H}u$. Since $\mathbb{H}$ is normal in $\mathbb{L}$, $a\mathbb{H}u = \mathbb{H}au$. The $\mathbb{H}$ action on U is algebraic, so the closure of an orbit must consist of the orbit plus sets of strictly lower dimension. Since all the sets $a\mathbb{H}u$ have the same dimension, this means that the closure of each $\mathbb{H}au$ in $\mathbb{L}u$ is $\mathbb{H}au$. Since $\mathbb{L}u$ is Zariski closed, so is $\mathbb{H}au$ and in particular $\mathbb{H}u$. That Hu is Hausdorff closed follows from the (proof of) the result of Bernstein-Zelevinsky as in the proof of Theorem 4.1. $\square$

5. Proof of Local Rigidity for Cocycles

In this section we prove Theorems 1.1 and 1.2. Actually we prove Theorem 5.1 below which implies Theorems 1.1 and 1.2. As mentioned in the introduction Theorem 5.1 is need to prove Theorem 1.8. We first fix some notations for the entire section. In the first subsection we formulate Theorem 5.1 and prove the theorem modulo a result on perturbations of cocycles taking values in compact groups. This last result Theorem 5.4, which holds for any compact valued cocycle over an action of a group with property T, is proven in the second subsection.

5.1. Theorem 5.1 and proof. We will let k be a local field of characteristic zero, $\mathbb{L}$ an algebraic k-group and $\mathbb{H}, \mathbb{A} < \mathbb{L}$ k-algebraic subgroups such that $\mathbb{L} = \mathbb{A} \ltimes \mathbb{H}$. We further assume that $\mathbb{A}$ is a connected semisimple k-group. We let L, A and H denote the k points of $\mathbb{L}, \mathbb{A}$ and $\mathbb{H}$ respectively. We will denote by D a group that is either G or Γ where G and Γ are as in the introduction. If π_1 and π_2 are two homomorphisms of a group B into a group C whose images commute, we will denote by (π_1, π_2) the homomorphism defined by $(\pi_1, \pi_2)(b) = (\pi_1(b), \pi_2(b))$ for all $b \in B$. We also fix a continuous homomorphism $\pi_0 : D \to L$ such that $\pi_0 = (\pi_A, \pi_H)$ where $\pi_A : D \to A$ and $\pi_H : D \to H$ are continuous homomorphisms such that $\pi_A(D)$ and $\pi_H(D)$ commute in L. If $D = \Gamma$, recall that π_0 is superrigid, which means that $\pi_0 = (\pi_0^E, \pi_0^K)$ where π_0^E is the restriction to Γ of a continuous homomorphism from G to H, and π_0^K is homomorphism from Γ to H with bounded image, and $\pi_0^E(\Gamma)$ and $\pi_0^K(\Gamma)$ commute. Similar statements are true for π_H and π_A. If $D = G$, we abuse notation by writing π_0^E for π_0 and π_0^K for the trivial homomorphism and similarly for π_A and π_H. We note that our assumption that $\pi_0 = (\pi_A, \pi_H)$ actually follows from the structure theory of algebraic group when D is G or the superrigidity theorems when D is Γ, though in both cases possibly only after replacing $\mathbb{A}$ by an isomorphic subgroup of $\mathbb{L}$.

THEOREM 5.1. *Let (S, μ) be a standard probability measure space, ρ a measure preserving action of D on S, and let $\alpha_{\pi_0} : D \times S \to L$ be the constant cocycle over the action ρ given by $\alpha_{\pi_0}(d, x) = \pi_0(d)$. Furthermore, let $\alpha : D \times S \to L$ be a Borel cocycle over the action ρ such that:*

1. *the cocycles α_{π_0} and α are L^∞ close*
2. *the projection of α to A is π_A.*

Then there exist measurable maps $\phi : S \to H$ and $z : D \times S \to Z$ where we define $Z = Z_L(\pi_0^E(D)) \cap H$ such that

1. *we have $\alpha(d, x) = \phi(dx)^{-1}(\pi_A(d), \pi_H^E(d)) z(d, x) \phi(x)$;*
2. *$\phi : S \to H$ is small in L^∞;*
3. *the map $(\pi_A^K(d), z(d, x))$ is a cocycle and is L^∞ close to the constant cocycle defined by π_0^K.*
4. *the cocycle (π_A^K, z) is measurably conjugate to a cocycle taking values in a compact subgroup K of Z and K is contained in a small neighborhood of $\pi_0^K(D)$.*

Furthermore if S is a topological space, $\mathrm{supp}(\mu) = S$ *and α and ρ are continuous then both ϕ and z can be chosen to be continuous.*

Remark: Theorem 5.1 is an immediate consequence of Theorem 5.4 if π_0 has bounded image, so we assume throughout that π_0 has unbounded image.
Remark: If $D = G$ or k is Archimedean or π_A^K is trivial, the map z is a cocycle. More generally, it is a *twisted cocycle*, as discussed in [**F**, Section 7.1].

As in the proof of Theorem 3.1 of [**MQ**], we deduce our result from the stability of partially hyperbolic vector bundle maps, though the details of the argument are quite different. To make the line of the argument clear, we outline it briefly for G actions assuming that the group A above is trivial and that the G action on S is ergodic. (In other words we sketch a proof of Theorem 1.1 from the introduction, with the additional assumption that the G action is ergodic.) It follows from Theorem 1.4 that $\alpha(g, x) = \phi(gx)^{-1} \pi'(g) c(g, x) \phi(x)$ where $\pi' : G \to L$ is a continuous homomorphism and $c : G \times S \to C$ is a cocycle taking values in a compact group

$C < Z_L(\pi'(G))$ and $\phi : X \to L$ is a measurable map. Since the set of polar elements of G is Zariski dense in G (see subsection 3.9) we can find a finitely generated Zariski dense subgroup $< g_1, \ldots, g_l >= F < G$ where each generator g_i has the property that it's image under any rational homomorphism $\pi : G \to L$ is uniquely determined by the characteristic numbers and subspaces of $\pi(g_i)$. We realize L as a subgroup of $GL(n, k)$ and study the dynamics of the skew product actions $\rho_{\alpha_{\pi_0}}$ and ρ_α on $S \times k^n$ determined by α_{π_0} and α. It follows from standard arguments on stability of partially hyperbolic vector bundle maps, see Lemma 5.2 below, that for any compact set $K < G$, if we pick α close enough to α_{π_0}, the characteristic numbers and subspaces for $\rho_{\alpha_{\pi_0}}(g)$ can be made arbitrarily close to those of $\rho_\alpha(g)$. This allows us to show that for almost any $x \in S$ and any i, the image of ${}^{\phi(x)}\pi'(g_i)$ is close to the image of $\pi_0(g_i)$. It then follows from Theorem 4.1 that π_0 and π' are conjugate as homomorphisms of F, and therefore as homomorphisms of G since $F < G$ is Zariski dense. Most of the remaining conclusions of the theorem are deduced by a more careful analysis of the data coming from Lemma 5.2 and Theorem 4.1. That z is measurably conjugate to a cocycle taking values in a compact group contained in a neighborhood of the identity uses Theorem 5.4. The general case follows more or less the same outline, using Theorems 3.22 and 3.23 in place of Theorem 1.4 and requiring somewhat more care due to the presence of many ergodic components. For Γ actions and cocycles there is an additional nuance since we cannot choose $F < \Gamma$ and here we use Lemma 3.21.

If $D = \Gamma$ then by Lemma 3.21 of subsection 3.8 there are elements $g_1, \ldots, g_l \in D$ and such that the group F generated by their polar parts is Zariski dense in G. If $D = G$, we pick a collection $g_1, \ldots, g_l$ of polar elements in G such that the group F generated by $g_1, \ldots, g_l$ is Zariski dense in G. In either case, we fix F and $g_1, \ldots, g_l$ for the remainder of the section. The reader is referred to subsection 3.8 for a discussion of polar elements.

Remark: Under any rational homomorphism $\pi : \mathbb{G}(k) \to GL_n(k)$ the image of a polar element is a polar element and $\mathrm{pol}(\pi(g)) = \pi(\mathrm{pol}(g))$.

As stated above, we will use a dynamical argument to show that $\mathrm{pol}(\tilde{\pi}_i(g_j))$ is close to $\mathrm{pol}(\pi_0(g_j))$ for a finite collection of g_j in Γ and then use this to conclude that $\tilde{\pi}_i^E$ and π_0^E are close in the compact open topology on homomorphisms.

We fix an almost faithful representation $\sigma : L \to GL(n, k)$. We can associate to any action ρ of D on a space X and any L valued cocycle α over ρ, an action $d_\alpha \rho$ of D on the trivial bundle $X \times k^n$ via $g(x, v) = (\rho(g)x, \sigma(\alpha(g, x))v)$. We use this to define two actions $d_{\alpha_{\pi_0}} \rho$ and $d_\alpha \rho$, both of which are linear extensions of ρ.

For any linear map $A : k^n \to k^n$, there is a finite field extension k' of k and a decomposition $\mathbb{R}^n = \bigoplus_{j=1}^l W_j$ such that the eigenvalues of $A|_{W_j \otimes k'}$ have the same absolute value λ_j for each j, and $\lambda_1 > \lambda_2 > \ldots \lambda_l$. We call W_j the *characteristic subspace* of A with *characteristic number* λ_j.

Let S, μ be a finite measure space, T a measure preserving transformation of S, and $\alpha : \mathbb{Z} \times S \to GL(n, k)$ a cocycle. We let

$$W_{\alpha,\varepsilon,\lambda}(x) = \{w \in W| \quad \limsup_{m \to \infty} |\frac{1}{m} \log \|\alpha(m, x)w\| - \lambda| < \varepsilon\} \cup \{0\}.$$

LEMMA 5.2. *Let $A \in GL(n, k)$ be linear transformation, (S, μ) a finite measure space, T a measure preserving transformation of (S, μ) and α_0 be the constant cocycle over the T action defined by A. Then there exists ε_0 depending only on A, such that for any $\varepsilon < \varepsilon_0$ and any cocycle $\alpha : \mathbb{Z} \times S \to GL(n, k)$ over the T action that* *is sufficiently L^∞ close (depending on ε) to A the spaces W_{A,λ_j} and $W_{\alpha,\varepsilon,\lambda_j}(x)$ are L^∞ close. Furthermore, the subspaces $W_{\alpha,\varepsilon,\lambda_j}(x)$ are measurable functions of $x \in S$ and if T and α are continuous, then $W_{\alpha,\varepsilon,\lambda_i}(x)$ is defined for all x and depends continuously on x, T and α.*

Remark: The number ε_0 is explicitly known and can be taken to be one third of $\min_{1 \le j < l}(\lambda_j - \lambda_{j-1})$.
This lemma follows from standard arguments on the stability of invariant distributions for partially hyperbolic vector bundle maps. There are many possible sources for such arguments, which go back at least as far as Anosov [**An**]. For a proof that is easily adapted to this setting see the proofs of Theorem 1 and Lemma 1 in [**P**].

Let $g_1, \ldots, g_k$ be as above. For π_0, we let λ_j^l be the characteristic numbers of $\pi_0(g_l)$ and let W_j^l be the corresponding subspaces. For α as in Theorem 5.1, we apply Lemma 5.2 to $g_1, \ldots, g_k$. We denote the resulting subspaces of W by $W_{\alpha,\varepsilon,g_l,\lambda_j^l}(x)$.

Before proceeding with the proof of Theorem 5.1, we record the information given by applying the results of section 3 to α.

PROPOSITION 5.3. *Let D, S, ρ, μ, α and π be as above. On each ergodic component μ_i of the measure μ the cocycle $\alpha : D \times S \to L$ is cohomologous to the product of a constant cocycle with a compact valued cocycle. More precisely there exist μ_i-measurable maps $\phi_i : S \to L$ such that*

$$\alpha(d,x) = \phi_i(dx)^{-1}(\pi_A, \pi_i)(d)c_i(d,x)\phi_i(x)$$

for all $d \in D$ and μ_i almost every x. Here π_i is the restriction to D of a continuous homomorphism $\pi_i : G \to H$ and $c_i : D \times S \to C_i$ are measurable maps taking values in compact subgroups $C_i < L$ commuting with $(\pi_A^E, \pi_i)(D)$. Furthermore, we can assume that there is a finite set Π of homomorphisms of G in H containing all π_i and that each ϕ_i takes values in H.

PROOF OF PROPOSITION 5.3. The proposition follows by applying Theorems 3.22 and 3.23 to α. To apply those theorems, we need to see that α is D-integrable. Since α is L^∞ close to a constant cocycle, it follows that $\ln^+ Q_{M,\alpha}(x)$ is essentially bounded for any precompact M and so in L^1. Therefore the cocycle is D-integrable for almost every ergodic component μ_i of μ.

Theorems 3.22 and 3.23 show that α is cohomologous to β where $p_A \circ \beta = \pi_A$ and $\beta = \bar{\pi}_i c_i$ where $\bar{\pi}_i : G \to L$ is a continuous homomorphism and $c_i : D \times S \to C$ is a cocycle taking values in a compact group $C_i < Z_L(\bar{\pi}_i(D))$. Furthermore, we have that $p_A \circ \beta = \pi_A$ and that the cohomology ϕ_i takes values in H. To prove the proposition, we need to see that we can write $\bar{\pi}_i = (\pi_A^E, \pi_i)$.

Let $\mathbb{H} = \mathbb{F} \ltimes \mathbb{U}$ be Levi decomposition, with $\mathbb{F}$ reductive and $\mathbb{U}$ unipotent. Since $\mathbb{A}$ is connected, it follows that $\mathbb{F} = \mathbb{F}_1\mathbb{F}_2$ an almost direct product, where $\mathbb{A}$ and $\mathbb{F}_2$ commute as subgroups of $\mathbb{L}$ and $\mathbb{A}$ acts on $\mathbb{F}_1$ by automorphisms via a homomorphism $\mathbb{A} \to \mathbb{F}_1$ composed with the adjoint action of $\mathbb{F}_1$ on itself. We write $\mathbb{L} = ((\mathbb{A} \ltimes \mathbb{F}_1)\mathbb{F}_2) \ltimes \mathbb{U}$ and as usual denote $F_1 = \mathbb{F}_1(k), F_2 = \mathbb{F}_2(k)$ and $U = \mathbb{U}(k)$. The map $\Delta(a,f) = (a, a^{-1}f)$ is an isomorphism between $\mathbb{A} \ltimes \mathbb{F}_1$ and $\mathbb{A} \times \mathbb{F}_1$. By replacing $\mathbb{A} < \mathbb{L}$ by $\Delta(\mathbb{A})$, we have that $\mathbb{A}$ and $\mathbb{F}$ commute. This does not effect π_0, since we have assumed that π_A and π_H commute which forces π_H to take values in F_2. After conjugating by some element of U, we may assume $\bar{\pi}_i$ takes values in $\mathbb{A} \times \mathbb{F}$. Writing $\bar{\pi}_i(d) = \pi_A^E(d)\pi_i(d)$ where π_i takes values in $\mathbb{F}$, it follows from the fact that $\mathbb{F}$ and $\mathbb{A}$ commute that π_i is a homomorphism.

That all π_i are contained in a finite collection Π follows from the fact that there are only finitely many (conjugacy classes of) homomorphisms of G into L. $\square$

We denote by $\tilde{\pi}_i$ the homomorphism (π_A, π_i) and call Π' the collection of all $\tilde{\pi}_i$. We fix the collection Π' of homomorphisms of G into L (or equivalently π_A and the collection Π of homomorphisms of G into H) for the remainder of the section.

Let $\tilde{\pi}_i(g_l)$ have characteristic numbers ${}^{i}a^l_1, ..., {}^{i}a^l_{s_i}$ and ${}^{i}W^l_j$ be the characteristic subspaces corresponding to ${}^{i}a^l_j$. Then for any $w \in W - \{0\}$ and almost every $x \in \operatorname{supp} \mu_i$, there is a j such that

$$\limsup_{m \to \infty} \frac{1}{m} \log \|\alpha(g_l^m, x) w\| = {}^{i}a^l_j.$$

Since the set $\{\lambda^l_m, {}^{i}a^l_j\}_{i,l,j,m}$ is finite, if we choose α close enough to π_0 and ε small enough, after re-indexing we have

$$\phi_i(x){}^{i}W^l_j = W_{\alpha,\varepsilon,g_l,\lambda^l_m}(x) \tag{3}$$

for μ_i almost all x and all i. Furthermore, we have that for each j there is an m such that ${}^{i}a^l_j = \lambda^l_m$ and that $\dim({}^{i}W^l_j) = \dim(W^l_m)$. This proof of equation 3 is essentially contained in [**MQ**] discussion preceding Lemma 3.4 or [**QZ**] proof of Theorem A.

Proof of Theorem 5.1. First we show that $\tilde{\pi}_i = (\pi_A, \pi^E_H)$ for all i. Recall that $\pi_0 = (\pi^E_0, \pi^K_0)$ and that $\tilde{\pi}_i = (\tilde{\pi}^E_i, \tilde{\pi}^K_i)$ where π^E_0 and π^K_0 (resp. $\tilde{\pi}^E_i$ and $\tilde{\pi}^K_i$) commute and $\pi^E_0, \tilde{\pi}^E_i$ are restrictions of continuous homomorphisms of G and $\tilde{\pi}^K_i, \pi^K_0$ have bounded image. Note that for any $g \in \Gamma$ we have $\operatorname{pol}(\tilde{\pi}_i(g)) = \operatorname{pol}(\tilde{\pi}^E_i(g))$ and that the same is true for π_0. Since $\pi_0 = (\pi_A, \pi_H)$ and $\tilde{\pi}_i = (\pi_A, \pi_i)$ and π_i is the restriction of a rational homomorphism from $\mathbb{G}$ to $\mathbb{L}$, it suffices to show that $\tilde{\pi}^E_i = \pi^E_0$.

By Lemma 5.2,by choosing ε small enough and α close enough to π_0, the space $W_{\alpha,\varepsilon,g_l,\lambda^l_m}(x)$ can be made arbitrarily close to the space W^l_m for almost every $x \in S$. By equation 3, we have that $\phi_i(x){}^{i}W^l_j = W_{\alpha,\varepsilon,g_l,\lambda^l_m}(x)$ for μ_i almost every $x \in S$, so W^l_m is close to $\phi(x){}^{i}W^j_l$ for almost every x. Furthermore, by the remark following equation 3, for each j there is an m such that the action of the polar part of ${}^{\phi(x)}\tilde{\pi}_i(\gamma_l)$ on $\phi(x){}^{i}W^l_j$ is the same as the action of the polar part of $\pi_0(\gamma_l)$ on W^l_m. This implies that, for α close enough to π_0, we can make $\operatorname{pol}({}^{\phi_i(x)}\tilde{\pi}_i(\gamma_l)) = {}^{\phi_i(x)}\tilde{\pi}_i(\operatorname{pol}(\gamma_l))$ arbitrarily close to $\operatorname{pol}(\pi_0(\gamma_l)) = \pi_0(\operatorname{pol}(\gamma_l)$ for every l and i and μ_i almost every x. Therefore, for almost every $x \in S$, the homomorphisms ${}^{\phi(x)}\tilde{\pi}^E_i$ and π^E_0 can be made arbitrarily close as homomorphisms of F by choosing α close enough to π_0. Since there are only finitely many homomorphisms in Π' and H orbits in $\operatorname{Hom}(F, L)$ are closed by Corollary 4.3, this implies that $\tilde{\pi}^E_i$ and π^E_0 are in the same H orbit in $\operatorname{Hom}(F, L)$ and so are conjugate by an element of H. Since F is Zariski dense in G and $\tilde{\pi}^E_i$ and π^E_0 are restrictions of rational homomorphisms, it follows that $\tilde{\pi}^E_i$ is conjugate to π^E_0 as homomorphisms of G.

By relabelling, we now have

$$\alpha(g, x) = \phi_i(gx)^{-1}(\pi_A, \pi^E_H)(g) c_i(g, x) \phi_i(x)$$

for each i and μ_i almost every x where each c_i is a cocycle taking values in a compact subgroup of $Z = Z_L((\pi^E_0)(D)) \cap H$.

We define a map $\bar{\phi}_h : S\to \mathrm{Hom}(F, L)$ by taking x to ${}^{\phi_i(x)}\tilde{\pi}_i|_F$. By Lemma 5.2 and the discussion above, this map is measurable and L^∞ small and has image contained in a single H orbit in $\mathrm{Hom}(F, L)$. Since F and D are Zariski dense in G, and π_0^E is rational, it follows that $Z_L(\pi_0^E(F)) = Z_L(\pi_0^E(G)) = Z_L(\pi_0^E(D))$ and so we may identify the H orbit in $\mathrm{Hom}(F, L)$ with H/Z where $Z = Z_L(\pi_0^E(D))\cap H$ as before. Therefore, choosing a point w in the image, we can define a measurable map $\phi_h : S\to H$ such that $\bar{\phi}_h(x) = \phi_h(x)w$. Furthermore, we can choose ϕ_h to be L^∞ small. This uses that we can choose a Borel section of $H\to H/Z$ which is continuous in a small neighborhood of $[1_H]$ and does not increase norms.

It follows from the definitions of ϕ_i and ϕ_h that for each i, we have that $\phi_i(x) = \phi_h(x)\phi_i^z(x)$ where ϕ_i^z takes values in Z. Then a simple computation shows that:

$$\alpha(g,x) = \phi_h(gx)\phi_i^z(gx)(\pi_A, \pi_H^E)(g)c_i(g,x)\phi_i^z(x)^{-1}\phi_h(x)^{-1}$$
$$= \phi_h(gx)(\pi_A, \pi_H^E)(g)z(g,x)\phi_h(x)^{-1}$$

where $z(g,x) = \pi_A^K(g)^{-1}\phi_i^z(gx)\pi_A^K(g)c_i(g,x)\phi_i^z(x)^{-1}$ takes values in Z.

Since α is measurable and L^∞ close to π_0 and ϕ_h is measurable and L^∞ small, it follows that $z(g,x) = \phi_h(gx)\alpha(g,x)\phi_h(x)^{-1}(\pi_A, \pi_i^E)(g)^{-1}$ is measurable and L^∞ close to π_H^K. It is clear that $(\pi_A^K(g), z(g,x))$ is a cocycle taking values in $Z_L(\pi_0^E)$, which implies that z is a cocycle when π_A^K is trivial. Theorem 5.4 in the next section shows that $(\pi_A^K, z(g,x))$ is measurably conjugate to a cocycle taking values in a group that is contained in a small open neighborhood of $(\pi_A^K, \pi_H^K)(D)$. When k is Archimedean, this implies that $(\pi_A^K, z(g,x))$ is measurably conjugate into the closure of $(\pi_A^K, \pi_H^K)(D)$ by Lemma 5.6. This implies that π_A^K and z take values in groups which commute, which by Lemma 3.3 implies that z is a cocycle.

When ρ and α are continuous, we deduce continuity of ϕ_h from the continuity of $W_{\alpha,\varepsilon,g_l,\lambda_m^l}(x)$ which follows from Lemma 5.2. Since we know that $\pi_i^E = \pi_0^E$, we can rewrite equation 3 as $\phi(x)W_m^l = W_{\alpha,x,g_l,\lambda_m^l}$ for each l. This implies that the polar part of ${}^{\phi(x)}\pi_0^E(g_l)$ depends continuously on x for each l. Combined with Lemma 3.21 this implies that ${}^{\phi(x)}\pi_0^E$ depends continuously on x. Therefore $\phi(x)$ is continuous modulo the stabilizer of π_0^E in H, or ϕ_h is continuous. That z is then continuous as well follows from the formula defining z two paragraphs above. □

5.2. Local rigidity of compact valued cocycles. In this subsection D will denote a locally compact group with property T of Kazhdan, A will be a locally compact group and $C < A$ will be a compact subgroup. As before (S, μ) will be a standard measure space and we will assume D acts on S preserving μ. We will say that a group C' is close to C if there is a small neighborhood U of C such that $C'\subset U$. Throughout ν will denote right Haar measure on A. We will say that two cocycles $\alpha, \alpha' : D\times S\to A$ are close if there is a compact generating set K for D and a small neighborhood U of 1 in A such that $\alpha(d,s)\alpha'(d,s)^{-1}$ is in U for all $d\in K$.

THEOREM 5.4. *Let $\alpha_0 : D\times S\to C$ be a cocycle over the D action. Any cocycle $\alpha : D\times S\to A$ which is close to α_0 is conjugate to a cocycle into a compact group C' that is close to C.*

Remark: The proof of Theorem 5.4 is simpler if one assumes the D action on S is ergodic.

We will need the following proposition in order to find C' satisfying the conclusions of the theorem.

PROPOSITION 5.5. *Let $C < A$ be a compact subgroup. Then given any small enough neighborhood U of C, there exists a compact group $C' < A$, contained in U, such that any subgroup $C'' < A$ contained in U is conjugate to a subgroup of C' by a small element of A.*

LEMMA 5.6. *Let $C < A$ be a compact subgroup such that A/C is a manifold. Then any compact subgroup C' sufficiently close to C is conjugate to a subgroup of C by a small element of A.*

PROOF. This follows from a barycenter argument that is similar to the one that shows that any two maximal compact subgroups in a semisimple group are conjugate. We let $X = A/C$ and take the C' orbit $\mathcal{O}$ of the coset of the identity $[1_A]$. Since C' is close to C, it follows that $\mathcal{O}$ is contained in a small neighborhood of $[1_A]$. We can then take the barycenter or center of gravity for $\mathcal{O}$. This is defined as the unique minimum of the function $d_{\mathcal{O}}(x) = \int_{\mathcal{O}} d(x,y)^2 d\mu(y)$ where μ is the push-forward of Haar measure on C' to $\mathcal{O}$. That a barycenter exists and is unique can be proven from convexity of the distance function on a small enough neighborhood $U_{[1_A]}$. Convexity of the distance function on this neighborhood can be proven by comparison with the sphere whose sectional curvature is the maximum of the sectional curvatures of two planes in $T(A/C)$, using the fact that for small enough neighborhoods on the sphere, the distance function is convex, see for example [**BH**, Exercises 2.3(1),p.176]. The barycenter is then a fixed point for the C' action and is close to $[1_A]$ since $c'[1_A]$ is close to $[1_A]$ for all $c' \in C'$. This implies that $C' < aCa^{-1}$ for $a \in A$ small. □

PROOF OF PROPOSITION 5.5. We let A^0 be the connected component of the identity in A, and $p : A \to A/A^0$ be the projection. Then, since A/A^0 is totally disconnected, there is an open subgroup $\bar{C}$ containing $p(C)$. Since $\bar{C}$ is open, if U is a sufficiently small neighborhood of C, then $p(U)$ is contained in $\bar{C}$. We will find $C' < p^{-1}(\bar{C})$ and so replace A by $A' = p^{-1}(\bar{C})$.

Given any open set U containing the identity in A' there is a compact normal subgroup $N \subset U$ such that A'/N has no small subgroups, and is therefore a manifold. This is an extension by Glushkov [**Gl**] of results due to Gleason, Montgomery and Zippin, see [**K**] for further discussion, particularly Theorem 18 and the remark following on page 137. We let $C' = CN$. Then $A/C' = (A/N)/(C/(C \cap N))$ and so is a manifold. It then follows from Lemma 5.6 that if U is small enough, any subgroup contained $C'' \subset U$ is conjugate to a subgroup of $C' = CN$. □

Given a non-negative, integrable function h on A and a unitary representation ρ of A on a Hilbert space $\mathcal{H}$, we define $\rho(h) = \int_A h(a)\rho(a)d\nu(a)$. If $\int_A h(a)d\nu(a) = 1$, then $||\rho(h)|| \leq 1$ as verified in [**M3**, III.1.0].

We recall that a locally compact group D has property T of Kazhdan if the trivial representation is isolated in the unitary dual. This has the following consequence, which can be seen as an effective version of the standard statement "any (ε, K)-invariant vector is close to a D invariant vector".

LEMMA 5.7. *Let D be a locally compact group with property T. Then for any compact generating set K for A there is a non-negative continuous function h with support contained in K^2 and $\int_A f d\mu_A = 1$ and a constant $B = B(K,h)$, such that for any unitary representation ρ of D on a Hilbert space $\mathcal{H}$ and for any vector $v \in \mathcal{H}$, we have*

1. $\lim_{n\to\infty} \rho(h)^n v = v_F$ *exists;*
2. v_F *is fixed by* D*;*
3. $d(v_F, v) \leq B \operatorname{supp}_{k\in K} d(kv, v)$.

PROOF. This follows from the definition of property T and from [**M3**, III.1.3], which shows that $\rho(h)$ is a contraction on the orthogonal complement of the D fixed vectors in $\mathcal{H}$. That h can be chosen with support in K^2 follows from the construction of h in the proof of [**M3**, III.1.1]. □

PROOF OF THEOREM 5.4. We let $\mathcal{H} = L^2(A)$ and write the natural action of A coming from the right regular representation as $v \to va^{-1}$. Fix a vector $v_0 \in \mathcal{H}$ whose stabilizer is C. We define two representations ρ_{α_0} and ρ_α of D on $L^2(S, \mathcal{H}, \mu)$ by $(\rho_{\alpha_0}(d)f)(x) = f(d^{-1}x)\alpha_0(d,x)^{-1}$ and $(\rho_\alpha(d)f)(x) = f(d^{-1}x)\alpha(d,x)^{-1}$. Then the function $f_0 : S \to \mathcal{H}$ defined by $f_0(x) = v_0$ for all x is ρ_{α_0} invariant. It is easy to see that $\operatorname{supp}_{k\in K} d(\rho_\alpha(k)f_0, f_0) < \varepsilon$ where ε only depends on how close α is to α_0.

First assume that the action of D on S is ergodic. By 2 and 3 of Lemma 5.7, there is a function $f \in L^2(S, \mathcal{H})$ such that f is $\rho_\alpha(D)$ invariant and $\|f - f_0\|_2$ is small. By the proof of [**Z2**, Lemma 9.1.2] one sees that the A action on $\mathcal{H}$ is tame and so ergodicity of the D action on S implies that f takes values in a single A orbit $\mathcal{O}$ in $\mathcal{H}$. This then implies by [**Z2**, Lemma 5.2.11] that α is equivalent to a representation into the stabilizer A_v of some vector $v \in \mathcal{O}$. It is easy to verify that A_v is compact. Since f is L^2 close to f_0, we can choose v to be close to v_0. This immediately implies that the stabilizer of A_v is Hausdorff close to the stabilizer of v_0.

If the action is not ergodic, we cannot conclude that f takes values in a single orbit. If one traces through the above argument, one sees that, on each ergodic component μ_i of μ, f takes values in a single A orbit we can view f as a μ measurable map $S \to S \times H/C_i$ where $C_i < A$ is a compact subgroup depending on μ_i. That we can find a μ measurable function conjugating α to a cocycle α' taking values in C_i for μ_i almost every x follows from the existence of a Borel section for the map $S \times H \to S \times H/C_i$. The existence of such a section can be deduced from [**Z2**, Theorem A.5]. However, since we only know that f is close to f_0 in $L^2(S, \mathcal{H}, \mu)$, it only follows from this argument that C_i is close to C on "most" ergodic components. We want to see that f is actually close to f_0 in $L^2(S, \mathcal{H}, \mu_i)$ for almost every ergodic component. This is deduced from 1 of Lemma 5.7, since $f = \lim_{n\to\infty} \rho(h)^n f$ and this equation holds in both $L^2(S, \mathcal{H}, \mu)$ and $L^2(S, \mathcal{H}, \mu_i)$. That f is close to f_0 in $L^2(S, \mathcal{H}, \mu_i)$ for almost every ergodic component then follows from 3 of Lemma 5.7. This implies that each C_i is contained in a small neighborhood of C, and so is conjugate into a subgroup $C' < A$ contained in a small neighborhood of C by an element a_i. Conjugating by a map $\phi : S \to A$ such that $\phi(s) = a_i$ for μ_i almost every s, we have that α is conjugate to a cocycle taking values in C'. □

6. Affine actions, perturbations and cocycles

In this section we prove Corollary 1.7 and Theorem 1.8. In order to do so, we need a more detailed description of the actions in Definition 1.6 when the group acting is G or Γ as above. The section is divided into two subsections, the first giving an algebraic description of affine actions, the second proving Corollary 1.7 and Theorem 1.8.

6.1. Description of affine actions. Let H be a connected Lie group and $\Lambda < H$ a discrete cocompact subgroup. We will let $\mathrm{Aff}(H/\Lambda)$ be the group of affine transformation of H/Λ. Any affine transformation f of H/Λ has a lift of the form $h_f \circ L_f$ where $h_f \in H$ and $L_f \in \mathrm{Aut}(H)$ where $\mathrm{Aut}(H)$ denotes the group of continuous automorphisms of H. Let $N_{\mathrm{Aut}(H)}(\Lambda)$ be the group of elements $L \in \mathrm{Aut}(H)$ such that $L \cdot \Lambda \subset \Lambda$. Then $L_f \in N_{\mathrm{Aut}(H)}(\Lambda)$ and we have a map $\phi : N_{\mathrm{Aut}(H)}(\Lambda) \ltimes H \to \mathrm{Aff}(H/\Lambda)$. There is a map $\Delta^{-1} : \Lambda \to N_{\mathrm{Aut}(H)}(\Lambda) \ltimes \Lambda$ given by $(\lambda) \to \mathrm{Ad}((\lambda)^{-1}), (\lambda))$. We denote the image of this map by $\Delta^{-1}(\Lambda)$. (If H has trivial center, then the image is in fact an anti-diagonal embedding of Λ.)

PROPOSITION 6.1. *The kernel of the map* $\phi : N_{\mathrm{Aut}(H)}(\Lambda) \ltimes H \to \mathrm{Aff}(H/\Lambda)$ *is* $\Delta^{-1}(\Lambda)$.

PROOF. First note that any diffeomorphism f of H/Λ gives rise to an element f_* of $\mathrm{Out}(\pi_1(H/\Lambda))$. If f is trivial, then f_* must be trivial as well. If $f = \phi((a, h_0))$ for $a \in N_\Lambda(\mathrm{Aut}(H))$ and $h_0 \in H$ and f_* is trivial, then a must be an inner automorphism of H preserving Λ. This implies that $a = z\,\mathrm{Ad}(\lambda)$ for some $\lambda \in \Lambda$ and $z \in Z_{\mathrm{Aut}(H)}(\Lambda)$, the centralizer in $\mathrm{Aut}(H)$ of Λ. But then $(a, h_0)[h] = [z \cdot (\lambda h_0 h \lambda^{-1})] = [z \cdot (\lambda(h_0)h)]$ for all $h \in H$. So $(a, h_0)[h] = [h]$ for all $h \in H$ if and only if $z \cdot (\lambda h_0)^{-1} z \cdot h = h$ for all $h \in H$. Since z centralizes λ this is equivalent to $\lambda(z \cdot h_0)(z \cdot h) = h$ for all $h \in H$. Picking $h = 1$ this forces $z \cdot h_0 = \lambda^{-1}$. This implies that $z \cdot h = h$ for all $h \in H$ which implies that $z = 1$ and so $h_0 = \lambda^{-1}$. □

Most of the difficulty in proving the theorems we need describing affine actions derive from tori in the reductive component of H. To deal with this difficulty we replace H and Λ by groups H' and Λ' such that the respective quotients are diffeomorphic and the affine groups are the same. First we note the a simple fact about covers.

LEMMA 6.2. *Let H be a real Lie group and $\Lambda < H$ a cocompact lattice. Let $p : H' \to H$ be a covering map and $\Lambda' = p^{-1}(\Lambda)$. Then*

1. *H/Λ is diffeomorphic to H'/Λ'*
2. $\mathrm{Aff}(H/\Lambda) < \mathrm{Aff}(H'/\Lambda')$

PROOF. The first claim of the lemma is immediate. To see the second, we note that any continuous automorphism A of H lifts to a continuous automorphism A' of H'. This uses the fact that the fundamental group of H is abelian and so any cover is a normal cover. If $A \cdot \Lambda = \Lambda$ then $A' \cdot \Lambda' = \Lambda'$. Also, given any element of h, we can choose an element h' in H' projecting to h. It is easy to verify that hA and $h'A'$ induce the same diffeomorphism of $H/\Lambda = H'/\Lambda'$. □

We now show how to replace H by a cover H', though we need to use an algebraic structure on H' so that the cover is not a rational map.

PROPOSITION 6.3. *Given a real algebraic group H and a cocompact lattice Λ there is a cover $p : H' \to H$ and a realization of H' as $\mathbb{H}'(\mathbb{R})$ for a connected $\mathbb{R}$ algebraic group $\mathbb{H}'$, such that*

1. *there is a finite index subgroup $\mathrm{Aut}^A(H') < \mathrm{Aut}(H')$ such that all elements of $\mathrm{Aut}^A(H')$ are rational automorphisms of H' and*
2. *$\mathrm{Aut}^A(H') \ltimes H'$ is the real points of a real algebraic group which we denote by $\mathrm{Aut}^A(\mathbb{H}') \ltimes \mathbb{H}'$.*

PROOF. We first define the group H'. By definition $H = \mathbb{H}(\mathbb{R})$ where $\mathbb{H}$ is an algebraic $\mathbb{R}$-group. We take a Levi decomposition $\mathbb{H} = \mathbb{L} \ltimes \mathbb{U}$ where $\mathbb{L}$ is reductive and $\mathbb{U}$ is unipotent. We first pass to a finite central extension $\tilde{\mathbb{H}}$ so as to be able to assume that $\mathbb{L}$ is a direct product of a torus $\mathbb{T}$ and a simply connected semisimple group $\mathbb{J}$. We let $\sigma : \mathbb{L} \to \mathrm{Aut}(\mathbb{U})$ be the representation defining the semidirect product. We let $\mathbb{T}_1 = \ker(\sigma) \cap \mathbb{T}$. This is a finite extension of a connected group $\mathbb{T}_1^0$, and $\mathbb{T}_1^0 < Z(\tilde{\mathbb{H}})$ and so $\tilde{\mathbb{H}} = \mathbb{T}_1^0 \times \mathbb{H}^*$. The universal cover of $\mathbb{T}_1^0(\mathbb{R})$ is isomorphic to $\mathbb{R}^n$ for $n = \dim(\mathbb{T}_1^0)$, and we can realize $\mathbb{R}^n$ as the real points of a unipotent algebraic group which we denote by $\mathbb{U}^*$. We replace $\mathbb{H}$ by $\mathbb{H}' = \mathbb{H}^* \times \mathbb{U}^*$ and H by $H' = \mathbb{H}'(\mathbb{R})$. There is a covering map $p_1^* : \mathbb{U}^*(\mathbb{R}) \to \mathbb{T}_1^0(\mathbb{R})$ which defines a covering map $p_1 : \mathbb{H}'(\mathbb{R}) \to \tilde{\mathbb{H}}(\mathbb{R})$ which we compose with the covering map $p_2 : \tilde{\mathbb{H}}(\mathbb{R}) \to \mathbb{H}(\mathbb{R})$ to define a covering map $p : \mathbb{H}'(\mathbb{R}) \to \mathbb{H}(\mathbb{R})$. We let $\Lambda' = p^{-1}(\Lambda)$.

To continue the proof, we will need a Levi decomposition of $\mathbb{H}' = \mathbb{L}' \ltimes \mathbb{U}'$. By the discussion above, we can write $\mathbb{L}' = \mathbb{J}' \times \mathbb{T}'$ where $\mathbb{J}'$ is semisimple and $\mathbb{T}'$ is a torus. We have constructed $\mathbb{H}'$ such that the homomorphism $\mathbb{T}' \to \mathrm{Aut}(\mathbb{U}')$ has finite kernel. (The attentive reader will note that $\mathbb{J}'$ is isomorphic to $\mathbb{J}$ above and that $\mathbb{T}$ above is $\mathbb{T}_1 \times \mathbb{T}'$, but we will not need this in the discussion that follows.) As usual, $L' = \mathbb{L}'(\mathbb{R}), U' = \mathbb{U}'(\mathbb{R}), J' = \mathbb{J}'(\mathbb{R})$ and $T' = \mathbb{T}'(\mathbb{R})$.

The group $\mathrm{Aut}^A(H')$ will consist of those automorphisms of H' which project to inner automorphisms of L. We first show this has finite index in $\mathrm{Aut}(H)$. The group of outer automorphisms of J is finite, so it suffices to show that the group of automorphisms of T' that extend to automorphisms of H' is finite. In fact, the group Ξ of automorphisms of T' which extend to $T' \ltimes U'$ is finite. Any such automorphism must induce a permutation of the finite collection Δ of weights defining the representation σ of $\mathbb{T}$ on $\mathbb{U}$, so by passing to a subgroup $\Xi' < \Xi$ of finite index, we may assume that Ξ' fixes Δ pointwise. Since the kernel of σ is finite, Δ forms a basis for the group of characters of $\mathbb{T}'$ which vanish on $\ker \sigma$. Therefore Ξ' acts trivially on a subgroup of finite index in the group of characters of $\mathbb{T}'$ and, since $\mathbb{T}'$ is connected, acts trivially on $\mathbb{T}'$.

We can write any element of $\phi \in \mathrm{Aut}^A(H')$ as a composition of three elements. First we translate by an element u of U' so that $u \circ \phi(L) = L$. Then we conjugate by an element l of L so that $\mathrm{Ad}(l) \circ u \circ \phi$ is trivial on L. The automorphism $\mathrm{Ad}(l) \circ u \circ \phi = a$ is clearly an automorphism of U which commutes with the action of L on U. We write $\phi = alu$. Viewing a as belonging to $Z_{\mathrm{Aut}(U)}(L)$, l as an element of the adjoint group of $\bar{L}$ of L, and u as an element of $U/Z_U(L)$, this decomposition is unique, and clearly makes $\mathrm{Aut}^A(H')$ the set of $\mathbb{R}$ point of an $\mathbb{R}$-variety. Writing the multiplication on $\mathrm{Aut}^A(H') \cong Z_{\mathrm{Aut}(U)}(L) \times \bar{L} \times (U/Z_U(L))$ it is clear that all factors commute pairwise except the last two. The product $\bar{L} \ltimes (U/Z_U(L))$ is clearly a quotient of the adjoint group of $J \ltimes U$ by the image of $Z_U(L)$ in the adjoint group of $J \ltimes U$, and so is the real points of an algebraic group defined over $\mathbb{R}$, and so $\mathrm{Aut}^A(H')$ is the real points of an algebraic group defined over $\mathbb{R}$.

It also follow easily from our description of $\mathrm{Aut}^A(H')$ that every element of $\mathrm{Aut}^A(H')$ is the restriction of a rational automorphism of $\mathbb{H}'$. This follows from the fact that $\mathrm{Aut}(\mathbb{U})$ acts rationally on $\mathbb{U}$ which follows from the fact that exp and ln are rational diffeomorphisms between U and $\mathfrak{u}$ by the Baker-Campbell-Hausdorff formula, and that any automorphism of $\mathfrak{u}$ is linear and therefore rational.

To show that $\mathrm{Aut}^A(H') \ltimes H'$ is the real points of an algebraic group defined over $\mathbb{R}$ only requires that we show that $\mathrm{Aut}^A(H') \ltimes H' \to H'$ is the restriction of a rational map. Using the coordinates on $\mathrm{Aut}^A(H')$ described above this reduces to showing that the map $Z_{\mathrm{Aut}(\mathbb{U})}(\mathbb{L})\ltimes\mathbb{U}\to\mathbb{U}$ is rational. This follows from the fact that $\mathrm{Aut}(\mathbb{U})\ltimes\mathbb{U}\to\mathbb{U}$ is rational which follows from the fact that $\mathrm{Aut}(\mathbb{U})$ is defined as an algebraic subgroup of $GL(\mathfrak{u})$ and from rationality of automorphisms of U discussed above. $\square$

THEOREM 6.4. *Let G be as in section 3.1, but with no assumption on the rank of G. Assume H is a connected real algebraic group and Λ a cocompact discrete subgroup. Let ρ be an affine action of G on H/Λ. Then the action ρ is given by $\rho(g)[h]=[\pi_0(g)h]$ where $\pi_0 : G\to H$ is a continuous homomorphism.*

THEOREM 6.5. *Let G be as in section 3.1 and let $\Gamma<G$ be a weakly irreducible lattice. Let H and Λ be as above. Let ρ be an affine action of Γ on H/Λ. Then there is a finite index subgroup $\Gamma'<\Gamma$ such that, possibly after replacing H by H' as in Proposition 6.3, the Γ' action on H/Λ is given by $\rho(\gamma)[h]=[\pi_H(\gamma)\cdot\pi_A(\gamma)h]$. Here $\pi_H : \Gamma'\to H'$ and $\pi_A : \Gamma'\to\mathrm{Aut}(H')$ are homomorphisms whose images commute as subgroups of $\mathrm{Aut}(H')\ltimes H'$. Furthermore, we can assume that $(\pi_A,\pi_H)(\Gamma')$ is contained in $\mathrm{Aut}^A(H')\ltimes H'$, an algebraic group.*

Remark: Using the results of [**C, GS, Rg1, Rg2, St**] in combination with the arguments in [**M3**], one can assume only that Γ projects to a dense subgroup of a rank one simple factors not locally isomorphic to F_4^{-20} or $Sp(1,n)$.

It is obvious from this description that the action of G or Γ on H/Λ lifts to H on a subgroup of finite index.

PROOF OF THEOREM 6.4. The action ρ defines a continuous homomorphism

$$\pi : G\to(N_{\mathrm{Aut}(H)}(\Lambda)\ltimes H)/(Z(H)\cap\Lambda).$$

Since the target is a Lie group, any simple factor $\mathbb{F}(k)$ of G which is defined over a non-Archimedean field k has trivial image, since is totally disconnected and topologically almost simple. Therefore it suffices to consider the case where G is a connected Lie group. We replace H by a group H' and Λ by Λ' as in Proposition 6.3. Let $\mathrm{Aut}^A(H')<\mathrm{Aut}(H')$ be the subgroup of finite index such that every element of $\mathrm{Aut}^A(H')$ is rational. Then since $\pi(G)$ is connected, $\pi(G)$ must be contained in the image of $(\mathrm{Aut}^A(H')\ltimes H')\cap(N_{\mathrm{Aut}(H')}(\Lambda)\ltimes H')$.

It follows from generalizations of Borel's density theorem that Λ' is Zariski dense in a cocompact normal subgroup of H', see for example [**D**] or [**Sh**, Theorem 1.1]. Therefore by Lemma 6.3 any automorphism of H' that fixes Λ' pointwise factors through an automorphism of a compact quotient $\bar{H}$ of H. It is easy to see that $\mathrm{Aut}(\bar{H})$ is a discrete extension of a compact group, since $\bar{H}$ is an almost direct product of compact simple groups and compact torii. Since Λ' is discrete $N_{\mathrm{Aut}(H')}(\Lambda')/Z_{\mathrm{Aut}(H')}(\Lambda')$ is discrete and therefore $N_{\mathrm{Aut}(H')}(\Lambda')$ is a discrete extension of a compact group. As remarked above $\pi(G)$ is contained in the connected component of $\mathrm{Aff}(H/\Lambda')$. Letting Z^0 be the connected component of $Z_{\mathrm{Aut}(H')}(\Lambda')$, the connected component of $\mathrm{Aff}(H/\Lambda)$ is $\phi(Z^0\ltimes H')$. Now $Z^0\ltimes H'\cap\ker(\phi)=Z(H')\cap\Lambda'$ so $\phi(Z^0\ltimes H')\cong(Z^0\ltimes H')/(Z(H')\cap\Lambda')$. Since G has no compact factors and Z^0 is compact, the map $\pi : G\to(Z^0\ltimes H')/(Z(H')\cap\Lambda')$ takes values in $H/(Z(H)\cap\Lambda')$.

Since G is either simply connected or simply connected as an algebraic group, we can lift π to a homomorphism $\tilde{\pi} : G\to H'$. This also define a homomorphism

$\pi : G \to H$, and it is easy to verify that $\tilde{\pi}$ and π define the same affine action on $H/\Lambda \cong H'/\Lambda'$. □

The proof for Γ actions is more complicated and requires the use of the superrigidity theorems. In addition to a direct application, we will also use the following consequence of the superrigidity theorems.

LEMMA 6.6. *Let G and Γ be as above. Let $\pi : \Gamma \to D$ be any Zariski dense homomorphism into a real algebraic group D. Let $\tilde{D}$ be a real algebraic group and $\tilde{D} \to D$ an isogeny. Then there is a finite index subgroup $\Gamma' < \Gamma$ and a homomorphism $\tilde{\pi} : \Gamma' \to \tilde{D}$ such that $p \circ \tilde{\pi} = \pi$ where $p : \tilde{D} \to D$ is the natural covering map.*

PROOF OF LEMMA 6.6. Since by [**M3**, IX.5.8], the image of any homomorphism from Γ into a real algebraic group has semisimple algebraic closure, it suffices to consider the case where D is semisimple. Since it also suffices to consider the case where $\mathbb{D}$ is simply connected as an algebraic group and simply connected semisimple algebraic groups are direct products of simple groups by [**M3**, I.1.4.10], it suffices to consider the case where $\mathbb{D}$ is simple and simply connected as an algebraic group.

We first assume $G = \prod_I G_i$ where each G_i is algebraic and $G_1 = \mathbb{G}_1(\mathbb{R})$. We have a homomorphism $\pi : \Gamma \to \mathbb{D}(\mathbb{R})$. It is clear that $\pi(\Gamma) < \mathbb{D}(k)$ where k is a finite extension of $\mathbb{Q}$, and we let $\bar{k}$ be the algebraic closure of k. Since Γ is Zariski dense in D, it follows that D is defined over k. A corollary of the superrigidity theorems, see [**M3**, Theorems VII.6.5 and VII.6.6], shows that, after passing to a subgroup of finite index, there is an embedding σ of $\bar{k}$ in $\mathbb{C}$, and a $\mathbb{C}$-rational map $\eta : \mathbb{G}_1 \to {}^{\sigma}\mathbb{D}$ such that $\pi(\gamma) = \sigma^{-1}(\eta(\gamma))$. Here ${}^{\sigma}\mathbb{D}$ is the $\sigma(k)$ algebraic group defined by the image under σ of the equations defining $\mathbb{D}$. Since $\mathbb{G}_1$ is simply connected as an algebraic group, it follows that we can lift η to a map to $\tilde{\mathbb{D}}$, and then define the lift of π by the same equation.

For G_1 a topological cover of a real algebraic group $\bar{G}_1$, the argument above gives the same conclusion concerning π, η and σ where η is a continuous homomorphism of G_1 which factors through $\bar{G}_1$. □

PROOF OF THEOREM 6.5. The action ρ is described by a homomorphism $\pi : \Gamma \to (N_{\mathrm{Aut}(H)}(\Lambda) \ltimes H)/\Delta^{-1}(\Lambda)$. We observe that a finite index subgroup in the group $(N_{\mathrm{Aut}(H)}(\Lambda) \ltimes H)/\Delta^{-1}(\Lambda)$ maps into $(\mathrm{Aut}^A(H) \ltimes H)/\Delta^{-1}(\Lambda)$ which maps onto $(\mathrm{Aut}^A(H) \ltimes H)/\Delta^{-1}(H)$. After passing to a subgroup of finite index in Γ and composing π with this inclusion and surjection, we obtain a homomorphism $\bar{\pi} : \Gamma \to (\mathrm{Aut}^A(H) \ltimes H)/\Delta^{-1}(H)$. Recall that $(\mathrm{Aut}^A(H) \ltimes H)$ is an algebraic group, and note that $\Delta^{-1}(H)$ is an algebraic subgroup, so the quotient $(\mathrm{Aut}^A(H) \ltimes H)/\Delta^{-1}(H)$ is an algebraic group. Applying the superrigidity theorems to π, we see that $\bar{\pi} = (\bar{\pi}_E, \bar{\pi}_K)$ where $\bar{\pi}_E$ is the restriction of a continuous homomorphism of G and $\bar{\pi}_K$ has bounded image. By [**M3**, IX.5.8], we know that $\bar{\pi}(\Gamma)$ has semisimple Zariski closure $\mathbb{J}$. Here $\mathbb{J} = \mathbb{J}_1\mathbb{J}_2$ where $\mathbb{J}_1$ is isotropic over $\mathbb{R}$ and $\mathbb{J}_2$ is anisotropic over $\mathbb{R}$.

We let $\mathrm{Aut}^A(H) \ltimes H = \mathbb{L}_1(\mathbb{R}) \ltimes \mathbb{U}_1(\mathbb{R})$ where $\mathbb{L}_1$ is reductive algebraic group and $\mathbb{U}_1$ is the unipotent radical. Similarly, we let $\Delta^{-1}(H) = \mathbb{L}_2(\mathbb{R}) \ltimes \mathbb{U}_2(\mathbb{R})$ and $(\mathrm{Aut}^A(H) \ltimes H)/\Delta^{-1}(H) = \mathbb{L}_3(\mathbb{R}) \ltimes \mathbb{U}_3(\mathbb{R})$ where $\mathbb{L}_2$ and $\mathbb{L}_3$ are reductive and $\mathbb{U}_2$ and $\mathbb{U}_3$ are unipotent. Then $\mathbb{L}_3 = \mathbb{L}_1/\mathbb{L}_2$ and these are all reductive, we can find a subgroup $\tilde{\mathbb{L}}_3$ such that $\mathbb{L} = \mathbb{L}_2\tilde{\mathbb{L}}_3$ is an almost direct product and the map $\tilde{\mathbb{L}}_3 \to \mathbb{L}_3$ is an isogeny. By Lemma 6.6, there is a finite index subgroup $\Gamma' < \Gamma$ on which we can lift $\bar{\pi}$ to a representation into $\tilde{\pi}$ into $\mathbb{L}_3(\mathbb{R})$ and therefore into $\mathrm{Aut}^A(H)\ltimes H$. It is easy to verify that $\tilde{\pi}$ and π define the same affine action of Γ'.

That π can be chosen to be of the form (π_A, π_H) requires a supplementary argument. Let $\mathbb{H} = \mathbb{L}\ltimes\mathbb{U}$ be Levi decomposition. It follows from the proof of Proposition 6.3 that the Levi complement of $\mathrm{Aut}^A(H)$ a direct product of the adjoint group $\bar{\mathbb{L}}$ of $\mathbb{L}$ and a reductive subgroup $\mathbb{L}'$ of $\mathrm{Aut}(\mathbb{U})$ that commutes with $\bar{\mathbb{L}}$. In the description above, one can take $\mathbb{L}_2 = \Delta^{-1}(\mathbb{L})$ and $\mathbb{L}_1 = \mathbb{L}'\times(\bar{\mathbb{L}}\ltimes\mathbb{L})$. Therefore $\mathbb{L}_3$ is isomorphic to $\mathbb{L}'\times\bar{\mathbb{L}}$, and if one chooses $\tilde{\mathbb{L}}_3$ to be $\mathbb{L}'\times\mathbb{L} < \mathbb{L}'\times(\bar{\mathbb{L}}\ltimes\mathbb{L})$ one has the desired conclusion. $\square$

6.2. Applications. We now proceed to prove the applications listed in the introduction. For the remainder of this section G is as defined in the second paragraph of the introduction and Γ is a lattice in G. For any affine action of G or Γ on H/Λ, or any associated quasi-affine or generalized affine action, we assume that H satisfies the conclusions of Proposition 6.3 and so we can describe the action by Theorem 6.4 or 6.5. We call the homomorphism defining the action $\pi_0 = (\pi_A, \pi_H)$ and let $\mathbb{A}$ be the Zariski closure of $\pi_A(\Gamma')$ in $\mathrm{Aut}^A(\mathbb{H})$. If we are concerned with Γ actions, π_0 only defines the action on a subgroup of finite index Γ'. For the proof of Theorem 1.8, we replace Γ' by a subgroup of finite index to assure that $\mathbb{A}$ is connected.

Proof of Corollary 1.7. Let $D = G$ or Γ and ρ be a generalized standard affine action of D on a manifold M. Since the entropy of an element is determined by the entropy of it's kth power, it suffices to prove the corollary for a subgroup D' of finite index.

We first prove the corollary for ρ affine and $M = H/\Lambda$ and then describe the modifications for the general case. For an affine action we have $TM = H/\Lambda\times\mathfrak{h}$, and by Theorems 6.4 and 6.5 there is a subgroup of finite index $D' < D$ and a homomorphism $\pi_0 : D\to Aut(H)\ltimes H$ such that the derivative cocycle of the D' action is $\tilde{\pi}_0 = (\mathrm{Ad}_{\mathrm{Aut}(H)\ltimes H}\circ\pi_0)|_{\mathfrak{h}}$.

Let ρ' be a C^2 action C^1 close to ρ. By a result of Seydoux, ρ' preserves a measure that is in the same measure class as Lebesgue measure [**S**]. Since the derivative cocycle $\alpha_{\rho'}$ of ρ' is C^0 close to the constant cocycle given by $\tilde{\pi}_0$ (which is the derivative cocycle for ρ), it follows from Theorem 5.1 that $\alpha_{\rho'}$ is cohomologous to $\tilde{\pi}_0^E{\cdot}c$ where $\tilde{\pi}_0^E$ is the extendable part of $\tilde{\pi}_0$, and c is cocycle over the D action taking values in a compact group that commutes with $\tilde{\pi}_0^E(D')$. The corollary now follows from the fact that the entropies $h_\rho(d)$ and $h_{\rho'}(d)$ can both be computed in the same manner from the eigenvalues of $\tilde{\pi}_0^E(d)$. This follows from Pesin's formula relating entropy to Lyapunov exponents as observed by Furstenberg in [**Fu3**], see also [**Z2**, Chapter 9].

We now pass to the case of a generalized affine action ρ of D on $K\backslash H/\Lambda$. By definition ρ is the quotient of an affine action D on H/Λ, so on a subgroup of finite index, ρ is given by a homomorphism $\pi_0 : D\to \mathrm{Aut}(H)\ltimes H$. Since $K < H$ commutes with D, and since K is compact and the Zariski closure of $\pi_0(D)$ in $\mathrm{Aut}(H)\ltimes H$ is semisimple, there is a splitting $\mathfrak{h} = \mathfrak{k}\oplus\mathfrak{m}$ invariant under $\mathrm{Ad}_{\mathrm{Aut}(H)\ltimes H}$ restricted to both K and $\pi_0(D)$. The tangent bundle to $K\backslash H/\Lambda$ can be identified with $K\backslash H/\Lambda\times\mathfrak{m}$ and, on a subgroup $D' < D$ of finite index, the derivative cocycle is $(\mathrm{Ad}_{\mathrm{Aut}(H)\ltimes H}\circ\pi_0)|_{\mathfrak{m}}$. The remainder of the proof follows as before. $\square$

We remark that the proof of the corollary only uses part of Theorems 1.1 and 1.2, namely the conclusion that the derivative cocycle for the perturbed action is cohomologous to $\pi_0^E{\cdot}c$ where c takes values in a compact group.

We now prove Theorem 1.8 from the introduction. Actually, we prove a slightly stronger statement. Let H be a connected real algebraic group and $\Lambda < H$ a discrete cocompact subgroup. Let ρ be a standard affine action of $D = G$ or Γ on $H/\Lambda{\times}M$. By Theorems 6.4 and 6.5 above, there is a finite index subgroup $D' < D$ such that the action of D' on $H/\Lambda{\times}M$ is given by $\rho(d)([h], m) = (\pi_0(d)h, \iota(d, h)m)$ where $\pi_0 : D{\to}\mathrm{Aut}(H){\ltimes}H$ is a homomorphism and $\iota : D{\times}H/\Lambda{\to}\mathrm{Isom}(M)$ is a cocycle. It is clear from this description that the action of D' lifts to $H{\times}M$. For G actions, we let $Z = Z_H(\pi_0(G))$. For Γ actions the description of Z is more complicated. Let Γ' be the subgroup of finite index given by Theorem 6.5. Recall that $\pi_0 = (\pi_A, \pi_H)$ and let A be the set of real points of the Zariski closure of $\pi_A(\Gamma')$, and define $L = A{\ltimes}H$. We view π_0 as taking values in L. We let $Z = Z_L(\pi_0^E(\Gamma)){\cap}H$.

THEOREM 6.7. *Let ρ be a standard affine action of $D = G$ or Γ on $H/\Lambda{\times}M$ as above. Let D' and Z be as above. Given any action ρ' sufficiently C^1 close to ρ, there is a cocycle $z : D{\times}H/\Lambda{\times}M{\to}Z$ and a continuous map $f : H/\Lambda{\times}M{\to}H/\Lambda$, such that*

1. *f is C^0 close to the natural projection*
2. *for any $d{\in}D'$ and any $([h], m){\in}H/\Lambda{\times}M$ we have*

$$f(\rho'(d)([h], m)) = (\pi_A(d), \pi_H^E(d)z(d, ([h], m)))f([h], m).$$

To prove Theorem 6.7, we recall, and generalize slightly, the construction, from [**MQ**] of a cocycle from a perturbation of an affine action. We will prove Theorem 6.7 by applying Theorem 5.1 to this cocycle.

Once again, we let D denote our acting group. For the construction of the cocycle, D can be more general than G or Γ above, as long as the D action is as in the conclusions of Theorems 6.4 or 6.5. First we define the cocycle for actions by left translations. Let the D action ρ on H/Λ be defined via a homomorphism $\pi_0 : D{\to}H$. Let ρ' be a perturbation of ρ. If D is connected it is clear that the action lifts to $\tilde{H}$ and therefore to H. If D is discrete, this lifting still occurs, since the obstacle to lifting is a cohomology class in $H^2(D, \pi_1(H/\Lambda))$ which does not change under a small perturbation of the action. (A direct justification without reference to group cohomology can be found in [**MQ**] section 2.3.) Write the lifted actions of D on H by $\tilde{\rho}$ and $\tilde{\rho}'$ respectively. We can now define a cocycle $\alpha : D{\times}H{\to}H$ by

$$\tilde{\rho}'(g)x = \alpha(g, x)x$$

for any g in D and any x in H. It is easy to check that this is a cocycle and that it is right Λ invariant, and so defines a cocycle $\alpha : D{\times}H/\Lambda{\to}H$. See [**MQ**] section 2 for more discussion.

Similarly if D acts on H/Λ affinely we can obtain a cocycle as well. Note this only occurs if D is discrete. For a finite index subgroup $D' < D$ the action is described by a homomorphism π_0 into a group $A{\ltimes}H = L$ as described before the statement of Theorem 6.7. This action D' lifts to H as noted above. We define the cocycle only after replacing D with D'. As before the D action defined by ρ' also lifts to H. We let π_A be the projection of π_0 on A. Recall that $\pi_A(D)$ must normalize Λ and that $\pi_A(D)$ is Zariski dense in A. Here we consider H/Λ as the space $\pi_A(D){\ltimes}H/\pi_A(D){\ltimes}\Lambda$, and then define the cocycle by the same equation as before, noting that we get a cocycle $\alpha : D{\times}H/\Lambda{\to}\pi(D){\ltimes}H$. Observe that there is a map $\alpha' : D \times H/\Lambda \to H$ such that $\alpha(d,x) = (\pi_A(d), \alpha'(d,x))$. It is possible to define α' directly by the same equation as α, but it is not a cocycle. The map α' is called a twisted cocycle in [**F**], further discussion can be found there and in [**MQ**] section 2, example 2.2.

Whenever we have the cocycle α defined as taking values in $\pi_A(D) \ltimes H$, we extend this to a cocycle into $L = A \ltimes H$ where A is as above.

If D acts on $H/\Lambda \times M$ we can also define a cocycle as above. For simplicity we discuss the definition of this cocycle when the D action on H/Λ is given by a homomorphism $\pi_0 : D \to H$. Let $p : H/\Lambda \times M \to H/\Lambda$ be the projection. In this case we define $\alpha : D \times H/\Lambda \times X \to H$ via the formula

$$\alpha(g,x)p(x) = p(\tilde{\rho}'(g)x).$$

The verification that this is a cocycle and defined on $D \times H/\Lambda \times M$ follows exactly as in [**FW**]. The case of more general π_0 follows exactly as above and again results in a cocycle into $A \ltimes H$. Note that to define this cocycle, we do not need to know that the skew product action is isometric on M.

In all three cases, we can define a cocycle α_{π_0} corresponding to the action ρ it is clear from the definition that $\alpha_{\pi_0}(g,x) = \pi_0(g)$ in all cases.

Proof of Theorem 6.7. Given ρ' a perturbation of ρ, we construct a cocycle $\alpha : D \times H/\Lambda \to A \ltimes H$ as described above. It is clear from the construction that α is continuous and close to the constant cocycle defined by π_0. Applying the result of Seydoux, we see that ρ' preserves a measure μ that is in the same measure class as Lebesgue measure. Therefore μ has full support. Applying Theorem 5.1 to α, we have that there is a map $\phi_h : H/\Lambda \times M \to H$ and a cocycle $z : H/\Lambda \times M \to H$ such that

1. $\alpha(g,x) = \phi_h(gx)^{-1}(\pi_A(g), \pi_H^E(g)z(g,x))\phi_h(x)$
2. ϕ_h is continuous, C^0 small and depends continuously on $\rho' \in \mathrm{Diff}^1(H/\Lambda \times M)$
3. z is continuous and C^0 close to the constant cocycle defined by π_H^K

We define $f : H/\Lambda \times M \to Z\backslash H/\Lambda$ by $f(x) = p(\phi_h(x)x)$ where we write $p : H/\Lambda \times M \to H/\Lambda$ for the natural projection. It then follows from the definition of α and our conclusions about ϕ that:

$$\begin{aligned} f(\rho'(g)x) &= p(\phi_h(gx)\rho'(g)x) = p(\phi_h(gx)\alpha(g,x)x) \\ &= p(\phi_h(gx)\phi_h(gx)^{-1}(\pi_A(g), \pi_H^E(g)z(g,x))\phi_h(x)x) \\ &= (\pi_A(g), \pi_H^E(g)z(g,x))p(\phi_h(x)x) \\ &= (\pi_A(g), \pi_H^E(g)z(g,x))f(x). \end{aligned}$$

□

Theorem 1.8 is an immediate consequence of Theorem 6.7 and the definition of Z.

References

[AB] N. A'Campo and M.Burger, Réseaux arithmétiques et commensurator d'après G.A.Margulis, *Invent. Math.* 116 (1994) 1-25.

[An] D.V.Anosov, Geodesic flows on compact manifolds of negative curvature, *Trudy Mat. Inst. Steklov* 90 (1967) .

[BZ] I.N.Bernstein and A.V.Zelevinsky, Representations of the group $GL(n, F)$ where F is a non-archimedean local field, Russ. Math. Surveys, 31.3 (1976), 1-68.

[B1] A.Borel, *Introduction aux groupes arithmtiques.* (French) Publications de l'Institut de Mathmatique de l'Universit de Strasbourg, XV. Actualits Scientifiques et Industrielles, No. 1341 Hermann, Paris 1969 125 pp.

[B2] A.Borel, *Linear Algebraic Groups*, Springer-Verlag, New York, 1991.

[BH] M.Bridson, A.Haefliger, *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.

[C] K.Corlette, Archimedean superrigidity and hyperbolic geometry. *Ann. of Math.* 135 (1992), no. 1, 165–182.

[CZ] K.Corlette and R.J.Zimmer, Superrigidity for cocycles and hyperbolic geometry. *Internat. J. Math.* 5 (1994) 273–290.

[D] S.G.Dani, A simple proof of Borel's density theorem, *Math. Z.* 174 (1980) 81-94.

[F] D. Fisher, On the arithmetic structure of lattice actions on compact spaces, *Erg. Th. Dyn. Sys.* (2002) 22, 1141-1168.

[FW] D. Fisher and K.Whyte, Continuous quotients for lattice actions on compact spaces, *Geom. Ded.* 2001 (87) 181-189.

[FM1] D.Fisher and G.A.Margulis, Local rigidity of standard actions of higher rank groups and their lattices, preprint.

[FM2] D.Fisher and G.A.Margulis, Almost isometric actions, Property T, and local rigidity, preprint.

[Fu1] H.Furstenberg, A Poisson formula for semisimple Lie groups, *Annals of Math.* 77(1963), 335-383.

[Fu2] H.Furstenberg, Boundary theory and stochastic processes on homogeneous spaces in: Harmonic analysis on homogeneous spaces. *Symposia Pure and Appl. Math.* 26 (1973) 193-229.

[Fu3] H.Furstenberg, Rigidity and cocycles for ergodic actions of semisimple Lie groups (after G.A.Margulis and R.Zimmer), *Seminaire Bourbaki*, 1979/80, 272-292.

[GR] H.Garland and M.S.Ragunathan, Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups. *Ann. of Math.* (2) 92 1970 279–326.

[Gl] V.M.Glushkov, The structure of locally compact groups and Hilbert's Fifth problem, *Uspehi Mat. Naut* 12 (1957) 3-41.

[GS] Gromov, Mikhail; Schoen, Richard. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. Inst. Hautes tudes Sci. Publ. Math. No. 76 (1992), 165–246.

[HV] P. de la Harpe, A.Valette, *La propriete (T) de Kazhdan pour les groupes localement compacts*, Asterisque 175, Soc. Math. de France, Paris, 1989.

[K] I.Kaplansky, *Lie algebras and locally compact groups.* Reprint of the 1974 edition. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995.

[La] S. Lang, *Algebra, Third Revised Edition*, Springer-Verlag, New York, 2002.

[L] J. Lewis, The algebraic hull of a measurable cocycle, to appear *Geom. Ded.*

[Li] L. Lifschitz, Superrigidity thoerems in positive characteristic, *J. Algebra* 229 (2000) 375-404.

[LMR] A. Lubotzky, S. Mozes, M.S. Raghunathan, The word and Riemannian metrics on lattices of semisimple groups, *Publ IHES* 91 (2000) 5-53.

[M1] G.A. Margulis, Discrete groups of motions of manifolds of non-positive curvature, *Proc. Int. Cong. Math.* Vancouver 1974. *AMS Translations* 109 (1977) 33-45.

[M2] G.A. Margulis, Arithmeticiy of irreducible lattices in semisimple gropus of rank greater than 1, appendix to the Russian translation of M.Ragunathan, *Discrete subgroups of Lie groups*, Mir, Moscow, 1977 (in Russian). English translation in *Invent. Math.* 76 (1984) 93-120.

[M3] G.A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, New York, 1991.

[MQ] G. Margulis and N. Qian, Local rigidity of weakly hyperbolic actions of higher real rank semisimple Lie groups and their lattices, *Ergodic Theory and Dynamical Systems* 21 (2001) 121-164.

[M] G.D.Mostow, *Strong rigidity of locally symmetric spaces*, Annals of Math. Studies 78, 1973, Princeton University Press, Princeton, NJ.

[P] Ya. B. Pesin, On the existence of invariant fiberings for a diffeomorphism of a smooth manifold, *Math. USSR Sbornik.*, 20 (1973), 213-222.

[PlRa] V. Platonov and A. Rapinchuk, *Algebraic Groups and Number Theory*, Academic Press, New York, 1994.

[PR] G.Prasad and M.S.Ragunathan, Cartan subgroups and lattices in semisimple groups, *Ann. Math.*, 96 (1972), 296-317.

[QZ] N.Qian and R.J. Zimmer, Entropy rigidity for semisimple group actions, *Israel J. Math.*, 99 (1997) 55-67.

[Rg1] M.S. Raghunathan, Cohomology of arithmetic subgroups of algebraic groups. I, II. *Ann. of Math.* 86 (1967), 409-424; ibid. (2) 87 (1967) 279–304.

[Rg2] M.S. Raghunathan, On the first cohomology of discrete subgroups of semisimple Lie groups. *Amer. J. Math.* 87 (1965) 103–139.

[S] Seydoux, G., Rigidity of ergodic volume-preserving actoins of semisimple groups of higher rank on compact manifolds, *Trans. Am.* 345 (1994) 753-776.

[Sh] Y.Shalom, Invariant measures for algebraic actions,Zariski dense subgroups and Kazhdan's property (T), *Trans. Am.* 351 (1999) 3387-3412.

[St] A.N.Starkov, Vanishing of the first cohomologies for lattices in Lie groups. *J. Lie Theory* 12 (2002), no. 2, 449-460.

[Stu] G.Stuck Cocycles of ergodic group actions and vanishing of first cohomology for S-arithmetic groups. *Amer. J. Math.* 113 (1991), no. 1, 1–23.

[V] T.N. Venkataramana, On superrigidity and arithmeticity of lattices in semsimple groups over local fields of arbitrary characteristic, *Invent. Math.* 92 (1988) 255-306.

[Z1] R.J, Zimmer Strong rigidity for ergodic actions of semisimple Lie groups. *Ann. of Math.* (2) 112 (1980), no. 3, 511–529.

[Z2] R.J. Zimmer,*Ergodic Theory and semisimple Groups*, Birkhäuser, Boston, 1984.

[Z3] R.J. Zimmer, Actions of semisimple groups and discrete subgroups, *Proc. Internat. Cong. Math.*, Berkeley, 1986, 1247-1258.

[Z4] R.J. Zimmer, On the algebraic hull of an automorphism group of a principal bundle, *Comment. Math. Helv.*, 65 (1995) 375-387.

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