Moment maps in differential geometry

S.K. Donaldson

More than twenty years ago, Atiyah and Bott observed that the curvature can be viewed as a moment map for the action of the gauge group on the space of connections [2]. Since then, this notion of a moment map—applied to infinite-dimensional symmetry groups underlying differential-geometric problems—has proved to be very fruitful. It yields a unified point of view on many different questions, and brings with it a package of standard theory which can either be applied directly or at least, in the deeper aspects, suggests what one ought to try to prove. In this article we will first survey briefly some of the well-established applications of these ideas in the literature. Then, in Section 2, we go on to study a new variant of the theme, for diffeomorphism groups acting on sections of bundles. We prove a general abstract result and then begin the study of a particular example, which leads to a hyperkahler extension of the Weil-Petersson metric on the moduli space of Riemann surfaces.

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1. Survey of some examples

We recall the definition of a moment map. Suppose a Lie group G acts on a symplectic manifold (X, ω). Thus the derivative of the action is a map

$$r : \mathfrak{g} \to \mathrm{Vect}(X).$$

A moment map for the action is a map

$$\mu : X \to \mathfrak{g}^*$$

whose derivative

$$d\mu : TX \to \mathfrak{g}^*$$

is the transpose of r, where TX is identified with T^*X using the form ω. We will always require that the moment map be an equivariant map, intertwining the G-actions on X and $\mathfrak{g}^*$. One of the main applications of this idea is to the construction of symplectic quotients. If c is an element of $\mathfrak{g}^*$ which is fixed by the co-adjoint action we form the G-invariant subset $Z_c = \mu^{-1}(c) \subset X$. If G acts freely on Z_c this set is a submanifold, and the quotient Z_c/G is a manifold with a natural induced symplectic form—the symplectic quotient of X. In the case when X is Kahler the quotient has a natural Kahler structure. In addition, there is a circle of ideas relating this symplectic quotient to a quotient of a set of stable points in X by the action of the complexified group, but we will not say much about this side of things in the present article.

1.1. Gauge theory. The original Atiyah and Bott example is this. Let K be a compact Lie group and let Σ be a compact oriented surface. We consider the space $\mathcal{A}$ of connections on a principal K-bundle P over Σ. This is an affine space, with tangent space the sections of $T^*\Sigma \otimes \text{ad } P$ where ad P is the bundle of Lie algebras associated to the adjoint representation of K. Fix an invariant inner product on $\mathfrak{k}$. The tensor product of this inner product and the wedge product on 1-forms gives a skew-symmetric map

$$(T^*\Sigma \otimes \text{ad } P) \otimes (T^*\Sigma \otimes \text{ad } P) \to \Lambda^2 T^*\Sigma,$$

and we define a symplectic form on $\mathcal{A}$ by integrating over the compact surface Σ. To make the notation more transparent, let us suppose that K is the unitary group $U(n)$, so we can work with a Hermitian vector bundle E over Σ and regard ad P as a sub-bundle of End E (the sub-bundle of skew-adjoint endomorphisms). Then the symplectic form is given by the formula

$$\Omega(a, b) = \int_\Sigma \text{Tr}(a \wedge b).$$

The gauge group $\mathcal{G}$ of automorphisms of P acts on $\mathcal{A}$, preserving the symplectic form. We can represent an element of $\mathcal{G}$ as a section of End E and the action is given by

$$g(A) = A - (d_A g) g^{-1},$$

where d_A is the covariant derivative on End E induced by the connection. The Lie algebra of $\mathcal{G}$ can be regarded as the sections of ad P and the infinitesimal action is given by

$$r(u)(A) = -d_A u.$$

The curvature $F(A)$ of a connection A is a 2-form with values in the bundle ad P and the variation of the curvature is given by

$$F(A + a) = F(A) + d_A a + a \wedge a,$$

where d_A is the coupled exterior derivative extending the covariant derivative on ad P. Thus the derivative of the curvature, regarded as a map

$$F : \mathcal{A} \to \Omega^2(\text{ad } P),$$

is just

$$d_A : \Omega^1(\text{ad } P) \to \Omega^2(\text{ad } P).$$

The space $\Omega^2(\text{ad } P)$ of bundle-valued 2-forms is regarded as embedded in the dual of the Lie algebra of $\mathcal{G}$ via the pairing

$$(u, F) \mapsto \int_\Sigma \text{Tr}(uF).$$

The assertion that F is a moment for the action is the statement that, for all u and a,

$$\int_\Sigma \text{Tr } (d_A u \wedge a) = -\int_\Sigma \text{Tr } (u d_A a),$$

which follows from Stokes' Theorem since

$$\text{Tr}(d_A u \wedge a + u d_A a) = d(\text{Tr } (ua)).$$

In this case the symplectic quotient $\mathcal{N}$ is a moduli space of (projectively) flat connections over Σ and one of the fruits of Atiyah and Bott's observation is that one immediately sees that this moduli space has a natural symplectic structure.

1.2. Coupled equations. Among the various generalisations of the set-up of the previous subsection, one of the most useful is to the study of connections combined with additional data. A very general formulation of this idea had been given by Mundet i Riera [12] and Cielibak *et al* [5]. In the set-up above, we consider an additional auxiliary symplectic manifold (X, ω) with a K-action and a moment map $\mu : X \to \mathfrak{k}^*$. Now we form the associated bundle $\underline{X} \to \Sigma$ with fibre X. There is a symplectic form, which we still denote by ω, on the vertical subbundle $T_V\underline{X}$ of the tangent bundle of $\underline{X}$. Likewise the moment map defines a bundle map, which we still denote by μ, from $\underline{X}$ to ad P. We consider the space $\Gamma(\underline{X})$ of sections of $\underline{X}$. A tangent vector in $\Gamma(\underline{X})$ is given by a section of the pull-back of the vertical subbundle. Thus if we have two such tangent vectors ξ, η we can form a function $\omega(\xi, \eta)$ on Σ. Now suppose that we fix an area form ρ on Σ. We define a symplectic pairing by

$$\Omega(\xi, \eta) = \int_\Sigma \omega(\xi, \eta)\rho.$$

It is straightforward to see that this yields a closed 2-form on the infinite-dimensional space $\Gamma(\underline{X})$. The gauge group $\mathcal{G}$ acts as fibrewise automorphisms of $\underline{X}$ and hence acts on Γ, and this action preserves the symplectic form. The moment map for the action is given merely by

$$\phi \mapsto \mu(\phi)\rho,$$

for a section ϕ of $\underline{X}$. Now consider the space $\mathcal{A} \times \Gamma(\underline{X})$ with the product symplectic form and the diagonal $\mathcal{G}$-action. The moment map is given by the equation

$$\underline{\mu}(A, \phi) = F(A) + \mu(\phi)\rho.$$

So we can form a symplectic quotient parametrising solutions (A, ϕ) of the equation

$$F(A) + \mu(\phi)\rho = c\rho \tag{1}$$

for a suitable fixed c. This equation, regarded as a PDE for (A, ϕ), is underdetermined and the symplectic quotient is infinite-dimensional. The main interest comes from combining this moment map equation with another equation, a coupled Cauchy-Riemann equation. Here we suppose that X is actually a Kahler manifold, with K acting by isometries, and that Σ is a Riemann surface. Given a connection A on P we can form the covariant derivative $\nabla_A\phi$ of a section ϕ of $\underline{X}$. This is an $\mathbf{R}$-linear map from the tangent space of Σ to the vertical tangent bundle of X and under our assumptions each of these spaces have natural complex structures. Thus we can write the covariant derivative as a sum of complex-linear and anti-linear parts

$$\nabla_A\phi = \partial_A\phi + \overline{\partial}_A\phi.$$

The coupled Cauchy-Riemann equation is the equation

$$\overline{\partial}_A\phi = 0. \tag{2}$$

The combined equations (1) and (2) form an elliptic system (when account is taken of the gauge group action) with a finite-dimensional moduli space of solutions.

The Cauchy-Riemann equation (2) can be thought of as cutting out an infinite-dimensional symplectic (in fact complex) submanifold of $\mathcal{A} \times \Gamma(\underline{X})$ and the general theory yields a symplectic (in fact Kahler) structure on the moduli space.

The examples of this general set-up which have been studied most extensively arise when X is a complex vector space, i.e. from a unitary representation of K. See [3],[4] for surveys of these developments. *Hitchin's equations* [10] arise when one takes the complexified adjoint representation. Reverting again to the case when K is $U(n)$, so we can formulate things in terms of a Hermitian vector bundle E, the auxiliary field ϕ—the "Higgs field"—is now a section of End$E$. Then the finite dimensional moment map is given by

$$\mu(\phi) = [\phi, \phi^*].$$

In fact it is more useful to consider a minor variant of the general set-up in which we take ϕ to be a section of the tensor product

$$T^*\Sigma \otimes_{\mathbf{C}} \text{End } E.$$

The preceding discussion goes through without essential change but the advantage is that the symplectic form on the space of Higgs fields can now be defined without fixing an area form on Σ. Likewise $[\phi, \phi^*]$ is a bundle valued 2-form and (1) is modified to

$$F(A) + [\phi, \phi^*] = c\rho \tag{3}$$

The appropriate constant c is determined by the first Chern class of the bundle E. If the bundle is trivial the constant is zero and the theory is entirely conformally invariant. In sum, Hitchin associates to a compact Riemann surface a moduli space $\mathcal{N}^c$ parametrising solutions to (2),(3). Clearly the Kahler moduli space $\mathcal{N}$ forms a submanifold of $\mathcal{N}^c$.

Hitchin found many remarkable properties of the solutions of these equations and of their moduli space $\mathcal{N}^c$. Many of these stem from the fact that the moduli space can be regarded as a "hyperkahler quotient". To review these ideas, recall that a hyperkahler structure on a manifold X is a Riemannian metric together with three complex structures I_1, I_2, I_3 satisfying the algebraic relations of the quaternions and such that X is Kahler with respect to each structure. Thus there are three symplectic forms $\omega_1, \omega_2, \omega_3$. If a group G acts on X, preserving all this structure, then in favourable situations there will be three moment maps $\mu_1, \mu_2, \mu_3 : X \to \mathfrak{g}^*$. If c_i are fixed by the co-adjoint action we can form the *hyperkahler quotient*

$$\frac{\mu_1^{-1}(c_1) \cap \mu_2^{-1}(c_2) \cap \mu_3^{-1}(c_3)}{G}.$$

It is a general fact that (provided the group acts freely on the indicated set) this quotient space has a natural hyperkahler structure.

Going back to Hitchin's equations: we can regard the space $\mathcal{A} \times \Omega^{1,0}(\text{End } E)$ as the cotangent bundle of the complex manifold $\mathcal{A}$. As such it has a canonical holomorphic symplectic structure, and the real and imaginary parts of this yield two real symplectic forms Ω_2, Ω_3. The first symplectic form Ω_1 is the natural Kahler form. These three forms are associated to an obvious flat hyperkahler structure on $\mathcal{A} \times \Omega^{1,0}(\text{End } E)$. The moment map for the gauge group action with this first symplectic structure is $\underline{\mu}_1 = \underline{\mu}$, as we have seen. The special feature now is that the other two moment maps can be identified with the real and imaginary parts of

$\overline{\partial}_A\phi$, that is

$$\overline{\partial}_A\phi = \underline{\mu}_2 + i\underline{\mu}_3.$$

So $\mathcal{N}^c$ has a natural hyperkahler structure, as the hyperkahler quotient of $\mathcal{A} \times \Omega^{1,0}(\Sigma)$.

The theory alluded to above, relating symplectic and complex quotients, gives different descriptions $\mathcal{N}^c$ and the different complex structures are visible in the different descriptions. This manifold can be described as

1. a moduli space of "stable pairs" which is a completion of the cotangent bundle $T^*\mathcal{N}$. Here the visible complex structure is I_1 and $\mathcal{N} \subset \mathcal{N}^c$ is a complex submanifold for this structure.
2. a moduli space of irreducible representations of $\pi_1(\Sigma)$ in $GL(n, \mathbf{C})$. Here the visible complex structure is I_2 and $\mathcal{N} \subset \mathcal{N}^c$ is a totally real submanifold (the unitary representations) for this structure.

We will not atempt to review any more of this rich theory except to recall that, given a solution (A, ϕ) of Hitchin's equations one constructs a flat $GL(n, \mathbf{C})$ connection

$$A + \Phi + \Phi^*.$$

(Here we are considering the case when the bundle is topologically trivial and the constant c is zero.) Conversely, given such a flat connection the Higgs field ϕ appears as the derivative of a section of the associated flat bundle with fibre the hyperbolic 3-space $H^3 = PGL(2, \mathbf{C})/PU(2)$. From this point of view, the first of Hitchin's equations is the requirement that the section be a *harmonic section.*

The extension of the Kahler manifold $\mathcal{N}$ to the hyperkahler "thickening" $\mathcal{N}^c$ is a general phenomenon. Indeed if N is any real-analytic Kahler manifold then it has been shown by Feix [8] and Kaledin [15] that there is a hyperkahler manifold N^c with a circle action, containing N as the fixed submanifold of the action, such that the first Kahler structure on N^c restricts to the given one on N. A neighbourhood of N in N^c can be viewed either as a neighbourhood of the zero section in the cotangent bundle T^*N or as a complexification of N. Further, any two such extensions are isometric, on small neighbourhoods of N. There is one point which we shall want to refer to in this general set-up. The circle action preserves the first symplectic form and so has a Hamiltonian $H : N^c \to \mathbf{R}$, vanishing along the fixed set $N \subset N^c$. It is easy to show that this function H is a *Kahler potential* for the metric with respect to the second complex structure

$$i\overline{\partial}^{I_2}\partial^{I_2}H = \omega_2. \tag{4}$$

1.3. Maps and diffeomorphisms. In [7], the author developed some analogues of this moment map theory where the relevant group is a diffeomorphism group, rather than a gauge group. We do not wish to repeat much of that discussion here but we will recall one of the main points since similar ideas will appear in Section 2 below.

Consider again a symplectic manifold (X, ω) and a compact manifold M. We work with the infinite dimensional space Maps (M, X) of smooth maps from M to X. If f is such a map, a tangent vector to Maps (M, X) at f is a section of the vector bundle $f^*(TX)$ over M. The symplectic form ω defines a symplectic structure on the fibres of this bundle, also denoted by ω. Thus if ξ and η are two tangent vectors to Maps (M, X) at f we have a function $\omega(\xi, \eta)$ on M. Now suppose that M has a fixed volume form ρ. Then we can define a skew pairing on these tangent vectors by

$$\Omega(\xi, \eta) = \int_M \omega(\xi, \eta)\rho. \tag{5}$$

This is essentially the same as the construction at the beginning of (1.2), in the case when the bundle P is trivial, and it is easy to see that Ω is, at least formally, a symplectic structure on Maps (M, X).

The group $\mathcal{G}$ which we will consider in this case is the group of "exact" volume-preserving diffeomorphisms. Going to the level of Lie algebras, a vector field v on M preserves the volume form ρ if

$$L_v(\rho) = d(i_v(\rho)) = 0.$$

A volume-preserving vector field is called exact if the closed form $i_v(\rho)$ can be written as the exterior derivative of an $(n-2)$-form. Of course, if $H_{n-1}(M; \mathbf{R})$ vanishes this condition is vacuous. The group $\mathcal{G}$ can be defined to be the set of volume-preserving diffeomorphisms which can be generated by integrating time-dependent families of such exact vector fields. It is a normal subgroup of the component of the identity $\mathcal{G}_0^+$ in the full group $\mathcal{G}^+$ of volume-preserving diffeomorphisms and the quotient $\mathcal{G}_0^+/\mathcal{G}$ can be identified with the torus

$$A_M = H^{n-1}(M; \mathbf{R})/H^{n-1}(M; \mathbf{Z}).$$

Now the group $\mathcal{G}$ obviously acts on Maps (M, X) by composition, preserving the symplectic form Ω, and so we can seek a moment map for the action. The moment map should take values in the dual of the Lie algebra of $\mathcal{G}$. This Lie algebra can be identified with the quotient of the $(n-2)$-forms on M by the closed $(n-2)$-forms. Thus, using the pairing by wedge product and integration over M, the dual is the space of closed 2-forms on M. (More precisely, we should take closed currents.) So a moment map should assign to a map $f : M \to X$ a closed 2-form over M. As shown in [7], the moment map is simply

$$\underline{\mu}(f) = f^*(\omega). \tag{6}$$

To give one illustration of how these ideas can be applied; consider the case when M is the 2-sphere and restrict to the $\mathcal{G}$-invariant open subset of Maps (X, M) consisting of embeddings with symplectic image, in a given homotopy class. Define the constant c by

$$\int_M f^*(\omega) = c \int_M \rho.$$

for any map f in this homotopy class. Then the symplectic quotient $\underline{\mu}^{-1}(c\rho)/\mathcal{G}$ can be identified with the set of embedded symplectic spheres in the given homotopy class and the general theory tells that this space has a natural symplectic structure. Thus the theory gives a tidy way of passing between parametrised and unparametrised symplectic submanifolds.

We will not say any more about this development here, except to point out that ideas in a similar spirit have been developed in a very exciting way by Thomas recently [15]; introducing the notion of "stable" Lagrangian submanifolds of Calabi-Yau manifolds.

2. Sections of bundles and diffeomorphisms

We now begin the main part of this article, developing a new variation of these ideas. We start with a general, abstract, discussion and then go on to consider two examples in more detail.

2.1. Abstract theory. Let (X, ω) be a symplectic manifold with an $SL(n, \mathbf{R})$-action and an equivariant moment map

$$\mu : X \to \mathfrak{sl}(n)^*.$$

Let M^n be a compact manifold with a fixed volume form $\rho \in \Omega^n(M)$. Thus there is a principal bundle of frames over M with structure group $SL(n, \mathbf{R})$ and we can form the associated bundle $\underline{X} \to M$, with fibre X. As in Section 1.2, there is a natural symplectic form, which we again denote by ω, on the vertical tangent bundle $T_V\underline{X}$ along the fibres over $\underline{X}$. We consider the infinite-dimensional space $\mathcal{S}$ of sections of $\underline{X} \to M$. Again, as in Section 1.2, there is a symplectic form Ω on $\mathcal{S}$ given at a point $\phi \in \mathcal{S}$ by

$$\Omega(\xi, \eta) = \int_M \omega(\xi, \eta)\rho, \tag{7}$$

where ξ and η are sections of $\phi^*(T_V\underline{X})$.

Now let $\mathcal{G}$ be the group of exact volume-preserving diffeomorphisms of M. This group acts naturally on $\mathcal{S}$, preserving the symplectic form, so we seek a moment map $\underline{\mu}$ for the action. As in Section 1.3, this moment map should assign to a section ϕ a closed 2-form over M. The author does not know of a really satisfactory definition of this moment map. The definition we give goes via the choice of an auxiliary structure; a torsion-free $SL(n, \mathbf{R})$-connection ∇ on the tangent bundle of M. Fixing such a connection, we associate three 2-forms over M to a section ϕ of $\underline{X}$, as follows.

1. Using the connection we can differentiate the section to obtain
$$\nabla\phi \in \Gamma(T^*M \otimes \phi^*T_V\underline{X}).$$
Then we define
$$\omega(\nabla\phi, \nabla\phi) \in \Omega^2(M),$$
by taking the tensor product of the wedge-product on the T^*M component and the form ω on the $T_V\underline{X}$ component.
2. The equivariant moment map μ induces a map from sections of $\underline{X}$ to sections of the vector bundle $TM \otimes T^*M$ with trace zero. (Using the standard identifiction of $\mathfrak{sl}(n, \mathbf{R})^*$.) Thus to a section ϕ we associate $\mu_\phi \in \Gamma(TM \otimes T^*M)$, written in index notation as μ^i_j. The curvature of the connection ∇ is a section $R^i_{j\ kl}$ of $TM \otimes T^*M \otimes \Lambda^2T^*M$. There is a natural pairing $R.\mu_\phi \in \Omega^2(M)$, written in index notation as $R^i_{j\ kl}\mu^j_i$.
3. The covariant derivative of μ_ϕ is a section of $T^*M \otimes TM \otimes T^*M$:
$$\nabla\mu_\phi = \mu^i_{j;k}.$$
We produce a 1-form $c(\nabla\mu_\phi)$ by contracting two indices
$$c(\nabla\mu_\phi) = \mu^i_{j;i}.$$
Then we have a 2-form $d\left(c(\nabla\mu_\phi)\right) \in \Omega^2(M)$.

With these three ingredients to hand, we define

$$\underline{\mu}(\phi) = \omega(\nabla\phi, \nabla\phi) + R.\mu_\phi + d\left(c(\nabla\mu_\phi)\right). \tag{8}$$

Thus $\underline{\mu}$ is a map from $\mathcal{S}$ to $\Omega^2(M)$. The main result of this section is as follows.

Theorem 9

1. *$\underline{\mu}$ is independent of the choice of the connection ∇, hence it is a $\mathcal{G}$-equivariant map from $\mathcal{S}$ to $\Omega^2(M)$.*
2. *For each ϕ, $\underline{\mu}(\phi)$ is a closed 2-form over M.*
3. *The map $\underline{\mu}$ is an equivariant moment map for the $\mathcal{G}$-action on $(\mathcal{S}, \Omega)$.*

Of course, all of these assertions are, at bottom, straightforward calculations. We begin with item (1). Suppose we make an infinitesimal change γ in the connection ∇. Thus, in index notation, we have a tensor γ^i_{jk}, symmetric in the indices j, k since we are considering torsion-free connections. The change in the curvature, to first order, is given by $\gamma^i_{jk;l} - \gamma^i_{jl;k}$. So the change in $R.\mu_\phi$ is

$$(\gamma^i_{jk;l} - \gamma^i_{jl;k})\mu^j_i. \tag{10}$$

The change in the covariant derivative $\nabla\mu_\phi$ is

$$\mu^l_j\gamma^i_{lk} - \mu^i_l\gamma^l_{jk}.$$

When we make the contraction one term vanishes, since $\gamma^i_{li} = 0$. Thus the change in $c(\nabla\mu_\phi)$ is the 1-form $-\mu^i_j\gamma^j_{ik}$, and the change in its exterior derivative is

$$\mu^i_{j;k}\gamma^j_{il} + \mu^i_j\gamma^j_{il;k} - \mu^i_{j;l}\gamma^j_{ik} - \mu^i_j\gamma^j_{ik;l}. \tag{11}$$

Let $r : \mathfrak{sl}(n;\mathbf{R}) \to \mathrm{Vect}(X)$ be the derivative of the action of $SL(n;\mathbf{R})$ on X. Any section δ of $TM \otimes T^*M$, with trace zero, defines a vertical vector field $r(\delta)$ over $\underline{X}$. That is, $r(\sigma)$ is a section of $T_V\underline{X}$ over $\underline{X}$. Taking the tensor product with T^*M we can define a section $r(\gamma)$ of $T_v\underline{X} \otimes \pi^*T^*M$. Then the infinitesimal change in $\nabla\phi$ is $\phi^*(r(\gamma))$. This means that, to first order, the variation in the term $\omega(\nabla\phi, \nabla\phi)$ is

$$\omega\left(r(\gamma), \nabla\phi\right) + \omega\left(\nabla\phi, r(\gamma)\right).$$

Now the defining property of the moment map μ asserts that

$$\omega(r(\delta), \eta) = \delta.D\mu(\eta),$$

where δ is any section of $TM \otimes T^*M$, η is any section of $T_V\underline{X}$ and $D\mu$ denotes the fibrewise derivative of μ—a bundle map from $T_V\underline{X}$ to the lift of $T^*M \otimes TM$. The covariant derivative $\nabla\mu$ is given by applying the bundle map $D\mu$ to the covariant derivative of ϕ. Thus the variation in $\omega(\nabla\phi, \nabla\phi)$ can be written as

$$\gamma^i_{jk}\mu^j_{l;i} - \gamma^i_{jl}\mu^j_{k;i}. \tag{12}$$

Putting together the three contributions (10),(11),(12) we see that the overall change in $\underline{\mu}$ vanishes. This completes the proof of item (1) of Theorem 9.

To prove item (2), that the 2-form $\underline{\mu}(\phi)$ is closed, we observe that this is a local property so by item (1) we can reduce to the case the connection ∇ to be flat, and we regard the section as a map from the base into X. Then the form $\omega(\nabla\phi, \nabla\phi)$ is nothing other than the pull back of the form ω on X to the base and hence is closed. The curvature term vanishes and the term $d\left(c(\nabla\mu_\phi)\right)$ is evidently closed.

Finally we come to the crucial property (3); the moment map condition. To derive this we digress to give another interpretation of the formula (8) which will also be useful for other purposes. Let us assume that there is an $SL(n, \mathbf{R})$-equivariant Hermitian line bundle $L \to X$, with connection, and that the moment map μ defines the lift of the action to L in the standard way. This means that there is a line bundle $\underline{L} \to \underline{X}$, defined as an associated bundle to the frame bundle. The line bundle $\underline{L}$ comes endowed with a connection in the vertical directions in $\underline{X}$; the choice of a conection ∇ on TM gives a natural extension of this to a conection $\nabla_{\underline{L}}$ on $\underline{L} \to \underline{X}$. The curvature of this connection is a closed 2-form $\underline{\omega}$ on the total space $\underline{X}$, restricting to ω in the vertical directions. (This form $\underline{\omega}$ exists without any assumption about the line bundle, but it seems easiest to explain the construction using this device. Another framework is provided by the theory of equivariant cohomology.)

Lemma 13 *For a section ϕ of $\underline{X}$ the pull-back $\phi^*(\underline{\omega})$ is*

$$\omega(\nabla\phi, \nabla\phi) + R.\mu_\phi.$$

The proof is a matter of unravelling the definitions which we mostly leave to the reader. At a given point of $\underline{X}$ we split the tangent bundle into horizontal and vertical parts

$$T\underline{X} \cong TM \oplus TX.$$

The 2-form $\underline{\omega}$ thus has *a priori* three components; in $\Lambda^2 T^*X, \Lambda^2 T^*M$ and $T^*X \otimes T^*M$. The first component is given by the form ω and the second by $R.\mu$. The fact that the connection ∇ is torsion-free implies that the third, mixed, component is identically zero. These two terms in $\underline{\omega}$ go over to the two terms in the statement of the lemma under pull-back by ϕ.

With this discussion in place we prove item (3) of Theorem 9. Let λ be a fixed $(n-2)$-form over M and let $\xi \in \Gamma(\phi^* T_V \underline{X})$ be an infinitesimal variation of a section ϕ. It is convenient to think of ξ as being the pull-back of a section of the vertical bundle over the whole of $\underline{X}$. We want to compute the derivative

$$G_1 = \delta_\xi \int_M \underline{\mu}(\phi) \wedge \lambda,$$

with respect to ϕ in the direction ξ. By Lemma 13, this is

$$G_1 = \delta_\xi \int_M \phi^*(\underline{\omega}) \wedge \lambda + \delta_\xi c(\nabla\mu_\phi) \wedge d\lambda,$$

where we have used integration by parts on the second term. For the first term we have

$$\delta_\xi \int_M \phi^*(\underline{\omega}) \wedge \lambda = \int_M \phi^*(i_\xi d + d i_\xi)(\underline{\omega}) \wedge \lambda,$$

which is

$$\int_M d\phi^*(i_\xi(\underline{\omega})) \wedge \lambda,$$

since $\underline{\omega}$ is closed. Integrating by parts on this term we have in sum

$$G_1 = \int_M (\phi^*(i_\xi(\underline{\omega})) + \delta_\xi c(\nabla\mu_\phi)) \wedge d\lambda.$$

Now write $d\lambda = i_v(\rho)$, where v is a volume-preserving vector field on M. Then we have a pointwise identity

$$\phi^*(i_\xi(\underline{\omega})) \wedge i_v(\rho) = \phi^*(\omega(\nabla_v\phi, \xi))\rho,$$

so

$$G_1 = \int_M \phi^*(\omega(\nabla\phi, \xi))\rho + \delta_\xi c(\nabla\mu_\phi) \wedge i_v(\rho).$$

Now consider the integral

$$G_2 = \int_M \omega(\delta_v\phi, \xi)\rho.$$

Here $\delta_v \phi$ denotes the derivative of the action of the volume-preserving vector fields on $\mathcal{S}$: the Lie derivative of ϕ along v. This is given by the formula

$$\delta_v\phi = \nabla_v\phi + r(\nabla v)(\phi),$$

where $r(\nabla v)$ is the vertical vector field defined by $\nabla v \in \Gamma(TM \otimes T^*M)$, as above. By the definition of the moment map μ we have

$$\omega(\xi, r(\nabla v)(\phi)) = \delta_\xi((\nabla v).\mu_\phi).$$

Thus

$$G_1 - G_2 = \delta_\xi \int_M (c(\nabla\mu_\phi) \wedge i_v + (\nabla v).\mu_\phi)\, \rho. \tag{14}$$

In index notation, the integrand on the right hand side of (14) is

$$\mu^j_{i;j} + v^i_{;j}\mu^j_i$$

which is the divergence $(v^i\mu^j_i)_{;j}$. Thus the integral over M vanishes and so $G_1 = G_2$, which is precisely the desired moment map identity.

2.2. The Weil-Petersson metric. The fundamental example of the set-up considered in the previous subsection arises when the base manifold M is an oriented surface Σ of genus genus $(\Sigma) \geq 2$ and X is the homogeneous space $H^2 = SL(2, \mathbf{R})/SO(2)$—the hyperbolic plane. Points of X can be viewed as complex structures on $\mathbf{R}^2$ and a section of the bundle $\underline{X} \to \Sigma$ is a complex structure on the surface. Thus we denote a section by J. The symplectic form on H^2 is uniquely determined by the $SL(2, \mathbf{R})$-invariance up to an overall scale. We fix this scale by decreeing that the moment map $\mu : H^2 \to \mathfrak{sl}(2, \mathbf{R})$ is just one half the natural inclusion, thinking of points of H^2 as trace-free endomorphisms of $\mathbf{R}^2$. A little calculation shows if we take the model of H^2 as the upper half plane in $\mathbf{C}$ the symplectic form we are using is

$$\frac{dxdy}{2y^2}.$$

To identify the moment map $\underline{\mu}$ in this case we can, given a complex structure J, choose the connection ∇ to be the Levi-Civita connection of the metric defined by J and ρ. Thus the covariant derivatives of J and μ_J vanish and the only term remaining in the formula (8) is that involving the curvature, which in this case is just $R = KJ \otimes \rho$, where K is the Gauss curvature. Since $|J|^2 = 2$ the moment map is $\underline{\mu} = K\rho$. Thus we recover the fact that the Gauss curvature furnishes a moment map for the action of the group of exact area preserving diffeomorphisms of a surface on the complex structures. As far as the author knows, this was first shown by Quillen, in about 1983, in answer to a question of Atiyah. (There is a straightforward variant of the whole theory of (2.1) to the case where X is a symplectic manifold with an action of the symplectic group $Sp(2n, \mathbf{R})$, the base manifold M is a symplectic manifold and we consider the action of the group of exact symplectomorphisms of M on sections of the resulting bundle $\underline{X} \to M$. The fundamental example here is when $X = Sp(2n, \mathbf{R})/U(n)$ so sections are compatible almost-complex structures on M. Then the general theory produces the moment-map calculation of [6] which—at least in the case of integrable structures—had been observed previously by Fujiki in [9].)

Just as for the moduli spaces of flat connections considered in Section 1, one simple application of this moment map calculation is the definition of a canonical Kahler metric on the symplectic quotient $\mathcal{M} = \underline{\mu}^{-1}(c\rho)/\mathcal{G}$. Here the appropriate constant c is fixed by Gauss-Bonnet to be

$$c = \frac{2\pi(2 - 2\text{genus }(\Sigma))}{\text{Area }(\Sigma, \rho)}. \tag{15}$$

A subtlety arises here because $\mathcal{M}$ is not quite the same as the usual moduli space $\mathcal{M}_0$ of complex structures which is the quotient of $\underline{\mu}^{-1}(c\rho)$ by the group $\mathcal{G}^+$ of all area-preserving diffeomorphisms. There are two aspects to this. First, the quotient of the identity component $\mathcal{G}_0^+$ of $\mathcal{G}^+$ by $\mathcal{G}$ is the $2g$-torus $A_\Sigma = H^1(\Sigma, \mathbf{R})/H^1(\Sigma, \mathbf{Z})$. This means that the torus A_Σ acts on $\mathcal{M}$ and the quotient is the Teichmuller space $\mathcal{T}$. Second, the quotient of $\mathcal{G}^+$ by its identity component $\mathcal{G}_0^+$ is the mapping class group Γ. This discrete group acts on $\mathcal{T}$ and $\mathcal{M}_0$ with quotient $\mathcal{M}_0$. It is the first aspect which is of a differential geometric nature. The difficulty is that there is no way to extend the moment $\underline{\mu}$ to an equivariant moment map for the full action of $\mathcal{G}_0^+$. Instead we can proceed as follows. Suppose in general that a torus T acts freely on a symplectic manifold (Y, ω) and that the T-orbits are *symplectic submanifolds* of Y. In this special case the naive quotient Y/T has a natural induced symplectic structure. To see this, we take the field of subspaces $H \subset TY$ defined as the anhilliator under ω of the tangent spaces to the T-orbits. Since the orbits are symplectic the subspaces H furnish complements to the orbit tangent spaces and ω is nondegenerate on H. Then identifying H with the pull-back of the tangent bundle of Y/T we can push the form down to define a nondegenerate 2-form on the quotient which one checks is closed. We can apply this to our situation, with the action of A_Σ on $\mathcal{M}$. The A_Σ-orbits are actually complex submanifolds of $\mathcal{M}$. In fact we can identify $\mathcal{M}$ with the moduli space of pairs consisting of a marked Riemann surface (Σ, J) and a choice of holomorphic line bundle of fixed degree over Σ and this moduli space has a natural complex structure. The A_Σ-orbits just arise from varying the line bundle and are obviously complex submanifolds. Using this device, we get a symplectic (Kahler) form on $\mathcal{T} = \mathcal{M}/A_\Sigma$. But then the whole construction is obviously invariant under the mapping class group so we finally obtain a Kahler metric on $\mathcal{M}$ which is of course nothing but the standard Weil-Petersson metric.

3. The hyperkahler extension of the Weil-Petersson metric

We will now consider a new example, in which the aim is to produce an explicit construction of the Feix-Kaledin hyperkahler thickening of the Weil-Petersson metric. As we outline in 3.1 below, we expct that this will be a hyperkahler metric on the "quasi-Fuchsian moduli space". We should mention here that I. Platis has constructed a complex symplectic structure on this space [13] and has investigated the hyperkahler geometry [14]. It will be interesting to compare the formulae of Platis with those derived from our moment map point of view. Our construction is obviously closely modelled on Hitchin's in the gauge theory case. Variants of the same idea, which we do not explore here, give a uniform framework for discussing other moduli spaces of "pairs" consisting of a Riemann surface with an additional tensor field.

3.1. The hyperkahler extnsion of the hyperbolic plane. We consider again the case when the base manifold is a compact surface Σ with a fixed area form ρ. As we have seen, the $SL(2,\mathbf{R})$-space H^2—the upper half plane— leads essentially to the standard Weil-Petersson metric on the moduli space. Now let X be the unit disc bundle in the cotangent bundle T^*H: this has a natural $SL(2;\mathbf{R})$ action induced by that on H^2 which commutes with action of S^1 given by rotation of the fibres.

Lemma 16.

There is an $SL(2;\mathbf{R}) \times S^1$-invariant hyperkahler metric g_X on X which on a fibre of $X \to H$ is given by

$$\frac{1}{4}\frac{d\sigma d\overline{\sigma}}{\sqrt{1-|\sigma|^2}}.$$

Here, in the formula for the metric on the fibre, we understand that the fibre is identified with standard disc in $\mathbf{C}$. The hyperkähler structure comprises complex structures I_1, I_2, I_3 on X and $SL(2;\mathbf{R})$ preserves each of I_1, I_2, I_3 whereas the circle action preserves I_1 but rotates I_2, I_3. This metric is nothing other than the Feix-Kaledin hyperkahler extension of the constant curvature metric on H^2 and the point of Lemma 16 is to find this extension explicitly. The metric g_X is the analogue of the well-known Calabi-Eguchi-Hanson metric on the cotangent bundle of the round 2-sphere. To find the metric from first principles one can proceed as follows. We consider $\mathbf{C}^2$ with the indefinite Hermitian metric $|z|^2 - |w|^2$ and make the standard identification of H^2 with the unit disc, the complex projectivization of the positive cone for this Hermitian form. In this model the symmetry group $SL(2;\mathbf{R})$ appears as the locally isomorphic group $SU(1,1)$. Then the set

$$\tilde{X} = \{(z,w) \in \mathbf{C}^2 : 0 < |z|^2 - |w|^2 < 1\}$$

is a bundle over H^2 with fibre $\mathbf{C}^*$ and it is in a natural way a double covering of X minus the zero section. Thus we can calculate in the more convenient model $\tilde{X}$. Here we seek a $U(1,1)$-invariant Calabi-Yau Kahler metric. Thus we take the metric to be of the form $i\overline{\partial}\partial F$ where $F = F(r)$ is a function of $r = |z|^2 - |w|^2$. A short calculation shows that

$$\overline{\partial}\partial F = (F' + |z|^2)d\overline{z}dz + (|w|^2F'' - F')d\overline{w}dw - F''(z\overline{w}d\overline{z}dw + \overline{z}wd\overline{w}dz).$$

The condition we need to satisfy is the Monge-Ampere equation

$$(\overline{\partial}\partial F)^2 = dzd\overline{z}dwd\overline{w}.$$

This reduces to the ODE

$$rF'F'' + (F')^2 = -1.$$

Setting $F' = G$, we get a first order equation for G which we can integrate to give

$$G(r) = \sqrt{\frac{b^2}{r^2} - 1},$$

for a constant of of integration b. The desired soluton is obtained when $b = 1$ (other values just give trivial rescaling). One can check that this induces the metric we fixed before on the zero-section of X. On the fibre of $\tilde{X} \to H$ given by $w = 0$ the metric is just

$$\frac{|z|^2}{\sqrt{1-|z|^4}} d\bar{z}dz.$$

To get the metric on the fibre of X we set $\sigma = z^2$, which yields

$$\frac{1}{4}\frac{d\sigma d\bar{\sigma}}{\sqrt{1-|\sigma|^2}}$$

as asserted in the Lemma.

We now fit into the general framework of Section 2, forming the bundle $\underline{X}$ over our surface with fibre X and the space of sections of $\underline{X}$, which we will now denote by $\mathcal{S}^c$. We use the symbol $\mathcal{S}$ to denote the sections of the bundle with fibres H^2 considered in the previous section, so we can obviously regard $\mathcal{S}$ as a subset of $\mathcal{S}^c$. Explicitly, a point ϕ in $\mathcal{S}^c$ is given by a pair (J, σ) where J is a complex structure on Σ and σ is a smooth quadratic differential with respect to this structure, with $|\sigma| < 1$ everywhere. (Here of course the norm $|\sigma|$ is computed using the metric defined by J and the area form ρ.) There is a hyperkahler structure on $\mathcal{S}^c$ induced from that on X, preserved by the group $\mathcal{G}$. Thus we are in the familiar general setting, sketched in (1.2), where we can take a hyperkahler quotient. The three symplectic forms $\omega_1, \omega_2, \omega_3$ on X induce forms $\Omega_1, \Omega_2, \Omega_3$ on $\mathcal{S}^c$ and we have moment maps $\underline{\mu}_1, \underline{\mu}_2, \underline{\mu}_3$.

Proposition 17.

The moment maps are given by

$$\underline{\mu}_1(J,\sigma) = \left(\sqrt{1-|\sigma|^2}K + \frac{1}{4\sqrt{1-|\sigma|^2}}(|\partial\sigma|^2 - |\bar{\partial}\sigma|^2) + i\bar{\partial}\partial(\sqrt{1-|\sigma|^2})\right)\rho$$

$$(\underline{\mu}_2 + i\underline{\mu}_3)(J,\sigma) = \bar{\partial}(\iota\bar{\partial}\sigma),$$

where K is the Gauss curvature and ι is the natural isomorphism from $\Omega^{0,1}(T^*\Sigma \otimes T^*\Sigma)$ to $\Omega^{1,0}$ defined by the metric.

To identify $\underline{\mu}_1$ we apply Theorem 9. A complex structure J is, by definition, a section of $T\Sigma \otimes T^*\Sigma$. We claim that the moment map $\mu_1 : X \to \mathfrak{sl}(2, \mathbf{R})$, for the form ω_1 on X, is given by

$$\mu(J,\sigma) = \frac{1}{2}\sqrt{1-|\sigma|^2}.J \tag{18}$$

To see this, observe that symmetry conditions dictate that the moment map has the form $f(|\sigma|)J$ for some function f. To find this function we need only consider the action of the circle subgroup generated by J. That is, we essentially have to find the Hamiltonian H for the rotations acting on the disc with the area form of Lemma 16. In polar co-ordinates this area form is

$$\frac{1}{2\sqrt{1-r^2}} r dr d\theta,$$

and contraction with $\frac{\partial}{\partial\theta}$ yields

$$\frac{r}{2\sqrt{1-r^2}} dr = -dH,$$

where $H(r) = \frac{1}{2}\sqrt{1-r^2}$. Taking account of the fact that the circle generated by J acts with weight 2 on the disc and that $|J|^2 = 2$, we deduce that $f(|\sigma|) = \frac{1}{2}\sqrt{1-|\sigma|^2}$ as required.

We can now identify the three terms in the formula (8). We use the Levi-Civita connection ∇ of the metric defined by J and ρ. The curvature R is thus $KJ \otimes \rho$, where K is the Gauss curvature, and so (since $|J|^2 = 2$):

$$R.\mu_{J,\sigma} = \sqrt{1-|\sigma|^2}K\rho.$$

The term $\omega(\nabla\phi, \nabla\phi)$ only has a contribution from $\nabla\sigma$, since J is parallel. Writing the derivative in holomorphic and anti-holomorphic parts and using the formula for the metric on the fibres of $X \to H$ we get

$$\frac{1}{4\sqrt{1-|\sigma|^2}}(|\partial\sigma|^2 - |\overline{\partial}\sigma|^2).$$

Finally, the term $\nabla\mu$ is just $\frac{1}{2}J \otimes d(\sqrt{1-|\sigma|^2})$ and the contraction $c(\nabla\mu)$ is $\frac{1}{2}J(d(\sqrt{1-|\sigma|^2})$, where J acts on the 1-forms on Σ in the standard way. So the exterior derivative is

$$dc(\nabla\mu) = \frac{1}{2}dJd(\sqrt{1-|\sigma|^2}) = i\overline{\partial}\partial\sqrt{1-|\sigma|^2}.$$

With these identifications we obtain the given formula for $\underline{\mu}_1$.

We could find $\underline{\mu}_2$ and $\underline{\mu}_3$ in the same manner, applying the general discussion of (2.1). However it is much simpler to proceed as follows. Since X is contained in the cotangent bundle of H^2, the space $\mathcal{S}^c$ can be viewed formally as an open set in the cotangent bundle of $\mathcal{S}$ and the form Ω_1 is nothing other than the standard symplectic form on the cotangent bundle. Now suppose, in general, that a group G acts on a manifold Q so we have a linear map $r : \mathfrak{g} \to \Gamma(TQ)$. The tranpose of r gives a map

$$r^T : T^*Q \to \mathfrak{g}^*,$$

and it is a simple fact that this is the moment map for the induced action of G on the cotangent bundle T^*Q. Thus we can apply this procedure to find the moment map $\underline{\mu}_2$. The Lie algebra of $\mathcal{G}$ is identified with the functions on Σ, modulo constants, and the infinitesimal action on $\mathcal{S}_0$ is given by

$$H \mapsto \overline{\partial}(v_H),$$

where v_H is the Hamiltonian vector field on Σ defined by H and ρ and $\overline{\partial}$ denotes the $\overline{\partial}$-operator on the tangent bundle. One readily checks that the transpose of this is given by

$$\sigma \mapsto \mathrm{Re}(\partial\iota\overline{\partial}\sigma),$$

whence the identification of μ_2. The formula for $\underline{\mu}_3$ then follows from the requirement of compatibility with the complex structure.

Our interpretation in Lemma 13 of the moment map formula shows that the integrals of the $\underline{\mu}_i$ over Σ are independent of the point (J, σ) in $\mathcal{S}$. Clearly these integrals vanish for $\underline{\mu}_2$ and $\underline{\mu}_3$ and for $\underline{\mu}_1$ we can find the integral by reducing to the case when $\sigma = 0$. Thus Gauss-Bonnet shows that

$$\int_\Sigma \underline{\mu}_1 = 2\pi(2 - 2\text{genus }(\Sigma)).$$

We have now reached the point where we can apply the general hyperkahler moment map theory to see that we have a manifold $\mathcal{M}^c$, parametrising solutions (J, σ) of the equations $\underline{\mu}_1 = c\rho, \mu_2 = \mu_3 = 0$ (where the constant c is again given by (15)), modulo the action of the group $\mathcal{G}$. The next issue we have to face is that $\mathcal{M}^c$ is larger than the space we really want. This is just the same point that we encountered in (2.2), except now we have to take care because the three symplectic forms behave in different ways. We would really like to take the quotient by the larger group $\mathcal{G}^+$ of all area-preserving diffeomorphism but, just as we have seen in (2.2), there is no way to define an equivariant moment map for the action of this space, extending $\underline{\mu}_1$. For $\underline{\mu}_2$ and $\underline{\mu}_3$ however the picture is different. Indeed it follows from the discussion in the proof of Proposition 17 that the map

$$(J,\sigma) \mapsto \iota(\overline{\partial}\sigma),$$

defines a moment map for the action of $\mathcal{G}^+$ (where we identify $\Omega^{0,1}$ with real 1-forms and pair these with vector fields). Likewise the map $(J,\sigma) \mapsto T(J\overline{\partial}\sigma)$ extends $\underline{\mu}_3$. To handle this, we consider the map

$$m : \mathcal{M}^c \to H^1(\Sigma; \mathbf{R})$$

defined by mapping a pair (J,σ) to the cohomology class of $\mathrm{Re}\iota(\overline{\partial}\sigma)$. General principles show that this is the moment map for the action of the torus $A_\Sigma = \mathcal{G}_0^+/\mathcal{G}$ on $\mathcal{M}^c$ for the symplectic form Ω_2. Thus we get an induced symplectic form Ω_2 on the symplectic quotient

$$\mathcal{M}_1^c = m^{-1}(0)/A_\Sigma.$$

Similarly the map $(J,\sigma) \mapsto Jm(J,\sigma)$, using the complex structure on $H^1(\Sigma;\mathbf{R})$ defined by the metric J, is a moment map for the same action with respect to the form Ω_2, so we also get a symplectic form Ω_2 induced on $\mathcal{M}_1^c$. For the first symplectic structure we need to proceed differently. Let V be the complex vector bundle over the moduli space $\mathcal{M}_0$ consisting of isomorphism classes of pairs (J,α), where J is a complex structure and α is a class in the cohomology group $H^{0,1}(\Sigma)$ defined using this complex structure. Thus the total space V is naturally a complex manifold. We have a map

$$\tilde{m} : \mathcal{M}^c \to V$$

defined by $\tilde{m}(J,\sigma) = (J, m(J,\sigma))$, where we use the complex structure J to identify $H^{0,1}$ with $H^1(\Sigma;\mathbf{R})$. One readily checks that this is a holomorphic map, with respect to the first complex structure on $\mathcal{M}^c$. This means that $m^{-1}(0) \subset \mathcal{M}^c$, which is the pre-image of the zero section of V, is a complex submanifold of $\mathcal{M}^c$ with respect to the first complex structure. In particular the form Ω_1 restricts to a nondegenerate form on $m^{-1}(0)$. Further, one checks much as in (2.2) that the orbits of the action of J on $m^{-1}(0)$ are complex submanifolds, thus we can use the same quotient construction as in (2.2) to obtain an induced symplectic form Ω_1 on $\mathcal{M}_1^c$. Again, everything is invariant under the mapping class group so we finally get three algebraically-compatible symplectic forms on $\mathcal{M}_0^c = \mathcal{M}_1^c/\Gamma$ and hence, by a lemma of Hitchin [10], a hyperkahler structure.

Now the points of $\mathcal{M}_0^c$ have a more straightforward geometric interpretation. If $\underline{\mu}_2$ and $\underline{\mu}_3$ and m all vanish on a pair (J,σ) we have $\partial\iota(\overline{\partial}\sigma) = 0$ and $\iota(\overline{\partial}\sigma) = \overline{\partial}f$ for some function f on Σ. But then $\partial\overline{\partial}f = 0$ so f is a constant, thus $\overline{\partial}\sigma = 0$. So the points of $\mathcal{M}_0^c$ correspond to equivalence classes of pairs (g,σ) where g is a Riemannian metric on Σ; σ is a *holomorphic* quadratic differential for the complex stucture defined by g which satisfy the equation $\underline{\mu}_1 = c\rho$. The equivalence classes are now taken under the action of the full group of diffeomorphisms of Σ. When σ is holomorphic we can write the moment map $\underline{\mu}_1$ in a neater way.

Lemma 18.

If σ is a holomorphic quadratic differential we have

$$\underline{\mu}_1(J,\sigma) = \left(K + \frac{1}{2}\Delta \log(1+\sqrt{1-|\sigma|^2})\right)\rho$$

To establish this formula we can work around a point where σ does not vanish. Let h be the smooth function $|\sigma|^2$, with the norm defined by the metric. Thus the curvature form of $T^*\Sigma$ is $\frac{1}{2}\overline{\partial}\partial \log h$ and

$$|\partial\sigma|^2 = h^{-1}|\partial h|^2.$$

The formula we need then follows from the identity, for any function h,

$$\sqrt{1-h}\;\overline{\partial}\partial \log h - \frac{1}{2h\sqrt{1-h}}\overline{\partial}h\partial h - 2\overline{\partial}\partial\sqrt{1-h} = \overline{\partial}\partial\log h - 2\overline{\partial}\partial\log(1+\sqrt{1-h}),$$

which we leave as an exercise for the reader. (The author's solution to this exercise goes via the identity:

$$-2\partial\log(1+\sqrt{1-h}) = \sqrt{1-h}\;\partial\log h + \frac{1}{\sqrt{1-h}}\partial h - \partial\log h.\ ) \tag{19}$$

To sum up then we have

Proposition 20 *The moduli space $\mathcal{M}_0^c$ is the quotient by the diffeomorphism group of the set of pairs (g,σ) where g is a Riemannian metric on the surface Σ and σ is a holomorphic quadratic differential on (Σ, g) such that*

$$K + \frac{1}{2}\Delta\log(1+\sqrt{1-|\sigma|^2}) = c,$$

where the constant c is given by (15). There is a hyperkahler metric on $\mathcal{M}_0^c$ extending the Weil-Petersson metric on the moduli space $\mathcal{M}_0 \subset \mathcal{M}_0^c$.

3.2. Geometric interpretation. The equation we have found in the previous section does not look very natural but we can transform it to a much more familiar shape. We make a start on this story here, although we have not yet been able to explore it in full detail.

Suppose we have a Riemannian surface (Σ, g) with a holomorphic quadratic differential σ and $|\sigma| < 1$ everywhere. We define a function F by

$$F = 1 + \sqrt{1-|\sigma|^2}.$$

Let g_1 be the scaled metric $g_1 = Fg$. Thus the curvature 2-form of (Σ, g_1) is

$$\Theta_1 = \Theta - \overline{\partial}\partial\log F,$$

where Θ is the curvature form of (Σ, g). On the other hand from the definition of F we have

$$(F-1)^2 = 1 - |\sigma|^2$$

which implies that

$$1 = \frac{2}{F} - \frac{|\sigma|^2}{F^2}. \tag{21}$$

If we write, for the original metric

$$K + \frac{1}{2}\Delta \log F = c + \chi,$$

then the Gauss curvature of g_1 is

$$K_1 = \frac{c+\chi}{F}.$$

On the other hand the norm of σ in the rescaled metric is

$$|\sigma|_1^2 = \frac{|\sigma|^2}{F^2}.$$

So

$$K_1 - \frac{c}{2}|\sigma|_1^2 = \frac{\chi}{F} + \frac{c}{2}.$$

using (21). We see then that the original metric satisfies the equation of Proposition 20 if and only if the rescaled metric satisfies the equation

$$K_1 - \frac{c}{2}|\sigma|_1^2 = \frac{c}{2}. \tag{22}$$

Notice that $|\sigma|_1 < 1$ and that the function

$$y = \frac{x}{1+\sqrt{1-x^2}}$$

gives a diffeomorphism from $[0,1)$ to itself, so the construction is invertible. Thus we can equally well regard $\mathcal{M}_0^c$ as a moduli space of pairs (g_1, σ) which satisfy the equation (22). It is now convenient to assume that the area of our original metric was normalised so that $c = -2$, thus the equation is

$$K_1 + |\sigma|_1^2 = -1. \tag{23}$$

The equation (23) is a reduced form of Hitchin's equation. In making this identification however we should be clear that the context is different. In Hitchin's case the complex structure on the underlying Riemann surface is fixed whereas in our case it is allowed to vary. Suppose we have any Riemannian surface (Σ, g_1) with a quadratic differential σ. We choose a square root L of the tangent bundle of Σ, so L is a holomorphic line bundle over Σ. Let a denote the $U(1)$-connection on L induced by the Levi-Civita connection. Now consider the vector bundle $L \oplus L^{-1}$ with the connection

$$A = \begin{pmatrix} a & \sigma \\ \overline{\sigma} & -a \end{pmatrix}.$$

Here $-a$ denotes the connection on L^{-1} and σ is regarded as an element of $\Omega^{1,0}(L^{-2}) = \Omega^{0,1}(\mathrm{Hom}(L, L^{-1}))$. We define the Higgs field $\Phi \in \Omega^{1,0}(\mathrm{End}(E))$ by

$$\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Here the element "1" is regarded as an element of $\Omega^{1,0}(L^2) = \Omega^{1,0}(\mathrm{Hom}(L^{-1}, L)$ by the natural isomorphism. Then

$$F(A) + [\Phi, \Phi^*] = \begin{pmatrix} K_1 + 1 + |\sigma|_1^2 & 0 \\ 0 & -(K_1 + 1 + |\sigma|_1^2) \end{pmatrix} \frac{i\rho}{2}$$

So the solutions to Hitchins equation of this shape precisely correspond to the solutions of equation (23). All this is very similar to, but not the same as, the special solutions of Hitchin's equation studied in [10] of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \Phi = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}$$

which lead to a parametrisation of Teichmuller space.

Now, given a solution of equation (22) we get a flat $SL(2,\mathbf{C})$-connection $A + \Phi + \Phi^*$ over Σ with a harmonic section of the associated bundle with fibre the hyperbolic 3-space $H^3 = PSL(2,\mathbf{C})/PU(2)$. In other words, we get a $\pi_1(\Sigma)$-equivariant harmonic map from the universal cover $\tilde{\Sigma}$ to H^3. The derivative of this map is represented by the Higgs field Φ, and the special form of this implies that the map is actually an *isometric immersion.* Thus the image is a minimal surface in H^3, by the standard relation between harmonicity and minimality. The quadratic differential σ appears now as the second fundamental form of the surface. Let us assume now that the quotient of H^3 by the action of $\pi_1(\Sigma)$ is a manifold: hence a hyperbolic 3-manifold Y. We then get an isometric minimal immersion of Σ in Y. Conversely, given such an immersion we can go backwards to recover the data (g_1, σ).

The overall picture that can be expected to emerge from this brings in the "quasi-Fuchsian" moduli space of Riemann surface theory. Let Σ_+, Σ_- be two compact Riemann surfaces of the same genus and fix a homotopy class $[f]$ of homeomorphisms between them. The simultaneous uniformisation theorem of Bers [1] asserts that there is a discrete subgroup $\pi \subset SL(2,\mathbf{C})$ and a Jordan curve whose complement has two components Ω_+, Ω_-, such that Ω_+/π is a uniformisation of Σ_+ and Ω_-/π is a uniformisation of $\overline{\Sigma_-}$ (the surface with the opposite complex structure). The group π also acts on the hyperbolic space H^3 and the quotient gives a hyperbolic 3-manifold $Y(\Sigma_+, \Sigma_-, [f])$ homeomorphic to $\Sigma_\pm \times \mathbf{R}$. This is a "hyperbolic cobordism" from Σ_+ to Σ_-, in that the conformal structures naturally induced on the two ends of the 3-manifold are the given ones. The quasifuschian moduli space $\mathcal{QF}$ is the moduli space of this data: it can be regarded either as an open subset of the moduli space of representations of $\pi_1(\Sigma_+)$ in $SL(2,\mathbf{C})$, modulo the mapping class group, or as the quotient of $\mathcal{T} \times \overline{\mathcal{T}}$ by the mapping class group (a bundle over the moduli space $\mathcal{M}_0$ with fibre $\overline{\mathcal{T}}$).

It is reasonable to hope that our hyperkahler manifold $\mathcal{M}_0^c$ can identified with $\mathcal{QF}$ via the correspondences above. What this would mean is that one could find a unique minimal surface, of a suitable kind, in any $Y(\Sigma_+, \Sigma_-, [f])$. This should be related to old work of Uhlenbeck [16]. Thus, in rough analogy with Hitchin's case, we would get three different descriptions of the same manifold:

1. as an open subset of the moduli space of representations of a surface group in $SL(2,\mathbf{C})$ modulo the mapping class group;
2. as an open subset in the cotangent bundle $T^*\mathcal{M}_0$ (viewed as pairs (J,σ));
3. as pairs of Riemann surfaces Σ_+, Σ_- with a given homotopy class of homeomorphisms between them.

Let us finally note that the last description gives a particularly attractive representation of the metric. Recall that, in the general framework of the Feix-Kaledin hyperkahler extensions, the Hamiltionian for the circle action with respect to the first structure gives a Kahler potential for the metric in the second complex structure, (see (4) above). Returning to our original picture in (3.1), it is easy to see that the Hamiltionian is given by

$$H(J,\sigma) = \int_{\Sigma} (\sqrt{1-|\sigma|^2} - 1)\ \rho.$$

But, up to a constant, this is just the *area* of the surface in the rescaled metric g_1. The second complex structure is the apparent structure in the third (conjectural) representation above. So we conjecture that there should be a hyperkahler metric on $\mathcal{T} \times \overline{\mathcal{T}}/\Gamma$ defined by taking the obvious complex structure and a Kahler potential $H(\Sigma_+, \Sigma_-, [f])$ given by the area of the preferred minimal surface in $Y(\Sigma_+, \Sigma_-, [f])$.

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