Degeneration of Einstein metrics and metrics with special holonomy

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0. Introduction

A a riemannian manifold, M^n, is called *Einstein* if for some constant, λ, the metric, g, and Ricci tensor, Ric_{M^n}, satisfy

$$\mathrm{Ric}_{M^n} = \lambda g\,.$$

In a local coordinate system in which the coordinate functions are harmonic, the Einstein condition becomes a quasi-linear elliptic equation on the metric; see (1.2).

This paper is not intended as a *general* survey of the subject of the Einstein metrics. A number of topics which are of central importance are not even mentioned e.g. [**Ca1**], [**Yau1**], [**Yau2**]. Rather, we try to give a detailed overview of one *specific* line of progress due to Colding, Tian and the author (in various combinations) which has shed light on the following question:

What sorts of objects arise as limits of sequences of badly behaved Einstein metrics?

It is natural to make the normalization,

$$|\mathrm{Ric}_{M^n}| \le n-1\,, \tag{0.1}$$

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or equivalently, $|\lambda| \leq n-1$, since this can be achieved by rescaling the metric; n-dimensional spaces with constant sectional curvature, ± 1, have $\lambda = \pm(n-1)$.

For $n \leq 3$, Einstein metrics have constant curvature; [**Be**]. Thus, we will assume $n \geq 4$.

In dimension 4, if collapsing does not take place and the topology remains bounded, the limiting objects are Einstein manifolds with orbifold singularities; [**An1**]–[**An3**], [**Nak1**], [**Nak2**], [**Ti1**], [**Ti2**].

Important features of the 4-dimensional case persist in higher dimensions. In many instances, the limiting objects are known to have singularities of codimension ≥ 4. Conjecturally, the codimension 4 piece of the singular set, consists of singular points of orbifold type. At the infinitesimal level, this is known hold. As in dimension 4, the L_2-norm of curvature plays a distinguished role. However, in higher dimensions, the L_2-norm fails to be scale invariant. This, together with the appearance of nonisolated singularities, causes substantial difficulties.

The approach we describe starts with the development of a structure theory for limits of sequences of manifolds satisfying the weaker assumption,

$$\mathrm{Ric}_{M^n} \geq -(n-1)\,, \tag{0.2}$$

where to be more precise, we should write $\mathrm{Ric}_{M^n} \geq -(n-1)g$. This theory, is outlined in Sections 1–4 below; for an exposition with proofs see [**Ch5**].

Later, additional conditions are added: an upper bound on the Ricci tensor, the assumption that the metric is Einstein, or of special holonomy (e.g. Kähler-Einstein) and finally, an L_p bound on the full curvature tensor. The case, $p = 2$, is particularly significant. For compact manifolds with special holonomy and Ricci tensor normalized as in (0.1), the L_2-norm of the curvature is majorized by a certain topological invariant $C(M^n)$; see (0.10).

Let $\{M_i^n\}$ denote a sequence of compact riemannian manifolds satisfying (0.2) for which the diameters are uniformly bounded, $\mathrm{diam}(M_i^n) \leq d$ for all i. By a fundamental observation of Gromov, any such sequence has a subsequence, $\{M_j^n\}$, which converges in a certain weak sense — the Gromov-Hausdorff sense — to some compact metric space Y. We write $M_j^n \xrightarrow{d_{GH}} Y$. Gromov's compactness theorem is a consequence of relative volume comparison, the control which the lower bound on Ricci curvature, (0.2), exerts over the volumes of metric balls.

Gromov-Hausdorff convergence of a sequence of metric spaces is the notion of convegence associated to the Gromov-Hausdorff distance between (isometry classes of) compact metric spaces.

Let (W_1, ρ_1), (W_2, ρ_2), denote compact metric spaces. The Gromov-Hausdorff distance, $d_{GH}((W_1,\rho_1),(W_2,\rho_2))$, is the infimum of those $\epsilon > 0$, for which there exists a metric, ρ, on the disjoint union, $W_1 \amalg W_2$, such that:

i) $\rho \,|\, W_k = \rho_k$, $k = 1, 2$.

ii) W_k is ϵ-dense in $W_1 \amalg W_2$, $k = 1, 2$.

Often, one just writes $d_{GH}(W_2, W_2)$, supressing the metrics ρ_1, ρ_2.

The Gromov-Hausdorff distance is an intrinsic generalization of the classical Hausdorff distance, $d_H(W_1, W_2)$, between compact subsets, W_1, W_2, of a metric space, (W, ρ):

$$d_H(W_1, W_2) := \max\left(\max_{w_1 \in W_1} \rho(w_1, W_2),\ \max_{w_2 \in W_2} \rho(w_2, W_1)\right)\,.$$

Intuitively, the Hausdorff distance between two subsets of a metric space, or Gromov-Hausdorff distance between two metric spaces, is very small if, to the naked eye, the objects appear to be identical.

There is an extension of Gromov-Hausdorff convergence to sequences of pointed metric spaces $\{(W_i, y_i, \rho_i)\}$. Namely, pointed Gromov-Hausdorff convergence means Gromov-Hausdorff convergence of the sequence of balls, $\{B_r(y_i)\}$, for all $r < \infty$. With this understanding, Gromov's compactness theorem has a natural generalization to the case in which the bound on the diameter is dropped. This is used in defining the concept of "tangent cone", which plays a key role in the sequel.

By starting with *any* sequence of manifolds satisfying (0.2), no matter how badly behaved, and passing to a suitable subsequence, we obtain a limit space, Y, whose regularity and singularity structure we can examine. Clearly, the properties of Y will closely reflect those of the manifolds in our sequence.

A length space is a metric space such that every pair of points is joined by a continuous curve whose length is equal to the distance between them. Here, the length of a continuous curve is defined using partitions as in elementary calculus. It is not difficult to see that the Gromov-Hausdorff limit of a sequence of length spaces is again a length space. This completely general result is one of the few easily established properties of Gromov-Hausdorff limit spaces satisfying

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,. \tag{0.3}$$

Since the elliptic equation satisfied by Einstein metrics is *nonlinear*, it is quite possible for limit spaces to have singularities. Many such examples of singular limit spaces are known; see e.g. [**Joy3**] and the references therein. In actuality, since there is no fixed background metric, the Einstein equation is *more* nonlinear than the partially analogous equations for minimal surfaces, harmonic maps and Yang-Mills fields. In and of itself, this makes the Einstein case harder to handle.

EXAMPLE 0.4. (Eguchi-Hansen manifolds) The unit sphere bundle of TS^2, the tangent bundle of the 2-sphere, is diffeomorphic to the real projective space $\mathbf{RP}(3)$. Thus, the complement of the unit disc bundle of TS^2 is diffeomorphic to $(1,\infty) \times \mathbf{RP}(3)$, and hence to a neighborhood of infinity in $\mathbf{R}^4/\mathbf{Z}_2$, where the action of $\mathbf{Z}_2$ is generated by the antipodal map.

In fact, TS^2 has a Ricci flat (hyper-Kähler) metric, g, the Eguchi-Hansen metric, which at infinity, becomes rapidly asymptotic to the space, $(1,\infty) \times \mathbf{RP}(3) \subset \mathbf{R}^4/\mathbf{Z}^2$, with its canonical flat metric; see [**EgHan**].

When $\epsilon \to 0$, the 1-parameter family, $(TS^2, \epsilon^2 g)$, converges in the pointed Gromov-Hausdorff sense, to the flat singular cone $\mathbf{R}^4/\mathbf{Z}_2$. The zero section, a homologically nontrivial 2-sphere, shrinks in the limit to the vertex of the cone. Since the condition of being Ricci flat is invariant under scaling of the metric, the space, $\mathbf{R}^4/\mathbf{Z}_2$, is the Gromov-Hausdorff limit of a sequence of Ricci flat manifolds. The singular set of this limit space has codimension 4.

Rescaling, rigidity and almost rigidity.

Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (0.3). Any *individual* manifold, M_i^n, looks locally like $\mathbf{R}^n$ on a sufficiently small scale. Thus, the formation of singularities in the limit reflects the absence of *uniform* control over the scale on which the metric becomes standard.

The theory we will describe gives restrictions on the *Hausdorff dimension* of the singular set of Y. From this it follows that uniform control of the metric does exist, except perhaps on a set which is very small in a definite sense. We will also give restrictions on the *structure* of the singular set. These imply that there is a weaker sort of uniform control at all points of all manifolds in the approximating sequence.

Imagine that a manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $\mathrm{diam}(M^n) = d$, is observed under a powerful microscope, so that all distances appear to have been multiplied by a large factor ϵ^{-1}. A ball of radius ϵ will appear to have radius 1 and the Ricci curvature will appear to be bounded below by $-(n-1)\cdot\epsilon^2$. Thinking more globally, we can just rescale the distance function on the whole manifold by a factor ϵ^{-1}. The Ricci curvature of the rescaled metric is bounded below by $-(n-1)\cdot\epsilon^2$. The rescaled diameter is $\epsilon^{-1}\cdot d$, which for ϵ^{-1} sufficiently large, is effectively infinite.

The rescaling idea, which is pervasive in science, strongly suggests that after suitable rescaling, the small scale properties of manifolds whose Ricci curvature has a definite lower bound should resemble the fixed scale properties of complete noncompact manifolds whose Ricci curvature is nonnegative.

Since ϵ, while very small, is nonetheless strictly positive, implementation of this approach requires quantitative versions of theorems on nonnegative Ricci curvature, allowing for small errors in both the hypotheses and conclusions. The relevant theorems are *rigidity* theorems, for example, the splitting theorem; [**Top2**], [**ChGl1**]. Their corresponding quantitative versions are called *almost rigidity theorems*; [**ChCo2**].

The implications of rescaling can be expressed directly in terms of limit spaces:

At the infinitesimal level, limit spaces have nonnegative Ricci curvature in a generalized sense.

The noncollapsed case.

Much, but not all, of our discussion will be restricted to the *noncollapsed* case, in which it is assumed apriori, that there exists $v > 0$, such that for all $m_i \in M_i^n$,

$$\mathrm{Vol}(B_1(m_i)) \geq v\,. \tag{0.5}$$

There are many specific examples of noncollapsed limit spaces with interesting properties.

The *collapsed* case is also of great interest, for example in connection with mirror symmetry; compare [**StYauZa**], [**GsWil**], [**KonSoi**]. Examples show that the situation in the collapsed case is much less constrained, and by the same token, much less is understood. To date, the main structure theorem in the collapsed case, asserting the rectifiability of Y with respect to any so-called renormalized limit measure, has not been significantly strengthened if (0.2) is replaced by the Einstein condition; see [**ChCo5**].

Concentration of curvature and formation of singular sets.

Behavior like that in Example 0.4 could not occur for noncollapsed Gromov-Hausdorff limit spaces with a uniform bound on *sectional* curvature. Such limit spaces are actually smooth manifolds and the metric space structure arises from a riemannian metric which, in harmonic coordinate systems, is $C^{1,\alpha}$, for any $\alpha < 1$; [**Ch1**], [**GvLP**], [**An2**], [**JoKar**].

For the family of Eguchi-Hansen metrics described in Example 0.4, the singular set consists of a single point at which the curvature concentrates in the limit in the L_2 sense. It does not concentrate in the L_p sense for $1 \leq p < 2$. In considerable generality, for noncollapsed limit spaces with a uniform lower Ricci curvature bound, the failure of curvature to concentrate sufficiently in the L_p sense prevents the formation of a singular set with positive $(n-2p)$-dimensional Hausdorff measure. The proof of this fact relies on certain previously established constraints on the the structure of singular sets in dimension $n-2p$.

Special holonomy.

Important examples of Einstein manifolds are furnished by manifolds with special holonomy.

By definition, the holonomy group, H_p of a riemannian manifold, M^n, at $p \in M^n$, is the group of the orthogonal transformations of the tangent space, M^n_p, obtained by parallel translating around smooth closed loops based at p. The holonomy group is always a Lie group; [**Yam**]. A smooth curve from p to q induces an isomorphism of corresponding holonomy groups. Up to conjugacy in the orthogonal group, this isomorphism is independent of the chosen path. ¿From now on we drop the dependence on the base point and just write H for the holonomy group.

The *restricted holonomy group*, H_0, is the identity component of H. Equivalently, it is the subgroup generated by loops which are contractible. In particular, H_0 is a normal subgroup of H.

A riemannian manifold, M^n, is said to have *special restricted holonomy*, if it is locally irreducible, H_0 is a proper subgroup of $\mathrm{SO}(n)$ and there exists some (possibly distinct) riemannian manifold, with restricted holonomy group H_0, which is not a symmetric space.

The special restricted holonomy groups, H_0, are:

$$\mathrm{U}(\frac{n}{2}), \;\; \mathrm{SU}(\frac{n}{2}), \; \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1) \;\; (n > 4)\,,$$

$$\mathrm{Sp}(\frac{n}{4}), \;\; \mathrm{G}_2, \; \mathrm{Spin}(7);$$

see [**Ber**]; see also [**Si**], [**Al**], [**BrGr**].

Each of these groups occurs as the holonomy group of a compact simply connected riemannian manifold; see [**Joy3**]. In all cases, the representation of the restricted holonomy group is unique up to conjugacy $\mathrm{O}(n)$, and the action of the holonomy group on the unit sphere is transitive; [**Si**].

We make the following convention:

A manifold (which might not be locally irreducible) has special holonomy if H is contained in one of the above groups.

If $H \subset \mathrm{U}(\frac{n}{2})$, the dimension, n, is even and there is an almost complex structure, J, which is parallel, and hence, integrable. Equivalently, the Kähler form, $\omega = g(J\,\cdot\,,\,\cdot\,)$, is closed. The first Chern class is represented by the 2-form,

$$c_1 = \frac{1}{2\pi}\mathrm{Ric}_{M^n}(J\,\cdot\,,\,\cdot\,)\,, \tag{0.6}$$

which, up to normalization, can also be viewed as the curvature of the anticanonical line bundle. The Einstein condition is equivalent to

$$c_1 = \frac{1}{2\pi}\lambda\omega\,. \tag{0.7}$$

If $H_0 = \mathrm{U}(\frac{n}{2})$, the underlying manifold is never Ricci flat. If $H_0 \subset \mathrm{SU}(\frac{n}{2})$, M^n it is always Ricci flat.

If $H \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ then n is a multiple of 4. In this case, there is a distinguished canonically oriented 3-dimensional *parallel sub-bundle*, E, of the endomorphism bundle of the tangent bundle, such that any oriented orthonormal basis of a fibre consists of a triple, I, J, K, of anticommuting almost complex structures satisfying $IJ = K$. The 4-form, $\omega_I^2 + \omega_J^2 + \omega_K^2$, the sum of the Kähler forms associated to I, J, K, is parallel and is independent of the specific choice of I, J, K.

If $H_0 \subset \mathrm{Sp}(\frac{n}{4})$, the bundle E is flat. Thus, manifolds of this type carry 3 locally defined parallel almost complex structures I, J, K. Hence, the individual Kähler forms, ω_I, ω_J, ω_K, are parallel in this case. More generally, an endomorphism of the form, $rI+sJ+tK$, with $r^2+s^2+t^2 = 1$, is a parallel almost complex structure. With respect to any such complex structure, $\mathrm{Sp}(\frac{n}{4}) \subset \mathrm{SU}(\frac{n}{2})$. Therefore, this case is Ricci flat.

If $H_0 = \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, the bundle, E, does not have parallel local sections and in general, a manifold with holonomy, $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, need not admit any integrable almost complex structure

For $n > 4$, M^n is Einstein, but not Ricci flat. The curvature tensor of E can be expressed in terms of the Ricci tensor and a triple of almost complex structures, I, J, K; compare (0.7) and see (10.3), (10.6). The curvature tensor of E is parallel and its norm is determined by the dimension, n, and Einstein constant.

The unit sphere bundle, $S(E)$, is called the *twistor space* associated to M^n. The natural metric for which the projection to M^n is a riemannian submersion has the property that the Ricci tensor and all of its covariant derivatives are bounded. In addition, there is a canonical integrable almost complex structure on $S(E)$, all of whose covariant derivatives are bounded. These facts allow many questions concerning manifolds with $H \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ to be reduced to the corresponding questions for a slight generalization of the $\mathrm{U}(\frac{n}{2})$ case.

For $n = 4$, we have the canonical isomorphism Sp(1)Sp(1)=SO(4). Hence, Sp(1)Sp(1) is *not* a special restricted holonomy group and the metric on the underlying manifold is not constrained. Einstein 4-manifolds with anti-self-dual curvature, are the analogs of higher dimensional manifolds with $H = \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$. For such 4-manifolds, the curvature properties of the bundle, E, are as in the higher dimensional case and the associated twistor space plays an analogous role.

We note the inclusions

$$\mathrm{Sp}(\frac{n}{4}) \subset \mathrm{SU}(\frac{n}{2}) \subset \mathrm{U}(\frac{n}{2})\,,$$

$$\mathrm{Sp}(\frac{n}{4}) \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)\,.$$

The cases, $H_0 \subset \mathrm{SU}(\frac{n}{2})$, $H_0 \subset \mathrm{Sp}(\frac{n}{4})$, can be viewed as limits of the $\mathrm{U}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ cases, in which the norm of the Ricci tensor tends to zero.

Manifolds with the restricted holonomy groups G_2, Spin(7), have dimension 7, 8, respectively. There are natural inclusions,

$$\mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7)\,.$$

Each of the subgroups, $\mathrm{SU}(3) \subset \mathrm{G}_2$, $\mathrm{G}_2 \subset \mathrm{Spin}(7)$, is realized as the isotropy group of a point on the unit sphere of its representation space.

Together with the twistor construction, the above relations indicate that for instance of special holonomy, there is a close connection with the $\mathrm{U}(\frac{n}{2})$ or $\mathrm{SU}(\frac{n}{2})$ case. This comes in to play in the proofs of the main results discussed in Sections 5, 6, 9.

The cases, $H \subset \mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, are called Kähler, respectively, special-Kähler In the $\mathrm{SU}(\frac{n}{2})$ case, the canonical bundle is flat. If the flat connection is actually globally trivial, the manifold is called Calabi-Yau.

The cases, $H \subset \mathrm{Sp}(\frac{n}{4})$, $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, are called hyper-Kähler, respectively quaternion-Kähler.

The cases, G_2, $\mathrm{Spin}(7)$, are called exceptional.

Anti-self-duality of curvature.

Let V denote a real inner product space space and let $W \subset \bigwedge^2(V)$. The orthogonal projection, $\pi_W : \bigwedge^2(V) \to W$, can be viewed as an element of the symmetric tensor product $S^2(\bigwedge^2(V))$. Let $A : S^2(\bigwedge^2(V)) \to \bigwedge^4(V)$ denote the canonical map given by antisymmetrization. With respect to an orthonormal basis, $\{w_j\}$, of W, we have

$$A(\pi_W) = \sum_j w_j \wedge w_j\,.$$

An oriented manifold, M^n, with holonomy group H_p, at $p \in M^n$ and Lie algebra, $\mathfrak{h}_p$, carries a canonical parallel $(n-4)$-form whose value at p is $A(\pi_{\mathfrak{h}_p})$. If the restricted holonomy group is $\mathrm{SO}(n)$, then $A(\pi_{\mathfrak{h}_p}) \equiv 0$.

If the holonomy group, H, is special, we put

$$* \,\Omega(p) = c(H) \cdot A(\pi_{\mathfrak{h}_p})\,. \tag{0.8}$$

where $c(H)$ is an appropriately chosen of the constant. A calculation shows that the (suitably defined) trace free part, R_0, of the curvature, R, satisfies the anti-self-duality relation,

$$* \, R_0 = -\Omega \wedge R_0\,; \tag{0.9}$$

compare [**Sa1**], [**Sa2**], [**BaHiSing1**], [**Ti3**].

If $H_0 \subset \mathrm{U}(\frac{n}{2})$, the form, $*\Omega$, is a multiple of the square of the Kähler form. If $H_0 \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, then $*\Omega$ is a multiple of the form $\omega_I^2 + \omega_J^2 + \omega_K^2$. The groups, G_2, $\mathrm{Spin}(7)$, can actually be characterized as the subgroups of all linear transformations which preserve the form $*\Omega$; see Section 10.

Let p_1 denote the first Pontrjagin class. Put

$$C(M^n) = -(p_1 \cup [\Omega])(M^n)\,. \tag{0.10}$$

A multiple, $c_0(n)$, of the topological invariant, $C(M^n)$, bounds the L_2-norm of R_0. This is a direct consequence of (0.9) and Chern-Weil theory; see Section 9. Given the normalization of the Ricci curvature in (0.1), it follows that a multiple, $c(n)$, of $C(M^n)$, bounds the L_2-norm of the full curvature tensor; compare Conjecture 0.15 and Theorem 9.4.

Degenerations.

M. Anderson has conjectured that if Y is the Gromov-Hausdorff limit of a noncollapsing sequence of Einstein manifolds, then Y is a smooth Einstein manifold off a closed singular set of Hausdorff dimension $\leq n-4$. At least for limits of manifolds with special holonomy, this holds.

THEOREM 0.11. ([**Ch3**], [**ChTi2**]) Let $M_i^n \xrightarrow{d_{GH}} Y$, where the manifolds, M_i, have special holonomy, H, or, in case $n=4$, are Einstein with anti-self-dual curvature. Assume

$$|\mathrm{Ric}_{M_i^n}| \leq n-1\,,$$
$$\mathrm{Vol}(M_i^n) \geq v > 0\,.$$

Then there is a closed set, $\mathcal{S}$, with $\dim \mathcal{S} \leq n-4$, such that $Y \setminus \mathcal{S}$ is a smooth n-manifold with holonomy contained in H, respectively an Einstein 4-manifold with anti-self-dual curvature.

The Kähler case of Theorem 0.11 is due to Cheeger and to Tian. It was first written up in [**Ch3**]. Ideas in the proof are similar to ones used in [**ChCoTi2**]. For the remaining cases, see [**ChTi2**]. The proof of Theorem 0.11 is described in Section 5.

REMARK 0.12. In Theorem 0.11, anti-self-duality of curvature does not enter and no assumption concerning the boundedness of the sequence of characteristic numbers, $\{C(M_i^n)\}$, is required.

Before stating the following corollary of Theorem 0.11 it is convenient to make a definition. (For the notion of "tangent cone", which appears in Corollary 0.13, see Section 3.)

A metric space, X, is said to have a singularity at $x \in X$ of *orbifold type* if some neighborhood of x is isometric to U/Γ, where U is a smooth riemannian manifold and Γ is a finite group of isometries of U.

Let P_c denote parallel translation along the smooth curve c.

For H a compact Lie group, the singularity will be called of *H-orbifold type* if the following additional condition holds. For all $p \in U$, there is a subgroup, H_p, conjugate in $\mathrm{O}(n)$ to H, such that for all $k \in \Gamma$ and all curves, c, from p to $k(p)$, the transformation, $dk^{-1} \circ P_c$, lies in H_p.

COROLLARY 0.13. ([**Ch3**], [**ChTi2**]) Let the assumptions be as Theorem 0.11. There exists $\mathcal{S}_{n-6} \subset \mathcal{S}$, with $\dim \mathcal{S}_{n-6} \leq n-6$, such that for all $y \in (\mathcal{S} \setminus \mathcal{S}_{n-6})$, there exists at least one tangent cone of H-orbifold type, $\mathbf{R}^{n-4} \times \mathbf{R}^4/\Gamma$, where $\Gamma \subset H \subset \mathrm{SO}(n)$ acts trivially on $\mathbf{R}^{n-4}$ and freely on $\mathbf{R}^4 \setminus \{0\}$.

REMARK 0.14. In Theorem 0.11 and Conjecture 0.15 below, if $H_0 \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ then $\mathcal{S}_{n-6}$ can be replaced by $\mathcal{S}_{n-8}$ (with $\dim \mathcal{S}_{n-8} \leq n-8$); see Theorem 6.1.

The following Conjecture 0.15, is suggested by additional facts which are known to hold in the 4-dimensional case. Its solution is one main goal of the program described in this paper.

CONJECTURE 0.15. ([**ChTi2**]) Let the sequence of manifolds, $M_i^n \xrightarrow{d_{GH}} Y$, with special holonomy group, H, satisfy

$$|\mathrm{Ric}_{M_i^n}| \leq n-1\,,$$
$$\mathrm{Vol}(M_i^n) \geq v > 0\,.$$

Assume in addition that the sequence, $\{C(M_i^n)\}$, is bounded above. Then there exists $\mathcal{S}_{n-6} \subset \mathcal{S}$, with dim $\mathcal{S}_{n-6} \leq n-6$, such that for all $y \in (\mathcal{S} \setminus \mathcal{S}_{n-6})$, the singularities are of H-orbifold type.

REMARK 0.16. If Conjecture 0.13 holds, then $\mathcal{S}_{n-4}$, the codimension 4 piece of the singular set, is *calibrated* by the $(n-4)$-form Ω; see [**HarLaw**].

For $H \subset \mathrm{U}(\frac{n}{2})$, Conjecture 0.15 is due to Cheeger-Colding-Tian, who also gave somewhat weaker and more technical versions for Einstein manifolds in general; compare Sections 7, 8.

In addition to the information contained in Theorem 0.11 and Corollary 0.13, the assertions of the conjecture are known to hold at the *infinitesimal* level, off a subset with $(n-4)$-dimensional Hausdorff measure 0.

Off such a subset, the tangent cone is unique and is of H-orbifold type, $\mathbf{R}^{n-4} \times \mathbf{R}^4/\Gamma$, where the finite group, $\Gamma \subset H \subset \mathrm{SO}(n)$, acts freely on $\mathbf{R}^4 \setminus \{0\}$ and trivially on $\mathbf{R}^{n-4}$; see Theorem 7.14.

Information concerning behavior on the *local* level is also available: Bounded subsets of $\mathcal{S}$, have finite $(n-4)$-dimensional Hausdorff measure and are actually $(n-4)$-rectifiable; see Theorems 8.1, 9.4.

These matters, including the relevant definitions, will be explained at greater length in the sequel. Although the above mentioned results represent strong progress, Conjecture 0.15 still seems quite difficult.

Organization.

In Sections 1–3, we assume only the lower bound on Ricci curvature, $\mathrm{Ric}_{M^n} \geq -(n-1)$.

In Sections 4–6, we assume the 2-sided bound, $|\mathrm{Ric}_{M^n}| \leq n-1$, or in addition, that M^n is Einstein. In Sections 5, 6, we also assume that M^n has special holonomy.

In Sections 7, 8, we assume $\mathrm{Ric}_{M^n} \geq -(n-1)$, and also, a bound on the L_p-norm of the full curvature tensor.

In Sections 9, all of our assumptions come into play, the 2-sided bound, $|\mathrm{Ric}_{M^n}| \leq n-1$, special holonomy, and a bound on the topological invariant $C(M^n)$.

In Section 10, the appendix, we review some standard facts concerning special holonomy, including the definitions of the special holonomy groups, the computation of the Ricci tensor and the relations between the cases $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, G_2, $\mathrm{Spin}(7)$ and the Kähler case.

Apart from Theorem 0.11 and Corollary 0.13, the main results on degenerations of Einstein metrics and metrics with special holonomy are Theorems 3.4, 4.2, 6.1, 7.11, 7.14, 9.4.

1. Preliminaries

We collect here, some basic properties of manifolds whose Ricci curvature is bounded below, on which the proofs of results described in later sections depend. Thus, although we will not supply proofs of those results, we will be able to give an indication of what the proofs involve. An exposition with complete proofs, of the results of Sections 1–4, is given in [**Ch4**].

Bochner's formula.

Let X^* denote the 1-form dual to the vector field X and let θ^* denote the vector field dual to the 1-form θ. Let $\Delta = \operatorname{div}\nabla = -\delta d$ denote the Laplacian on functions, ∇, the riemannian covariant derivative and $d\delta + \delta d$, the Hodge Laplacian on 1-forms.

Bochner's formula asserts:

$$\begin{aligned}\frac{1}{2}\Delta|X|^2 &= \langle \Delta X, X\rangle + \langle \nabla X, \nabla X\rangle \\ &= |\nabla X|^2 - \langle X, ((d\delta + \delta d)X^*)^*\rangle + \operatorname{Ric}(X,X)\,.\end{aligned}$$

Since $d(d\delta + \delta d) = (d\delta + \delta d)d$ and $df^* = \nabla f$, it follows that if $X = \nabla f$ is the gradient of a function, f, then

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}_f|^2 + \langle \nabla f, \nabla \Delta f\rangle + \operatorname{Ric}(\nabla f, \nabla f)\,,$$

where $\mathrm{Hess}_h = \nabla(\nabla h)$.

For h harmonic,

$$\frac{1}{2}\Delta|\nabla h|^2 = |\mathrm{Hess}_h|^2 + \operatorname{Ric}(\nabla h, \nabla h)\,. \tag{1.1}$$

Note that if $\operatorname{Ric}_{M^n} \geq 0$ (e.g. $M^n = \mathbf{R}^n$) and $|\nabla h| \equiv 1$, we get $\nabla(\nabla h) \equiv 0$. By the de Rham decomposition theorem, every $p \in M^n$, is contained in a neighborhood, U, which is isometric to a subset of a riemannian product, $\mathbf{R}\times \overline{M}^{n-1}$, by an isometry with carries ∇h to the unit tangent vector field to the $\mathbf{R}$ factor. We think of such functions as locally linear in a generalized sense.

Local harmonic coordinates.

In a neighborhood of any point, $m \in M^n$, there exist many local coordinate systems with the property that the coordinate functions are harmonic. It a simple consequence of Bochner's formula that in such a local coordinate system, the Ricci tensor is given by

$$\operatorname{Ric}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = -\frac{1}{2}\Delta(g_{i,j}) + Q_{i,j}\left(g^{s,t}, \frac{\partial g_{k,\ell}}{\partial x_m}\right), \tag{1.2}$$

where $Q_{i,j}$ is quadratic in both $\frac{\partial g_{k,l}}{\partial x_m}$ and $g^{s,t}$. In each product, $\frac{\partial g_{k,l}}{\partial x_m} \cdot \frac{\partial g_{k',l'}}{\partial x_{m'}}$, one of k, ℓ, m, equals i, and one of k', ℓ', m' equals j.

In view of (1.2), in harmonic coordinates, the Einstein condition becomes a quasi-linear elliptic equation on the metric.

Relative Volume comparison.

Let $\underline{M}^n_H$ denote the complete, simply connected, n-dimensional space of constant curvature $\equiv H$. Let $\underline{p} \in \underline{M}^n_H$,

THEOREM 1.3. ([**Bish**], [**GvLP**]) If $\operatorname{Ric}_{M^n} \geq (n-1)H$, then the following functions are nondecreasing:

$$\frac{\operatorname{Vol}(\partial B_r(p))}{\operatorname{Vol}(\partial B_r(\underline{p}))} \downarrow,$$

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(B_r(\underline{p}))}\downarrow .$$

Monotonicity inequalities play a key role in geometric analysis. They are assertions to the effect that behavior of a certain quantity on a fixed scale, controls its behavior on all smaller scales. The relations in Theorem 1.3 are the *only* monotonicity inequalities known to hold for manifolds with Ricci curvature bounded below. In particular:

Volume behavior for a given ball of a fixed size, controls from below, volume behavior for all smaller concentric balls.

Relative volume comparison is a consequence of the formula for the first variation of area, together with so-called *mean curvature* comparison off the set, $\{p\}\cup\mathcal{C}_p$, where $\mathcal{C}_p$ denotes the *cut locus* of p; see below.

Gromov's compactness theorem.

THEOREM 1.4. ([**GvLP**]) The collection of (isometry classes of) compact riemannian manifolds satisfying $\mathrm{Ric}_{M^n} \geq -(n-1)$, $\mathrm{diam}(M^n) \leq d$, is precompact with respect to the Gromov-Hausdorff distance.

By a standard argument based on the doubling property for the riemannian measure implied by Theorem 1.3, the manifolds in Theorem 1.4 form a uniformly totally bounded collection i.e. for all $\epsilon > 0$, there exists $N(\epsilon, d, n)$, such that every such M^n, contains an ϵ-dense set with at most $N(\epsilon, d, n)$ members. (Any maximal set for which each pair of distinct points has distance at least ϵ, is easily seen to have this property.)

A pigeon hole argument shows that any uniformly totally bounded collection of compact metric spaces is totally bounded with respect to the Gromov-Hausdorff distance.

Finally, from a construction like that which associates to a metric space its completion, it follows that the ideal points which must be added to complete the above collection with respect to the Gromov-Hausdorff distance, are actually realized by compact metric spaces.

Mean curvature comparison and Laplacian comparison.

Let $m(x)$ denote the mean curvature in the the direction of inward normal of distance sphere, $\partial B_r(p)$, at the point $x \in \partial B_r(p)$. Let $\underline{m}(r)$ denote the mean curvature of the distance sphere, $\partial B_r(\underline{p})$, where $\underline{p} \in \underline{M}^n_H$. Mean curvature comparison states that $\mathrm{Ric}_{M^n} \geq (n-1)H$ implies

$$m(x) \leq \underline{m}(r).$$

Initially, mean curvature comparison only makes sense at points at which the distance function is smooth i.e. off the set $\{p\} \cup \mathcal{C}_p$. This version of mean curvature comparison is proved by elementary arguments from the theory of ordinary differential equations.

Let $\tilde{\Delta}$ denote the Laplacian on $\partial B_r(p)$, with respect to the induced metric. In view of the formula for the Laplacian in geodesic polar coordinates, at smooth points of $\partial B_r(p)$, we have

$$\Delta = \frac{\partial^2}{\partial r^2} + m\frac{\partial}{\partial r} + \tilde{\Delta}.$$

It follows that mean curvature comparison has an immediate generalization to Laplacian comparison:

$$\Delta f(r) \le \underline{\Delta} f(r) \qquad (\text{if } f' \ge 0)\,,$$
$$\Delta f(r) \ge \underline{\Delta} f(r) \qquad (\text{if } f' \le 0)\,.$$

Mean curvature comparison is the special case of Laplacian comparison in which $f(r) = r$.

It is of crucial importance that Laplacian comparison has a sense and is valid at *all* points of M^n, including the set $\{p\} \cup \mathcal{C}_p$. It holds in the distribution sense and in the sense of barriers, which concept is very closely related to (what subsequently became known as) *viscosity solutions*. Since distance functions have natural barriers and the strong maximum principle extends to the barrier setting, Laplacian comparison is a powerful tool in the study of Ricci curvature. These fundamental results were obtained by Calabi in 1957; [**Ca2**].

Let the metric on $\underline{M}^n_H$ be given in geodesic polar coordinates by

$$\underline{g} = dr^2 + \underline{k}^2 g^{S^{n-1}}\,,$$

where

$$\underline{k} = \begin{cases} \sin\sqrt{H}r/\sqrt{H} & H > 0 \\ r & H = 0 \\ \sinh\sqrt{-H}r/\sqrt{-H} & H < 0\,. \end{cases}$$

Denote the Laplacian in geodesic polar coordinates on $\underline{M}^n_H$ by

$$\underline{\Delta} = \frac{\partial^2}{\partial r^2} + \underline{m}\frac{\partial}{\partial r} + \underline{\tilde{\Delta}}\,.$$

The following functions on the spaces, $\underline{M}^n_H$, are particularly useful for applying Laplacian comparison.

For $n \ge 3$, the function,

$$\underline{G}(r) = \frac{1}{(n-2)\mathrm{Vol}(S^{n-1})}\int_r^\infty \underline{k}^{1-n}(s)ds\,,$$

satisfies $\underline{\Delta}\,\underline{G} = 0$, $\underline{G}(0) = \infty$ and $\underline{G}' < 0$.

The function, $\underline{G}(\overline{x,p})$, is the Green's function with singularity at $\underline{p} \in \underline{M}^n_H$. (For $n = 2$, the definition must be slightly modified.)

If $H \equiv 0$, then $\underline{k} = r$ and

$$\underline{G} = \frac{1}{(n-2)\mathrm{Vol}(S^{n-1})} r^{2-n}\,.$$

The smooth function,

$$\underline{U} = \int_0^r \underline{k}^{-(n-1)}(s)\left(\int_0^s \underline{k}^{n-1}(u)du\right)ds\,,$$

satisfies $\underline{\Delta}\,\underline{U} = 1$, $\underline{U}(0) = 0$, $\underline{U}' \ge 0$ and $|\nabla\underline{U}(r)| = \mathrm{Vol}(B_r(\underline{p}))/\mathrm{Vol}(\partial B_r(\underline{p}))$.

If $H \equiv 0$, then

$$\underline{U} = \frac{r^2}{2n}\,.$$

Given $R > 0$, put $\underline{G}_R = \underline{G} - \underline{G}(R)$ and $\underline{U}_R = \underline{U} - \underline{U}(R)$.

Set $c_1 = -\frac{\underline{U}'(R)}{\underline{G}'(R)} > 0$. We have $\underline{G}''_R \geq 0$, $\lim_{r\to 0} \underline{G}'_R = -\infty$, and $\underline{U}'' \geq 0$, $\underline{U}'(0) = 0$. Thus, if we let

$$\underline{L}_R = c_1\, \underline{G}_R + \underline{U}_R\,,$$

then we have $\underline{\Delta}\underline{L}_R = 1$, $\underline{L}'_R \leq 0$ on $(0, R]$, and $\underline{L}_R(R) = 0$.

The function, $\underline{L}_R$, plays a role in the Abresch-Gromoll inequality discussed below.

The segment inequality.

The segment inequality is a sharpened version of the Poincaré inequality. It is of particular importance in proving almost rigidity theorems in the collapsed case.

Given a nonnegative function, $g \geq 0$, put

$$\mathcal{F}_g(x_1, x_2) = \inf_\gamma \int_0^\ell g(\gamma(s))ds\,,$$

where the inf is taken over all minimal geodesics, γ, from x_1 to x_2, and s denotes arclength.

THEOREM 1.5. ([**ChCo3**]) Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let $A_1, A_2 \subset B_r(p)$, with $r \leq R$. Then

$$\int_{A_1\times A_2} \mathcal{F}_g(x_1, x_2) \\ \leq c(n, R)r\big(\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2)\big) \times \int_{B_{2R}(p)} g\,.$$

One specific consequence of the segement inequality is a lower bound on the first eigenvalue of the Laplacian for Dirichlet problem on a metric ball $B_R(p)$; compare [**B**], [**LiSch**], [**Li**]. To see this, let $f : B_R(p) \to \mathbf{R}$ satisfy $f \,|\, B_R(p) \equiv 0$ and extend f to vanish identically outside $B_R(p)$. Take $g = |\nabla f|$, $A_1 = A_2 = B_R(p)$.

The proof of the segment inequality employs the techniques which lead to relative volume comparison.

The Cheng-Yau gradient estimate.

THEOREM 1.6. ([**CgYau**]) Let

$$\mathrm{Ric}_{M^n} \geq (n-1)H \qquad (\text{on } B_{R_2}(p))\,,$$

and let $u : B_{R_2}(p) \to \mathbf{R}$ satisfy

$$u \geq 0\,,$$
$$\Delta u = K(u)\,.$$

Then for $R_1 < \mathrm{R}_2$, on $B_{R_1}(p)$,

$$\frac{|\nabla u|^2}{u^2} \leq \max\big(2u^{-1}K(u), c(n, R_1, R_2, H) + 2u^{-1}K(u) - 2K'(u)\big)\,.$$

The Cheng-Yau gradient estimate, which we have not stated in utmost generality, can be further specialized to the particularly important case in which u satisfies an equation of the form $\Delta u = a\cdot u + b$. This class of functions includes harmonic functions and eigenfunctions of Δ.

Note that by slightly changing the form of the function, $K(u)$, the assumption, $u \geq 0$, can be achieved by adding a suitable constant to the function u. The resulting estimate then depends on the size of this constant. In particular, if u is harmonic, then so is $u + c$, for any constant c.

The Abresch-Gromoll inequality.

Let $\overline{A_{R_1,R_2}(p)}$ denote the metric annulus, $B_{R_2}(p) \setminus \overline{B_{R_1}(p)}$.

Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let $f : B_{R_2}(p) \to \mathbf{R}$. The Abresch-Gromoll inequality provides a bound for $f(p)$ under the assumptions:

i) $f \,|\, \partial B_{R_2}(p) \geq 0$,

ii) Δf has a definite upper bound on $\overline{A_{R_1,R_2}(p)}$,

iii) $f(x)$ has a definite upper bound at some $x \in A_{R_1,R_2}$,

iv) the Lipschitz constant of f has a definite bound on $B_{R_1}(p)$.

At the heart of Abresch-Gromoll inequality is a beautiful quantitative extension of the maximum principle.

Let the functions, $\underline{L}_R$, $\underline{G}_R$ be defined as in the subsection on Laplacian comparison.

THEOREM 1.7. ([**AbGl**]) Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let $\overline{B_{R_2}(p)} \subset \Omega$. For $\delta > 0$, let $f : \overline{B_{R_2}(p)} \to \mathbf{R}$ satisfy $f \,|\, \partial B_{R_2}(p) \geq 0$ and

$$\Delta f \leq \delta \qquad (\text{on } \overline{A_{R_1,R_2}(p)})\,,$$

$$\mathrm{Lip}\, f \leq c \qquad (\text{on } B_{R_1}(p))\,.$$

If for some $x \in A_{R_1,R_2}(p)$ and $t > 0$,

$$f(x) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \qquad (R = \overline{x,p})\,,$$

then

$$f(p) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1) + cR_1\,.$$

Since we assume $\mathrm{Lip}\, f \leq c$ on $B_{R_1}(p)$, to get the bound on $f(p)$, it suffices to show that somewhere on $\partial B_{R_1}(p)$, we have $f \leq (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1)$. This in turn, is a quantitive extension of the maximum principle (on the annulus $A_{R_1,R_2}(p)$). It is proved with the aid of Laplacian comparison. To see the relevance of the maximum principle, consider the limiting case $\delta = 0$.

Abresch and Gromoll used their inequality in proving a weakened quantitative extension of the splitting theorem. For this, the metric is rescaled so that the lower bound on Ricci curvature is close to 0. Later, this weakened version played a significant role in the proof of the full quantitative extension of the splitting theorem; see Theorem 2.3 of Section 2.

The cutoff function with bounded Laplacian.

The the proof of the following theorem, which gives the existence of a cutoff function with bounds on the Laplacian, is based on ideas which are related to those used in proving the Abresch-Gromoll inequality. It is required for the proof of the almost splitting theorem.

THEOREM 1.8. ([**ChCo3**]) Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let $B_{R_2}(p) \subset M^n$. Then for all $R_1 < R_2$, there exists $\phi : M^n \to [0,1]$, such that

$$\phi \,|\, B_{R_1}(p) \equiv 1\,,$$

$$\operatorname{supp}\phi \subset B_{R_2}(p),$$

$$|\nabla\phi| \leq c(n, R_1, R_2)\,,$$

$$|\Delta\phi| \leq c(n, R_1, R_2)\,.$$

2. Almost rigidity and rescaling

Typically, geometric properties of a riemannian manifold which are excluded if the curvature satisfies a certain strict inequality, can occur only under very restricted circumstances if the strict inequality is relaxed to a weak one. Theorems to this effect are known as *rigidity theorems.*

The splitting theorem.

Nonnegative Ricci curvature tends to oppose noncompactness. In particular, according to Myers' theorem, if the Ricci curvature has a positive lower bound, $\mathrm{Ric}_{M^n} \geq (n-1)H > 0$, then

$$\operatorname{diam}(M^n) \leq \frac{\pi}{\sqrt{H}}\,.$$

While manifolds with nonnegative Ricci curvature (e.g. 2-dimensional paraboloids) can certainly be noncompact, the situation becomes highly constrained if the manifold is noncompact in a sufficiently strong sense.

A geodesic, $\gamma : (-\infty, \infty) \to M^n$, is called a *line*, if each finite segment of γ is minimal.

THEOREM 2.1. ([**ChGl2**]) If M^n is complete, $\mathrm{Ric}_{M^n} \geq 0$, and M^n contains a line, then M^n splits as an isometric product, $M^n = \mathbf{R} \times \overline{M}^{n-1}$, for some $\overline{M}^{n-1}$.

In proving Theorem 2.1, one starts by considering a *ray*, $\gamma : [0, \infty) \to M^n$. By definition, each finite segment of γ is minimal. Associated to γ is its *Buseman function*, defined by

$$b_\gamma(x) = \lim_{s\to\infty} \overline{x, \gamma(s)} - s\,.$$

By using the triangle inequality, it is easily seen that the limit exists. The pointwise Lipschitz constant of b_γ is $\equiv 1$.

In the special case in which $\gamma : (-\infty, \infty) \to \mathbf{R}^n$ is the x-axis, b_γ is the linear harmonic function $-x$. In the general case, one shows by means of Laplacian comparison that b_γ is superharmonic.

For γ a line, let $b_\pm$ denote the Buseman functions associated to the rays $\gamma \,|\, [0, \infty)$, $-\gamma \,|\, [0, -\infty)$, which form the two halves of γ. The functions, b_+, b_-, are superharmonic and by the triangle inequality $b_+ + b_- \geq 0$. Since γ is a line, it also follows that $b_+ + b_- \,|\, \gamma \equiv 0$.

From the strong maximum principle, we now get $b_+ + b_- \equiv 0$. Hence, $b_+ = -b_-$ is both superharmonic and subharmonic. So b_+, b_-, are harmonic functions. In particular, they are smooth.

Clearly, $|\nabla b_+| \equiv 1$ and by Bochner's formula, $\mathrm{Hess}_{b_+} \equiv 0$ i.e. ∇b_+ is parallel. Now, the existence of an isometric splitting, given by the level surfaces of b_+ and the integral curves of ∇b_+, follows from the de Rham decomposition theorem.

REMARK 2.2. For the case in which the sectional curvature is nonnegative, the splitting theorem is due to Toponogov, whose proof relied on his comparison theorem for geodesic triangles; see [**Top1**], [**Top2**] and compare [**ChGl1**]; compare also [**EschHei**].

Making the discussion quantitative.

The quantitative generalization of the splitting theorem is most easily stated as an extension of the splitting theorem to pointed Gromov-Hausdorff limit spaces whose Ricci curvature is nonnegative in a generalized sense.

THEOREM 2.3. ([**ChCo2**]) *Let the sequence of complete manifolds, M_i^n, satisfy, $(M_i^n, m_i) \xrightarrow{d_{GH}} (Y, y)$,*

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i \qquad (\text{where } \delta_i \to 0)\,. \tag{2.4}$$

If Y contains a line, then Y splits as an isometric product, $Y = \mathbf{R} \times \overline{Y}$, for some length space $\overline{Y}$.

By arguing by contradiction, it is a simple exercise to translate Theorem 2.3 into an equivalent quantitative statement concerning manifolds whose Ricci curvature is bounded below by a small negative constant.

Note in this connection, that a line in Y need not be the limit of a sequence of lines in the manifolds, M_i^n, of the approximating sequence. However, a line in Y can always be viewed as the limit of a sequence of triangles in the manifolds, M_i^n, the excesses of which converge to zero. (The excess of a triangle is the minimum of the three numbers which are expressible as the sum of the lengths two sides minus the length of a third.)

To prove Theorem 2.3, one establishes the equivalent statement for smooth manifolds, in which the hypotheses of the splitting theorem are only satisfied up to small errors. The idea is to make quantitative, each of the steps in the argument which was outlined above. This turns out to be less straightforward than one might initially suppose.

Let $\mathrm{Ric}_{M^n} \geq -(n-1)\delta$, for some very small δ. Assume that $p, q_+, q_- \in M^n$ determine a triangle with small excess. Specifically let

$$\overline{p,q_-} + \overline{p,q_+} - \overline{q_-,q_+} \leq \delta\,,$$

$$\overline{p,q_\pm} \geq \delta^{-1}\,.$$

Put $b_+(x) = \overline{x,q_+} - \overline{p,q_+}$. Fix a ball, $B_R(p)$, where R can be taken arbitrarily large if δ is sufficiently small. Let $\mathbf{b}_+$ denote the harmonic function on $B_R(p)$ whose boundary values coincide with those of b_+. In the ideal case, that of Theorem 2.1, we would have $b_+ = \mathbf{b}_+$. So we would like to know that $b_+, \mathbf{b}_+$ are close in a suitably strong sense.

With the help of the Abresch-Gromoll inequality, it follows that $b_+ + b_- - \overline{q,q_+}$ is small on $B_R(p)$. By a straightforward argument based on Laplacian comparison and the maximum principle, we get that $b_+, \mathbf{b}_+$ are uniformly close on $B_R(p)$. From this, together with the lower bound for the smallest eigenvalue of the Dirichlet problem on $B_R(p)$ (see Theorem 1.5) we find that the gradients, $\nabla b_+, \mathbf{b}_+$ are close in the L_2 sense. (Here and below, the riemannian measure on $B_R(p)$ is normalized so as to make the volume of $B_R(p)$ equal to 1.)

In particular, $|\nabla \mathbf{b}_+|$ is close to the constant function, 1 in the L_2 sense. From Bochner's formula, used in concert with the cutoff function with bounded Laplacian (of Theorem 1.8), and the fact that $|\nabla \mathbf{b}_+|$ has small oscillation, it follows that $|\mathrm{Hess}_{\mathbf{b}_+}|$ is small in the the L_2 sense on $B_{\frac{R}{2}}(p)$; compare (1.1).

Finally, and this is the most significant step, by using the segment inequality, the gradient estimate and the information established above, one shows by an argument based on the first variation formula, that $B_{\frac{R}{2}}(p)$ is close in the Gromov-Hausdorff sense, to a ball in some product space, $\mathbf{R} \times Y$, where b_+ corresponds to the $\mathbf{R}$ coordinate; see **[ChCo3]** and compare **[Co1]**–**[Co4]**.

REMARK 2.5. A lower bound on volume is not required in Theorem 2.1, nor is the noncollapsing assumption is needed in Theorem 2.3.

Volume cones are metric cones.

Next, we consider the equality cases in the relative volume comparison theorem, Theorem 1.3. As before, let the metric on $\underline{M}^n_H$ be given in geodesic polar coordinates by $dr^2 + \underline{k}^2 g^{S^{n-1}}$.

THEOREM 2.6. *Let $\mathrm{Ric}_{M^n} \geq (n-1)H$ on the annulus, $A_{r_1,r_2}(p)$, and assume that*

$$\frac{\mathrm{Vol}(\partial B_{r_1}(p))}{\mathrm{Vol}(\partial B_{r_2}(p))} = \frac{\mathrm{Vol}(\partial B_{r_1}(\underline{p}))}{\mathrm{Vol}(\partial B_{r_2}(\underline{p}))}\,.$$

Then the metric on $A_{r_1,r_2}(p)$ is of the form,

$$dr^2 + \underline{k}^2 \tilde{g}\,,$$

for some smooth riemannian metric, $\tilde{g}$, on $\partial B_{r_1}(p)$.

In essence, to prove Theorem 2.6, one follows the proof of relative volume comparison and notes that in the resulting string of inequalities, equality must hold at every stage. For instance, the Schwarz inequality applied to the second fundamental form of a distance sphere enters in the proof of mean curvature comparison. Since we must be in the equality case, we get the key fact that everywhere on such a distance sphere, the second fundamental form must be diagonal.

So far, we have only considered the equality case of the first inequality in Theorem 1.3. In actuality, given Theorem 2.6, it is easily seen that the equality case of the second inequality in Theorem 1.3 can only occur for annuli in the space of constant curvature $\underline{M}^n_H$ i.e. $\tilde{g} = g^{S^{n-1}}$. However, if as in Theorem 2.3, we include limit spaces, then new examples arise.

We will assume nonnegative Ricci curvature in the extended sense of (2.4), since this is the condition which figures in the applications. The generalization for arbitrary lower bounds on Ricci curvature is straightforward.

EXAMPLE 2.7. Let $Y^2 \subset \mathbf{R}^3$ denote some 2-dimensional cone. By rounding off the cone tip, one sees that Y^2 is the Gromov-Hausdorff limit of a sequence of 2-dimensional surfaces which are convex and hence of nonnegative curvature. Thus, Y^2 has nonnegative Ricci curvature in the sense of (2.4). The volume function of a ball, $B_r(y)$, centered at the cone tip, y, satisfies

$$\frac{\mathrm{Vol}(B_r(y))}{\mathrm{Vol}(B_r(\underline{p}))} \equiv \frac{\mathrm{Vol}(B_1(y))}{\pi},$$

or equivalently,

$$\frac{\mathrm{Vol}(B_r(y))}{\mathrm{Vol}(B_1(y))} \equiv \frac{\mathrm{Vol}(B_1(\underline{p}))}{\mathrm{Vol}(B_1(\underline{p}))}.$$

Here, $\mathrm{Vol}(B_r(y))$ is 2-dimensional area, the model space, $\underline{M}_0^2$, is $\mathbf{R}^2$ with its usual flat metric and $\mathrm{Vol}(B_1(\underline{p})) = \pi$.

For Z a metric space, denote by $C(Z)$, the *metric cone*, on Z. By definition, $C(Z)$ is the completion of metric space, $(0,\infty) \times Z$, with metric,

$$\overline{(r_1, z_1),\,(r_2, z_2)} = r_1^2 + r_2^2 - 2r_1 r_2 \cdot \cos \overline{z_1, z_2} \qquad (\text{if } \overline{z_1, z_2} \leq \pi)\,,$$

and

$$\overline{(r_1, z_1),\,(r_2, z_2)} = r_1 + r_2 \qquad (\text{if } \overline{z_1, z_2} > \pi)\,.$$

To see that the definition is reasonable, think of the case of 3-dimensional cones with arbitrary cross-section in Euclidean space $\mathbf{R}^3$.

Let z^* denote the vertex of $C(Z)$. Thus, z^* is the unique point which is added when $(0,\infty) \times Z$, with the cone metric, is completed to get $C(Z)$.

Note that if there exist z_1, z_2, with $\overline{z_1, z_2} \geq \pi$, then $(1, z_1)$, $(1, z_2)$ lie on a line which passes through z^*. If $\overline{z_1, z_2} > \pi$, this line does *not* split off as an isometric factor. Thus, by Theorem 2.3, cones for which there exist such pairs of points do not arise as limits of sequences of spaces satisfying (2.4).

EXAMPLE 2.8. For $n > 1$, the n-fold ramified covering space of the plane, $\mathbf{R}^2$ (ramified at a single point) is a cone with cross-section the circle, $\mathbf{S}_{2n\pi}$, with intrinsic diameter $n\pi$ (i.e. the circle of circumference $2n\pi$). These cones does not arise as limit spaces satisfying (2.4).

The following theorem asserts that for limit spaces with nonnegative Ricci curvature in the sense of (2.4), volumes cones are metric cones.

THEOREM 2.9. ([**ChCo3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (2.4). For $\underline{p} \in \mathbf{R}^n$ and all $s \in [0, r]$, assume

$$\frac{\mathrm{Vol}(B_s(p))}{\mathrm{Vol}(B_1(p))} = \frac{\mathrm{Vol}(B_s(\underline{p}))}{\mathrm{Vol}(B_1(\underline{p}))}.$$

Then for some length space Z, with

$$\mathrm{diam}(Z) \leq \pi\,,$$

the ball, $B_r(p)$, is isometric to the ball, $B_r(z^*)$, in the cone $C(Z)$.

As in the case of the generalized splitting theorem, Theorem 2.3, an argument by contradiction allows Theorem 2.9 to be reformulated as an almost rigidity theorem for smooth manifolds whose Ricci curvature is bounded below by a small negative constant $-(n-1)\delta$. It is this equivalent formulation which is actually proved. The proof is similar in spirit to that of Theorem 2.3, but the technical details are somewhat more complicated.

Rescaling and almost splitting.

We return to the rescaling idea introduced in Section 0, which suggested a program for understanding the small scale structure of manifolds with $\mathrm{Ric}_{M^n} \geq -(n-1)$.

Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and $\mathrm{diam}(M^n) \geq 2$. Let $\gamma : [-L, L] \to$ be a geodesic segment of length $2L \geq 2$. If we rescale distance by a factor ϵ^{-1}, where $0 < \epsilon << 1$, and $1 << \epsilon^{-1}L$, then the ball, $B_\epsilon(\gamma(0))$, in the original metric, now looks to the naked eye, like a ball of radius 1, with center on in a line in a noncompact manifold with nonnegative Ricci curvature — or better — in a noncompact limit space, whose Ricci curvature is nonnegative in the sense of (2.4).

The generalized splitting theorem, Theorem 2.3, implies that this rescaled ball appears to be a ball in some isometric product space, $\mathbf{R} \times X$. Unless the space, X, is indistinguishable from a point, we can repeat the construction in some direction tangent to X, etc. In this sense, the geometry has a self-regulating feature:

The more directions, the more small scale approximate splitting.

Rescaling, almost volume cones and almost metric cones.

Theorem 2.9 above, is particularly useful in the noncollapsed case.

Let $\mathrm{Ric}_{M^n} \geq -(n-1)$ and let

$$\mathrm{Vol}(B_1(p)) \geq v\,.$$

The decreasing function, $\mathrm{Vol}(B_s(p))/\mathrm{Vol}(B_s(\underline{p}))$, satisfies

$$1 \geq \frac{\mathrm{Vol}(B_s(p))}{\mathrm{Vol}(B_s(\underline{p}))} \geq \frac{v}{\mathrm{Vol}(B_1(\underline{p}))}\,.$$

It follows that for $r \leq 1$ and all $\delta > 0$, the relation,

$$(1-\delta)\frac{\mathrm{Vol}(\partial B_{\frac{1}{2}r}(p))}{\mathrm{Vol}(\partial B_{\frac{1}{2}r}(\underline{p}))} \leq \frac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial B_r(\underline{p}))}\,,$$

is violated for the end points of at most a definite number, $N(\delta)$, of disjoint intervals $[\delta r, r]$.

When combined with rescaling, this leads to the conclusion that tangent cones at points of noncollapsed limit spaces are metric cones.

3. The structure of limit spaces

Let

$$\mathrm{Ric}_{M_i^n} \geq -(n-1)\,, \tag{3.1}$$

and let

$$(M_i^n, \overline{m}_i) \xrightarrow{d_{GH}} (Y, \overline{y})\,.$$

Tangent cones.

The basic notion for studying the infinitesimal structure of such limit spaces is that of a *tangent cone*, Y_y, at $y \in Y$. Intuitively, tangent cones are obtained by blowing up the space at a point and taking a limit, possibly after passing to a subsequence. In case Y happens to be a smooth riemannian manifold, this proceedure recovers the usual tangent space with its canonical metric.

Let ρ denote the distance function (i.e. the metric) of Y. Gromov's compactness theorem implies that every pointed sequence, $\{(Y, y, r_i^{-1}\rho)\}$, has a subsequence which converges in the pointed Gromov-Hausdorff sense, to some space $(Y_y, y_\infty, \rho_\infty)$. The limit, Y_y, of such a subsequence is called a *tangent cone* at y.

For a given $y \in Y$, the tangent cone need not be unique.

The regular and singular sets.

The *regular set*, $\mathcal{R}$, is the set of those points, $y \in Y$, such that there exists $k \leq n$ for which every tangent cone is isometric to $\mathbf{R}^k$.

As expected, in the noncollapsed case, it turns out that only $k = n$ is possible. Even if Y is collapsed, almost all points are regular, provided "almost all" is defined in an appropriate measure theoretic sense.

By definition, the *singular set*, $\mathcal{S}$, is the complement of the regular set.

The regular set need *not* be open; equivalently, the singular set need not be closed. One can easily construct convex 2-dimensional surfaces, Y^2, which are limits of suitable sequences of polyhedral surfaces, for which $\mathcal{S}(Y^2)$ is a countable set, with limit points that are regular points. It follows by "rounding the corners", that such Y^2 are limit spaces with positive curvature in the sense of (2.4); compare Example 2.7.

Iterated tangent cones have nonnegative Ricci curvature.

Let ρ denote the distance function of the riemannian manifold, M^n. An obvious diagonal argument gives the key fact that every tangent cone is itself a pointed limit space,

$$(M^n_{k(j)}, p_{k(j)}, r^{-1}_{k(j)}\rho_{k(j)}) \xrightarrow{d_{GH}} (Y_y, y_\infty, \rho_\infty)\,,$$

where the rescaled manifolds, $(M^n_{k(j)},\, r^{-1}_{k(j)}\rho_{k(j)})$, satisfy

$$\liminf_{k(j)\to\infty}\ \mathrm{Ric}_{M^n_{k(j)}} \geq 0\,;$$

compare (2.4). By a similar argument, it follows that any tangent cone, Y_{y_1}, at $y_1 \in Y_y$, is a limit space satisfying (2.4). More generally, if we pass to tangent cones of tangent cones an arbitrary number of times, then such *iterated* tangent cones have nonnegative Ricci curvature in the generalized sense.

From the generalized splitting theorem, Theorem 2.3, we get:

THEOREM 3.2. (**[ChCo3]**) *If $M_i^n \xrightarrow{d_{GH}} Y$ satisfies (3.1), then any iterated tangent cone which contains a line, splits off this line as an isometric factor.*

The noncollapsed case.

From now on we assume the noncollapsing condition

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0\,. \tag{3.3}$$

The ϵ-regular set.

The ϵ-regular set, $\mathcal{R}_\epsilon \supset \mathcal{R}$, consists of those points, $y \in Y$, such that for every tangent cone, Y_y, we have

$$d_{GH}(B_1(y_\infty), B_1(0)) < \epsilon\,,$$

where $B_1(0)$ denotes the unit ball in $\mathbf{R}^n$.

There exists $\epsilon(n) > 0$, such that for $\epsilon \leq \epsilon(n)$, the interior of $\mathcal{R}_\epsilon$ is homeomorphic to a smooth connected topological manifold which contains the regular set $\mathcal{R}$. The proof depends on a fundamental theorem of Colding to the effect that if $\mathrm{Ric}_{M^n} \geq (n-1)H$, then a ball, $B_r(p) \subset M^n$, is Gromov-Hausdorff close to the ball, $B_r(\underline{p}) \subset \underline{M}^n$, if and only if $\mathrm{Vol}(B_r(p))$ is close to $\mathrm{Vol}(B_r(\underline{p}))$; see [**Co2**], [**Co3**].

The proof that the interior of $\mathcal{R}_\epsilon$ is manifold also uses an *intrinsic* version of a classical result of Reifenberg; see [**ChCo3**]. Reifenberg's theorem says that if $C \subset \mathbf{R}^N$ is a closed subset with the property that for all $p \in C$ and all $r \leq 1$, the ball, $B_r(p)$, is sufficiently Hausdorff close (on its own scale) to a ball in some affine subspace, $V^n_{p,r} \subset \mathbf{R}^N$, then C is a topological n-manifold; see [**Reif**].

In the Einstein case, we can assume that $\epsilon(n)$ has been chosen such that for $\epsilon \leq \epsilon(n)$, we have $\mathcal{R}_\epsilon = \mathcal{R}$. In this case, $\mathcal{R}$ is a smooth Einstein manifold; see Theorem 4.2.

Noncollapsed tangent cones are metric cones.

From Theorem 2.9 and the previous discussion (compare also the discussion at the end of Section 2) we obtain:

THEOREM 3.4. ([**ChCo3**]) Let the noncollapsed limit space, $M^n_i \xrightarrow{d_{GH}} Y^n$, satisfy (3.1), (3.3). Then every (possibly iterated) tangent cone is a metric cone, $C(Z)$, on some length space, X, with $\mathrm{diam}(X) \leq \pi$.

Blow up arguments.

Theorems 3.2, 3.4, provide two respects in which noncollapsed iterated tangent cones are better behaved than arbitrary noncollapsed limit spaces satisfying (0.3). This enables certain statements concerning arbitrary noncollapsed limit spaces with $\mathrm{Ric}_{M^n_i} \geq -(n-1)$, to be proved by "blow up" arguments; [**Fe**]. These are proofs by contradiction, in which the main step consists of showing that if a desired property were ever to fail, it would already fail for some tangent cone. After repeating this step sufficiently many times, with *suitably chosen* iterated tangent cones, one arrives at a situation in which the geometry of the iterated tangent cone has improved to such an extent, that the desired property actually holds. This contradiction establishes that the property holds in general.

Theorem 3.7 below is proved by a blow up argument.

Volume convergence.

We recall the definition of ℓ-dimensional Hausdorff measure.

Let W denote a metric space. For $A \subset W$, consider the collection of countable coverings, $\{B_{r_i}(w_i)\}$, of A, such that $\sup_i r_i \leq \eta$. Here we allow $\eta = \infty$. Put

$$\mathcal{H}^\ell_\eta(A) = \inf_{\{B_{r_i}(w_i)\}} \omega_\ell \sum_i r_i^\ell \,,$$

where $\omega_\ell > 0$ is a certain explicit constant, which for ℓ an integer, is equal to the volume of the unit ball in $\mathbf{R}^\ell$.

It is clear that $\mathcal{H}^\ell_\eta(A)$ is a nonincreasing function of η. Define the ℓ-dimensional spherical Hausdorff measure of A by

$$\mathcal{H}^\ell(A) = \lim_{\eta \to 0} \mathcal{H}^\ell_\eta(A) \,,$$

where the value, $\mathcal{H}^\ell(A) = \infty$, is permitted.

It follows easily that there is a unique value, dim A, the *Hausdorff dimension* of A, with $0 \leq \dim A \leq \infty$, such that, $\mathcal{H}^\ell(A) = \infty$ for $\ell < \dim A$, and $\mathcal{H}^\ell(A) = 0$ for $\dim A < \ell$.

THEOREM 3.5. ([**ChCo3**]) Let the noncollapsed limit spaces, Y_i^n, satisfy (3.1), (3.3), for all i. If $(Y_i^n, y_i) \xrightarrow{d_{GH}} (Y^n, y)$, then for all $r < \infty$,

$$\lim_{i\to\infty} \mathcal{H}^n(\overline{B_r(y_i)}) = \mathcal{H}^n(\overline{B_r(y)})\,.$$

In follows from Theorem 3.5 that a noncollapsed limit space, Y^n, has Hausdorff dimension, n and (by Theorem 2.1) that n-dimensional Hausdorff measure on Y^n satisfies relative volume comparison.

REMARK 3.6. The special case of Theorem 3.5 in which Y_i^n, Y^n are smooth riemannian manifolds was conjectured by Anderson-Cheeger and proved by Colding; [**Co4**]. This was very significant for the development of the theory.

The natural filtration on the singular set.

For $0 \leq k \leq n-1$, let $\mathcal{S}_k \subset \mathcal{S}$ consist of those points for which no tangent cone splits off a factor, $\mathbf{R}^{k+1}$, isometrically. (As motivation, think of the k-skeleton of a simplicial complex.)

We have $\mathcal{S}_{n-1} = \mathcal{S}$. Equivalently, if some tangent cone at y is isometric to $\mathbf{R}^n$, then so is every other tangent cone at y.

By Theorem 3.7 below, $\dim \mathcal{S}_k \leq k$. Hence, $\dim \mathcal{S} \leq n-1$. From this and (3.3), it follows in particular that the regular set, $\mathcal{R}$, has full measure with respect to $\mathcal{H}^n$. In fact, Theorem 3.8 below gives $\mathcal{S} = \mathcal{S}_{n-2}$. Together with Theorem 3.7, this implies that the singular set has Hausdorff codimension 2.

THEOREM 3.7. ([**ChCo3**]) Let $M_i^n \xrightarrow{d_{GH}} Y^n$ satisfy (3.1), (3.3). Then

$$\dim \mathcal{S}_k \leq k\,.$$

The blow up argument used in proving Theorem 3.7 depends on Theorems 3.2, 3.4. For a closely related result (with a very different proof) for subsets of $\mathbf{R}^n$, see [**Wh**], which also contains additional related references.

THEOREM 3.8. ([**ChCo3**]) Let Y^n satisfy (3.1), (3.3). Then

$$\mathcal{S} = \mathcal{S}_{n-2}\,,$$

and

$$\dim \mathcal{S} \leq n-2\,.$$

If $y \in \mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2}$, then by definition, some tangent cone, Y_y, splits off a factor $\mathbf{R}^{n-1}$. Since $y \in \mathcal{S}$, it follows that Y_y is not isometric to $\mathbf{R}^n$. It is easy to see that the only remaining possibility is that Y_y is closed half space $\mathbf{R}^{n-1} \times \mathbf{R}_+$. So to prove Theorem 3.8, it suffices to show that this potential tangent cone does not actually occur. This is done by an argument which is essentially topological. The idea is that a limit of closed manifolds should not have a nonempty topological boundary.

The complex filtration in the Kähler case.

Here and in Sections 7, 8, results on Kähler manifolds which depend only on the lower bound $\mathrm{Ric}_{M_i^n} \geq -(n-1)$, will be included in our general discussion of that case. Thus, although $\mathrm{U}(\frac{n}{2})$ is a special holonomy group, only those theorems on Kähler manifolds which depend on the 2-sided bound, $|\mathrm{Ric}_{M_i^n}| \leq n-1$, will be bracketed with results on special holonomy.

By definition, if M^n is Kähler, there is a parallel almost complex structure J. Thus, $J^2 = -1$, $\nabla J = 0$.

Suppose M^n splits off a line isometrically, $M^n = \mathbf{R} \times \overline{M}^{n-1}$. If v denotes the parallel vector field tangent to the $\mathbf{R}$ factor of this splitting, then $\nabla v = 0$. Hence, $\nabla J(v) = 0$ as well.

By the de Rham decomposition theorem, it follows that the vector fields, $v, J(v)$, are tangent to the $\mathbf{R}^2$-factor of a *local* isometric splitting of the metric. In addition, the factor, $\mathbf{R}^2$, can be naturally identified with the complex plane $\mathbf{C}$. However, the local factor, $\mathbf{C}$, need *not* be global. To see this, consider the case of a 2-dimensional cylinder $\mathbf{R} \times \mathbf{S}^1 = \mathbf{C}^2/\mathbf{Z}$.

If we view the real line, $\mathbf{R}$, as the collapsed Gromov-Hausdorff limit of a sequence of cylinders for which the $\mathbf{S}^1$ factor shrinks to a point, we see that the local isometric factor corresponding to $J(v)$ (the $\mathbf{S}^1$ factor) can disappear in the limit.

It turns out that for noncollapsed limit spaces satisfying (2.4) which are metric cones, the above complications do not occur. Roughly speaking, an almost parallel vector field on a cone is always almost the gradient of the coordinate function corresponding to the factor, $\mathbf{R}$, of some (global) almost isometric splitting. The proof of this fact relies on a technical theorem from the general theory of metric measure spaces for which the measure satisfies a doubling condition and for which a (suitably formulated) Poincaré inequality holds; see Theorem 16.32 of [**Ch2**].

THEOREM 3.9. ([**ChCoTi2**]) Let $M_i^n \xrightarrow{d_{GH}} Y^n$ satisfy (3.1), (3.3) and assume that M_i^n is Kähler for all i. Then $\mathcal{S}_{2i+1} = \mathcal{S}_{2i}$ for all i. In particular, all strata have even codimension. Moreover, every tangent cone, Y_y, has a parallel almost complex structure. If $Y_y = \mathbf{R}^j \times C(X)$ denotes the isometric splitting for which j is maximal, then $\mathbf{R}^j = \mathbf{C}^{\frac{j}{2}}$.

In Theorem 3.9, we were intentionally vague about the sense in which there exists a parallel almost complex structure on the cone $C(X)$. In the Kähler-Einstein, in which the regular part is a smooth manifold, the exact meaning is clear.

We will give a version of Theorem 3.9 which covers all cases of special holonomy; see Theorem 6.1.

4. Einstein limit spaces

We now specialize the discussion to the Einstein case,

$$\mathrm{Ric}_{M_i^n} = \lambda g_i\,,$$

where we make the normalization

$$|\mathrm{Ric}_{M_i^n}| \leq n-1\,. \tag{4.1}$$

THEOREM 4.2. ([**ChCo3**]) There exists $\epsilon(n) > 0$, such that if the sequence of Einstein manifolds, $M_i^n \xrightarrow{d_{GH}} Y^n$, satisfies (3.3), (4.1), then for $\epsilon < \epsilon(n)$, the

ϵ-regular set, $\mathcal{R}_\epsilon$, is a smooth Einstein manifold. In particular, the singular set, $\mathcal{S} = \mathcal{S}_{n-2}$, is a closed set with dim $\mathcal{S} \leq n-2$.

Theorem 4.2 is a quite direct consequence of Theorem 3.7 and the following ϵ-regularity theorem of M. Anderson, which states that balls of almost maximal volume are standard.

THEOREM 4.3. ([**An2**]) There exists $\delta = \delta(n) > 0$ and constants, $K_k(n) < \infty$, such that if M^n is Einstein, with

$$|\mathrm{Ric}_{M^n}| \leq n-1\,,$$

and $B_r(p) \subset M^n$, satisfies

$$\mathrm{Vol}(B_r(p)) \geq (1-\delta)\mathrm{Vol}(B_r(\underline{p}))\,,$$

then $B_{\frac{r}{2}}(p)$ is the domain of a harmonic coordinate system in which

$$|g_{i,j}|_{C^k} \leq K_k r^{-k}\,,$$
$$\det(g_{i,j}) > K_0 > 0\,.$$

Anderson's theorem is proved by a blow up argument which depends on the fact that in a local harmonic coordinate system, the Einstein equation is a quasi-linear elliptic equation on the metric; see (1.2).

REMARK 4.4. Anderson also shows that if the Einstein condition is removed in Theorem 4.3, then there exists a harmonic coordinate system as above, in which the metric satisfies corresponding $C^{1,\alpha}$-bounds, for all $\alpha < 1$. Thus, if the Einstein condition is dropped in Theorem 4.2, the ϵ-regular set, $\mathcal{R}_\epsilon$, is a $C^{1,\alpha}$-riemannian manifold for all $\alpha < 1$. Essentially, this is the only difference between the case in which both (4.1) *and* the Einstein condition are assumed, and that in which only (4.1) is assumed.

From Theorem 4.3 and the behavior of (4.1) under scaling we get:

THEOREM 4.5. ([**ChCo3**]) Let the sequence, $M_i^n \xrightarrow{d_{GH}} Y^n$, satisfy (3.3), (4.1). Then the regular part of any iterated tangent cone is a smooth Ricci flat Einstein manifold.

The elementary fact that in dimension 3, Einstein manifolds have constant curvature, immediately implies:

COROLLARY 4.6. Let the sequence, $M_i^n \xrightarrow{d_{GH}} Y$, satisfy (3.3), (4.1). If for $k \leq 3$, an interated tangent cone splits isometrically as $\mathbf{R}^{n-k} \times C(X)$, then the regular part, $\mathcal{R}(Y_y)$, is flat.

Consider an iterated tangent cone of the form, $\mathbf{R}^{n-3} \times C(X)$, for which. the cross-section, X, is smooth. From Corollary 4.6, it follows that, $\mathbf{R}^{n-3} \times \mathbf{R}^3/\mathbf{Z}_2 = \mathbf{R}^{n-3} \times C(\mathbf{RP}(2))$ is the only nontrivial possibility. However, a slicing argument (like the one explained in Section 5) shows that if such a tangent cone actually did exist, then the manifold, $\mathbf{RP}(2)$, would bound some (nonorientible) 3-manifold. Since, the Euler characteristic of $\mathbf{RP}(2)$ satisfies $\chi(\mathbf{RP}(2)) = 1$ and the Euler characteristic of an even dimensional manifold which bounds is even, this is impossible.

Thus, we get:

COROLLARY 4.7. If $M_i^n \xrightarrow{d_{GH}} Y$ satisfies (3.3), (4.1) and $\mathcal{S}_{n-2}(Y_y)\backslash\mathcal{S}_{n-3}(Y_y) = \emptyset$, for all iterated tangent cones, Y_y, then Y satisfies $\mathcal{S} = \mathcal{S}_{n-4}$.

Given our previous discussion, it is natural to sharpen the conjecture of Anderson which was mentioned in Section 0.

CONJECTURE 4.8. If $M_i^n \xrightarrow{d_{GH}} Y^n$ satisfies (3.3), (4.1), then $\mathcal{S} = \mathcal{S}_{n-4}$.

Proving Conjecture 4.8 amounts to ruling out the occurance of tangent cones of the form, $\mathbf{R}^{n-2} \times C(\mathbf{S}^1_{2d})$, where $\mathbf{S}^1_{2d}$ denotes the circle of circumference $2d < 2\pi$. At least for the case of special holonomy, this can be done; see Theorem 0.11, whose proof is discussed in Section 5.

CONJECTURE 4.9. If $M_i^n \xrightarrow{d_{GH}} Y^n$ satisfies (3.3), (4.1), then $\mathcal{R}$ has finite topological type.

The conclusion of Conjecture 4.9 fails if the 2-sided bound on Ricci curvature, (3.3), is replaced by the lower bound (3.1); see [**Men1**].

5. Special holonomy and codimension 4 singularities

In this section we indicate the proof of Theorem 0.11.

Theorem 0.11 states that for noncollapsed Gromov-Hausdorff limits of sequences of manifolds with special holonomy (or Einstein 4-manifolds with anti-self-dual curvature) satisfying

$$|\mathrm{Ric}_{M_i^n}| \le n-1\,,$$
$$\mathrm{Vol}(M_i^n) \ge v\,,$$

we have

$$\mathcal{S} = \mathcal{S}_{n-4}\,.$$

By Corollary 4.7, to prove Theorem 0.11, it suffices to rule out tangent cones of the form $\mathbf{R}^{n-2} \times C(\mathbf{S}^1)$, which are not isometric to $\mathbf{R}^n$.

Those cases in which H is contained in one of the groups, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4})$, G_2, $\mathrm{Spin}(7)$ are not difficult to handle. It suffices pass to the limit and examine the situation away from the singular set $\mathbf{R}^{n-2} \times x^* \subset \mathbf{R}^{n-2} \times C(\mathbf{S}^1)$. In the cases, $H \subset \mathrm{U}(\frac{n}{2})$, $H \subset \mathrm{Sp}(\frac{n}{2})\mathrm{Sp}(1)$, it is necessary to take into account what is happening near the singularity which is assumed to be developing as the limit is approached.

The easier cases; $\mathcal{S} = \mathcal{S}_{n-4}$.

We will explain the argument for the case $H \subset \mathrm{SU}(\frac{n}{2})$. The cases, G_2, $\mathrm{Spin}(7)$, can be treated by essentially the same argument; see [**ChTi2**] and compare Theorem 6.1.

Suppose Y_y is a tangent cone of the form $\mathbf{R}^{n-2} \times C(\mathbf{S}^1)$. Since $H \subset \mathrm{U}(\frac{n}{2})$, it follows easily from Anderson's ϵ-regularity theorem, Theorem 4.3, that there is a limiting parallel almost complex structure, J, on the flat open manifold, $\mathcal{R}(Y_y)$, the regular part of Y_y. The holonomy group, $H(\mathcal{R}(Y_y))$, satisfies $H(\mathcal{R}(Y_y)) \subset \mathrm{SU}(\frac{n}{2})$, where $\mathrm{SU}(\frac{n}{2})$ is defined with respect to this J. Given the riemannian product structure, it now follows that the isometric factor, $\mathbf{R}^{n-2}$, is actually a complex subspace, $\mathbf{C}^{\frac{n}{2}-1}$, and in addition, that $C(\mathbf{S}^1) = \mathbf{C}$. Otherwise, the complex determinant of a holonomy transformation would not be equal to 1.

The remaining cases; $\mathcal{S} = \mathcal{S}_{n-4}$.

By means of the twistor space construction, the case, $H \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, can be reduced to a slight generalization of the case, $H \subset \mathrm{U}(\frac{n}{2})$, discussed below; see [**ChTi2**] for details.

In the $\mathrm{U}(\frac{n}{2})$ case ([**Ch3**]), Theorem 0.11 comes down to an ϵ-regularity theorem. The basic idea behind it and our subsequent ϵ-regularity theorems, is illustrated by the following elementary 2-dimensional example. Since $n = 2$, technical issues concerning slicing which are present in higher dimensions do not enter.

A model example.

Consider a conical piece of 2-dimensional surface, Y^2, in the shape of a paper cup (of the sort which is often dispensed at a water cooler). Assume that except at the cone tip, Y^2 is smooth and intrinsically flat. We ask if it is possible to replace Y^2 by a smooth manifold with boundary, which coincides with Y^2 except in a tiny neighborhood, U, of the cone tip, and for which the total amount of curvature contained in U is very small. Although U might be too small to observe directly, we can conclude that the proposed smoothing is impossible, by noting that the resulting metric would not obey the Gauss-Bonnet formula for manifolds with boundary.

Let M^2 denote the purported smooth manifold. The interior term in the Gauss-Bonnet formula is a very small number, ϵ, which is the sum contributions from an "observable region" and a "too small to be directly observable region". The boundary term, x, which is equal to the circumference of the bounding circle divided by 2π, satisfies $0 < x < 1$, and is a definite amount away from both 0 and 1 (since we imagine a cone in the shape of cup that could actually hold water). Thus, we get $0 < \epsilon + x < 1$. However, the Euler characteristic of M^2 is an integer; a contradiction.

REMARK 5.1. Note that in obtaining the above contradiction, it was *not* sufficient to assume that the amount of the curvature (measured in the L_1-norm) in a neighborhood of the rounded cone tip was small. In applying the Gauss-Bonnet formula, we also used the cone structure of the observable region.

Slicing.

For ϵ very small, let the metric on M^n satisfy $\mathrm{Ric}_{M^n} \geq -(n-1)\epsilon$. Assume that for some $\ell >> 1$, $\epsilon << 1$, the ball, $B_\ell(m)$, is ϵ-close in the Gromov-Hausdorff sense, to a ball, $B_\ell((\underline{0}, x^*))$, in $\mathbf{R}^{n-2} \times C(\mathbf{S}^1)$.

We want to "slice" M^n in such a way that the generic slice resembles the manifold M^2 in the model example. By bringing in the first Chern form, c_1, this slicing allows the $\mathrm{U}(\frac{n}{2})$ case of Theorem 0.11 to be treated by an argument which is very similar to the one used in the model example.

Since the same slicing strategy is also employed in our subsequent ϵ-regularity theorems, we begin more generally with a cone of the form $\mathbf{R}^k \times C(X)$, where $0 \leq k \leq n-2$.

From the proof of the generalized splitting theorem, Theorem 2.3, it follows that there exist harmonic functions, $\mathbf{b}_1, \ldots, \mathbf{b}_k$, on $B_1(m)$, which are very "close" to the coordinate functions of the factor, $\mathbf{R}^k$, on the ball, $B_3((\underline{0}, x^*))$. The notion of closeness is in the "Gromov-Hausdorff sense" which is meaningful since the functions involved are uniformly continuous.

Let u denote the distance function from the closed set $\mathbf{R}^k \times x^* \subset \mathbf{R}^k \times C(X)$. The function, u^2, satisfies the equation $\Delta u^2 = 2(n-k)$.

From the proof of Theorem 2.9, the "almost volume annulus implies almost metric annulus" theorem, there exists a function, $\mathbf{u} : B_3(m) \to \mathbf{R}_+$, which is very close to the function, u, in the Gromov-Hausdorff sense, such that $\Delta \mathbf{u}^2 = 2(n-k)$

We have maps, $\Phi : B_3(m) \to \mathbf{R}^k$, $\Lambda : B_3(m) \to \mathbf{R}^k \times \mathbf{R}_+$, given by $\Phi = (\mathbf{b}_1, \dots, \mathbf{b}_k)$, $\Lambda = (\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{u})$.

We would like to know that when intersected with sublevel sets of the the function, $\mathbf{u}$, generic fibres of the map, Φ, resemble (k-dimensional versions of) the surface, M^2, in the model example.

The first step is to show that the notion of "generic" makes good sense i.e. that the images of the maps, Φ, Λ, have almost full measure. Here, the point of departure is the theorem on volume convergence; see Theorem 3.5.

The next step uses the coarea formula and the Cheng-Yau gradient estimate to establish that the volumes of almost all of the fibres are close to the volumes of the corresponding sets in the space $\mathbf{R}^k \times C(X)$.

In proving the above assertions (and additional ones which play a role in Section 7) the fact that the functions, $\mathbf{b}_j$, $\mathbf{u}$, satisfy suitable elliptic equations is crucial for obtaining the required estimates. Among other things, Bochner formulas are used; for details, see [**ChCoTi2**].

At this point, we specialize to the case, $k = n-2$, $X = \mathbf{S}^1$, $H \subset \mathrm{U}(\frac{n}{2})$. (For the ϵ-regularity theorems of Section 7, it will be necessary to continue the discussion of the general case.)

Let S_{2d} denote the circle of circumference $2d$. Assume that $H \subset \mathrm{U}(\frac{n}{2})$ and that there exists a tangent cone of the form, $\mathbf{R}^{n-2} \times C(\mathbf{S}^1_{2d})$, which is not isometric to $\mathbf{R}^n$ i.e. $2d < 2\pi$. Then we can find a Kähler manifold, M_i^n, for which a rescaled ball, $B_1(m_i)$, is as close as we like to the ball, $B_1((\underline{0}, x^*))$, in the Gromov-Hausdorff sense.

As above, the 2-dimensional area of a generic slice, Σ_i^2, of M_i^n, is close to the area of the ball $B_1(x^*) \subset C(\mathbf{S}^1)$. Since the metric on M_i^n has been rescaled, the pointwise norm of the first Chern form, c_1 is very small, and we have

$$\lim_{i \to \infty} \int_{\Sigma_i^2} c_1 = 0\,.$$

On the other hand, according the theory of differential characters, the above integral is equal mod $\mathbf{Z}$, to the associated differential character, $\widehat{c}_1$, evaluated on $\partial \Sigma_i^2$; [**ChSi**]. The crucial point is that the quantity, $\widehat{c}_1(\partial \Sigma_i^2) \in \mathbf{R}/\mathbf{Z}$, is determined entirely from the geometry of TM_i^n at $\partial \Sigma_i^2$. In fact, as $i \to \infty$, the secondary geometric invariant, $\widehat{c}_1(\partial \Sigma_i^2)$, converges to the mod $\mathbf{Z}$ reduction of the boundary term in Gauss-Bonnet formula for $C(S^1_{2d})$. As in the model example, this is nonzero mod $\mathbf{Z}$, unless $2d = 2\pi$ i.e. unless $C(S^1_{2d}) = \mathbf{R}^2$. Thus, we get a contradiction.

To see why the boundary term in the Gauss-Bonnet formula appears, note that near $\partial \Sigma_i^2$, as $i \to \infty$, the convergence of M_i^n to $\mathbf{R}^{n-2} \times C(\mathbf{S}^1_{2d})$, takes place in the C^∞ topology (since we are away from the singularity which is purported to be developing). Thus, the tangent bundle, TM_i^n, converges to the Whitney sum of a flat trivial bundle of dimension $n-2$ and the tangent bundle to $C(\mathbf{S}^1)$.

By the Whitney sum formula in Chern Weil theory, the Chern form, $c_1(TM_i^n)$, converges, to the Chern form $c_1(\Sigma_i^2)$. Since dim $\Sigma_i^2 = 2$, the first Chern form is the Euler form i.e. the Gauss-Bonnet form of $C(\mathbf{S}^1_{2d})$. According to [**ChSi**], these relations continue to hold at the more refined level of differential characters. Moreover, when evaluated on $\partial\Sigma_i^2$, the Euler character is just the boundary term in the Gauss-Bonnet formula.

REMARK 5.2. In the above argument, we used the first Chern form. This required a Kähler structure and a 2-sided bound on Ricci curvature. Had we instead attempted to use the Gauss-Bonnet formula for the induced metric on a slice, then given our assumptions, it would not have been possible to show that the interior term approached zero as $i \to \infty$; compare the discussion in Section 7.

6. Special holonomy and tangent cones

In this section we indicate the proof of Corollary 0.13 and give some additional results on the classification of tangent cones for the case of special holonomy; for details see **[Ch3]**, **[ChTi2]**.

Corollary 0.13 states that if $M_i^n \xrightarrow{d_{GH}} Y$, where the manifolds, M_i^n, have special holonomy and satisfy the 2-sided bound, $|\mathrm{Ric}_{M_i^n}| \leq n-1$, then for all $y \in \mathcal{S}_{n-4} \setminus \mathcal{S}_{n-6}$, there exists at least one tangent cone, Y_y, of H-orbifold type,

$$Y_y = \mathbf{R}^{n-4} \times \mathbf{R}^4/\Gamma\,.$$

By definition, if $y \in \mathcal{S}_{n-4} \setminus \mathcal{S}_{n-5}$, then there exists some tangent cone of the form, $\mathbf{R}^{n-4} \times C(X^3)$. By Corollary 4.6, the regular part of this cone is Ricci flat. From Theorem 0.11, it follows that X^3 is smooth. It follows that $Y_y = \mathbf{R}^{n-4} \times \mathbf{R}^4/\Gamma$ as above.

It remains only to show

$$\mathcal{S}_{n-5} \setminus \mathcal{S}_{n-6} = \emptyset\,,$$

and if $H \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, then

$$\mathcal{S}_{n-5} \setminus \mathcal{S}_{n-8} = \emptyset\,.$$

This is a consequence of the following Theorem 6.1.

Recall the inclusions $\mathrm{SU}(\frac{n}{2}) \subset \mathrm{U}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4}) \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, and $\mathrm{SU}(3) \subset \mathrm{G}_2 \subset \mathrm{Spin}(7)$.

THEOREM 6.1. (**[ChTi2]**) Let the sequence, $M_i^n \xrightarrow{d_{GH}} Y^n$, of manifolds with special holonomy, H, satisfy (3.3), (4.1).

i) If $H \subset \mathrm{U}(\frac{n}{2})$, then

$$\mathcal{S}_{n-i} \setminus \mathcal{S}_{n-i-1} = \emptyset \qquad (i \neq 2k, \text{ for some } k \in \mathbf{Z})\,.$$

ii) If $H \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, then

$$\mathcal{S}_{n-i} \setminus \mathcal{S}_{n-i-1} = \emptyset \qquad (i \neq 4k, \text{ for some } k \in \mathbf{Z})\,.$$

iii) If $H \subset \mathrm{Spin}(7)$, then

$$\mathcal{S}_{n-i} \setminus \mathcal{S}_{n-i-1} = \emptyset \qquad (i \neq 4, 6, 7, 8).$$

7. Integral bounds on curvature

In this section (and the next) we describe the effect of adding an integral bound on curvature to the hypotheses (3.1), (3.3). Recall that (3.1) specifies only a lower bound on Ricci curvature and not a 2-sided bound as in Sections 5, 6. However, the results apply in particular to Einstein manifolds.

The integral bound is an extra apriori assumption. But in the case of special holonomy, an L_2 bound on curvature is implied by a bound on the topological invariant, $\dot{C}(M^n)$, of (0.10); see Section 9.

Let $1 \leq p \leq \frac{n}{2}$. In the presence of an L_p bound on curvature, there are two basic statements which hold without further qualification:

i) The singular set has Hausdorff dimension $\leq (n-2p)$. In fact, $\mathcal{H}^{n-2p}(\mathcal{S}) = 0$ unless p is an integer.

ii) If p is an integer, the tangent cone is unique for $\mathcal{H}^{n-2p}$-a.e. $y \in \mathcal{S}$ and is of orbifold type $Y_y = \mathbf{R}^{n-2p} \times \mathbf{R}^{2p}/\Gamma$.

At least in the Kähler case and for manifolds with special holonomy, an additional fact holds:

iii) For p an integer, bounded subsets of the singular set have have finite $(n-2p)$-dimensional Hausdorff measure and are actually $(n-2p)$-rectifiable.

If the Kähler or special holonomy assumption is omitted in iii), rectifiability of bounded subsets is known for that part, $\mathcal{N}$, of $\mathcal{S}$, where "nonexceptional" tangent cones are present. This awkward point might just be a reflection of the present state of our technology. If $p = 1$ or $p = \frac{n}{2}$, then $\mathcal{N} = \mathcal{S}$.

There are three main steps in the proofs of i) and the statement in iii) concerning the finiteness of Hausdorff measure. The first of these is also the starting point for proving ii).

a) Show that if $k \leq 2p$, then for $\mathcal{H}^{n-2p}$-a.e. $y \in \mathcal{S}_{n-k}$, there exists at least one tangent cone of type $\mathbf{R}^{n-k} \times C(\mathbf{S}^{k-1}/\Gamma)$, $(k > 2)$, or $\mathbf{R}^{n-2} \times C(S^1_{2d})$, $(k = 2)$.

b) Prove an ϵ-regularity theorem to the effect that if a sequence satisfying (3.1), (3.3), converges to a tangent cone, Y_y, as in a), then when measured in the L_k sense, a definite amount of curvature must concentrate in the limit at y.

c) Show that for bounded subsets of $\mathcal{S}$, the curvature concentration required in b) can only occur on a subset with finite $(n-2p)$-dimensional Hausdorff measure, whose intersection with $\mathcal{S} \setminus \mathcal{S}_{n-2p}$ has vanishing $(n-2p)$-dimensional Hausdorff measure.

Step c) is proved by a standard covering argument. The assertion in step b) should be compared to that of Corollary 0.13.

L_1 curvature bounds.

Step a) is trivial for $p = 1$.

For $p = 1$, the ϵ-regularity theorem, step b), incorporates a direct generalization of the argument in the model case which was described in Section 5. For simplicity, we consider balls of radius 1; the case of arbitrary radius follows by a simple scaling argument.

THEOREM 7.1. ([**ChCoTi2**]) For all $v, \eta > 0$, there exists $\epsilon = \epsilon(\eta, v, n) > 0$, $\ell = \ell(\eta, v, n) < \infty$, $\delta = \delta(\eta, v, n) > 0$, such that if M^n satisfies

$$\mathrm{Ric}_{M^n} \geq -(n-1)\epsilon\,, \tag{7.2}$$

$$\mathrm{Vol}(B_1(m)) \geq v\,, \tag{7.3}$$

$$\fint_{B_{3\epsilon}(m)} |\mathrm{R}| \leq \delta\,, \tag{7.4}$$

and for some $0 < d \leq \pi$ and $(\underline{0}, x^*) \in \mathbf{R}^{n-2} \times C(\mathbf{S}^1_{2d})$,

$$d_{GH}(B_\ell(m), B_\ell((\underline{0}, x^*))) \leq \ell^{-1}\,, \tag{7.5}$$

then for $B_\epsilon(0) \subset \mathbf{R}^n$, we have

$$d_{GH}(B_1(m), B_1(0)) \leq \eta\,. \tag{7.6}$$

As in the proof of Theorem 0.11, the idea is to reduce to the 2-dimensional case by employing a slicing argument. Since we don't have a first Chern form at our disposal, we want to use the intrinsic Gauss-Bonnet formula on a generic slice. So we must continue the discussion of Section 5.

The next step in that discussion is to show that the second fundamental form of a generic level set of the map Φ is small in the L_2-sense. This, together with (7.4) and the Gauss curvature equation, gives that the L_1-norm of the intrinsic curvature on a generic slice is small. Similar properties hold for the map Λ.

At this point, the proof can be completed by employing on a generic slice, essentially the same argument as in the model example explained in Section 5.

Suppose we add to assumptions, (3.1), (3.3), the integral bound,

$$\fint_{B_1(m_i)} |\mathrm{R}| \leq c\,. \tag{7.7}$$

Since c) above follows by a standard covering argument, we get:

THEOREM 7.8. ([**ChCoTi2**]) Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (3.1), (3.3), (7.7). Then bounded subsets of the singular set, $\mathcal{S}$, have finite $(n-2)$-dimensional Hausdorff measure.

L_p curvature bounds; $p > 1$.

Assume

$$\fint_{B_1(m_i)} |\mathrm{R}|^{2p} \leq c\,. \tag{7.9}$$

Flatness of $\mathcal{R}(Y_y)$.

The proof of step a) begins by establishing a weaker statement. To avoid inessential technical difficulties, we explain the argument in the Einstein case where, by Theorem 4.2, the regular part, $\mathcal{R}$, is a smooth Einstein manifold.

Relation (7.9), together with a covering argument, easily implies that for $\mathcal{H}^{n-2p}$-a.e. $y \in Y$, if we rescale the ball, $B_r(y)$, to a ball of radius 1, and let $r \to 0$, then

$$\fint_{B_1(y)\cap\mathcal{R}} |\mathrm{R}|^p \to 0 \qquad (\text{as } r \to 0)\,. \tag{7.10}$$

To see this, let $\{U_i\}$ denote a sequence of neighborhoods whose intersection is some bounded subset of $\mathcal{S}$. Then

$$\lim_{i\to\infty}\int_{U_i\cap\mathcal{R}}|R|^p = 0\,,$$

and since we let $r \to 0$, in the covering argument, we need only consider balls which are contained in the U_i.

From (7.10) it follows that there exists $A \subset \mathcal{S}$, with $\mathcal{H}^{n-2p}(\mathcal{S}\setminus A) = 0$, such that $\mathcal{R}(Y_y)$ is flat, for all $y \in A$.

Existence of tangent cones of orbifold type.

If $y \in (\mathcal{S}_{n-k}\setminus\mathcal{S}_{n-k-1})$, there exists some tangent cone of the form $\mathbf{R}^k \times C(X)$. By induction on p, the cross-section, X can be assumed to be smooth. Otherwise, we would have $\dim\mathcal{S} \geq n-k+1 > n-2p$. (Here, we use $\dim\mathcal{S}(C(X)) = \dim\mathcal{S}(X)+1$.) It follows that X is of the form $\mathbf{S}^{k-1}/\Gamma$ which gives step a).

The ϵ-regularity theorem.

For p a half integer, an argument which closely resembles the one given at the end of Section 4 can be applied away from the singularity.

For n even and $p = \frac{n}{2}$ there is no problem in generalizing Theorem 7.1, since there is no second fundamental form to contend with.

As of this moment, for p an integer, with $1 < p < \frac{n}{2}$, the proof of Theorem 7.1 has not been generalized to the case in which an L_p curvature bound is assumed and the cone, $\mathbf{R}^{n-2}\times C(\mathbf{S}^1_{2d})$, is replaced by a cone $\mathbf{R}^{n-2p}\times C(X)$, where $X = \mathbf{S}^{n-2p-1}/\Gamma$. A technical difficulty arises when one attempts to estimate L_{2p}-norm of the second fundamental form of a generic slice.

The difficulty can be illustrated by considering the case, $\mathbf{R}\times C(X)$, in which there is just a single harmonic function $\mathbf{b}$. By a standard formula, the second fundamental form of a level surface of $\mathbf{b}$ satisfies

$$|II|^2 = \frac{|\mathrm{Hess}_{\mathbf{b}}|^2 - |\mathrm{Hess}_{\mathbf{b}}(\,\cdot\,,N)|^2}{|\nabla\mathbf{b}|^2}\,,$$

where $N = \nabla\mathbf{b}/|\nabla\mathbf{b}|$. So to estimate the L_{2p}-norm of the second fundamental form, one must bound an expression with $|\mathrm{Hess}_{\mathbf{b}}|^{2p}$ in the numerator and $|\nabla\mathbf{b}|^{2p}$ in the denominator.

The L_p bound on curvature, together with a higher order Bochner formula, yields an L_{2p} bound on $|\mathrm{Hess}_{\mathbf{b}}|$. This bound, which is also crucial for the results on rectifiability of singular sets, is explained at greater length in Section 8.

To get the required bound on the denominator, it would suffice show that at *all* points of a generic slice, $|\nabla b|$ is close to 1. Thinking (a bit loosely) in terms of the Sobolev imbedding theorem, we note that the L_{2p} bound on $|\mathrm{Hess}_{\mathbf{b}}|$ gives an L_{2p} bound on the norm of $\nabla|\nabla\mathbf{b}|$. However, since the slices have dimension $2p$, it would seem that what is actually needed is a bound on the $L_{2p'}$-norm of $\nabla|\nabla\mathbf{b}|$, for some $p' > p$. Such a bound is lacking; compare the discussion of Section 8.

If on the other hand, we are given an $L_{p'}$ bound on curvature, with $p' > p$, then we do get that $|\nabla\mathbf{b}|$ is close to 1, everywhere on a generic slice.

By a covering argument like that which plays a role in Theorem 7.8, we obtain:

THEOREM 7.11. ([**ChCoTi2**], [**Ch3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (3.1), (3.3), (7.9). Then

$$\dim \mathcal{S} \leq n - 2p\,,$$
$$\mathcal{H}^{n-2p}(\mathcal{S} \setminus \mathcal{S}_{n-k}) = 0 \qquad (k < 2p)\,,$$
$$\mathcal{H}^{n-2p}(\mathcal{S}) = 0 \qquad (p \text{ not an integer})\,.$$

Finiteness of Hausdorff measure.

Because of the difficulty in generalizing Theorem 7.1 to the case, $p > 1$, at present there is only a partial analog of Theorem 7.11 in the real case. In the complex case, one has an analog of Theorem 7.1 in which higher Chern forms, c_p, are used in the same way as c_1 is used in proving Theorem 0.11; compare Sections 5, 9. This leads to:

THEOREM 7.12. ([**ChCoTi2**], [**Ch3**]) Suppose that the sequence of Kähler manifolds, $M_i^n \xrightarrow{d_{GH}} Y$, satisfy (3.3), (4.1), (7.9). Then bounded subsets of the singular set, $\mathcal{S}$, have finite $(n-2p)$-dimensional Hausdorff measure.

The real case.

In the real case, the best we can do is to use Pontrjagin forms in place of Chern forms. This requires our assuming that $p = 2k$ is an *even* integer. However, the ϵ-regularity theorem still does not work perfectly. The reason is that there are certain tangent cones, $\mathbf{R}^{n-4k} \times C(\mathbf{S}^{4k-1}/\Gamma)$, for which all secondary geometric invariants associated to Pontrjagin polynomials of degree k vanish; see [**ChSi**]. This means that we can only deal with the subset, $\mathcal{N} \subset \mathcal{S}$, for which there exists a cone, $\mathbf{R}^{n-4k} \times C(\mathbf{S}^{4k-1}/\Gamma)$, which is *not* of this type.

If, for instance, $k = 1$, then the tangent cones which must be excluded are those of the form $\mathbf{R}^{n-4} \times C(\mathbf{S}^3/\Gamma)$, where $\mathbf{S}^3/\Gamma$ is a lens space, $L_{p,q}$, with $q^2 \equiv -1 \bmod p$.

As in Theorem 7.12, we get:

THEOREM 7.13. ([**ChCoTi2**], [**Ch3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$, satisfy (3.3), (4.1), (7.9), for $p = 2k$. Then bounded subsets of the subset, $\mathcal{N} \subset \mathcal{S}$, have finite $(n-4k)$-dimensional Hausdorff measure.

Uniqueness of tangent cones.

The statement concerning the existence of tangent cones of orbifold type can be strengthened.

THEOREM 7.14. ([**ChCoTi2**], [**Ch3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (3.3), (4.1), (7.9). Then there exists $A \subset \mathcal{S}$, with $\mathcal{H}^{n-2p}(\mathcal{S} \setminus A) = 0$, such that for all $y \in A$, the tangent cone is unique and of orbifold type $\mathbf{R}^{n-2p} \times \mathbf{R}^{2p}/\Gamma$.

For $p > 1$, Theorem 7.14 is proved by a deformation argument.

It is not difficult to see that (in all situations in geometric analysis in which tangent cones arise) the space of all tangent cones is connected in a suitable topology. So in the context of Theorem 7.14, one can start with a tangent cone of the form $\mathbf{R}^{n-2p} \times C(\mathbf{S}^{2p-1}/\Gamma)$ and try to show that it cannot be deformed within the space of tangent cones.

If we knew that *every* tangent cone split off an isometric factor, $\mathbf{R}^{n-2p}$, it would follow as in the proof of existence of tangent cones of orbifold type, that every tangent cone is of the form $\mathbf{R}^{n-2p} \times C(\mathbf{S}^{2p-1}/\Gamma)$. Although apriori, Γ might depend on the particular tangent cone, since spherical space forms cannot be deformed, we would be done. (Here, the assumption, $p > 1$, enters.)

A volume comparison argument.

Since we *can't* assume that every tangent cone splits off $\mathbf{R}^{n-2p}$ isometrically, we must bring in some additional information.

A general consequence of relative volume comparison is that, for all tangent cones at a given point, the $(n-1)$-dimensional Hausdorff measure of the cross-section is the same. Using this and a volume comparison argument which depends on knowing that for every tangent cone, Y_y, the regular part, $\mathcal{R}(Y_y)$, is flat, the proof of Theorem 7.11 can be completed.

Specifically, one shows that among all tangent cones which are sufficiently close to one of the form $\mathbf{R}^{n-2p} \times C(\mathbf{S}^{2p-1}/\Gamma)$, the cone, $\mathbf{R}^{n-2p} \times C(\mathbf{S}^{2p-1}/\Gamma)$, is the *unique* one for which the $(n-1)$-dimensional Hausdorff measure, of the cross-section is maximal.

REMARK 7.15. The volume comparison argument discussed above also uses the rigidity of spherical space forms (for $p > 1$), to locate inside of Y_y, a copy of $\mathbf{S}^{2p-1}/\Gamma$. This being done, one views Y_y as a tube around $\mathbf{S}^{2p-1}/\Gamma$ and uses the flatness of $\mathcal{R}(Y_y)$.

REMARK 7.16. If the conclusion of the volume comparison argument could be shown to hold without without appealing to the flatness of the regular part, $\mathcal{R}(Y_y)$, it would follow that the tangent cone is unique at *all* points of $\mathcal{S}_{n-4} \setminus \mathcal{S}_{n-5}$. Alternatively, this would follow if the flatness of $\mathcal{R}(Y_y)$ could be established without using the integral bound on curvature (7.9). Knowing uniqueness of the tangent cone at all points of $\mathcal{S}_{n-4} \setminus \mathcal{S}_{n-6}$ would be helpful in proving Conjecture 0.15.

REMARK 7.17. For the case $p = 1$, there is a completely different proof of uniqueness of tangent cones, which is closely related to the proof of rectifiablity of singular sets discussed in Section 8. This argument is valid for all p. However, it does not work in the absence of the integral bound (7.9).

8. Rectifiability of singular sets

A metric space, W, is called d-rectifiable if $0 < \mathcal{H}^d(W) < \infty$ and there exists a countable collection of subsets, C_j, with $\mathcal{H}^d(W \setminus \bigcup_j C_j) = 0$, such that each C_j is bi-Lipschitz to a subset of $\mathbf{R}^d$.

Rectifiability can be thought of as a weak measure theoretic generalization of the property of being a Lipschitz manifold. In particular, first order calculus makes sense on rectifiable spaces.

Proving that a space is rectifiable can be a step in the process of establishing additional regularity properties; compare [**HarShiff**], [**Ti3**]. It is hoped that the rectifiability theorems of this section and Section 9 will be useful in proving Conjecture 0.15.

While rectifiability plays a central role in geometric measure theory, most of the classical theory is concerned with subsets of $\mathbf{R}^N$; compare [**Fe**], [**Ma**], [**Sim1**]–[**Sim3**]. As a consequence, the standard criteria for establishing that a set is rectifiable do not apply in our (intrinsic) context.

Our main results are strengthenings of Theorems 7.8, 7.12, 7.13.

THEOREM 8.1. ([**Ch3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy (3.3), (4.1), (7.9). Then bounded subsets of the singular set, $\mathcal{S}$, are $(n-2)$-rectifiable.

THEOREM 8.2. ([**Ch3**]) Let the sequence of Kähler manifolds, $M_i^n \xrightarrow{d_{GH}} Y$, satisfy (3.3), (4.1), (7.9). Then bounded subsets of the singular set, $\mathcal{S}$, are $(n-2p)$-rectifiable.

THEOREM 8.3. ([**Ch3**]) Let $M_i^n \xrightarrow{d_{GH}} Y$, satisfy (3.3), (4.1), (7.9), for $p = 2k$. Then bounded subsets of the subset, $\mathcal{N} \subset \mathcal{S}$, are $(n-4k)$-rectifiable.

Theorems 8.1–8.3 are proved by directly constructing the sets, C_j, and bi-Lipschitz maps from the C_j to a Euclidean space of the appropriate dimension. Apart from the differences in the hypotheses of the underlying ϵ-regularity theorems, the proofs in all three cases are essentially the same.

To fix notation, we suppose that our integral bound is on the L_p-norm of the curvature and hence, that $\dim \mathcal{S} = n-2p$.

To grasp the main ideas, there is no harm in thinking of the Einstein case, in which we have a closed singular set and a smooth regular part.

Tangent cones and locally defined Lipschitz maps.

Consider $y \in A$, where $A \subset \mathcal{S}$ is as in Theorem 7.14. As in the discussion of slicing in Section 5, a tangent cone of the form $\mathbf{R}^{n-2p} \times C(\mathbf{S}^{2p-1}/\Gamma)$, gives rise to a locally defined map $\Phi_i = (\mathbf{b}_{i,1}, \dots, \mathbf{b}_{i,n-2p})$, from each approximating manifold, M_i^n, to $\mathbf{R}^{n-2p}$, where the $\mathbf{b}_{i,j}$ are harmonic functions.

By the Cheng-Yau gradient estimate, the maps, Φ_i, have a uniform gradient bound. Thus we can pass to the limit and get a locally defined Lipschitz map, $\Phi = (\mathbf{b}_1, \dots, \mathbf{b}_{n-2p})$, from a neighborhood of $y \in Y$, to $\mathbf{R}^{n-2p}$.

If we can show that when restricted to $\mathcal{S}$, the map, Φ, is *bi-Lipschitz* on some subset, $C \subset \mathcal{S}$, with $\mathcal{H}^{n-2p}(C) > 0$, then by standard soft argument, the proof can be completed.

Intuitively, we would like to find $C \subset \mathcal{S}$, with $\mathcal{H}^{n-2p}(C) > 0$, such that $\Phi\,|\,C$ is nondegenerate in the sense that the "gradients" of our Lipschitz functions, $\mathbf{b}_j$, are linearly independent. In the simplest case, $n - 2p = 1$, we want the "gradient" of $\mathbf{b}\,|\,C$ to be nonvanishing. (We use quotes because the sense in which $\mathbf{b}_j\,|\,\mathcal{S}$ has a gradient is not immediately clear.)

Volume nondegeneracy of Φ.

The first step is to show that $\Phi\,|\,\mathcal{S}$, is nondegenerate in a weaker sense. Namely, $(n-2p)$-dimensional measure of the range of $\Phi\,|\,\mathcal{S}$ has a definite lower bound. This statement entails kind of "gap" phenomenon: If A is nonempty, then $\mathcal{H}^{n-2p}(A) > 0$.

In fact, the gap theorem is just a quantitative version of the ϵ-regularity theorem. If near $y \in A$, most slices (i.e. fibres of Φ) did not intersect the singular set, we could apply the ϵ-regularity theorem directly on the limit space (to the union of such fibres) and conclude that $y \notin \mathcal{S}$.

Reduction to the existence of $\nabla\Phi$ on $\mathcal{S}$.

We observed above that $\Phi\,|\,\mathcal{S}$ *does not* totally compress volume. To complete the argument, it would suffice to know that $\Phi\,|\,\mathcal{S}$ has the additional property that if for $\mathcal{H}^{n-2p}$-a.e. $y \in A$, it compresses distance in some direction (i.e. if Φ is nowhere bi-Lipschitz) then it *does* compress volume.

Nice maps such as linear maps, or more generally, differentiable maps, do have this property. For instance, if a linear map defined on $\mathbf{R}^n$ has a nontrivial kernel, then its image is lower dimensional and hence, has vanishing n-dimensional volume. So we would like to know that near generic points of A, the map, Φ looks asymptotically (generalized) linear i.e. differentiable, in some generalized sense; compare (1.1).

In essence, we have reduced the problem of showing that the "gradient" of Φ does not vanish on a subset of A with positive measure, to showing that $\nabla\Phi$ exists on such a subset!

It is not difficult to see that rather than working directly on the singular set, it suffices to show that for $y \in A$, the restriction of Φ to the *regular part*, $\mathcal{R}$, looks asymptotically generalized linear as we approach y.

A higher order Bochner formula.

In our situation a function is (approximately) generalized linear if its Hessian is *small* in the integral sense; compare the discussion of Bochner's formula in Section 1 and the proofs of the splitting theorem and in its generalization, described in Section 2.

Let $f_{m,1}$ denote the average of f over the ball of radius 1.

Let $h : B_1(m) \to \mathbf{R}$ be harmonic and put

$$\sup_{B_{\frac{3}{4}}(m)} |\nabla h| = V\,.$$

By using in tandem the Bochner formulas for $\Delta|\nabla h|^2$, $\Delta|\mathrm{Hess}_h|^2$, one can derive an estimate for the quantities,

$$⨍_{B_{\frac{1}{2}}(m)} |\mathrm{Hess}_h|^{2p}\,, \qquad ⨍_{B_{\frac{1}{2}}(m)} |\nabla \mathrm{Hess}_h|^2 \cdot |\mathrm{Hess}_h|^{2p-4}\,, \tag{8.4}$$

in terms of

$$V, \qquad ⨍_{B_1(m)} \big||\nabla h|^2 - (|\nabla h|^2)_{m,1}\big|\,. \qquad ⨍_{B_1(m)} |R|^p\,; \tag{8.5}$$

see [**Ch3**],

For fixed V, the quantities in (8.4) will be as small as one likes, if the second two quantities in (8.5) are sufficiently small.

If we apply the above estimate to the function, $\mathbf{b}_{i,j}$, then the first term in (8.5) is bounded and the second vanishes in the limit as $i \to \infty$. However, since as $i \to \infty$, we are converging to a (rescaled) neighborhood of a *singular* point, $y \in A$, by the ϵ-regularity theorem, the curvature must be concentrating. Thus, while bounded, the third term in (8.5) is definitely *not* going to zero as $i \to \infty$.

A removability property of the singular set.

The above estimate can be passed to the limit space, Y, and applied to the functions $\mathbf{b}_j$ on the regular part $\mathcal{R}$. However, since apparently, a contribution from the curvature which has concentrated on the singular set must be included, what this accomplishes is not immediately clear.

The same considerations which led in Section 7, to the conclusion that tangent cones are flat on their regular parts, show that near $y \in A$, after suitable rescaling, the integral of $|R|^p$ over the *regular set*, $\mathcal{R}$, is small.

Thus, it would suffice to know that if near $y \in A$, we want to control the L_{2p}-norm of $\mathrm{Hess}_{\mathbf{b}_j}$ on $\mathcal{R}$, then only the curvature on $\mathcal{R}$ contributes to the estimate i.e. the curvature which concentrates in the limit on $\mathcal{S}$ can be ignored.

This assertion, which says that in a certain sense, the singular set is "removable", turns out to hold. To prove it, we attempt to repeat the derivation of (8.4), (8.5), while working on the smooth incomplete manifold $\mathcal{R}$.

Apart from a several integrations by parts, the argument is purely local. The whole point is to show is that the integrations by parts don't lead to a "residue term" attached to the singular set.

Given that bounded subsets of $\mathcal{S}$ have finite $(n-2p)$-dimensional Hausdorff measure, the estimates which were gotten by passing the estimates in the smooth case to the limit, provide enough control on the functions, $\mathbf{b}_j$, to prove the absence of a residue attached to the singular set. Here, although our ultimate interest is in $|\mathrm{Hess}_{\mathbf{b}_j}|_{L_{2p}}$, the estimate on the term involving $|\nabla \mathrm{Hess}_{\mathbf{b}_j}|$ is needed as well.

9. Anti-self-duality of curvature and the structure of singular sets

In dimension 4, the notion of anti-self-dual (or self-dual) curvature tensor, one which satisfies $*F = -F$ (or $*F = F$) is of crucial importance in gauge theory and of significance in riemannian geometry. The key point is the identity

$$\mp p_1(M^4) = \frac{1}{8\pi^2}\int_{M^4} |F|^2\,, \tag{9.1}$$

where p_1 denotes the first Pontrjagin class. This relation, which follows directly from Chern-Weil theory, implies that the underlying connection is an absolute minimum for the Yang-Mills functional.

The class of riemannian manifolds with special holonomy provides a framework for extending anti-self-duality to higher dimensions. For such manifolds, there is a parallel $(n-4)$-form Ω; compare [**Joy3**], [**Sa1**], [**Sa2**] and see (0.8). The curvature tensor of a connection on a bundle whose base space, M^n, has such a parallel $(n-4)$-form, is called *Ω-anti-self-dual* if

$$*\,F = -F \wedge \Omega\,; \tag{9.2}$$

compare [**Ti3**], [**BaHiSing1**], [**BaHiSing2**]. From now on, we will supress the dependence on Ω and say F satisfying (9.2) is anti-self-dual.

In [**Ti3**], a program was initiated, whose aim is to extend anti-self-duality and its consequences to higher dimensional gauge theory. Strong information was obtained on the degeneration problem, and a compactification of the moduli space was introduced, for the case in which the anti-self-dual connection on an auxilliary bundle varies, while the riemannian connection stays fixed. In the present paper, by contrast, we are concerned with the situation in which the anti-self-dual connection is the riemannian connection itself.

As noted in [**Ti3**], for certain purposes, it is useful to require only that the suitably defined trace free part, F_0, satisfies (9.2), and sometimes, that the trace of F is harmonic.

If M^n has special holonomy, then the trace free part, R_0, of its riemann curvature tensor, R, satisfies

$$* R_0 = -R_0 \wedge \Omega\,; \tag{9.3}$$

compare [**Joy3**], [**Sa1**], [**Sa2**].

With the exception of the cases, $H_0 = \mathrm{U}(\frac{n}{2})$, $H_0 = \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, we have $R_0 = R$.

In the $\mathrm{U}(\frac{n}{2})$ case, the part, $R - R_0$, is determined by the Ricci tensor and the almost complex structure.

In the $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ case, $(n > 4)$, $R - R_0$ is determined locally by the anti-commuting almost complex structures on the fibre (generated by I, J, K) and the Ricci tensor $\mathrm{Ric}_{M^n} = \lambda g$.

As in the case of 4-manifolds with anti-self-dual curvature, for manifolds with special holonomy, the L_2-norm of the full curvature tensor can be bounded in terms of a topological invariant, $C(M^n)$, and a bound on the norm of $F - F_0$, where in the riemannian case, the norm of $R - R_0$ can be bounded in terms of the Ricci tensor. The characteristic number, $C(M^n)$, is defined by

$$C(M^n) = (-p_1 \cup [\Omega])(M^n)\,.$$

Just as in Section 8 (compare also Section 7) we have:

THEOREM 9.4. ([**ChTi2**]) Let the sequence, $M_i^n \xrightarrow{d_{GH}} Y$, of manifolds with special holonomy and $\{C(M_i^n)\}$ bounded above, satisfy (3.1), (3.3). Then bounded subsets of the singular set, $\mathcal{S}$, are $(n-4)$-rectifiable.

REMARK 9.5. In light of Theorems 0.11, 6.1 and Corollary 0.13, the $(n-4)$-rectifiablity of $\mathcal{S}$ can be thought of as asserting that in a weak sense, $\mathcal{S}_{n-4}$ is calibrated by the $(n-4)$-form Ω; see [**HarLaw**].

REMARK 9.6. The statements of Conjecture 0.15 and Theorem 9.4 can be generalized by replacing the sequence of manifolds, $\{M_i^n\}$, by a sequence of H-orbifolds with codimension 4 singularities. Part of the interest of this degree of generality pertains to the quaternion-Kähler case with *positive* scalar curvature. As Blaine Lawson pointed out to us, there are many examples of orbifolds with this property, while conjecturally, in higher dimensions, the only smooth examples are symmetric spaces; see[**GaLaw**], [**HerHer**].

10. Appendix; Review of special holonomy

In this appendix, we recall some material concerning aspects of special holonomy which enter in the body of the paper. Specifically we discuss the Einstein condition and the role played by complex structures (or partial complex structures) in the various cases.

The cases: $\mathbf{U}(\frac{n}{2}), \mathbf{SU}(\frac{n}{2})$.

Let M^n be Kähler, with Hermitian almost complex structure, I, satisfying $\nabla I = 0$. Let $e_1, I(e_1), \ldots, e_{\frac{n}{2}}, I(e_{\frac{n}{2}})$ denote an orthonormal basis for the tangent space, M_p^n, at $p \in M^n$. The Kähler form, ω, is given by

$$\omega = e_1^* \wedge I(e_1^*) + \cdots + e_{\frac{n}{2}}^* \wedge I(e_{\frac{n}{2}})\,.$$

By the holonomy theorem of Ambrose-Singer (or by direct calculation) the curvature transformations are contained in $\mathfrak{u}(\frac{n}{2})$ and the riemann curvature tensor can be regarded as an element of $\bigwedge^2 \otimes \mathfrak{u}(\frac{n}{2})$. With this understanding, the relation $R(x,y,u,v) = R(x,y,I(u),I(v))$ and the Jacobi identity give the well known formula

$$\begin{aligned}
\langle\omega, R\rangle(x,y) &= \sum_{i=1}^{\frac{n}{2}} R(e_i, I(e_i), x, y) \\
&= -\sum_{i=1}^{\frac{n}{2}} R(x, e_i, I(e_i), y) - \sum_{i=1}^{\frac{n}{2}} R(I(e_i), x, e_i, y) \\
&= +\sum_{i=1}^{\frac{n}{2}} R(x, e_i, e_i, I(y)) + \sum_{i=1}^{\frac{n}{2}} R(x, I(e_i), I(e_i), y) \\
&= \mathrm{Ric}(x, I(y))\,. \qquad (10.1)
\end{aligned}$$

Thus, if we regard $R \in S^2(\mathfrak{u}(\frac{n}{2}))$, then M^n is Kähler-Einstein if and only if relative to the decomposition of $S^2(\mathfrak{u}(\frac{n}{2}))$ induced by the decomposition, $\mathfrak{u}(\frac{n}{2}) = \mathfrak{su}(\frac{n}{2}) \oplus \mathbf{R}\cdot\omega$, the mixed components of R vanish. Similarly, M^n is Ricci flat if and only if $R \in S^2(\mathfrak{su}(\frac{n}{2}))$.

The cases: $\mathbf{Sp(\frac{n}{4})Sp(1)}$, $\mathbf{Sp(\frac{n}{2})}$.

Let M^n be hyper-Kähler, with parallel anti-commuting parallel almost complex structures I, J, K, satisfying $IJ = K$. In this case, n is a multiple of 4. The group, $\mathrm{Sp}(\frac{n}{4})$ consists of those orthogonal transformations which commute with I, J, K.

Let $e_1, I(e_1), J(e_1), K(e_1), e_2, \ldots, K(e_{\frac{n}{4}})$ denote an orthonormal basis for the tangent space, M^n_p, at $p \in M^n$.

Let ω_I, ω_J, ω_K denote the the Kähler forms of I, J, K, respectively.

Since, M^n is Kähler with respect to I, by (10.1), we have $\langle\omega_I, R\rangle(x,y) = \mathrm{Ric}(x, I(y))$. Since $IJ = -JI$ and M^n is Kähler with respect to J,

$$\begin{aligned}
\langle\omega, R\rangle(x,y) &= \sum_{i=1}^{\frac{n}{4}} R(e_i, I(e_i), x, y) + \sum_{i=1}^{\frac{n}{4}} R(J((e_i), IJ(e_i)), x, y) \\
&= \sum_{i=1}^{\frac{n}{4}} R(e_i, I(e_i), x, y) - \sum_{i=1}^{\frac{n}{4}} R(J((e_i), JI(e_i)), x, y) \\
&= \sum_{i=1}^{\frac{n}{4}} R(e_i, I(e_i), x, y) - \sum_{i=1}^{\frac{n}{4}} R(((e_i), I(e_i)), x, y) \\
&= 0,. \qquad (10.2)
\end{aligned}$$

Thus, $\mathrm{Ric}_{M^n} \equiv 0$ in the hyper-Kähler case.

Let $\mathrm{Id} \in \mathrm{SO}(n)$ denote the identity transformation. Linear transformations of the form $r\cdot I + s\cdot J + t\cdot K + u\cdot \mathrm{Id}$, with $r^2 + s^2 + t^2 + u^2 = 1$, form a subgroup of $\mathrm{SO}(n)$. This subgroup, which is isometric to $\mathrm{Sp}(1)$, commutes with the action of $\mathrm{Sp}(\frac{n}{4})$ and intersects $\mathrm{Sp}(\frac{n}{4})$ in the element, $-\mathrm{Id}$. The group generated by $\mathrm{Sp}(\frac{n}{4})$ and $\mathrm{Sp}(1)$ is denoted $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$.

The Lie algebra of $\mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$ has the orthogonal orthogonal direct sum decomposition $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$, where $\mathfrak{sp}(1) = \mathbf{R}\cdot\omega_I \oplus \mathbf{R}\cdot\omega_J \oplus \mathbf{R}\cdot\omega_K \oplus \mathbf{R}$.

We have $\langle\omega_I,\omega_J\rangle = \langle\omega_I,\omega_K\rangle = \langle\omega_J,\omega_K\rangle = 0$. Thus, if the group, $\mathrm{U}(\frac{n}{2})$, is defined with respect to any one of the almost complex structures, I, J, K, then the remaining two almost complex structures are contained in $\mathfrak{su}(\frac{n}{2})$.

If M^n is a riemannian manifold with $H_0 \subset \mathrm{Sp}(\frac{n}{4})\mathrm{Sp}(1)$, the splitting, $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$, gives rise to orthogonal parallel sub-bundles of $\bigwedge^2(M^n)$. The second of these, which we denote by E, is spanned locally by I, J, K.

As above, let R denote the curvature tensor of M^n, and let R denote the curvature of E. It follows directly that $\mathrm{R}(L) = [R, L]$, for all $L \in E$.

In fact, for $n > 4$,

$$\begin{aligned}
\mathrm{R}_{J,K} &= \frac{8}{n+8}\cdot \mathrm{Ric}_{M^n}(I(x),y)\,, \\
\mathrm{R}_{I,K} &= -\frac{8}{n+8}\cdot \mathrm{Ric}_{M^n}(J(x),y)\,, \\
\mathrm{R}_{I,J} &= \frac{8}{n+8}\cdot \mathrm{Ric}_{M^n}(K(x),y)\,.
\end{aligned} \tag{10.3}$$

To see this, fix an almost complex structure, say I, and attempt to repeat (10.2), which in the hyper-Kähler case, gave $\mathrm{Ric}_{M^n} \equiv 0$. Using (10.3) in place of $\langle\omega_I, R\rangle(x,y) = \mathrm{Ric}_{M^n}(x, I(y))$ yields

$$\langle\omega_I, R\rangle(x,y) = \mathrm{Ric}_{M^n}(x, I(y)) - \mathrm{R}_{I,J}(x, J(y)) + \mathrm{R}_{I,K}(x, K(y))\,, \tag{10.4}$$

and instead of (10.2), we get

$$\mathrm{Ric}_{M^m}(x,y) = -\frac{n}{4}(x,y)\mathrm{R}_{J,K}\,. \tag{10.5}$$

Relations (10.4), (10.5) imply

$$\mathrm{Ric}_{M^m}(x,y) = \frac{n}{4}\mathrm{R}_{J,K}(x, I(y)) - \mathrm{R}_{I,K}(x, J(y)) + \mathrm{R}_{I,J}(x, K(y))\,, \tag{10.6}$$

which together with the corresponding relations for J, K, can be solved for $\mathrm{R}_{J,K}(x, I(y))$, $\mathrm{R}_{I,K}(x, J(y))$, $\mathrm{R}_{I,J}(x, K(y))$. This gives (10.3). (Note that for $n = 4$, the above system of equations is degenerate, but we are assuming $n > 4$.)

From the last equation in (10.4), we get

$$\mathrm{R}_{J,K}(x,y)|z|^2 = R(x,y,z,I(z)) + R(x,y,J(z),K(z))\,. \tag{10.7}$$

Employing (10.7) four times gives

$$\begin{aligned}
&R(x,I(x),z,I(z)) + R(x,I(x),J(z),K(z)) + R(J(x),K(x),z,I(z)) \\
&\quad + R(J(x),K(x),J(z),K(z)) \\
&\qquad = \frac{16}{n+8}\mathrm{Ric}_{M^n}(x,x)|z|^2 \\
&\qquad = \frac{16}{n+8}\mathrm{Ric}_{M^n}(z,z)|x|^2\,.
\end{aligned} \tag{10.8}$$

Relation (10.8) implies that M^n is Einstein.

In (10.2)–(10.8), we have followed [**Be**], which is based on [**Ish**].

Recall that the twistor space associated to M^n is by definition, the unit sphere bundle, $S(E)$, of E. When endowed with the natural riemannian submersion metric induced by the connection on E, the fibration,

$$\mathbf{S}^2 \to S(E) \xrightarrow{\pi} M^n\,,$$

has totally geodesic fibres and, by (10.2), apriori bounded parallel integrability tensor.

Given $L \in \pi^{-1}(p)$, the tangent space at L splits as a direct sum of its horizontal and vertical parts. Since L defines an Hermitian almost complex structure on M^n_p, there is an induced Hermitian almost complex structure on the horizontal subspace of $S(E)$ at L. The tangent space to the fibre is an oriented 2-dimensional inner product space. Thus, it carries a natural Hermitian almost complex structure as well. Hence, there is an induced almost complex structure, $\mathcal{I}$, on $S(E)$ which, as can be checked directly, is actually integrable.

In general, the almost complex structure, $\mathcal{I}$ is *not* parallel with respect to the riemannian connection on the tangent bundle to $S(E)$. However, there is a naturally associated Hermitian orthogonal connection for which $\mathcal{I}$ *is* parallel. This connection is characterized by the condition that its torsion tensor is given by $T = -\frac{1}{2}d^\nabla\mathcal{I}$, where $\mathcal{I}$ is viewed as 1-form with values in the tangent space and the exterior derivative, d^∇, is defined by using the riemannian connection, ∇, on $S(E)$.

The case: $\mathbf{G_2}$.

Let V^7 denote a real 7-dimensional inner product space, equipped with a 3-form, ϕ, for which there exists an orthonormal basis, $e_1, \ldots, e_7$, such that for $e_1^*, \ldots, e_7^*$ the corresponding dual basis,

$$\begin{aligned}\phi =& e_1^* \wedge e_2^* \wedge e_3^* + +e_1^*\wedge e_4^*\wedge e_5^* + e_1^*\wedge e_6^*\wedge e_7^* \\ &+ e_2^*\wedge e_4^*\wedge e_6^* - e_2^*\wedge e_5^*\wedge e_7^* - e_3^*\wedge e_4^*\wedge e_7^* - e_3^*\wedge e_5^*\wedge e_6^*\,. \end{aligned}\tag{10.9}$$

The Lie group, G_2, is the subgroup of $GL(V^7)$ that fixes ϕ. (A calculation shows that up to normalization, ϕ can be identified with $*\Omega$ of (0.8).)

Let $a = a_1e_1 + \cdots + a_7e_7$, $b = b_1e_1^* + \cdots + b_7e_7^*$, and put $\mathrm{Vol} = e_1^*\wedge\cdots\wedge e_7^*$. It is easy to check that

$$b\wedge i_a\phi\wedge i_a\phi\wedge i_a\phi = b(a)\cdot|a|^2\cdot\mathrm{Vol}\,.\tag{10.10}$$

Since, g fixes ϕ, by choosing b such that $b(a) = 1$, it follows that $|a|^2 = |g^{-1}(a)|^2\det(g)$, for all $g \in\mathrm{G}_2$. Since g^i fixes ϕ for all i, by replacing g by g^i and letting $i \to \infty$, it follows $\det(g) = 1$. Thus, $\mathrm{G}_2 \subset\mathrm{SO}(7)$.

Let i_e denote interior product with e. When restricted to the orthogonal complement, $[e_i]^\perp$, of the span of e_i, the form, $\omega_{e_i} := i_{e_i}\phi$, is clearly symplectic. Define the corresponding almost complex structure, I_{e_i}, on $[e_i]^\perp$, by $\omega_{e_i}(x,y) = \langle I_{e_i}(x), y\rangle$.

Let $H_e \subset G_2$ denote the subgroup fixing the unit vector e and put $H_{e_i} = H_i$. For all i, the subgroup $H_i \subset G_2$, consists of those elements of the unitary group, $U([e_i]^\perp)$, which also fix the 3-form, $\phi - e_i\wedge i_{e_i}\phi \in \bigwedge^3([e_i]^\perp)$. From this it is easily checked $H_{e_i} = \mathrm{SU}([e_i]^\perp)$. In particular, $\dim H_i = 8$.

It is not difficult to verify that the action of G_2 is transitive on the unit sphere. Hence, we get the fibration,

$$\mathrm{SU}(3) \to \mathrm{G}_2 \to \mathbf{S}^6\,,$$

from which it follows that G_2 is simply connected.

Let M^n denote a riemannian manifold with restricted holonomy contained in G_2.

To see that $\mathrm{Ric}_{M^7} \equiv 0$, note that since ϕ is G_2 invariant, it follows that ϕ is in the kernel of the action (by derivation) of the Lie algebra $\mathfrak{g}_2$. Writing this out in terms of the basis, $e_1, \ldots, e_7$, yields the 7 independent equations which characterize $\mathfrak{g}_2$. If we view $R \in \bigwedge^2(V^7) \otimes \mathfrak{g}_2$, then (written asymmetrically) these equations are

$$\begin{aligned}
R_{e_1,e_2} &= R_{e_4,e_7} + R_{e_5,e_6}\,, \\
R_{e_1,e_3} &= R_{e_4,e_6} - R_{e_5,e_7}\,, \\
R_{e_1,e_4} &= R_{e_2,e_7} - R_{e_3,e_6}\,, \\
R_{e_1,e_5} &= -R_{e_2,e_6} + R_{e_3,e_7}\,, \\
R_{e_1,e_6} &= R_{e_2,e_5} + R_{e_3,e_4}\,, \\
R_{e_1,e_7} &= R_{e_2,e_4} - R_{e_3,e_5}\,.
\end{aligned} \tag{10.11}$$

Let both sides of the i-th equation in (10.11) act on e_i, take the innner product with e_1, and apply the Jacobi identity to the right-hand of the resulting equation. If we sum over i, then terms the right-hand side cancel in pairs. Thus, $\mathrm{Ric}_{M^7}(e_1, e_1) \equiv 0$. Since G_2 acts transitively on the unit sphere, this shows that $\mathrm{Ric}_{M^n} \equiv 0$.

The case: Spin(7).

Let V^8 denote a real 8-dimensional inner product space, equipped with a 4-form, λ, for which there exists an orthonormal basis, $e_1, \ldots, e_8$, such that for $e_1^*, \ldots, e_8^*$ the corresponding dual basis,

$$\lambda = \phi_{e_8} \wedge e_8^* + *_{e_8} \phi_{e_8}\,, \tag{10.12}$$

where ϕ_{e_8} is the form in (10.9) and $*_{e_8}$ denotes the $*$ operator on the subspace spanned by $e_1^*, \ldots, e_7^*$. It is easy to check that the 4-form, λ, has a corresponding description with respect to any of the basis vectors, $e_1, \ldots, e_8$. The subgroup, $D \subset \mathrm{GL}(V^8)$, for which the adjoint action fixes λ, turns out to be $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ (and a calculation shows that up to normalization, λ can be identified with Ω of (0.8)). This can be seen as follows.

Let K_i denote the subgroup fixing e_i and let $\mathfrak{k}_i$ denote its Lie algebra. Clearly, K_i is isomorphic to G_2, for all i.

The Lie algebra generated by $\mathfrak{k}_8$ and the standard almost complex structure, I, given by $I(e_{2i-1}) = e_{2i}$, $I(e_{2i}) = -e_{2i-1}$, is $\mathfrak{su}(4)$ (where $\dim \mathfrak{su}(4) = 15$). It is easy to check that λ is in the kernel of the action of $\mathfrak{su}(4)$ (by derivation). Hence, $\mathrm{SU}(4) \subset D$.

Suppose there exists $g \in D$ such that $g(e_8) = c \cdot e_8$, with $c \neq 1$. The action of g^* on $(V^8)^*$ fixes the subspace spanned by $e_1^*, \ldots, e_7^*$. In addition, $g^*(e_8^*) = c^{-1} e_8^* + v^*$, where v^* is in the subspace spanned by $e_1^*, \cdots, e_7^*$. From this it is trivial to check that g fixes $*_{e_8}\phi_{e_8}$. Hence $g \in K_8$ and $c = 1$.

Since the subgroup, $\mathrm{SU}(4) \subset D$ acts isometrically and transitively on S^7, it follows that $D \subset \mathrm{O}(8)$; otherwise, there would exist g as above with $c \neq 1$. Since D preserves $\lambda^2 \neq 0$, this shows that $D \subset \mathrm{SO}(8)$.

The existence of the fibration,

$$G_2 \to D \to S^7\,,$$

implies that D is simply connected, with dim D=21. (This fibration plays a role in the proof of Theorem 9.4, as does the corresponding fibration for the group G_2.)

The subalgebra spanned by $\mathfrak{k}_7, \mathfrak{k}_8$ and I, is easily checked to be isomorphic to $\mathfrak{so}(7) = \mathfrak{spin}(7)$. Thus, D is isomorphic to Spin(7).

The remainder of the discussion of the Spin(7) case can be completed in a manner strictly analogous to the case G_2.

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