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NONCOMMUTATIVE YANG-MILLS THEORY AND STRING THEORY

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I review recent work on the relation between string theory and Yang-Mills theory on noncommutative spaces. In the long wavelength limit, string theory has a conventional string perturbation expansion in terms of ordinary Yang-Mills theory with small, α'-dependent corrections. On the other hand, in a certain limit, where the B-field is effectively large, the stringy excitations drop out and the string theory admits a systematic description in terms of non-commutative Yang-Mills theory. Compatibility of the two decriptions rests on a surprising mathematical fact: though the gauge group of noncommutative Yang-Mills theory is different from the conventional Yang-Mills gauge group, the equivalence relations generated by the two groups are the same, modulo a change of variables. Open string field theory might offer a systematic framework for describing open strings in terms of noncommutative associative algebras, with all of the excited string states included, but this description has not yet been useful.

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In this lecture, I will describe recent results with N. Seiberg [8] aiming to systematically describe the role in string theory of "noncommutative Yang-Mills theory" in the sense of A. Connes. Noncommutative Yang-Mills theory was first shown to give the solution of a string theory problem by Connes, Douglas, and Schwarz in the context of matrix model compactification on a torus [2]. There have been many subsequent contributions. Some of the new contributions that are most directly relevant to today's lecture are the work of Nekrasov and Schwarz [6] on instanton solutions of noncommutative Yang-Mills theory, the work of Schomerus

[7] on open strings in a background B-field, and Kontsevich's work on deformation quantization [4] together with its quantum field theory interpretation by Cattaneo and Felder [1]. See also [3] for a recent review. In the course of describing some of these results, I will also try to explain how they relate to a wider picture.

First of all, I think that the main reason that string theory will interest mathematicians in the long run is that at its core it is based on a new kind of geometry, a successor to Riemannian geometry. But this is very hard to convey, for two reasons. First of all, physicists do not really understand the new geometry in any systematic way. Second, the pieces of the story that we do understand are based on quantum field theory constructions that are very difficult and typically inaccessible mathematically.

Roughly speaking, in the new stringy geometry, the role of the Einstein equations is played by the requirement that a certain two-dimensional quantum field theory should be conformally invariant. This gives equations that are, in a certain limit, asymptotically close to the Einstein equations, but do not coincide with them. Both parts of this assertion are important.

If two-dimensional conformal invariance did not give the Einstein equations to very high accuracy under ordinary conditions, string theory would be in trouble, as a theory of nature. For Einstein's theory is certainly very successful experimentally.

On the other hand, as Einstein's theory apparently cannot be quantized, there is a need for a new theory that reduces to it in a suitable limit. If two-dimensional conformal invariance gave the Einstein equations on the nose, string theory would fall short of providing this new theory.

String theory in fact gives equations that differ from those of Einstein's theory in a very characteristic way. Einstein's classical equations are invariant under rescalings of length. If g denotes the spacetime metric and t is a positive real number, then the Einstein equations in vacuum are invariant under $g \to tg$. So, for example, classical black holes can come in any size.

In string theory, this scale invariance is lost. There is a characteristic length scale $\sqrt{\alpha'}$; in the most straightforward way of trying to relate string theory to the real world, this length is about 10^{-32} cm. (The value is found by using the string theory formulas for the fine structure constant and Newton's constant.) For objects much bigger than this, the Einstein equations are a good approximation. For small objects, they are not.

By analyzing the conditions for two-dimensional conformal invariance, one can make a systematic expansion of the equations in powers of the curvature. The expansion reads schematically

$$(1) \quad 0 = R_{IJ} + \alpha' R_{IKLM} R_J{}^{KLM} + (\alpha')^2 D_T R_{IKLM} D^T R_J{}^{KLM} + \dots$$

where only a few illustrative terms have been written, and the ellipses denote terms of higher order in α'. The generic term on the right-hand side of (1) is of the form $(\alpha')^s$ (for some integer $s \geq 0$) times a polynomial in the Riemann tensor and its covariant derivatives that is homogeneous of degree $2s + 2$ (here the Riemann tensor is considered to be of degree two, and a covariant derivative to be of degree one). In the small curvature limit, the equation is dominated by the leading term $0 = R_{IJ}$. For simplicity, I have here considered only the vacuum Einstein equations and their stringy extension. One can also incorporate matter; in fact, on the string theory side, one is forced to do so, and the matter takes a very definite form.

It is important for our story that the corrections to the Einstein equations that appear in equation (1) are ordinary, local, covariant terms. Einstein omitted them from his theory primarily on grounds of simplicity, but otherwise they obey most of his criteria. (The one general criterion formulated by Einstein that the corrections violate is, I believe, that they contain higher derivatives while Einstein looked for second order PDE's. As I have tried to explain elsewhere [10] in a lecture that was in a similar spirit to the one I am giving today, the higher derivative terms indicate that some additional "fast" variables have been averaged out of the equations. This is an important part of the story, but one that I will not describe today.)

Though the string theory corrections to the Einstein equations are usually negligible for large objects, for small objects these corrections are typically big. A relatively simple example is a Calabi-Yau threefold X. To use such a threefold for physical applications, one takes spacetime to be $\mathbf{R}^4 \times X$ (where $\mathbf{R}^4$ is intepreted as four-dimensional Minkowski space). When X is large compared to the stringy scale $\sqrt{\alpha'}$, it can be treated by classical Ricci-flat Kähler geometry, but when X becomes small, the classical description breaks down and wild things happen, such as mirror symmetry. It is very difficult to give a full account of all of the strange things that happen in string theory for Calabi-Yau threefolds.

I am going to talk today about a case where we can come closer to understanding what is going on in the stringy regime where the familiar classical equations fail. This will be the problem of Yang-Mills instantons on $\mathbf{R}^4$. How does this problem arise?

Gauge fields, and therefore instantons, can be incorporated in string theory in several different ways (which are nowadays often related to each other by nonperturbative dualities). In a previous lecture [10], I considered instantons mainly from the point of view of the heterotic string, but today we will use the older approach where gauge fields are associated with open strings while gravity is associated with closed strings. The most significant known physical application of the discussion is to D-branes in Type II superstrings. For today, all that one needs to know about D-branes is that a D-brane corresponds to a submanifold Y of spacetime, and that we will be doing gauge theory on Y. Moreover, for our purposes we can take Y to be a copy of $\mathbf{R}^4$ with its flat metric. $\mathbf{R}^4$ is linearly embedded in the spacetime, which for today's lecture we can take to be a flat $\mathbf{R}^{10}$.

Like the Einstein equations, the Yang-Mills equations receive corrections in string theory which are unimportant for large objects but very important for small ones. If F is the Yang-Mills curvature, $*$ the Hodge star operator, and D the gauge-covariant extension of the exterior derivative, the classical Yang-Mills equations read $0 = D * F$, or equivalently $0 = D^I F_{IJ}$. The stringy extension of these equations reads schematically

$$0 = D^I F_{IJ} + \alpha'[F_{KL}, D_J F^{KL}] + \dots \tag{2}$$

with higher order terms that are local, gauge invariant polynomials in F and its covariant derivatives, multiplying suitable powers of α'. As always, for large objects, the stringy corrections are small, and for small objects, they are large.

The classical instanton equations, in particular, are scale-invariant, so a classical Yang-Mills instanton can have any size. For a large instanton, the classical Yang-Mills equations are a good approximation; for a small instanton, they are not. So far, this is the usual story. The specific problem of instantons on $\mathbf{R}^4$, however, has some additional features.

The flat metric on $\mathbf{R}^4$ is, of course, essentially unique. However, the problem of string instantons on $\mathbf{R}^4$ depends not only on this flat metric but on an additional microscopic parameter $\theta \in \wedge^2\mathbf{R}^4$; I will say a word about its origin later. The self-dual projection of θ will be called θ^+.

If $\theta \neq 0$, then the rotation symmetry of $\mathbf{R}^4$ is broken to a subgroup. Thus, the case $\theta = 0$ is most similar to the classical instanton problem. Indeed, one can show using the hyper-Kähler structure of $\mathbf{R}^4$ that if $\theta^+ = 0$, then the instanton moduli space is the same in string theory as in classical Yang-Mills theory. The string theory instantons of size $\leq \sqrt{\alpha'}$ are not well approximated by classical instantons, but they have the same moduli space, if $\theta^+ = 0$. I will let $\mathcal{M}^\theta_{k,n}$ denote the string theory moduli space of based instantons on $\mathbf{R}^4$ of rank N and instanton number k for given θ.

In particular, the stringy instantons of $\theta^+ = 0$ have the familiar "bubbling" singularities that bedevil Donaldson theory. In "bubbling," an instanton becomes small and collapses to a delta function. Oddly, the term "bubbling," which was certainly coined long before instantons were studied in string theory, seems particularly appropriate in this stringy situation. Our instantons are supported on $\mathbf{R}^4 \subset \mathbf{R}^{10}$, but an instanton that shrinks to a point in $\mathbf{R}^4$ can literally "bubble away" into the higher dimensional world. The bubbled instanton is a point-like object (called technically a "-1-brane") in $\mathbf{R}^{10}$. The bubbling phenomenon in string theory is described by the ADHM construction of instantons.

If $\theta^+ \neq 0$, the instanton moduli space is modified from what it is in classical gauge theory. For $\theta^+ \neq 0$, there is a "no bubbling theorem," which is proved by using the fact that there is an energetic barrier to separating the -1-brane from $\mathbf{R}^4$. The barrier exists because a state with such a separated -1-brane would not be supersymmetric. Hence, for $\theta^+ \neq 0$, the moduli space $\mathcal{M}^\theta_{k,N}$ lacks the bubbling singularities. As a result, in fact, $\mathcal{M}^\theta_{k,N}$ is smooth if k and N are relatively prime. $\mathcal{M}^\theta_{k,N}$ still inherits a hyper-Kähler structure from the hyper-Kähler structure of $\mathbf{R}^4$, and it is independent of θ in the limit that the instantons are extremely large.

What hyper-Kähler manifold has those properties? According to the ADHM construction of instantons, the classical instanton moduli space is a "hyper-Kähler quotient" $\mu^{-1}(0)/G$, where μ is the hyper-Kähler moment map for a linear action of $G = U(k)$ on a flat hyper-Kähler manifold $\mathbf{R}^{4k^2+4kN}$. The relevant action of G preserves a hyper-Kähler structure on $\mathbf{R}^{4k^2+4kN}$, and μ is the associated hyper-Kähler moment map. μ takes values in $S = \wedge^{2,+}\mathbf{R}^4 \otimes \mathbf{g}$, with $\mathbf{g}$ the Lie algebra of G.

Because the center of G is $U(1)$, there is a natural embedding of $\wedge^{2,+}\mathbf{R}^4$ in S. A hyper-Kähler manifold that lacks bubbling singularities and is smooth if $(k, N) = 1$ is $\mu^{-1}(\theta^+)/G$, for nonzero $\theta^+ \in \wedge^{2,+}(\mathbf{R}^4)$. Taking $\theta^+ \neq 0$ does not change the behavior of the big instantons, but it eliminates bubbling for small instantons. This is what we want. The hyper-Kähler manifold $\mu^{-1}(\theta^+)/G$ has been studied mathematically as a partial desingularization of the usual instanton moduli space on $\mathbf{R}^4$ [5]. But what sort of objects does it parametrize?

This old question, which has been with us since the discovery of the ADHM construction of instantons and the hyper-Kähler quotient construction of hyper-Kähler manifolds, was neatly answered by Nekrasov and Schwarz [6]. They identified $\mu^{-1}(\theta^+)/G$ as the "moduli space of instantons on noncommutative $\mathbf{R}^4$" with the given θ^+.

To describe the appropriate notion of gauge theory on noncommutative $\mathbf{R}^4$, we begin with a bivector $\theta \in \wedge^2\mathbf{R}^4$. (The definition of the theory will depend on an arbitrary bivector. It can be shown, for instance via the ADHM construction, that the instanton moduli space depends only on the self-dual projection θ^+ of θ.) θ determines a Poisson bracket of functions on $\mathbf{R}^4$:

$$\{f,g\} = \sum_{i,j=1}^{4} \theta^{ij} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}. \tag{3}$$

One can deform the algebra of functions on $\mathbf{R}^4$ to an associative algebra $\mathcal{A}$, with multiplication $*$, such that

$$f * g - g * f = \{f,g\} + \dots \tag{4}$$

where the ellipses denote terms that in a suitable sense are small. This notion is often captured by introducing a formal deformation parameter $\hbar$, and writing $f*g-g*f = \hbar\{f,g\}+O(\hbar^2)$. For our present purposes, though, it is more pertinent to consider the behavior under scaling of $\mathbf{R}^4$. If we set $f_t(x) = f(x/t)$, keeping f fixed as $t \to \infty$, then $\{f_t, g_t\} \sim 1/t^2$. The property of the $*$ product that we want, apart from associativity, is that

$$f_t * g_t - g_t * f_t = \{f_t, g_t\} + O(1/t^4). \tag{5}$$

The $*$ product with these properties is essentially unique (up to automorphism of the algebra $\mathcal{A}$) and can be described by a very explicit formula:

$$f * g(x) = \left(\exp\left(\frac{1}{2}\sum_{i,j}\theta^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right) f(y)g(z)\right)\Bigg|_{y=z=x}. \tag{6}$$

Now let us move on to gauge theory, which we will formulate in the most elementary possible way. A gauge field, in the rank one case, is given by a "one-form"

$$A = \sum_{i=1}^{4} A_i dx^i, \tag{7}$$

where the A_i are elements of the algebra $\mathcal{A}$. The gauge-covariant exterior derivative is $D = d + iA$. The gauge transformation law is the statement that under an infinitesimal gauge transformation, $\delta D = i[D, \epsilon]$, with $\epsilon \in \mathcal{A}$. We get for noncommutative gauge fields of rank one

$$\delta A = d\epsilon + iA * \epsilon - i\epsilon * A. \tag{8}$$

For rank N gauge fields, one would use the same formulas, with A_i and ϵ regarded as elements of $\mathcal{A} \otimes \mathrm{Mat}(N)$, where $\mathrm{Mat}(N)$ is the algebra of $N \times N$ complex matrices. The gauge-covariant curvature is

$$\widehat{F} = \frac{1}{2} \sum_{i,j} F_{ij} dx^i \wedge dx^j, \tag{9}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + iA_i * A_j - iA_j * A_i. \tag{10}$$

The instanton equation is

$$\widehat{F}^+ = 0, \tag{11}$$

where $\widehat{F}^+$ is the self-dual projection of $\widehat{F}$. Nekrasov and Schwarz showed that solutions of this equation can be obtained by an ADHM construction, and that the moduli space of solutions so obtained is $\mu^{-1}(\theta^+)/G$.

This gives an interpretation of the deformed hyper-Kähler quotient, but is it what we want for string theory? So far, I have described two theories that both have classical Yang-Mills theory as a limiting approximation. In fact, in each case, the deformation has small effects for large objects, and large effects for small objects. In string theory the characteristic length, above which the theory reduces to classical Yang-Mills theory, is $\sqrt{\alpha'}$. In the case of the noncommutative Yang-Mills theory, a similar role is played by $\sqrt{|\theta|} = (\theta, \theta)^{1/4}$, where $(\ , \ )$ is the natural inner product on bivectors in $\mathbf{R}^4$, and we take a fourth root because (θ, θ) has dimensions of $(\text{length})^4$. If the functions f and g have characteristic scale of variation much greater than $\sqrt{|\theta|}$, then the Poisson bracket $\{f, g\}$ is small and the noncommutative Yang-Mills theory reduces to ordinary Yang-Mills theory.

So far, so good. There is a rough parallel between these two theories with α' corresponding to $|\theta|$. But if we probe a bit more closely, we find what at first sight appears to be an insuperable obstacle to matching up these two theories. Both string theory and noncommutative Yang-Mills theory can be systematically expanded in powers of $(\text{length})^{-1}$. In the string theory case, the general form of the expansion is schematically indicated in (2). In noncommutative Yang-Mills theory, one obtains an analogous expansion by expanding in powers of the Poisson bracket. In each case, one is expanding in powers of a quantity (α' or $|\theta|$) with dimensions of length squared.

But there appears to be a crucial difference between the two. In the string case, the expansion involves more and more complicated terms that are written in the standard framework of classical gauge invariance. The higher order corrections involves increasingly complicated terms that are all written in the standard framework. For noncommutative Yang-Mills theory, by contrast, the expansion seems to involve a change in the rules: it involves an expansion of the multiplication law (in the definition of the curvature $\widehat{F}$) in powers of the Poisson bracket. These two expansions sound very different. How can they agree?

Here we meet a surprise, described more fully in [8]. These is a sense in which these two types of expansion do agree. To draw out the essential issue in the sharpest way, consider the case of gauge fields of rank one. Let us contrast two types of gauge fields and gauge invariances. In the first case, we have a gauge field A that takes values in the space $\mathcal{B}$ of classical rank one connections. In the second case, the gauge field A' takes values in the space $\mathcal{B}'$ of noncommutative rank one connections. The respective gauge invariances are:

(A) Classical abelian gauge invariance: $\delta A_i = \partial_i \epsilon$.

(B) "Non-commutative" gauge invariance: $\delta A'_i = \partial_i \epsilon' + iA'_i * \epsilon' - i\epsilon' * A'_i$.

These infinitesimal gauge transformation laws generate group actions. The two groups involved are in fact different. The first is abelian and the second is non-abelian. No change of variables will establish an isomorphism between an abelian group and a nonabelian one.

It seems, therefore, that it is impossible for these two types of gauge theory to be equivalent. But that is not the right conclusion. To do physics with gauge theory, we do not need to know what the gauge group is; we only need to be able to identify its orbits. In other words, we need to know when two gauge fields should be considered equivalent. We need the equivalence relation that is generated by the infinitesimal gauge invariances, but we do not need to make a particular choice of generators of this equivalence relation.

It turns out that, though no change of variables could convert the commutative group (A) into the noncommutative group (B), there is a change of variables that maps one equivalence relation into the other. To identify only the equivalence relation, and not the group, one has more flexibility in the change of variables. A change of variables that would map one group into the other would take the general form

$$\epsilon \to \epsilon'(\epsilon, d\epsilon, \dots) \tag{12}$$

$$A \to A'(A, dA, \dots). \tag{13}$$

Here, in other words, one transforms the group generator ϵ to a new group generator ϵ' which (in a formal series expansion in powers of θ) can be a general local functional of ϵ and its derivatives. But ϵ' is independent of A: to show that two groups are isomorphic, one should establish an isomorphism that is independent of any details about the space that the groups act on. Likewise, in claiming an equivalence between commutative and noncommutative gauge theory, one would want a mapping between the two spaces $\mathcal{B}$ and $\mathcal{B}'$ of connections, so $\widetilde{A}$ should be a function of A and its derivatives, independent of ϵ. Of course, a mapping of the type (12) does not exist; an abelian group cannot be equivalent to a nonabelian one.

To show not that the two gauge groups are the same, but only that the two equivalence relations are the same, modulo a change of variables, one has more freedom. For this, one looks at a change of variables of the form

$$\epsilon \to \epsilon'(\epsilon, d\epsilon, \dots; A, dA, \dots) \tag{14}$$

$$A \to A'(A, dA, \dots). \tag{15}$$

There is no change in the second equation: we want to define a definite map from $\mathcal{B}$ to $\mathcal{B}'$, so A' depends on A only and not on ϵ. The change is in the first equation: ϵ' may depend on A as well as ϵ, as we are not aiming to identify the two gauge groups, but only the orbits they generate in $\mathcal{B}$ and $\mathcal{B}'$. Existence of a change of variables of the form (14) from the classical to the noncommutative theory has the following implication: if A is a classical gauge field, g a classical gauge transformation, and A^g the transform of A by g, then the corresponding noncommutative gauge fields A' and $(A^g)'$ are gauge equivalent in the noncommutative sense, but the gauge transformation g' that establishes this equivalence will generally depend on A as well as g. Such a transformation from classical to noncommutative gauge invariance does exist, and can be found in a completely elementary way once one is persuaded to look for it [8].

Thus, the general framework of classical gauge invariance is equivalent to the general framework of noncommutative gauge invariance. The question is thus not which of these is correct in describing a given problem, but which is more useful. In particular, in string theory, one wants to know which framework for describing the corrections to Yang-Mills theory is more convenient in a given situation.

The answer to this question turns out to be as follows. String theory has both α' and θ. It can be usefully described as noncommutative Yang-Mills theory in a certain limit in which effectively $|\theta| >> \alpha'$. For $|\theta| \leq \alpha'$, the noncommutative Yang-Mills framework is still perfectly correct, but does not appear to be particularly useful.

For a hint of how this comes about (for more detail see [8]), we will finally have to look at the two dimensional quantum field theories that stringy geometry actually comes from. The action for a string with worldsheet Σ is

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{IJ}dX^I \wedge *dX^J - \frac{i}{2}\int_\Sigma \sum_{I,J} B_{IJ}dX^I \wedge dX^J. \tag{16}$$

Here X^I, $I = 1,\ldots,4$ are coordinates on $\mathbf{R}^4$ that we use to describe a map $X : \Sigma \to \mathbf{R}^4$; g_{IJ} is the flat metric on $\mathbf{R}^4$; and $*$ is the Hodge star, using a conformal structure on Σ. The B-field for our purposes is a two-form with constant coefficients B_{IJ}.

This theory leads in general to the full complexity of string theory. There is, however, a limit of the theory in which the excited states of the strings drop out and the string theory can be described systematically in terms of noncommutative Yang-Mills theory. This is the limit in which, by taking g/α' to zero with fixed B (or by scaling things in various other ways to get a similar result) the second term in the action dominates, so that the action reduces to

$$S' = -\frac{i}{2}\int_\Sigma \sum_{I,J} B_{IJ}dX^I \wedge dX^J = -i\int_\Sigma X^*(B). \tag{17}$$

Actually, to be more precise, this limit does not exist for closed strings, for indeed if Σ is a closed surface, then S' always vanishes, since the two-form B is exact. However, if Σ has a boundary, S' is nontrivial. For the important case that Σ is a disc, S' is a functional only of the boundary values of X. If B is nondegenerate, then S' is the usual action functional for maps of a circle (namely $\partial\Sigma$) to the symplectic manifold $\mathbf{R}^4$ with symplectic form B, and is hence intimately connected with quantization of particle motion on $\mathbf{R}^4$. Note that I said "particle motion" rather than "string motion": in the limit that the full action functional S reduces to S', the strings effectively reduce to particles on $\mathbf{R}^4$, and that is why things become simple. This is also tied up with the fact that S' has more symmetry than S: it does not depend on the conformal structure of Σ, and so is invariant under arbitrary diffeomorphisms of Σ.

At any rate, in the limit that S' dominates, the string theory can be analyzed systematically [8] in terms of noncommutative Yang-Mills theory, with the noncommutativity parameter being the bivector $\theta = B^{-1}$. Actually, this limit is closely related to the content of many important recent papers. An example in which S' dominates is the limit of toroidal compactification (small area with fixed period of B) studied in the original application [2] of noncommutative Yang-Mills theory to string theory. Also, as Cataneo and Felder explain [1], the action they use in reinterpreting Kontsevich's results on deformation quantization reduces in the symplectic case to S'.

Thus, in this limit, which one can think of roughly as $|\theta| >> \alpha'$, the string theory remains nonclassical but can be described in great detail in terms of noncommutative Yang-Mills theory. The simplicity of this limit is tied with the fact that the characteristic excited states of the string drop out, and the conformal action S is replaced by the topological action S'. Is there a systematic framework, which somehow reduces to this description in the relevant limit, for using noncommutative, associative algebras to study the full-fledged string theory, with all the excited string states?

String field theory provides such a framework, at least for the open strings [9], but is regrettably messy. Here one looks not at functions on spacetime, but at functions on the path space of spacetime (suitably enriched with ghosts), and one defines a multiplication law for such functions using a gluing law for the paths. This description includes all of the stringy degrees of freedom, and is based on an elegant concept with an abstract Chern-Simons action $\int \left(A * QA + \frac{2}{3} A * A * A\right)$. But it is messy in detail and not much useful in practice. Indeed, the limit I have sketched, in which the stringy excitations drop out and the string theory can be described via noncommutative Yang-Mills theory, is the only known limit in which the open string field theory reduces to something nonclassical yet tractable.

But the purpose of the open string field theory, or whatever replaces it, should be precisely to incorporate the excited string states in the noncommutative framework. Many mathematicians and physicists have felt that the messiness of open string field theory comes from trying to shoehorn the more elegant two-dimensional worldsheet quantum field theory into an associative algebra framework that does not naturally fit. It has, in particular, been suggested that one should use an A_∞ algebra rather than an ordinary associative algebra, but this suggestion has not yet been accompanied by a suggestion of how to use an A_∞ algebra to write a Lagrangian.

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