Surveys in Differential Geometry, 2000
Vol. VII ©2000, International Press
pp. 673-684

UNIFYING THEMES IN TOPOLOGICAL FIELD THEORIES

CUMRUN VAFA

We discuss unifying features of topological field theories in 2, 3 and 4 dimensions. This includes relations among enumerative geometry (2d topological field theory) link invariants (3d Chern-Simons theory) and Donaldson invariants (4d topological theory). (Talk presented in conference on Geometry and Topology in honor of M. Atiyah, R. Bott, F. Hirzebruch and I. Singer, Harvard University, May 1999).

1. Introduction

There has been many exciting interactions between physics and mathematics in the past few decades. Many of these developments on the physics side are captured by certain field theories, known as topological field theories. The correlation function of these theories compute certain mathematical invariants. Even though the original motivation for introducing topological field theories was to gain insight into these mathematical invariants, topological field theories have been found to be important for answers to many questions of interest in physics as well.

The aim of my talk here is to explain certain connections that have been discovered more recently among various topological field theories. I will first briefly review what each one is, and then go on to explain some of the connections which has been discovered between them.

The main examples of topological field theories that have been proposed appear in dimension two [26] known as topological sigma models, in dimension three [27] known as Chern-Simons theory and in dimension four [28] known as topological Yang-Mills theory. The 2d and the 4d topological theory are related to an underlying supersymmetric quantum field theory, and there is no difference between the topological and standard version on the flat space. The difference between conventional supersymmetric theories and topological ones in these cases only arise when one considers curved spaces. In such cases the topological version, is a modified version of the supersymmetric theory on flat space where some of the fields have different Lorentz transformations properties (compared to the conventional choice). This modification of Lorentz transformation properties is also known as twisting, and is put in primarily to preserve supersymmetry on curved space. In particular this leads to having at least one nilpotent supercharge Q as a scalar quantity, as opposed to a spinor, as would be in the conventional spin assignments. The physical observables of the topological theory are elements of the Q cohomology. The path integral is localized to field configurations which are annihilated by Q and this typically leads to some moduli problem which lead to mathematical invariants.

In these theories the energy momentum tensor is Q trivial, i.e.,

$$T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$$

which (modulo potential anomalies) leads to the statement that the correlation functions are all independent of the metric on the curved space, thus leading to the notion of topological field theories (i.e. metric independence).

The case of the 3d topological theory, is somewhat different. In this case, namely the case of Chern-Simons theory, one starts from an action which is manifestly independent of the metric on the 3 manifold, and thus topological nature of the field theory is manifest.

The organization of this paper is as follows: In Section 2 I briefly review each of the three classes of topological theories and discuss how in each case one goes about computing the correlation functions. In section 3 I discuss relations between 2d and 4d topological theories. In section 4 I discuss relations between 2d and 3d topological theories.

2. A brief review of topological field theories

In this section I give a rather brief review of topological field theories in dimensions 2, 3 and 4.

2.1 TFT in $d = 2$: topological sigma models

Topological sigma models are based on $(2, 2)$ supersymmetric theories in 2 dimensions. These typically arise by considering supersymmetric sigma models on Kahler manifolds. In other words, we consider maps from 2 dimensional Riemann surfaces Σ to target spaces M which are Kahler manifolds (together with fermionic degrees of freedom on the Riemann surface which map to tangent vectors on the Kahler manifold). The topological theory in this case localizes on holomorpic maps from Riemann surfaces to the target:

$$X: \qquad \Sigma \to M$$
$$\bar{\partial} X = 0$$

If we get a moduli space of such maps we have to evaluate an appropriate class over it. This class is determined by the topological theory one considers (for precise mathematical definitions see [5]. Also there are two versions of this topological theory: coupled or uncoupled to gravity. Coupling to gravity in this case means allowing the complex structure of Σ to be arbitrary and looking for holomorphic curves over the entire moduli space of curves. The case coupled to gravity is also sometimes referred to as 'topological strings'.

A particularly interesting class of sigma modelds both for the physics as well as for mathematics, corresponds to choosing M to be a Calabi-Yau threefold, and considering topological strings on M. In this case the virtual dimension of the moduli space of holomorphic maps is zero. If this space is given by a number of points, the topological string amplitude just counts how many such points there are, weighted by $e^{-k(.)}$ where $k(.)$ is the area of the holomorphic map (pullback of the Kahler form integrated over the surface) times λ^{2g-2}, where g denotes the genus of the Riemann surface and λ denotes the string coupling constant. More generally the space of holomorphic maps will involve a moduli space. This space comes equipped with a bundle with the same dimension as the tangent bundle (the existence of this bundle and the fact that its dimensions is the same as the tangent bundle follows from the fact that the relevent index is zero). Topological string computes the top Chern class of such bundles again weighted by $e^{-k(.)}\lambda^{2g-2}$. These have to be defined carefully, due to singularities and issues of compactifications, and lead in general to rational numbers. The sum of these numbers for a given class $v \in H_2(M, \mathbf{Z})$ and fixed genus g, which we will denote by $r_{g,v}$, is known as Gromov-Witten invariant. We thus have the full partition function of topological string given by

$$F(\lambda, k) = \sum_{v \in H_2(M,\mathbf{Z})} r_{g,v} e^{-k(v)} \lambda^{2g-2}$$

here k denotes the Kahler class of M. Even though the numbers $r_{g,v}$ are not integers, it has been shown, by physical arguments that F can also be expressed in terms of other *integral* invariants [8]. These integral invariants are related to certain aspects of cohomology classes of moduli of holomorphic curves *together with flat bundles.*

These invariants associate for each $v \in H_2(M, \mathbf{Z})$ and eash positive (including 0) integer s a number $N_{v,s}$ which denotes the 'net' number of BPS membranes with charge in class v and 'spin' s (for precise definitions see [8]). Then we have

$$F(\lambda, k) = \sum_{n>0, v \in H_2(M,\mathbf{Z})} \frac{1}{n} N_{v,s} e^{-nk(v)} [2Sin(n\lambda/2)]^{2s-2} \tag{1}$$

For all cases checked thus far the Gromov-Witten invariants $r_{g,v}$ has been shown to be captured by these simpler integral invariants $N_{v,s}$ through the above map. In particular the checks made for constant maps [6] and for contribution of isolated genus g curves to all loops [21] as well as some low genus computations for non-trivial CY 3-folds [14] all support the above identification.

Let us illustrate the above results in the case of a simple non-compact Calabi-Yau threefold, which we will later use in this paper. Consider the total space of the rank 2 vector bundle $O(-1) + O(-1) \rightarrow \mathbf{P}^1$. This space has vanishing c_1, and is a non-compact CY 3-fold. In this case the only BPS state is a membrane wrapping $\mathbf{P}^1$ once. This state has spin $s = 0$. If we denote the area of $\mathbf{P}^1$ by t, then we have from (1)

$$F = \sum_{n>0} \frac{1}{n[2Sin(n\lambda/2)]^2} e^{-nt} \tag{2}$$

For this particular case this has also been derived using the direct definition of topological strings in [6], [21].

2.2 Topological field theory in 3d: Chern-Simons theory

The 3d topological theory we consider is Chern-Simons theory, which is given by the Chern-Simons action for a gauge field A:

$$S_{CS} = \frac{k}{4\pi} \int_M Tr[AdA + \frac{2}{3}A^3]$$

where M is a 3-manifold and k is an integer which is quantized in order for $exp(iS)$ to be well defined. As is clear from the definition of the above action, S does not depend on any metric on M and in this sense the theory is manifestly topological (i.e., metric independent).[1]

[1] At the quantum level there is a metric dependence which can be captured by a gravitational Chern-Simons term [27] [2].

Thus the partition function of Chern-Simons theory gives rise to topological invariants for 3-manifolds for each group G. In other words

$$Z_M(G) = exp(-F_M(G)) = \int \mathcal{D}A exp[iS_{CS}]$$

where A is a connection on M for the gauge group G and the above integral is over all inequivalent G-connections on M. The simplest way to compute such invariants is to use the relation between Hilbert space of Chern-Simons theory on a Riemann surface Σ and the chiral blocks of WZW model on Σ with group G and level k. For example the partition function on S^3 can be computed by viewing S^3 as a sum of two solid 2-tori, which are glued along T^2 by an order 2 element of $SL(2, \mathbf{Z})$ on T^2. In this way the partition function gets identified with

$$Z_{S^3}(G, k) = S_{00}(G, k)$$

where $S_{00} = \langle 0|S|0\rangle$ is a particular element of the order 2 operation of $SL(2, \mathbf{Z})$ on chiral characters, and is well studied in the context of WZW models. In particular for $G = SU(N)$ it is given by:

$$\begin{aligned} Z_{S^3}(SU(N), k) &= exp(-F) \\ &= e^{i\pi N(N-1)/8} \frac{1}{(N+k)^{N/2}} \sqrt{\frac{N+k}{N}} \\ &\quad \cdot \prod_{j=1}^{N-1} (2\sin\frac{j\pi}{N+k})^{N-j}. \end{aligned} \tag{3}$$

One can also consider knot invariants: Consider a knot γ in M and choose a representation R of the group G and consider the character of the holonomoy of A around the knot γ, i.e.,

$$P[\gamma, R] = Tr_R P\exp(i \int_\gamma A)$$

By the equation of motion for Chern-Simons theory, which leads to flatness of A, we learn that the above operator only depends on the choice of the knot type and not the actual knot[2]. One then obtains a knot invariant by computing the correlation function

$$< \prod_i P[\gamma_i, R_i] >= \int \mathcal{D}A \prod_i P[\gamma_i, R_i]\exp(iS_{CS})$$

Again these quantities can be computed by the braiding properties of chiral blocks in 2 dimensional WZW models and leads in particular to HOMFLY polynomial invariants for the knots.

[2]In the quantum theory one also needs to choose a framing for the knot.

2.3 Topological field theories in 4-dimensions

If one consider $N = 2$ supersymmetric Yang-Mills, with an unconventional spin assignments, one finds a topological field theory. The partition function is localized on the moduli space of instantons and the observables of this theory are given by intersection theory on the moduli space of instantons. More precisely each d-cycle on the four manifold M will lead to a $4-d$ cohomology element on the moduli space of instantons (obtained by integrating out $\int \mathrm{Tr} F \wedge F$ over the corresponding cycle on the universal moduli space of instantons), and the wedging of the cohomology classes gives rise to the observables in Donaldson theory. This does not depend on the metric in M (except when $b_2^+(M) = 1$) but will depend on the choice of smooth structure on M.

The computations in this case can be done for many choices of M by finding an equivalence of this theory and a simpler abelian theory. In this case studying the moduli space of non-abelian instantons gets replaced with the study of an abelian system known as the Seiberg-Witten equation. The relevant geometry for the case of $SU(N)$ Yang-Mills is captured by a certain geometric data related to a Jacobian variety over an $N-1$ dimensional family of genus $N-1$ Riemann surface, known as Seiberg-Witten geometry [22]. For topological field theory aspects and how the Seiberg-Witten geometry leads to computation of the topological correlation functions see [25], [19].

There is another topological theory in 4 dimensions which has been studied [24] and is related to twisting the maximal supersymmetric gauge theory in 4 dimensions. This theory computes the Euler characteristic of moduli space of instantons. In particular for each group G and each complex parameter q one considers

$$Z_M(G) = q^{-c(M,G)} \sum_k q^k \chi(\mathcal{M}_k)$$

for some universal constant c (depending on M and G), where k denotes the instanton number and $\chi(\mathcal{M}_k)$ denotes the euler characteristics (of a suitable resolution and compactification) of $\mathcal{M}_k$, the moduli space of anti-self dual G-connections with instanton number k on M. Moreover, according to Montonen-Olive duality conjecture one learns that the above partition function is expected to be modular with respect to some subgroup of $SL(2, \mathbf{Z})$ acting in the standard way on τ where $q = exp(2\pi i\tau)$. For certain M (such as $K3$) the above partition function has been computed and is shown to be modular in a striking way. For recent mathematical discussion on this see [11] and references therein.

3. Connections between 2d $\leftrightarrow$ 4d TFT's

There are three different links between 4 dimensional and 2 dimensional TFT's that I would like to discuss. In all three links the common theme is that the moduli space of instantons are mapped to moduli space of holomoprhic curves on appropriate spaces.

3.1 Topological reduction of $4d$ to $2d$

The simplest link between the two theories involves studying the 4d TFT on a geometry involving the product of two Riemann surfaces $\Sigma_1 \times \Sigma_2$, which was studied in [4]. In the limit where Σ_1 is small compared to Σ_2 one obtains an effective theory on Σ_2 which is the topological sigma model with target space given by moduli space of flat connections on Σ_1, in case one considers $N = 2$ topological field theories in 4 dimensions or the Hitchin space associated with Σ_1 if one considers $N = 4$ topological field theories. This is natural to expect because studying light supersymmetric modes in either case gives rise to the corresponding space of solutions, which thus behaves from the viewpoint of the space Σ_2 as a target space. In particular the moduli space of 4d instantons get mapped to moduli space of holomorphic maps for these target spaces. Thus quantum cohomology rings of moduli of flat connections on a Riemann surface, which are encoded in 2d topological correlation functions capture the corresponding topological correlation functions of the 4 dimensional $N = 2$ theory. Similarly in the $N = 4$ case the reduction to 2 dimensions yields a sigma model on the Hitchin space (which can also be viewed as a Jacobian variety). In this context the Montonen-Olive duality of $N = 4$ theory gets mapped to mirror symmetry of this 2d sigma model (by a fiberwise application of T-duality to Jacobian fibers).

3.2 A more subtle 2d $\leftrightarrow$ 4d link

For the $N = 2$ topologically twisted theory, an important role is played by the Seiberg-Witten geometry, which is an abelian simplification of the non-abelian gauge theory. This geometry is a quantum deformation of the classical one, due to pointlike four dimensional instantons. This geometry was first conjectured based on consistency with various properties of $N = 2$ quantum field theories and its deformation to $N = 1$ quantum field theories with mass gap, where plausible properties of $N = 1$ theories were assumed.

With the recent advances in our understanding of string theory, the same 4-dimensional gauge theories have been obtained by considering particular geometries where strings propagate in. This procedure is known as geometric engineering of QFT's (see [14], [12] and references therein). These geometries involve a non-compact Calabi-Yau threefold geometry which is a blow up of a geometry with some loci of A-D-E singularities (locally modelled by $\mathbf{C}^2/G$ where G is a discrete subgroup of $SU(2)$), giving rise to the corresponding gauge theory in 4 dimensions. Depending on the detailed structure of singularities one can obtain various interesting gauge groups and various matter representations.

It turns out that in this description of gauge theory, the guage theory instantons are mapped to stringy instantons, which are just worldsheet instantons. Thus being able to compute worldsheet instantons, i.e., counting of holomorphic curves in these target geometries, captures the geometry of 4-dimensional gauge theory instantons. Counting of holomrophic curves is precisely what the (A-model) topological string computes and thus in this way the geometry of vacua of 4-dimensional gauge theory gets mapped to solving topological amplitudes in 2d. This in turn can be done by using (local) mirror symmetry. For a physical derivation of mirror symmetry and some references on this subject see the recent work [10]. In this way, not only the Seiberg-Witten geometry has been rederived, but also other geometries which describe other $N = 2$ systems with various kinds of gauge groups and intricate matter representations have been obtained [15].

3.3 $N = 4$ Yang-Mills on elliptic surfaces and $2d$ topological theories

If we consider an $N = 4$ supersymmetric $SU(N)$ topological theory on an elliptic surface, with base B, the stable bundles get mapped to spectral covers of B on a dual elliptic surface M (where the Kahler class of the elliptic fiber is inverted). This uses the fact that in the limit of small tori, the stable bundles become flat fiberwise and flat bundles on tori are related to points on the dual tori. See [7], [3] for a discussion of how this arises. In particular a rank N stable bundle with instanton number k gets mapped to a spectral curve which is a holomorphic curve wrapping the base N times and the elliptic fiber k times. Thus the topological $N = 4$ amplitude on M, denoted by $Z_M(SU(N))$ which computes the Euler characteristic of moduli space of $SU(N)$ instantons on M gets mapped to computing Euler characteristic of moduli space of holomorphic curves (together with a flat bundle) which in turn is captured by genus zero topological string amplitudes, and can be computed using mirror symmetry. This idea has been implemented in great detail for the case of rational elliptic surface (also known as "half $K3$") [15], [18], [17]. The results for the case of rank 2 and its implications for the Euler characteristic of moduli space of instantons on rational elliptic surface has been confirmed using rigorous mathematical methods in [30].

4. Connections between 2d $\leftrightarrow$ 3d TFTs

Over two decades ago 't Hooft conjectured that $SU(N)$ gauge theories with large N look alot like string theories. In particular the partition function for these theories can be organized in terms of Riemann surfaces where each Riemann surface is weighted with N^{χ} where χ denotes the Euler characteristic of the Riemann surface. In particular the low genera dominate in the large N limit. The weight factor N^{χ} follows simply from the combinatorics of Feynman diagrams, and the Riemann surface can be identified with the Feynamn diagrams where the would be holes have been filled.

The main difficulty in the conjecture of 't Hooft is to identify precisely which string theory one obtains. In the past few years for serveral interesting gauge theories and in particular some in 4 dimensions the corresponding string theory has been identified [1]. Even though it has not been possible to actually compute the string theory amplitudes in these cases, due to the complicated background strings propagate in, there has been mounting evidence for the validity of the identification. One would like to have a similar conjecture in a setup which is more computable. An ideal setup for this is topological guage theories, and in particular the topological Chern-Simons theory.

If we consider $SU(N)$ Chern-Simons theory on S^3 in the limit of large N, one could hope to get a string theory. It has been conjectured in [9] that this is indeed the case. In particular it has been conjectured that $SU(N)$ Chern-Simons theory at level k on S^3 is equivalent to topological string with target being a non-compact Calabi-Yau threefold which is the total space of $O(-1)+O(-1) \to \mathbf{P}^1$, where the (complexified) size of $\mathbf{P}^1$ is given by $t = 2\pi i N/(k+N)$ and the string coupling constant $\lambda = \frac{2\pi i}{N+k}$. This is a natural conjecture in the following sense: The Chern-Simons theory on S^3 can itself be viewd as an open string theory with target T^*S^3 [29] By open string we mean considering Riemann surfaces with boundaries, where the boundaries are mapped to S^3. The geometry $O(-1)+O(-1) \to \mathbf{P}^1$ can be obtained from the T^*S^3 geometry by shrinking S^3 to zero size and blowing $\mathbf{P}^1$ instead. This kind of transition is also very similar to what is observed to happen in the other cases where large N string theory description was discovered [1]. In fact one can determine [9] the map of the parameters t and λ given above using this picture (and recalling the metric dependence anomaly in Chern-Simons theory).

This conjecture has been checked at the level of the partition function (which we have briefly reviewed for both the Chern-Simons theory on S^3 and for $O(-1)+O(-1) \to \mathbf{P}^1$ in section 2). The implications of this conjecture for knot invariants has been explored in [20] and provides a reformulation of knot invariants in terms of integral invariants which again capture the degeneracy of spectrum of (BPS) particles in the corresponding string theory. This involves considering a Largrangian submanifold which intersects T^*S^3 along the knot and following it through the transition to $O(-1)+O(-1) \to \mathbf{P}^1$ where it corresponds to a Lagrangian submanifold. The corresponding computation on the topological string side will now involove Riemann surfaces with boundaries, where the boundary can lie on this Lagrangian submanifold in $O(-1)+O(-1) \to \mathbf{P}^1$. The results for the unknot [20] as well as the integrality properties of the torus knots [16] are in perfect agreement with the conjecture.

5. Conclusions

We have seen some intricate relations among topological theories in 2, 3 and 4 dimensions and in some ways these connections parallel the discovery of duality symmetries in superstring theories (see [23] for a review of some mathematical aspects of string dualities). These topological examples provide a simpler version of superstring dualities, which one could hope to understand more deeply and which might provide a hint as to how to think about dualities in general.

This research was supported in part by NSF grants PHY-9218167 and DMS-9709694.

References

[1] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri & Y. Oz, *Large N field theories, string theory and gravity,* Phys. Rept. **323** (2000) 183.

[2] S. Axelrod & I. Singer, *Chern-Simons perturbation theory. II,* J. Differential Geom. **39** (1994) 173.

[3] M. Bershadsky, A. Johansen, T. Pantev & V. Sadov, *On four dimensional compactifications of F-theory,* Nucl. Phys. **B505** (1997) 165.

[4] M. Bershadsky, A. Johansen V. Sadov & C. Vafa, *Topological reduction of 4D SYM to 2D sigma models,* Nucl. Phys. **B448** (1995) 166.

[5] D.A. Cox & S. Katz, *Mirror symmetry and algebraic geometry*, Math. Surveys and Monographs **68** (ASMS, 1999).

[6] C. Faber & R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* math. AG/9810173.

[7] R. Friedman, J. Morgan & E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9707004.

[8] R. Gopakumar & C. Vafa, *M-Theory and topological strings. I,II,* hep-th/9809187, hep-th/9812127.

[9] ———, *On the gauge theory/geometry correspondence*, hep-th/9811131.

[10] K. Hori & C. Vafa, *Mirror symmetry*, hep-th/0002222.

[11] M. Kapranov, *The elliptic curve in the S-duality theory and Eisenstein series for Kac-Moody groups*, math.AG/0001005.

[12] S. Katz, P. Mayr & C. Vafa, *Mirror symmetry and exact solution of 4D $N = 2$ gauge theories,* Adv. Theor. Math. Phys. **1** (1998) 53.

[13] S. Katz, A. Klemm & C. Vafa, *Geometric engineering of quantum field theories,* Nucl. Phys. **B497** (1997) 173.

[14] bysame, *M-Theory, topological strings and spinning black holes,* hep-th/9910181.

[15] A. Klemm, P. Mayr & C. Vafa, *BPS states of exceptional non-critical strings,* hep-th/9607139.

[16] J. M. F. Labastida & M. Marino, *Polynomial invariants for torus knots,* hep-th/0004196.

[17] J. A. Minahan, D. Nemeschansky, C. Vafa & N. P. Warner, *E-Strings and $N = 4$ topological Yang-Mills theories,* Nucl. Phys. **B527** (1998) 581.

[18] J. A. Minahan, D. Nemeschansky & N. P. Warner, *Partition functions for BPS states of the non-critical E_8 string,* Adv. Theor. Math. Phys. **1** (1998) 167.

[19] G. Moore & E. Witten, *Integration over the u-plane in Donaldson theory,* hep-th/9709193.

[20] H. Ooguri & C. Vafa, *Knot invariants and topological strings,* hep-th/9912123.

[21] R. Pandharipande, *Hodge integrals and degenerate contributions,* math. AG/9811140.

[22] N. Seiberg & E. Witten, *Electric-magnetic duality, monopole, condensation, and confinement in N=2 supersymmetric Yang-Mills theory,* Nucl. Phys. **B426** (1994) 19.

[23] C. Vafa, *Geometric Physics,* Proc. ICM-98, hep-th/9810149.

[24] C. Vafa & E. Witten, *A Strong coupling test of S-duality,* Nucl. Phys. **B431** (1994) 3.

[25] E. Witten, *On S-duality in Abelian gauge theory,* hep-th/9505186.

[26] ———, *Topological sigma model,* Comm. Math. Phys. **118** (1988) 411.

[27] ———, *Quantum field theory and the Jones polynomial,* Comm. Math. Phys. **121** (1989) 351.

[28] ———, *Topological quantum field theory,* Comm. Math. Phys. **117** (1988) 353.

[29] ———, *Chern-Simons gauge theory as a string theory,* hep-th/9207094.

[30] K. Yoshioka, *Euler characteristics of $SU(2)$ instanton moduli spaces on rational elliptic surfaces,* Comm. Math. Phys. **205** (1999) 501.

JEFFERSON PHYSICAL LABORATORY, HARVARD UNIVERSITY