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SEIBERG-WITTEN INVARIANTS, SELF-DUAL HARMONIC 2-FORMS AND THE HOFER-WYSOCKI-ZEHNDER FORMALISM

CLIFFORD HENRY TAUBES

1.Introduction

Suppose that $\underline{X}$ is a compact, oriented 4-manifold with $b^{2+} \geq 1$. A symplectic form on $\underline{X}$ is a closed, non-degenerate 2-form whose square provides the given orientation. Little is known by way of sufficient conditions which guarantee the existence of such a form. However, there are smooth, closed forms on $\underline{X}$ which are symplectic off of a disjoint union of embedded circles, with the latter being the vanishing locus of the form. Indeed, if a sufficiently generic Riemannian metric is chosen for $\underline{X}$, then some of the self-dual, harmonic 2-forms on $\underline{X}$ have the aforementioned property. Moreover, the given metric, with such a form, defines a compatible almost complex structure on the complement of the form's zero set. Thus, the complement, $X \subset \underline{X}$, of the zero set of the given closed, self-dual 2-form has a natural pseudoholomorphic geometry, the 'Riemannian pseudoholmorphic geometry'. This geometry seems worthy of study if, for no other reason, then the following:A sufficient condition for the zero set of the form to homologically bound a pseudoholomorphic subvariety in its compliment is for $\underline{X}$ to have non-trivial Seiberg-Witten invariants [16].

Prior to the discovery of the Seiberg-Witten invariants, Hofer introduced [5] and then Hofer, Wysocki and Zehnder [9], [10], [11] (see [6]) systematically developed an elegant formalism for studying a particular

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version of pseudoholomorphic geometry on symplectic manifolds with tubular ends. In particular, the complement, $X \subset \underline{X}$, of the zero set of a form as just described provides a nice example for the Hofer, Wysocki and Zehnder formalism. These pseudoholomorphic geometries on X will be called 'HWZ pseudoholomorphic geometries'. In this regard, it is important to note the the HWZ pseudoholomorphic geometry near the zero set of the given form is not the same as the Riemannian one. In particular, the relationship between the HWZ pseudoholomorphic geometry and the Seiberg-Witten invariants must still be sorted out, and this article provides the first step in doing so with a theorem (Theorem 5.4, below) which implies the following:

- Suppose that $\underline{X}$ has a non-vanishing Seiberg-Witten invariant. Then, there is a finite set of irreducible, HWZ pseudoholomorphic subvarieties in X whose union, with positive integer weights, homologically bounds the zero locus of the given self-dual, harmonic 2-form. This is to say that the weighted union has algebraic intersection number 1 with each linking 2-sphere of the form's zero set.
- Moreover, X has its own Seiberg-Witten invariants from which the Seiberg-Witten invariants of $\underline{X}$ can be computed, and if just the former are non-trivial, then X still has an HWZ pseudoholomorphic subvariety as described in the preceding point.

(1.1)

Note that a Seiberg-Witten based existence proof for pseudoholmorphic subvarieties of compact symplectic manifolds has already been established [17] (but see the revised version in [18] which corrects some arguments in Section 6e of [17]). Moreover, in the case where X is a compact symplectic manifold, the complete Seiberg-Witten invariant of X can be computed completely in terms of the associated pseudoholomorphic geometry (see [19], [20]). This is to say that there is a symplectic invariant, Gr, which is obtained as a count of pseudoholomorphic subvarieties [21] in X and which turns out to be the same as the Seiberg-Witten invariant of X.

In the present context, there is a candidate for a version of Gr which is defined for the cylindrical end manifold $X \subset \underline{X}$, is computable completely in terms of the HWZ pseudoholomorphic geometry of X, and may well be equal to the Seiberg-Witten invariants of $\underline{X}$. This candidate Gr and its relation to the Seiberg-Witten invariants of $\underline{X}$ is the subject of a planned sequel to this article.

By the way, there is some circumstantial evidence to the effect that the pseudoholomorphic geometry of $X \subset \underline{X}$, either Riemannian or HWZ, provides 4-manifold information which goes beyond the Seiberg-Witten invariants (see, e.g. [22]). In this regard, the HWZ pseudoholomorphic geometry may prove the more tractible tool for the study of 4-manifold differential topology. For example, it turns out that the singularities of pseudoholomorphic subvarieties in the HWZ geometry are not hard to classify. In contrast, the singularities of the pseudoholomorphic subvarieties in the Riemannian pseudoholomorphic geometry has only been partly sorted out [23], and may turn out to be very complicated.

The remainder of this article provides the details to (1.1), and is organized along the following lines: Section 2 summarizes the basic features of HWZ pseudoholomorphic geometry, with a special focus on those manifolds which arise as the complement of the zero set of a generic, closed, self-dual 2-form on a compact 4-manifold.This is to say that each end of such a manifold is symplectically concave and is diffeomorphic to $[0, \infty) \times (S^1 \times S^2)$. Section 3 summarizes the Seiberg-Witten story on compact 4-manifolds, and Section 4 summarizes the analgous story for the class of non-compact manifolds 4-manifolds with $[0, \infty) \times (S^1 \times S^2)$ ends. Then, Section 5 points out some of the basic relationships between the Seiberg-Witten story and the HWZ geometry on the class of manifolds under consideration. The results in Section 5 are summarized by Theorems 5.4 and 5.5. The final two sections are devoted to the proofs of these last two theorems.

2. The HWZ pseudoholomorphic geometry

The HWZ geometry is designed for studying symplectic manifolds with contact boundary. The general context for this is described in Hofer [5] and Hofer, Wysocki and Zehnder [9], so attention here will be restricted to the case where the manifold in question is 4-dimensional. With this understood, the purpose of this section is to review the relevant portions of the HWZ geometry.After a general review in the first three subsections, the remaining subsections of Section 2 describe this HWZ geometry in the restricted context that is used in the remainder of this article.

a) Contact boundaries

Let X_0 denote the 4-manifold in question, ω the given symplectic form, and Y a component of the 3-manifold boundary of X_0. The convention here is to orient Y using the restriction to Y of the 3-form

$(\omega \wedge \omega)(v, \cdot, \cdot, \cdot)$ in the case where v is a tangent vector to X_0 along Y which is outward pointing.

The manifold Y is a contact type boundary when there exists a smooth 1-form α on Y such that

- $d\alpha = \iota^* \omega$.
- $\alpha \wedge d\alpha$ *is nowhere zero.*

(2.1)

In this regard, note that $\alpha \wedge d\alpha$ can either agree or disagree with the orientation of Y. In the former case, the boundary is called 'convex' and in the latter case, it is called 'concave'.

In any event, if Y has contact type, then there exists an orientation preserving embedding $\varphi : (0, 1] \times Y \to X_0$ with the following properties:

- $\varphi : \{1\} \times Y \to Y$ is the identity.
- $\varphi^* \omega = du \wedge \alpha \pm u \, d\alpha$ on some neighborhood of $\{1\} \times Y$.

(2.2)

Here, u is the Euclidean coordinate on $(0, 1]$. Also, the + sign is used when Y is convex, and the minus sign when Y is concave. (The concave case will be the case of interest in later sections.)

Write $u = e^{\varepsilon s}$ with $\pm\varepsilon > 0$ depending on whether the contact structure is convex (+) or concave (−). Then,

$$\varphi^* \omega = e^{\varepsilon s}(\varepsilon \, ds \wedge \alpha + d\alpha). \tag{2.3}$$

This form is defined for s non-positive and near zero, but it evidently extends to all positive values of s. This is to say that the form ω extends from X_0 to the noncompact manifold

$$X \equiv X_0 \cup_Y ([0, \infty) \times Y). \tag{2.4}$$

Note that when measured with the product metric on $([0, \infty) \times Y)$, the form in (2.3) either grows or shrinks in size exponentially fast as $s \to \infty$ depending on whether Y is convex or concave.

b) Pseudoholomorphic geometry

In all that follows, assume that all components of ∂X_0 are of contact type. An almost complex structure on X is an endomorphism, J, of TX whose square is -1. Such a J will be called 'ω-compatible' in the case where the bilinear form $\omega(\cdot, J(\cdot))$ defines a Riemannian metric. It proves useful to impose some further requirements on J's restriction to each end $[0, \infty) \times Y$ of X. In particular, the HWZ geometry considers ω-compatible almost complex structures which restrict to $[0, \infty) \times Y$ so that:

- J is invariant under the 1-parameter semi-group of translations $(s, x) \to (s + a, x)$ for $a \geq 0$.
- $J \cdot \partial_s$ is annihilated by $d\alpha$.
- J preserves the kernel of α.

(2.5)

Because the space of ω-compatible almost complex structures is contractible, there is no problem with finding such almost complex structures which also obey the requirements in (2.5). With this last point understood, the almost complex structures henceforth under consideration will be implicitly assumed to satisfy (2.5) as well as being ω-compatible.

c) Pseudoholomorphic subvarieties

A subvariety $C \subset X$ will be called 'pseudoholomorphic' when the following conditions are met:

- C is closed and locally compact.
- There is a non-accumulating set $\Lambda \subset C$ of at most a countable number of points such that $C - \Lambda$ is an embedded submanifold of X whose tangent space is J-invariant.

(2.6)

A pseudoholomorphic subvariety $C \subset X$ will be called an 'HWZ subvariety' when, in addition to (2.6),

$$\int_{C \cap ((0,\infty) \times M)} d\alpha < \infty. \tag{2.7}$$

By the way, when integrating either $d\alpha$ or $\varepsilon\, ds \wedge \alpha$ over a domain in a pseudoholomorphic subvariety C, keep in mind that both restrict to C as non-negative 2-forms. This is a consequence of C being pseudoholomorphic for the almost complex structure in (2.5). Here is a simple consequence of this last fact:

Lemma 2.1. *If all boundary components of X_0 are concave, then every pseudoholomorphic subvariety in X satisfies (2.7). That is, all pseudoholomorphic subvarieties are HWZ subvarieties.*

This subsection ends with the.

Proof of Lemma 2.1. It proves useful to make a short, preliminary digression to choose, for each $R \geq 4$, a function σ_R on R with the following properties:

$$
\begin{array}{lll}
 & \bullet \quad \sigma_R = 0 & \text{where } s \leq 0, \\
 & \bullet \quad \sigma_R = \sigma_1 \ \text{ and } \ 0 \leq \sigma_1' \leq 1 & \text{where } 0 \leq s \leq 2, \\
 & \bullet \quad \sigma_R = 1 & \text{where } 2 \leq s \leq R, \\
 & \bullet \quad -1 \leq \sigma_R' \leq 0 & \text{where } R \leq s \leq R+2, \\
(2.8) & \bullet \quad \sigma_R = 0 & \text{where } s \geq R+2,
\end{array}
$$

Thus, σ_R vanishes until $s = 0$, then increases to 1 by $s = 2$, stays equal to 1 until $s = R$ and finally decreases to zero by $s = R + 2$. Moreover, its derivative is nowhere greater than 1 or less than -1.

With the digression now over, remember that $d\alpha$ on C is non-negative as is σ_R; and as $\sigma_R = 1$ where $s \in [2, R]$, the demonstration of an R-independent upper bound for the integral over C of $\sigma_R \, d\alpha$ proves that C is an HWZ subvariety. With this last point understood, remark that $d(\sigma_R \alpha)$ is also integrable over C. Stokes' theorem finds this integral equal to zero, and so

$$
\int_C \sigma_R d\alpha = \int_C -d\sigma_R \wedge \alpha. \tag{2.9}
$$

Thus, it is enough to find an R independent upper bound to the integral over C of $-d\sigma_R \wedge \alpha$.

To achieve the latter task, remark first that $-d\sigma_R \wedge \alpha$ has support in two disjoint sets, the first where $0 \leq s \leq 2$ and the second where $R \leq s \leq R + 2$. Moreover, if all components of ∂X_0 are concave,then $-d\sigma_R \wedge \alpha$. is non-positive on C where $s \geq R$ because σ_R' is non-positive where $s \geq R$ while $-ds \wedge \alpha$ is non-negative on C. Thus,

$$
\int_C \sigma_R d\alpha \leq \int_{C \cap \{0 \leq s \leq 2\}} \sigma_1'(-ds \wedge \alpha). \tag{2.10}
$$

As the right-hand side of (2.8) is finite and independent of R, the desired bound follows.

d) The $S^1 \times S^2$ example

The relevant example for this article takes $Y = S^1 \times S^2$. To write the relevant contact form α, take standard spherical coordinates $(\theta, \varphi) \in$

$[0,\pi]\times[0,2\pi]$ for S^2 and a coordinate $t\in[0,2\pi]$ for S^1. In terms of these coordinates,

$$\alpha = -(1-3\cos^2\theta)dt - \sqrt{6}\cos\theta\sin^2\theta d\varphi. \tag{2.11}$$

A computation finds

$$d\alpha = 6\cos\theta\sin\theta dt d\theta + \sqrt{6}(1-3\cos^2\theta)\sin\theta d\theta d\varphi; \tag{2.12}$$

thus $\alpha\wedge d\alpha$ is seen not to vanish:

$$\alpha\wedge d\alpha = -\sqrt{6}(1+3\cos^4\theta)dt\sin\theta d\theta d\varphi. \tag{2.13}$$

By the way, take the metric on $S^1\times S^2$ to be the product of the standard round metrics and introduce the resulting Hodge star operator, $*$, on differential forms. Then α in (2.11) obeys

- $*d\alpha = -\sqrt{6}\alpha$.
- $d^*\alpha = 0$.

(2.14)

This last point is mentioned in as much as it implies that the 2-form

$$\omega \equiv e^{-\sqrt{6}s}(-\sqrt{6}ds\wedge\alpha + d\alpha) \tag{2.15}$$

on $\mathbb{R}\times S^1\times S^2$ is symplectic, and self-dual with respect to the Hodge star operator from the product metric.

The compact, integral curves in the foliation that is defined by the kernel of $d\alpha$ are of prime importance in the story. In this regard, the kernel of $d\alpha$ is the linear span of the vector field

$$v \equiv -(1-3\cos^2\theta)\partial_t - \sqrt{6}\cos\theta\partial_\varphi. \tag{2.16}$$

Thus, all integral curves of v can be parameterized by

$$\tau\to(t=t_0-\tau(1-3\cos^2\theta_0), \theta=\theta_0, \varphi=\varphi_0-\sqrt{6}\tau\cos\theta_0) \tag{2.17}$$

where t_0, θ_0 and φ_0 are constants. Note that the integral curve in (2.17) is compact if and only if

$$\frac{\sqrt{6}\cos\theta_0}{(1-3\cos^2\theta_0)} \in Q\cup\{\infty\}. \tag{2.18}$$

By the way, the kernel of the contact form α in (2.11) is spanned by the vectors

$$\{\partial_\theta, \sqrt{6}\cos\theta\sin\theta\partial_t - (1-3\cos^2\theta)(\sin\theta)^{-1}\partial_\varphi\}. \tag{2.19}$$

When considering almost complex structures on $\mathbb{R}\times(S^1\times S^2)$ which obey (2.5), there is one which is especially useful. To describe this J, it proves convenient to first digress for the purpose of introducing auxilliary functions f and h on $\mathbb{R}\times(S^1\times S^2)$ as follows:

- $f \equiv e^{-\sqrt{6}s}(1-3\cos^2\theta)$.
- $h \equiv e^{-\sqrt{6}s}\sqrt{6}\cos\theta\sin^2\theta$.

(2.20)

In terms of the 'coordinates' (t, f, h, φ), the form ω and the product metric on $\mathbb{R}\times(S^1\times S^2)$ are as follows:

- $\omega = dt\wedge df + d\varphi\wedge dh$.
- $ds^2+dt^2+d\theta^2+\sin^2\theta d\varphi^2 = dt^2+g^{-2}(df^2+\sin^{-2}\theta dh^2)+\sin^2\theta d\varphi^2$.

(2.21)

Here, $g \equiv \sqrt{6}e^{-\sqrt{6}s}(1+3\cos^4\theta)^{1/2}$.

With the preceding understood, define J by

- $J\cdot\partial_t = g\partial_f$,
- $J\cdot\partial_\varphi = \sin^2\theta\ g\partial_h$.

(2.22)

It is an exercise to verify that the almost complex structure so defined obeys the constraints in (2.5). (One reason for appreciating this particular J is that it acts as an orthogonal transformation with respect to the standard product metric on $\mathbb{R}\times(S^1\times S^2)$.)

The final comment on this example concerns orientations for the homology of $S^1\times S^2$. In this regard, orient $H_3(S^1\times S^2;\mathbb{Z})$ by requiring $\alpha\wedge d\alpha$ to have negative integral. (This is equivalent to the requirement that $(\omega\wedge\omega)(\partial_s,\cdot,\cdot,\cdot)$ have positive integral.) To orient the 2-dimensional homology, first remark that $d\alpha$ is non-degenerate on the kernel of α, and thus orients this $\mathbb{R}^2$ bundle by requiring $d\alpha$ to be positive on a positively oriented frame. The Euler class of this oriented bundle is twice a generator of $H^2(S^1\times S^2;\mathbb{Z})\approx\mathbb{Z}$ and so orients the latter by making this Euler class positive. Use this last orientation to orient the

2-dimensional homology by requiring the oriented generator of the latter to pair with that of H^2 to give 1. Finally, an orientation on H_1 is induced from that on H^2 via Poincaré duality. In less prosaic terms, $S^1 \times S^2$ is oriented by the form $dt \sin\theta d\theta d\varphi$, while $\sin\theta d\theta d\varphi$ orients the S^2 factor and dt orients the S^1 factor.

e) The context

This example arises in the following context: Let $\underline{X}$ be a compact, oriented 4-manifold with $b^{2+} \geq 1$. Put a Riemannian metric, g, on $T\underline{X}$ and Hodge-DeRham theory provides a b^{2+} dimensional vector space of closed, self dual 2-forms. In this regard, note that a 2-form ω is self-dual when

$$\omega \wedge \omega = |\omega|^2 d\,\mathrm{vol}\,. \tag{2.23}$$

Thus, closed, self-dual 2-forms are symplectic except where they vanish.

If the metric is chosen in a suitably generic fashion [7] (from a Baire set of smooth forms), then there exist closed, self-dual 2-forms which vanish transversely as sections of the $\mathbb{R}^3$ bundle of self dual 2-forms. Thus, if ω is such a form, then

$$Z \equiv \omega^{-1}(0) \tag{2.24}$$

is a disjoint union of embedded circles and ω is symplectic on $\underline{X} - Z$.

To consider the behavior of ω near a component circle $Z_0 \subset Z$, note that a neighborhood of Z_0 is diffeomorphic to the product of S^1 with a centered 3-ball $B^3 \subset \mathbb{R}^3$. In particular, a coordinate $t \in [0, 2\pi]$ for S^1 and $x = (x_1, x_2, x_3)$ for B^3 can be chosen so that with respect to these coordinates, the form ω is given by

$$\omega = dt \wedge A_{ij} x_i dx_j + 2^{-1} A_{ij} x_i \varepsilon_{ijk} dx_j \wedge dx_k + o(|x|^2), \tag{2.25}$$

where $A_{ij}(t)$ is a matrix valued function on S^1. Moreover,

- $A_{ij} = A_{ji}$,
- $\sum_j A_{jj} = 0$,
- $\det(A) < 0$.

(2.26)

Here, the first two properties are consequences of the fact that $d\omega = 0$, and the third is a consequence of the fact that ω vanishes transversally along Z_0. (To be precise, this just insures that $\det(A) \neq 0$; the sign is arranged through a choice of orientation for Z_0.)

Now, the fact that A_{ij} is traceless and symmetric and has negative determinant implies that it has, at each $t \in Z_0$, two positive eigenvalues and one negative eigenvalue. Let $\underline{\varepsilon} \to Z_0$ denote the real line bundle whose fiber at each $t \in Z_0$ is the negative eigenspace of the matrix $A_{ij}(t)$. This line bundle can either be orientable or not. In this regard, the following result of Gompf [4] is fundamental:

Lemma 2.4. *The parity (even or odd) of the number of components of Z where $\underline{\varepsilon}$ is orientable equals the parity of $b^{2+} - b^1 + 1$.*

In the case where $\underline{\varepsilon}$ is orientable, the form ω can be modified near Z_0 so that the resulting new form has the following properties:

- The new form is symplectic on $\underline{X} - Z_0$ and agrees with the old form outside of some previously specified neighborhood of Z_0.
- There are coordinates (t, x) for an $S^1 \times B^3$ tubular neighborhood of Z_0 in which the new form is equal to

$$\begin{aligned} dt \wedge (x_1 dx_1 + x_2 dx_2 - 2x_3 dx_3) + x_1 dx_2 \wedge dx_3 \\ -x_2 dx_1 \wedge dx_3 - 2x_3 dx_1 \wedge dx_2 \end{aligned} \tag{2.27}$$

In the case where $\underline{\varepsilon}$ is not orientable, there is a modification of ω near Z_0 so that the first point above holds, and so that the second point with (2.27) holds on a non-trivial, $S^1 \times B^3$ double cover of a tubular neighborhood of Z_0. An equivalent assertion in the $\underline{\varepsilon}$ non-orientable case is the following: The form ω has a modification on the original $S^1 \times B^3$ neighborhood of Z_0 so that the resulting new form obeys the first point in (2.27) and so that there are coordinates (t', x') on some $S^1 \times B^3$ tubular neighborhood of Z_0 in which the new form is equal to the form in (2.27) after the substitutions

- $t = t'/2$,
- $x_1 = x'_1$,
- $x_2 = \cos(t'/2)x'_2 - \sin(t'/2)x'_3$,
- $x_3 = \sin(t'/2)x'_2 + \cos(t'/2)x'_3$.

(2.28)

Now note that $B^3 - \{0\}$ is diffeomorphic to $S^2 \times [0, \infty)$ via the map which sends the centered sphere in B^3 with radius $a > 0$ to $S^2 \times \{s = -2^{-1} \ln a\}$ in $S^2 \times [0, \infty)$. The form in (2.27) pulls back under this diffeomorphism to

$$-d(e^{-s} 2^{-1}(1 - 3\cos^2\theta)dt + e^{-3s/2} \cos\theta \sin^2\theta d\varphi). \tag{2.29}$$

This form is not described by (2.3) for any contact form α on $S^1 \times S^2$.

However, it can be modified so that a constant multiple (2.3) is accurate at large s with α given by (2.11). In particular, consider:

Lemma 2.3. *Fix $R > 1$ and there are constants $c_1, c_2 > 0$ with the following significance: Let R' denote either ∞ or a number greater than $2R$. There is a symplectic form ω on $[0, \infty) \times S^1 \times S^2$ which is described by (2.22) on $[0, R/2] \times S^1 \times S^2$, and by c_1 times the form in (2.22) on $[R', \infty)$. Meanwhile, on $[R, R']$, the form is described by c_2 times the form in (2.15).*

(The proof of this lemma is left as an exercise save for the following hint: The numbers $c_{1,2}$ are on the order of e^{-R}. See also [8].)

Lemma 2.3 implies that a disjoint, finite set of embedded circles can be removed from any b^{2+} positive 4-manifold so that the resulting non-compact manifold is described by the HWZ formalism. Here, each boundary component is a copy of $S^1 \times S^2$; and after possibly passing to the non-trivial double cover, there are coordinates where the relevant contact form is given in (2.11).

f) A more general context

The subsequent discussions of the HWZ geometry takes place on a connected, non-compact manifold X which splits as $X = X_0 \cup ([0, \infty) \times \partial X_0)$, where X_0 is a compact, 4-manifold with boundary where the latter is a disjoint union of some number $N \geq 0$ copies of $S^1 \times S^2$. Furthermore, it will be assumed that X_0 has a symplectic form, ω, for which each boundary component is contact type and concave. Finally, it will be assumed that each boundary component of X_0 is described by at least one of the following points:

- There are coordinates in which the contact form is given by α in (2.11).
- There are coordinates on the non-trivial 2-fold cover in which the pull-back of the contact form is given by α in (2.11).

(2.30)

A component of ∂X_0 will be said to have *orientable z-axis line bundle* when the first point in (2.30) holds. Otherwise, it will be said to have *non-orientable z-axis line bundle.*

Note that (2.15) with (2.30) provides an extension of the symplectic form ω on X_0 to the whole of X. This extension of ω will be implicit in what follows.

The HWZ geometry of X will be defined by a choice of ω-compatible almost complex structure, $J : TX \to TX$ whose restriction to $[0,\infty) \times \partial X_0$ is as follows: When a given component of ∂X_0 has orientable z-axis line bundle, then take J as in (2.30) on the corresponding component of $[0,\infty) \times \partial X_0$. Otherwise, take J so that its lift to the non-trivial double cover of the corresponding component of $[0,\infty) \times \partial X_0$ is given by (2.30).

As the boundaries of X_0 are all concave with respect to the induced contact form, Lemma 2.1 finds all of the pseudoholomorphic subvarieties to be HWZ subvarieties. Moreover, these subvarieties are all reasonably well behaved, as indicated by the following lemma:

Lemma 2.4.*Let X, its symplectic form and its almost complex structure be as described at the beginning of this subsection. Now, let $C \subset X$ be an HWZ subvariety. Then the set $\Lambda \subset C$ of non-manifold points is a finite set at worst; infact, C intersects the complement of a compact subset of X as a properly embedded, disjoint union of cylinders. In particular, this implies that for sufficiently large s, the intersection of C with $\{s\} \times Y$ is transversal and a disjoint union of circles; and, these s-dependent circles in Y converge in the C^∞ topology as $s \to \infty$ to a disjoint union of smooth circles whose tangent lines lie in the kernel of $d\alpha$.*

As this lemma plays only a peripheral role in this article, its proof will be given elsewhere. (Given that the ends of C are embedded cylinders, the implication concerning the intersection of C with $\{s\} \times Y$ for large s follow from Theorem 1.2 in [9].)

The preceding lemma and theorems from HWZ (see [11]) provide a natural topology on the set of HWZ subvarieties in X which makes this set into a reasonable topological space.In particular, a neighborhood of a given HWZ subvariety C in this topological space is homeomorphic to the inverse image of zero for some smooth map between a ball in one finite dimensional Euclidean space to another such Euclidean space. In addition, the components of the space of HWZ subvarieties have natural compactifications as stratified spaces where the extra strata are also spaces of HWZ subvarieties. In short, these spaces of HWZ subvarieties are much like the moduli spaces of pseudoholomorphic subvarieties on compact symplectic manifolds with compatible almost complex structures.

Before preceding to the next subsection, a two part digression is in order to discuss issues which relate to the existence and uniqueness of the coordinates in (2.30).

Part 1. (Existence). Let ν be a concave contact form on $S^1 \times$

S^2. Then ν can be either tight or overtwisted. (A contact form ν is overtwisted or tight if there is or is not an embedded, closed disk which is transversal to the kernel of ν along its boundary, but whose boundary is tangent to the kernel ν.) The contact form α in (2.11) is overtwisted, witness the t = constant disk in S^2 where $\cos^2\theta \geq 1/3$. Thus, both cases of (2.30) require overtwisted contact structure.

Meanwhile, a fundamental theorem of Eliashberg [1] asserts that the overtwisted (concave) contact structures (up to homotopy through contact structures) on a compact, oriented 3-manifold are in 1-1 correspondence with the homotopy classes of oriented, 2-dimensional subbundles of the manifold's tangent bundle. In the case of $S^1 \times S^2$, the latter are classified in part by the degree of the Euler class of the subbundle.In both cases of (2.30), the Euler class in question has minimal (in absolute value) non-zero degree. (This can be verified by examining the zeros of the product of one of the vectors in (2.19) with $\sin\theta$.) Moreover, there are precisely two homotopy classes of 2-plane fields on $S^1 \times S^2$ whose Euler class is $+2$, and these two are not permuted by the diffeomorphisms group of $S^1 \times S^2$. On the otherhand, those with Euler class -2 can be mapped to the corresponding $+2$ classes by an orientation preserving diffeomorphism of $S^1 \times S^2$. Thus, up to homotopy through overtwisted contact structures, any overtwisted contact structure on $S^1 \times S^2$ whose kernel has Euler class with absolute value 2 obeys (2.30).

By way of contrast, another theorem of Eliashberg (see Theorem 4.1.4 in [2]) implies that there are no tight contact structures on $S^1 \times S^2$ whose contact 2-plane field has non-zero degree. Note that there is a unique (up to diffeomorphism) tight contact structure on $S^1 \times S^2$ [2].

The preceding observations directly imply the following:

Lemma 2.5. *Suppose that X_0 is a compact manifold with boundary where each boundary component is diffeomorphic to $S^1 \times S^2$. Let ω be a symplectic form on X_0 for which ∂X_0 is contact type, concave, and such that the kernel of the correponding contact structure has Euler class with absolute value 2 on each component of ∂X_0. Then ω can be homotoped through symplectic forms for which ∂X_0 is contact type so that the resulting form induces a contact structure on ∂X_0 which obeys one of the two points in (2.30) on each component of ∂X_0.*

Part 2. (Uniqueness). Given that a contact form α satisfies (2.30), it is by no means the case that the coordinates which realize (2.30) are unique. Even so, there are certain features of such a coordinate system which are invariant under diffeomorphisms which preserve the form in (2.11). One of these features relates to the closed integral curves in (2.17) of the vector field v in (2.16).In particular, the curves in (2.17) are, but with a single one parameter family of glaring exception, all non-trivial in $H_1(S^1 \times S^2)$. The homologically trivial, closed integral curves are characterized by the condition that $\cos^2\theta_0 = 1/3$. The union of the 1-parameter family of such curves is a pair of embedded tori in $S^1 \times S^2$ whose compliment is the disjoint union of three pieces,

- $a_0 \equiv S^1 \times \{(\theta, \varphi) : -1/3 < \cos^2\theta < 1/3\}$,
- $a_\pm \equiv S^1 \times \{(\theta, \varphi) : \pm\cos^2\theta > 1/3\}$.

(2.31)

With regard to (2.31), note that the component a_0 is the product of the circle with an annulus, while the other two components are the products of the circle with a disk.

The following lemma (the proof is self-evident) concludes the digression:

Lemma 2.6. *Any diffeomorphism of $S^1 \times S^2$ which preserves the form α in (2.11) must map a_0 to itself and either map $a_\pm$ to themselves or to each other.*

g) SW-admissable HWZ subvarieties

Let X with its symplectic form ω be as described in the previous subsection. As every component of ∂X_0 is concave, Lemma 2.1 finds all pseudoholomorphic subvarieties in X to be HWZ subvarieties. Even so, this term will be employed as a reminder that (2.7) is obeyed along with (2.6). Of particular concern are those HWZ subvarieties which are 'SW-admissable', a term which is specified in Defintion 2.9, below. This definition requires a three part, preliminary digression.

Part 1. This first part of the digression presents:

Lemma 2.7. *The contact form on ∂X_0 canonically orients the homology of ∂X_0.*

Proof of Lemma 2.7. This follows from the discussion in the final paragraph of Section 2d.

Remember this lemma when considering the definition of SW-admissable, below.

Part 2. Let $C \subset X$ denote an HWZ subvariety. Then, an irreducible component of C is, by definition, the closure of a component of the complement in C of the set Λ from the second point of (2.6). If C is an HWZ pseudoholomorphic subvariety, then so are its irreducible components. Note that an HWZ subvariety has only finitely many irreducible components. (If not, then the first point of (6) would force most to lie entirely in $(0, \infty) \times \partial X_0$. But, this possibility is ruled out by the fact that $d\alpha$ restricts as a non-negative form with finite total integral.)

Part 3. The symplectic form ω orients psuedoholomorphic subvarieties in X, and so any such variety, C, determines, by restriction, a class $[C] \in H_2(X_0, \partial X_0; \mathbb{Z})$.

Definition 2.8. A *generalized* HWZ subvariety is a finite collection $c \equiv \{(C_\alpha, m_\alpha)\}$ with the C_α's pairwise distinct, irreducible, HWZ subvarieties and the corresponding m_α's non-negative integers. (The integer m_α is called the mulitiplicity of the corresponding C_α). A generalized HWZ subvariety $\{(C_\alpha, m_\alpha)\}$ is called *SW-admissable* when the connecting homomorphism from $H_2(X_0, \partial X_0; \mathbb{Z})$ to $H_1(\partial X_0; \mathbb{Z})$ of the long exact homology sequence for the pair $(X_0, \partial X_0)$ sends $\sum_\alpha m_\alpha [C_\alpha]$ to the sum of the oriented generators of $H_1(\partial X_0; \mathbb{Z})$.

The following lemma offers some perspective on this definition:

Lemma 2.9. *Let C be an HWZ pseudoholomorphic subvariety with two properties: First, the pair $(C, 1)$ is SW-admissable. Second, there exists $s_0 > 0$ such that the intersection of C with each component of $\{s_0\} \times \partial X_0$ is path connected. Then, the following conclusions can be drawn:*

- *C intersects any large, constant s slice of any given component of $[0, \infty) \times \partial X_0$ as a circle which is an oriented generator of the first homology of the corresponding component of ∂X_0.*
- *If a particular component of ∂X_0 has orientable z-axis line bundle,then C intersects the large s portion of the corresponding component of $[0, \infty) \times \partial X_0$ in $[0, \infty) \times a_0$.*
- *If a particular component of ∂X_0 has non-orientable z-axis line bundle, then C intersects the corresponding component of $[0, \infty) \times \partial X_0$ in the image of $[0, \infty) \times a_0$ from the non-trivial 2-fold cover.*
- *Moreover, in this last case: As $s \to \infty$ on C's inverse image in the non-trivial double cover of the component in question the function θ converges to $\pi/2$ while φ converges either to $\pi/2$ or to $-\pi/2$.*

Proof of Lemma 2.9. Given the orientations of the homology of ∂X_0, the lemma is a direct consequence of Theorem 1.2 in [9].

3. The Seiberg-Witten invariants on compact 4-manifolds

This section consists of a summary of some of the relevant properties of the Seiberg-Witten invariants of a compact 4-manifold. These invariants, first introduced by Witten [24], are now discussed in a number of books (see, e.g. [12]) to which the reader is referred for more details.

In this section, X is a compact, connected, oriented 4-manifold with $b^{2+} \geq 1$. Let S denote the set of equivalence classes of Spin^C structures on X. This set is a principal bundle over a point for the additive group $H^2(X;\mathbb{Z})$. After a choice of orientation for the line $\det^+ \equiv \Lambda^{\mathrm{top}}(H^1(X;\mathbb{R})) \otimes \Lambda^{\mathrm{top}}(H^{2+}(X;\mathbb{R}))$, and also $H^{2+}(X;\mathbb{R})$ in the case where $b^{2+} = 1$, the Seiberg-Witten invariants constitute a map

$$(3.1)\quad SW_X : S \to \Lambda^*(H^1(X;\mathbb{Z})) = \mathbb{Z} \oplus H^1(X;\mathbb{Z}) \oplus \Lambda^2(H^1(X;\mathbb{Z})) \oplus \cdots$$

The map SW is defined as an algebraic count of solutions to a certain differential equation defined on X.

a) The Seiberg-Witten equations

The definition of the Seiberg-Witten equations has four parts.

Part 1. Fix a Riemannian metric on X. The latter specifies the principal $SO(4)$ bundle $\mathrm{Fr} \to X$ of oriented, orthonormal frames in TX. By definition, a Spin^C structure is a lift of Fr to a principal

$$(3.2)\qquad \mathrm{Spin}^C(4) = (SU(2) \times SU(2) \times U(1))/\{\pm 1\}$$

bundle. In this regard, note that $SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$; and with this understood, the homomorphisms from $\mathrm{Spin}^C(4)$ to $SO(4)$ simply forgets the $U(1)$ factor in (3.2).

In any event, let $F \to X$ denote a lift of Fr to a principal $\mathrm{Spin}^C(4)$ bundle.

Part 2. Associated to F are two canonical $\mathbb{C}^2$ bundles, $S_\pm$. Here, the association is via the representations of $\mathrm{Spin}^C(4)$ to $U(2) = (SU(2) \times U(1))/\{\pm 1\}$ which forget either the first factor of $SU(2)$ or the second. By convention, the projective plane bundles PS_- and PS_+ are the unit sphere bundles in the respective $\mathbb{R}^3$ bundles $\Lambda_\pm$ of anti-self dual and self dual 2-forms. The latter are associated to Fr via the two homomorphisms from $SO(4)$ to $SO(3) = SU(2)/\{\pm 1\}$ which forget one or the other factor $SU(2)$. Note that both $S_\pm$ inherit canonical Hermitian metrics.

There is also an associated $U(1)$ principal bundle, $L \to X$ which is defined via the homomorphism from $\mathrm{Spin}^C(4)$ to $U(1)$ which forgets both factors of $SU(2)$. In this regard, remember that $U(1)/\{\pm 1\}$ is isomorphic to $U(1)$.

As remarked above, the set of Spin^C structures on X is a principal bundle over a point for the group $H^2(X;\mathbb{Z})$. The action of this group on S can be simply described in terms of its effect on the bundles $S_\pm$ and L. Here, remember that $H^2(X;\mathbb{Z})$ is in 1 to 1 correspondence with the set of equivalence classes of complex hermitian line bundles where the correspondence associates a line bundle E to its first Chern class, $c_1(E) \in H^2(X;\mathbb{Z})$. With this understood, remark that when $\mathfrak{s} \in S$ and $e \in H^2(X;\mathbb{Z})$, then the Spin^C structure $e \cdot s$ is characterized by the condition that $S_\pm(e \cdot \mathfrak{s}) = E \otimes S_\pm(\mathfrak{s})$ where $E \to X$ is a complex line bundle with $c_1(E) = e$. Meanwhile, $L(e \cdot \mathfrak{s})$ is characterized by the property that its associated first Chern class equals $c_1(L(\mathfrak{s})) + 2e$.

Part 3. The Seiberg-Witten equations are defined with the help of the Clifford multiplication map

$$cl : TX \to \mathrm{Hom}(S_+, S_-) \tag{3.3}$$

Indeed, cl is a canonical bundle isomorphism between TX_C and $\mathrm{Hom}(S_+, S_-)$ which is defined by viewing the latter bundle as an associated bundle to Fr. The map in (3.3) has the following key property: When $v \in TX$, then $cl(v)^\dagger cl(v)$ and $cl(v)^\dagger cl(v)$ are equal $-|v|^2$ times the identity endomorphism of S_+ and S_-, respectively. Note that cl can also be viewed as a homomorphism

$$\underline{cl} : S_+ \otimes T^*X \to S_-. \tag{3.4}$$

Also required is the extension of cl to

$$cl_+ : \Lambda_+ \to \mathrm{End}(S_+) \tag{3.5}$$

The map cl_+ sends Λ^+ to the traceless, anti-hermitian endomorphisms of S_+. It is defined by the requirement that it send the self-dual projection of $w \wedge w'$ to

$$2^{-1}(cl(w')^\dagger cl(w) - cl(w)^\dagger cl(w')). \tag{3.6}$$

The adjoint of cl_+ maps $S_+^\dagger \otimes S_+$ to the imaginary valued sections of Λ^+. This adjoint will be denoted by cl^+.

Part 4. The data for the Seiberg-Witten equations consists of a pair (A, Ψ), where A is a connection on L and where Ψ is a section of S_+. The Seiberg-Witten equations involve the curvature 2-form F_A of the connection A and its projection, F_A^+, in Λ_+. These equations also involve the covariant derivative ∇_A on sections of S_+ which A induces with the help of the Levi-Civita conection on TX. Indeed, the Levi-Civita connection provides a connection on the principle $SO(3) \times SO(3)$ bundle $\mathrm{Fr}/\{\pm 1\}$. Thus, A and the Levi-Civita connection together provide a connection on the principle $(SO(3) \times SO(3) \times U(1))/\{\pm 1\}$ bundle $\underline{F}$ associated to F. As F is, fiberwise, a 4-fold cover of $\underline{F}$, the connection on $\underline{F}$ induces a unique connection on F. The covariant derivative of the latter connection is ∇_A.

With the preceding understood, the Seiberg-Witten equations read

- $D_A \Psi \equiv \underline{cl}(\nabla_A \Psi) = 0,$
- $F_A^+ = cl^+(\Psi \otimes \Psi^\dagger) + i\mu.$

(3.7)

Here, μ denotes a fixed, favored self-dual 2-form.

b) Properties of the space of solutions to the Seiberg-Witten equations

Fix a Spin^C structure s and so define the principal $U(1)$ bundle $L \to X$ and the $\mathbb{C}^2$ bundle S_+. The set of connections on L is naturally an affine space which is modeled on the space of smooth, imaginary valued 1-forms, $i \cdot C^\infty(T^*X) \subset C^\infty(T^*X) \otimes \mathbb{C}$. This affine structure endows the space of connections, $\mathrm{Conn}(L)$, with the structure of a smooth Frechet space manifold. Meanwhile, the space of sections of S_+ has its linear, C^∞ Frechet space structure.

Now, let $m \subset \mathrm{Conn}(L) \times C^\infty(S_+)$ denote the space of solutions to (3.7) for a given choice of μ. (Thus, m depends on the triple (s, g, μ) of Spin^C structure, Riemannian metric and self-dual 2-form.) Topologize m with the subspace topology.

The space m is always infinite dimensional because the equations in (3.7) are invariant under a certain smooth action on $\mathrm{Conn}(L) \times C^\infty(S_+)$ of the group $C^\infty(X; S^1)$ of smooth maps from X to the circle. Indeed, a map $\eta \in C^\infty(X; S^1)$ acts by sending the pair $c \equiv (A, \Psi)$ of connection on L and section of S_+ to $\eta \cdot c \equiv (A - 2\eta^{-1} d\eta, \eta \cdot \Psi)$. For future reference, note that this action is free except at pairs of the form $(A, 0)$ where the stabilizer is the circle of constant maps to S^1. By the way, such pairs $(A, 0)$ are termed *reducible*. In any event, let M denote the quotient $m/C^\infty(X; S^1)$ which will be viewed as a topological space using the quotient topology.

It also proves useful to introduce the space, $\underline{M}$, which is the quotient of $X \times m$ by the relation $(x, c) \sim (x', c')$ if and only if $x = x'$ and $c = \varphi \cdot c'$ where $\varphi \in C^\infty(X; S^1)$ obeys $\varphi(x) = 1$. Away from reducible points, the obvious projection from $\underline{M}$ to $X \times M$ has fiber S^1.

The following proposition lists some of the salient features of M and $\underline{M}$: (This proposition was known to Witten [24]; and proofs of its assertions can be found in [12].)

Proposition 3.1. *Fix a* Spin^C *structure* $\mathfrak{s}$*, a Riemannian metric* g *and a self-dual 2-form* μ*. Use this data to define the space* M *and* $\underline{M}$*. Then the following are true:*

- *M and $\underline{M}$ are compact.*
- *Each irreducible $c \in M$ has a neighborhood which is homeomorphic to the zero set of a real analytic map between balls about the origin in finite dimensional Euclidean spaces. In particular, the domain ball lies naturally in the kernel of a first order, elliptic operator δ_c and the range of this map is the cokernel of this same operator. Here, the index of δ_c is equal to*

$$(b^1 - 1 - b^{2+}) + 4^{-1}(-\tau_X + c_1(L) \bullet c_1(L)); \tag{3.8}$$

  where τ_X is the signature of X and the symbol '$\bullet$' between a pair of 2-dimensional cohomology classes signifies the value of their cup product on the fundamental class of X.
- *In general, the subspace $M_{\mathrm{reg}} \subset M$ of irreducible orbits where* $\mathrm{cokernel}(\delta_c) = 0$ *is open in M and has the structure of a smooth, orientable manifold whose dimension is given by (3.8). Moreover, an orientation of the line* $\det^+ \equiv \Lambda^{\mathrm{top}}(H^1(X;\mathbb{R}))\otimes\Lambda^{\mathrm{top}}(H^{2+}(X;\mathbb{R}))$ *provides M_{reg} with a canonical orientation. Meanwhile, the the inverse image in $\underline{M}$ of $X \times M_{\mathrm{reg}}$ has the structure of a smooth, principal S^1 bundle.*
- *Suppose that $b^{2+} > 0$. Fix the metric, and there is a Baire set $\mathcal{U} \subset C^\infty(X;\Lambda_+)$ of self dual 2-forms μ for which $M = M_{\mathrm{reg}}$, and so M has the structure of a smooth manifold of dimension given by the number in (3.8). In particular, for $\mu \in \mathcal{U}$, the operator δ_c has trivial cokernel for all $c \in M$.*

By the way, this operator δ_c is defined for any $c \in \mathrm{Conn}(E) \times C^\infty(X;S_+)$ and maps $i \cdot C^\infty(T^*X) \oplus C^\infty(S_+)$ to

$$i \cdot (C^\infty(X) \oplus C^\infty(\Lambda_+)) \oplus C^\infty(S_-).$$

In this regard, δ_c sends a pair $(b,\eta) \in i \cdot C^\infty(TX^*) \oplus C^\infty(S_+)$ to the triple in $i \cdot (C^\infty(X) \oplus C^\infty(\Lambda_+)) \oplus C^\infty(S_-)$ with the components

- $d^*b + 4^{-1}(\eta^\dagger\Psi - \Psi^\dagger\eta)$
- $d^+b - cl^+(\eta\otimes\Psi^\dagger + \Psi\otimes\eta^\dagger)$
- $D_A\eta + 2^{-1}cl(b)\Psi$.

(3.9)

Here are some observations about the preceding assertions:

- The number in (3.8) is either even or odd; its parity is the same as that of $1 - b^1 + b^{2+}$.
- When $b^{2+} > 0, \mu \in U$, and the integer in (3.8) is negative, the proposition asserts that $M = \emptyset$ since there are no negative dimensional manifolds.
- When $b^{2+} > 0,\ \mu \in U$ and (3.8) is zero, then M consists of a finite set of points. In this case, an orientation of M is an association of a sign to each point in M. (Note that a point has a canonical orientation since $H_0(point;\mathbb{Z})$ has a canonical generator.)
- When $b^{2+} > 0,\ \mu \in U$ and (3.8) is positive, let $\underline{c} \in X \times M$ denote the first Chern class of the principal S^1 bundle $\underline{M} \to X \times M$. Then slant product with $\underline{c}$ defines a canonical map, ϕ, from $H_*(X;\mathbb{Z})$ to $H^{2-*}(M;\mathbb{Z})$.

With regard to this map ϕ, note that the image under ϕ of a class $\gamma \in H^1(X;\mathbb{Z})$ has an alternate definition which goes as follows: Choose a map, $\underline{\gamma} : S^1 \to X$, which pushes forward the fundamental class of S^1 to give γ. The association to $c = (A, \Psi) \in \mathrm{Conn}(L) \times C^\infty(X; S_+)$ of the holonomy of $\underline{\gamma}^*A$ around S^1 defines a smooth map, $h_{\underline{\gamma}} : \mathrm{Conn}(L) \times C^\infty(X; S^1) \to S^1$. Then $\phi(\gamma)$ is the same class as the pull-back via h_γ of the generator of $H^1(S^1)$.

c) The Seiberg-Witten invariants

Here is the definition of the invariant SW:

Definition 3.2. Let X be a compact, conected, oriented 4-manifold with $b^{2+} > 0$. Fix an orientation for the line $\det^+$, and, in the case $b^{2+} = 1$, also fix an orientation of the line $H^{2+}(X;\mathbb{R})$. Fix a Riemannian metric on X and a Spin^C structure. Also, fix $\mu \in U$ in (3.7) to define M, but in the case when $b^{2+} = 1$, make the following additional requirement: Let ω denote a non-trivial, closed, self-dual 2-form whose direction provides the orientation for $H^{2+}(X;\mathbb{R})$. Now require that $r \equiv i\int_X \mu \wedge \omega$ be positive and very large. The value of

$$SW \in \mathbb{Z} \oplus H^1(X;\mathbb{Z}) \oplus \Lambda^2 H^1(X;\mathbb{Z}) \oplus \cdots$$

on the given Spin^C structure is computed using M as follows: Let d denote the integer in (3.8).

- If $d < 0$, then $SW = 0$.
- If $d = 0$, then M is a finite set of points and the chosen orientation for det^+ defines a map, ε, from M to $\{\pm 1\}$. With ε understood, then

$$SW = \sum_{c \in M} \varepsilon(c) \in Z. \tag{3.10}$$

- When $d > 0$, then SW has non-zero projection into $\Lambda^p H^1(X;\mathbb{Z})$ only if p has the same parity as $1 - b^1 + b^{2+}$. In this case, SW is defined by its values on the set of decomposable elements in $\Lambda^p(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$; and here, SW sends $\gamma_1 \wedge \cdots \wedge \gamma_p$ to

$$\int_M \phi(\gamma_1) \wedge \cdots \phi(\gamma_p) \wedge \phi(*)^{(d-p)/2}, \tag{3.11}$$

  where $* \in H_0(X;\mathbb{Z})$ is the class of a point.

The apparent dependence of SW on the Riemannian metric and on μ is spurious as the next proposition asserts:

Proposition 3.3. *Let X be a compact, connected, oriented 4-manifold with $b^{2+} \geq 1$. Then the values of SW on the elements of s are independent of the choice of Riemannian metric and form μ. In fact, SW depends only on the diffeomorphism type of X in the sense that it pulls back naturally under orientation preserving diffeomorphisms. This is to say that if $\varphi : X \to X$ is a diffeomorphism, then φ pulls back the chosen orientation of det^+ (and of H^{2+} when $b^{2+} = 1$), it pulls back $\Lambda^* H^1(X;\mathbb{Z})$ and it pulls back* Spin^C *structures (because metrics pull back). With this understood, then $SW(\varphi * (\cdot)) = \varphi^*(SW(\cdot))$.*

See, e.g. [12] for a proof of this Proposition.

4. The Seiberg-Witten invariants on manifolds with $S^1 \times S^2$ boundaries

There is now a well developed theory of the Seiberg-Witten invariants for manifolds with boundary. Here are a few relevant references:[13], [14],

[15]. In principle, the general story is well understood, though the details may be quite complicated if the boundary is a complicated 3-manifold. Fortunately, the case where the boundary components are all $S^1 \times S^2$'s is fairly simple to describe, and this section provides a description of the salient features.

To begin, let X_0 now denote a connected, compact, oriented 4-manifold with boundary such that $\partial X_0 = \cup_{1\leq j\leq N} S^1 \times S^2$. And, let S denote the set of equivalence classes of pairs $(\mathfrak{s}, \sigma)$ where $\mathfrak{s}$ is a Spin^C structure on X_0 whose associated line bundle L is trivial on ∂X_0, while σ is a class in $H^2(X_0, \partial X_0; \mathbb{Z})$ which maps to $c_1(L) \in H^2(X_0; \mathbb{Z})$.

After a choice of orientation for certain canonical lines, there is a well defined Seiberg-Witten invariant for X_0 which consitutes a map $SW : S \to \mathbb{Z} \oplus H^1(X_0; \mathbb{Z}) \oplus \Lambda^2 H^1(X_0; \mathbb{Z}) \oplus \cdots$. The definition of this map and a description of its properties occupies the various subsections of this Section 4.

By the way, the map SW which is defined below is very much like the Seiberg-Witten invariant defined in [13] for manifolds with boundary $S^1 \times \Sigma$, where Σ is a surface of genus greater than 1. Indeed, the arguments that justify assertions in this section are almost entirely slightly modified or simplified versions of arguments from [13]. Thus, the proofs of the various propositions and lemma to come will simply refer to the sections in [13] where analogous statements are proved, leaving it to the reader to make the necessary modifications.

a) Geometric preliminaries

To begin, consider X_0 as the complement of an open set in the non-compact manifold (without boundary) X, which is defined by identifying $\partial X_0 \subset X_0$ with $\partial X_0 \times \{0\} \subset \partial X_0 \times [0, \infty)$. Thus, $X \equiv X_0 \cup (\partial X_0 \times [0, \infty))$ is a manifold with 'tubular ends'.

Fix a metric with positive scalar curvature on ∂X_0 and then fix a metric on X which restricts to a neighborhood of $\partial X_0 \times [0, \infty)$ as the product of this standard metric on ∂X_0 and the Euclidean metric on the half line. With such a metric chosen, fix a pair $(\mathfrak{s}, \sigma) \in \mathfrak{s}$ and thus a lift, $F \to X$ of the frame bundle Fr. Use F to define (as in the compact case) the bundles $\Lambda_\pm$ of self and anti-self dual 2-forms, the complex $\mathbb{C}^2$ bundles $S_\pm$, and the principle S^1 bundle L.

Let $\mathrm{Conn}_e(L)$ now denote the space of connections, A, on the bundle L with the property that the norm of the associated curvature 2-form F_A has exponential decay on all ends of X and so that $(2\pi i)^{-1} F_A$ represents the class τ. Meanwhile, let $C_e^\infty(S_+)$ denote the space of sections of S_+ whose norms have exponential decay on all ends of X. To be explicit about this exponential decay condition, fix a function $s : X \to (-1, \infty)$

whose restriction to $\partial X_0 \times [0, \infty)$ is the projection onto the half line factor. Then a pair consisting of a connection A on L and a section Ψ of S_+ are in $\mathrm{Conn}_e(L) \times C_e^\infty(L)$ when there exists $\delta > 0$ such that

$$e^{\delta s}|F_A| + e^{\delta s}|\Psi| \tag{4.1}$$

is bounded on X, and when $(2\pi i)^{-1} F^A$ represents σ.

Note that $C_e^\infty(S_+)$ is a linear Frechet manifold where a neighborhood of zero is labeled by data $(\delta, \varepsilon, n, K)$ which consist of positive numbers δ and ε, a non-negative integer n, and a compact set $K \subset X$. The neighborhood labeled by this data consists of those $\Psi \in C_e^\infty(S_+)$ with the property the following two properties: First, all covariant derivatives of Ψ to order n are bounded by ε on K. Here, the covariant derivatives are defined by some hermitian connection on S_+ which is fixed in advance. Second, $e^{\delta s}|\Psi| < \epsilon$ on X.

Meanwhile $\mathrm{Conn}_e(L)$ is a fiber bundle over $\times_N S^1$ whose fiber is an affine Frechet manifold. With regard to this last point, remember that N denotes the number of components of ∂X_0, and with this understood, the k'th coordinate of this fibering map sends $A \in \mathrm{Conn}_e(L)$ to the limit as s tends to ∞ on the k'th end of X of the holonomy of A around any circle in $S^1 \times S^2$ which generates the latter's first homology. The boundedness of (4.1) for some $\delta > 0$ insures the existence of this limit.

By the way, as the curvature 2-form of each $A \in \mathrm{Conn}_e(L)$ has exponential decay along the ends of X, the 4-form $-(4\pi 2)^{-1} F_A \wedge F_A$ is integrable on X. Moreover, the value of the ensuing integral can be argued to be independent of the particular choice of A from $\mathrm{Conn}_e(L)$ and thus depends only on s. Indeed,

$$-(4\pi 2)^{-1} \int_X F_A \wedge F_A = c_1(L) \bullet c_1(L). \tag{4.2}$$

where the right hand side denotes the evaluation on the fundamental class in $H_4(X_0, \partial X_0; \mathbb{Z})$ of the cup product with itself of any lift of $c_1(L)$ to $H^2(X_0, \partial X_0; \mathbb{Z})$. (In particular, σ is such a lift.)

The point here is that the bilinear pairing, $\bullet$, on $\otimes_2 H^2(X_0, \partial X_0; \mathbb{Z})$ is symmetric, but not perfect and the kernel of this pairing is precisely the image of $H^1(\partial X_0)$ under the connecting homomorphism for the long exact cohomology sequence of the pair $(X_0, \partial X_0)$. Thus, $\bullet$ can be viewed, equivalently, as a non-degenerate pairing on the kernel of the restriction induced homomorphism from $H^2(X_0; \mathbb{Z})$ to $H^2(\partial X_0; \mathbb{Z})$. (The fact is that Poincaré duality provides only a perfect bilinear pairing from $H^2(X_0) \otimes H^2(X_0, \partial X_0)$ to $\mathbb{Z}$.) It is convenient to view the pairing $\bullet$ at times from one or the other of these view points.

In any event, for the time being, view $\bullet$ as a symmetric pairing on $H^2(X_0, \partial X_0; \mathbb{Z})$ and let $b^{2+}(X)$ denote the maximum of the dimensions of those subspaces $V \subset H^2(X_0, \partial X_0)$ on which it is positive definite.

b) Properties of the solutions

The Seiberg-Witten invariant for a fixed pair $(\mathfrak{s}, \sigma) \in S$ is computed via an appropriate count of the solutions $(A, \Psi) \in \mathrm{Conn}_e(L) \times C_e^\infty(S_+)$ of the equation in (3.7) where μ is a fixed self-dual form on X whose norm has exponential decay on the ends of X. (That is, $e^{\delta s}|\mu|$ is bounded on X for some $\delta > 0$.) Thus, of prime interest is the set $m \subset \mathrm{Conn}_e(L) \times C_e^\infty(S_+)$ of pairs (A, Ψ) which obey (3.7) on X. The lemma below lists some basic properties of elements $c \in m$.

Lemma 4.1. *Let m be as just described. Then, there exist constants $\kappa > 0$ and, for each $n \geq 0$, there exists $\zeta_n \geq 1$ with the following significance: Let $(A, \Psi) \in m$. Then*

- $e^{\kappa s}(|F_A| + |\Psi|) \leq \zeta_0$
- *For each $n \geq 1$, $e^{\kappa s}(|\nabla^n F_A| + |(\nabla_A)^n \Psi|) \leq \zeta_n$.*
- *On each end of X, the connection A has exponential decay to some flat connection in the following sense: There is a flat connection A_0 on L's restriction to the given end such that $a \equiv A - A_0$ obeys $e^{\kappa s}|\nabla^n a| \leq \zeta_n$ for all n.*

Proof of Lemma 4.1. The proof is obtained by modifying the arguments in Section 6.4 of [13]. Here, the fact that the scalar curvature of the metric on $S^1 \times S^2$ is positive plays the role played in[13] by the assumption that the solutions to the Seiberg-Witten equations on $S^1 \times \Sigma$ are non-degenerate (See Section 5 of [13]). (The positivity of the scalar curvature also implies that the only solutions to the unperturbed 3-manifold Seiberg-Witten equations for the same metric on $S^1 \times S^2$ have the form $(A, 0)$, where A is flat.) Note that the arguments in Section 6.4 of [13] give directly an exponential decay assertion in the L_1^2 Sobolev norm. However, standard elliptic estimates can be used to prove that there is exponential decay in the C_k norms for all k.

c) The moduli space

The group $C^\infty(X; S^1)$ acts on $\mathrm{Conn}_e(L) \times C_e^\infty(L)$ and this action is smooth when $C^\infty(X; S^1)$ is viewed as a Frechet lie group using the C^∞ Frechet topology. Here, the open neighborhoods of the constant map 1 are indexed by triples (ε, n, K) where $\varepsilon > 0, n \in \{0, 1, \ldots\}$, and $K \subset X$ is compact; and the corresponding open set consists of those maps φ which obey $|\nabla^k\varphi| < \varepsilon$ on K for all $k \in \{0, 1, \ldots, n\}$. Give $m \subset \mathrm{Conn}_e(L) \times C_e^\infty(S_+)$ the subspace topology and then give $M \equiv m/C^\infty(X; S^1)$ the quotient topology.

As in the compact X case, it proves useful to introduce the somewhat larger space $\underline{M}$ which is the quotient of $X \times m$ by the equivalence relation $(x, c) \sim (x', c')$ when $x = x'$ and when $c = \varphi \cdot c'$ where $\varphi \in C^\infty(X; S^1)$ obeys $\varphi(x) = 1$. This space $\underline{M}$ admits an continuous action of S^1 and the quotient space is $X \times M$.

Key properties of M and $\underline{M}$ are described in Proposition 4.2, below. The statement of this proposition uses τ_X to denote the signature of the pairing $\bullet$ on $H^2(X_0, \partial X_0)/\,\mathrm{Image}(H^1(\partial X_0)$ and it uses $H^{2+}(X_0)$ to denote any choice of b^{2+} dimensional subspace of

$$H^2(X_0, \partial X_0)/\,\mathrm{Image}(H^1(\partial X_0)$$

to which $\bullet$ restricts as a positive definite pairing.

Proposition 4.2. *Fix a pair $(\mathfrak{s}, \sigma) \in S$, a Riemannian metric g and a self-dual 2-form μ which exponentially decays on the ends of X. Use this data to define the space M and $\underline{M}$. Then the following are true:*

- *M and $\underline{M}$ are compact. In fact, given the* Spin^C *structure $\mathfrak{s}$, there are only finitely many pairs $(\mathfrak{s}, \sigma') \in S$ with M non empty.*

- *Each irreducible $c \in M$ has a neighborhood which is homeomorphic to the zero set of a real analytic map between balls about the origin in finite dimensional Euclidean spaces. In particular, the domain ball and the range ball lie naturally in the respective kernel and cokernel of a Fredholm operator, δ_c, between separable Hilbert spaces. Here, the index of δ_c is equal to*

$$b^1 - 1 - b^{2+} + 4^{-1}(-\tau_X + c_1(L) \bullet c_1(L)).$$

- *In general, the subspace $M_{\mathrm{reg}} \subset M$ of irreducible orbits where* $\mathrm{cokernel}(\delta_c) = 0$ *is open in M and has the structure of a smooth, orientable manifold whose dimension is equal to the index of δ_c. Moreover, the inverse image in $\underline{M}$ of $X \times M_{\mathrm{reg}}$ has the structure of a smooth, principal S^1 bundle.*

- *Fix attention on an end of X. Then, the assignment to $c = (A, \Psi) \in M$ of the $s \to \infty$ limit of the holonomy of A around the loop $S^1 \times$* {point} *$\times \{s\}$ in the given end of X defines a continuous map from M to S^1 which is smooth on M_{reg}.*

- *Suppose that $b^{2+} > 0$. Then, with the metric on X fixed, there is a Baire subset U of choices for μ in (3.7) which have exponential decay on the ends of X and are such that the following hold:*
  - *a) M contains no reducible pairs.*
  - *b) $M = M_{\mathrm{reg}}$.*
  - *c) Fix an end of X and the corresponding map from M to S^1 is generic in the sense that it has at most a finite number of critical points, and each is non-degenerate. Moreover, the critical values of this map can be assumed to miss any previously specified countable set in S^1.*
  - *d) The critical points for the maps to S^1 defined by distinct ends of X are distinct.*
- *An orientation of the line $\Lambda^{\mathrm{top}} H^1(X;\mathbb{R}) \otimes \Lambda^{\mathrm{top}} H^{2+}(X;\mathbb{R})$ canonically orients M_{reg}.*

Proof of Proposition 4.2. Except for the final point about orientations, all of the assertions can be proved by slightly modified versions of arguments from Section 8 of [13]. The final point can be proved by modifying arguments from Section 9.1 of [13].

Two key examples take $X_0 = S^1 \times \underline{B^3}$ and $X_0 = \underline{B^2} \times S^2$, where $\underline{B^p} \subset \mathbb{R}^p$ denotes the closed, unit radius ball centered at the origin. In the $S^1 \times \underline{B^3}$ case, take the metric to be one with non-negative scalar curvature which restricts to a product neighborhood of the boundary $S^1 \times S^2$ as the product of the Euclidean metric on the line with a metric having positive scalar curvature. With such a metric and with $\mu = 0$ in (3.7),

$$M = S^1. \tag{4.3}$$

In fact, M consists solely of reducible pairs $(A, 0)$, where A is pulled back from S^1. In particular, the map from M to S^1 given by the holonomy of A about $S^1 \times \{\text{point}\}$ provides the identification in (4.3).

In the $\underline{B^2} \times S^2$ case, take the product of the round metric on S^2 with a metric on $\underline{B^2}$ which has non-negative scalar curvature and restricts to a product neighborhood of the boundary circle as a flat, product metric. With such a metric and with $\mu = 0$ in (3.7),

$$M = \{\text{point}\}, \tag{4.4}$$

where the point in M is the reducible pair $(A, 0)$ with A a trivial connection.

d) The Seiberg-Witten invariants

Assume now that $b^{2+}(X_0) > 0$. Orient the line

$$\Lambda^{\mathrm{top}} H^1(X_0;\mathbb{R}) \otimes \Lambda^{\mathrm{top}}(H^{2+}(X_0;\mathbb{R});$$

and if $b^{2+} = 1$, also orient $H^{2+}(X_0;\mathbb{R})$. Having done so, the moduli space M can be used to define a map $SW : S \to \mathbb{Z} \oplus H^1(X_0;\mathbb{Z}) \oplus \cdots$ which is suitably invariant under diffeomorphisms of X_0. Indeed, Definition 3.2 translates verbatim to define SW in the present case. This is to say that the value of SW on a element $(\mathfrak{s}, \sigma) \in S$ is computed as follows: First, fix a suitable Riemannian metric on X and then fix μ from Proposition 4.2's set U to define M, but in the $b^{2+} = 1$ case, make the following added requirement on μ: Let ω denote a non-trivial, closed, compactly supported 2-form whose class in $H^2(X_0, \partial X_0; \mathbb{R})$ class has positive square and defines the given orientation for $H^{2+}(X_0)$. Then, require that $r \equiv i \int_X \mu \wedge \omega$ be positive and very large. Note that resulting M is a compact, oriented, smooth manifold of dimension

$$d \equiv \dim M = b^1 - 1 - b^{2+} + 4^{-1}(-\tau_X + c_1(L) \bullet c_1(L)).$$

With this last point understood, set $SW(\mathfrak{s}, \sigma) = 0$ when $d < 0$ as $M = \emptyset$ in this case. In the case when $d = 0$, then M is a finite set of signed points and $SW(\mathfrak{s}, \sigma)$ is the integer which is obtained by summing the signs which are associated to the points of M. Finally, when $d > 0$, then the component of $SW(\mathfrak{s}, \sigma)$ in $\Lambda^p H^1(X; Z)$ is defined by the condition that it send the decomposable element

$$\gamma_1 \wedge \cdots \wedge \gamma_p \in \Lambda^p(H_1(X;\mathbb{Z})/\,\mathrm{Torsion})$$

to the value of the expression in (3.1).

As just defined, SW has the following properties:

Proposition 4.3. *The values of SW as just defined are independent of the choice of Riemannian metric and form μ subject to the aforementioned constraints on $[0, \infty) \times \partial X_0$. Infact, SW depends only on the diffeomorphism type of X_0 as a manifold with boundary in the sense that it pulls back naturally under orientation preserving diffeomorphisms of the pair $(X_0, \partial X_0)$.*

Proof of Proposition 4.3. The arguments in Section 9.2 of [13] carry over directly. By the way, these arguments do not require a lemma to the effect that the space of oriented diffeomorphisms of $S^1 \times S^2$ is path connected, nor does it require a lemma to the effect that the space of metrics on $S^1 \times S^2$ with positive scalar curvature is path connected. An argument for Proposition 4.3 can be made given only that the space of all metrics on $S^1 \times S^2$ is path connected.

The latter argument is made along the following lines: First, establish a restricted version of Proposition 4.3 which limits the diffeomorphisms under consideration to those whose boundary restriction is isotopic to the identity, and which limits metric variation to that which changes the boundary metric along a path in the Frechet space of positive scalar curvature metrics. Then, establish an appropriate analog of the product formula in Theorem 9.5 of [13] for the resulting restricted Seiberg-Witten invariants. In fact, the analog of Theorem 9.5 from [13] for the restricted Seiberg-Witten invariants can be phrased to read like a modified version of Proposition 4.5, below; with the major modification occuring in the assumptions about $\underline{X}$. In particular, the modified Proposition 4.5 takes $\underline{X}$ diffeomorphic to X_{0+} and X_{0-} diffeomorphic to $[-1,1]\times\partial X_{0+}$. In any event, the analog of Theorem 9.5 from [13] will imply the full invariance of SW as stated in Proposition 4.3.

e) The invariant for X_0 and for compact 4-manifolds

Let $\underline{X}$ now denote a compact, connected oriented, smooth 4-manifold with $b^{2+} \geq 1$, and suppose that φ is an embedding of the disjoint union, Y, of some $N \geq 1$ copies of $(S^1 \times S^2)$ into $\underline{X}$, each of which separates $\underline{X}$. Then, $\underline{X}$ can be written as $\underline{X} \equiv X_{0-} \cup_Y X_{0+}$, where $X_{0\pm}$ are compact, oriented manifolds with boundary Y. In this case, the invariant SW for X can be computed in terms of that for X_{0-} and X_{0+}. The story in the general case is somewhat outside the scope of this paper. However, there are three special cases where the story is quite simple:

- $b^{2+} > 0$ for both $X_{0\pm}$.
- $b^{2+} > 0$ for X_{0+} while $X_{0-} \subset \underline{X}$ is the closure in $\underline{X}$ of a tubular neighborhood of the disjoint union of embedded circles and 2-spheres with self-intersection number zero. Furthermore, at least one of these 2-spheres gives a non-zero class in $H_2(\underline{X};\mathbb{R})$.
- $b^{2+} > 0$ for X_{0+} while $X_{0-} \subset \underline{X}$ is the closure in $\underline{X}$ of a tubular neighborhood of the disjoint union of embedded circles and 2-spheres, where the latter are all inessential in the real, second homology of $\underline{X}$.

(4.5)

The story for the first two cases in (4.5) is simply stated as follows:

Proposition 4.4. *Let $\underline{X}$ be as described above and suppose either the first or the second case in (4.5) holds. Then $SW_{\underline{X}} \equiv 0$.*

To describe $SW_{\underline{X}}$ in the third case of (4.5), digress first to introduce the set $\mathcal{Y}$ of $B^2 \times S^2$ components of X_{0-}. Then, note that the boundary of each $Y \in \mathcal{Y}$ is also a component of the boundary of X_{0+} and thus picks out a distinguished homology class, $\gamma_Y \in H_1(X_{0+}; \mathbb{Z})$. (This class is non-zero in $H_1(X_0; \mathbb{Z})/\operatorname{Torsion}$ because the core S^2 in Y is non-zero on $H_2(\underline{X}; \mathbb{Z})/\operatorname{torsion}$.) Digress again to note that the inclusion induced map $\iota : H_1(X_{0+}; \mathbb{Z}) \to H_1(\underline{X}; \mathbb{Z})$ is surjective.

With these last points understood, consider:

Proposition 4.5. *Suppose that $\underline{X}$ is a compact, oriented 4-manifold with $b^{2+} > 1$ which decompose as $X_{0-} \cup_Y X_0$ where $b^{2+}(X_0) > 0$ and where X_{0-} is a tubular neighborhood of a disjoint union of embedded circles and 2-spheres with zero self-intersection number. Let $\mathcal{Y}$ denote the set of components of X_{0-} which contain these 2-spheres. If $\mathcal{Y} = \emptyset$, set $\kappa \equiv 1$, and otherwise, order the components of $\mathcal{Y}$ and set $\kappa \wedge_{Y \in \mathcal{Y}} \gamma_Y$ where the order of the terms in this exterior product conforms to the chosen ordering of $\mathcal{Y}$. Then,*

$$SW_{\underline{X}}(\iota(\gamma_1) \wedge \cdots \wedge \iota(\gamma_p)) = \pm SW_{X_{0+}}(\kappa \wedge \cdots \wedge \gamma_p),$$

where the $\pm$ here is independent of $\{\gamma_j\}_{1 \le j \le p}$.

Proof of Propositions 4.4 and 4.5. The proof of Theorem 9.1 of [13] can be modified in the present context to provide a natural description of the extended moduli space $\underline{M}_X$ for a suitable metric on $\underline{X}$ in terms of the corresponding moduli spaces $\underline{M}_\pm$ for $X_{0\pm}$. In order to write this formula, let $\underline{M}_X$ denote the extended moduli space $\underline{M}$ for $\underline{X}$. Recalling that the latter comes with a canonical projection to $\underline{X}$, let $\underline{M}_X|_Y \subset \underline{M}_X$ denote the subset which lies over Y. Define $\underline{M}_\pm|_Y$ analogously. Moreover, use $(\underline{M}_-|_Y \times \underline{M}_+|_Y)|_\Delta \subset \underline{M}_-|_Y \times \underline{M}_+|_Y$ to denote the subspace which lies over the diagonal, $\Delta \subset Y \times Y$. With these definitions understood, the analog here of Theorem 9.1 from [13] asserts that there are certain metrics on $\underline{X}$ for which $\underline{M}_X|_Y$ has a natural, S^1-equivariant identification with $(\underline{M}_-|_Y \times \underline{M}_+|_Y)|_\Delta$. Here, S^1 acts on the product via the diagonal action. (The metrics in question admit, for some large constant T, an isometric embedding of $[-T,T] \times Y$ into $\underline{X}$ which map $\{0\} \times Y$ to $Y \subset \underline{X}$ via the identity map. Here, $[-T,T] \times Y$ has a metric which is the product of a Euclidean metric on $[-T,T]$ and a positive scalar curvature metric on Y.) The assertions of both propositions follow with a little algebraic topology from this picture of $\underline{M}_X|_Y$.

f) An important corollary

Proposition 4.6, below, summarizes the features of the map SW which are relevant for the subsequent discussion of the assertions in (1.1).

Note that this proposition is an immediate corollary to Propositions 4.3 and 4.5.

Proposition 4.6. *Let X_0 be as described in Section 4a, and suppose that $(\mathfrak{s}, \sigma) \in S$ has a non-zero value of SW. Let μ be any self-dual 2-form on X with exponential decay on $[0, \infty) \times \partial X_0$. Then, there exists at least one solution to the $((\mathfrak{s}, \sigma), \mu)$ version of (3.7). In particular, if $\underline{X}$ and X_0 are as described in Proposition 4.5 and $\underline{X}$ has non-trivial Seiberg-Witten invariant, then so does X_0 and so there exists $(\mathfrak{s}, \sigma) \in S$ such that the $((\mathfrak{s}, \sigma), \mu)$ version of (3.7) has at least one solution for every choice of exponentially decaying, self-dual 2-form μ.*

5. SW-admissable HWZ subvarieties and the Sieberg-Witten invariants

This section returns to the milieu of Sections 2f and 2g. In particular, suppose that the manifold with boundary X_0 and the corresponding manifold X with its symplectic form ω are as described in the aformentioned parts of Section 2. Likewise, endow X with an ω-compatible almost complex structure, J, which restirct to the components of $[0, \infty) \times \partial X_0$ as follows: If a component of ∂X_0 has orientable z-axis line bundle, take coordinates on this copy of $S^1 \times S^2$ so that the contact form is given by (2.11). Then, use (2.22) to define J. If the component in question does not have an orientable z-axis line bundle, take coordinates on the non-trivial 2-fold cover so that the pull-back of the contact form is given by (2.11). Then, take J so that its pull-back to this same double cover is given by (2.22) in these same coordinates.

Note that J defines the Riemannian metric $g = \sqrt{2}\omega(\cdot, J(\cdot))/|\omega|$, and with respect to this metric, ω is self-dual and J is an orthogonal transformation. Moreover, the metric on $[0, \infty) \times \partial X_0$ is given by the second line in (2.21) in the coordinates on a component or its double cover where J is given by (2.22) and α by (2.11).

The metric here will be used to define the Seiberg-Witten equations on X; thus it plays a role in the definition of SW for X_0. Meanwhile, the almost complex structure will be used, as in Section 2g, to define the SW-admissable subvarieties. The task for this section is to point out a relationship between the map SW on the one hand, and SW-admissable subvarieties on the other.

a) SW-admissable subvarieties and the set S

The relationship between the Seiberg-Witten invariants and SW-admissable subvarieties begins with the fact that a Spin^C structure on

X_0 of the sort which arises in the definition of SW can be canonically assigned to each SW-admissable, generalized HWZ subvariety:

Proposition 5.1. *An SW-admissable, generalized HWZ subvariety $c = \{(C_\alpha, m_\alpha)\}$ canonically defines a* Spin^C *structure $\mathfrak{s}_c$ with associated line bundle L that is trivial over ∂X_0. Moreover, an SW-admissable, generalized HWZ subvariety of the form $(C, 1)$ which satisfies the four points in Lemma 2.9 canonically defines a pair, $(\mathfrak{s}_C, \sigma_C) \in S$.*

The remainder of this section is occupied with the

Proof of Proposition 5.1. To start, fix a Spin^C structure on X_0 whose corresponding line bundle L is trivial on ∂X_0. Introduce the endomorphism cl_+ in (3.5); then $cl_+(\omega)$ is a skew hermitian endomorphism whose eigenvalues are $\pm i \cdot \sqrt{2}|\omega|$. These eigenspaces of $cl_+(\omega)$ decompose S_+ as $L_+ \oplus L_-$, a direct sum of complex line bundles where, $cl_+(\omega)$ acts as $\imath \cdot \sqrt{2}|\omega|$ on L_+. Note that the line bundle L_+ is non-trivial over any component of ∂X_0; its first Chern class is the oriented generator of the second cohomology of each component of ∂X_0. (The orientation of this cohomology is describe in Lemma 2.7 and the discussion in the final paragraph of Section 2d.)

Meanwhile, duality identifies $H_2(X_0, \partial X_0; \mathbb{Z})$ with $H^2(X_0; \mathbb{Z})$ and so an SW-admissable, generalized HWZ subvariety c defines a canonical element, $e_c \in H^2(X_0; \mathbb{Z})$ which is the Poincaré dual to $\sum_{(C,m)\in c} m[C]$. And, according to Definition 2.8, this class must restrict to each component of ∂X_0 as the oriented generator of the second cohomology of the component in question. Thus, e_c and $c_1(L_+)$ differ by an element in $H^2(X_0, \partial X_0; \mathbb{Z})$ and so one obtains

Lemma 5.2. *Let c denote an SW-admissable, HWZ subvariety. Then, there exists a unique* Spin^C *structure, $\mathfrak{s}_c$, over X_0 with the following properties:*

- *The associated line bundle L is trivial over ∂X_0.*
- $c_1(L_+) = e_c$.

The next task is to find the class $\sigma_C \in H^2(X_0, \partial X_0; \mathbb{Z})$ so that $(\mathfrak{s}_C, \sigma_C)$ are in S under the assumption that C satisfies the four points in Lemma 2.9. For this purpose, introduce the line bundle

$$K \equiv \mathrm{Hom}(L_-, L_+).$$

Now there are four claims to be made about K:

Lemma 5.3. *Define the line bundle K as above. Then K has the following properties:*

- *K admits a canonical section, κ, over ∂X_0.*
- *The zero set of κ is canonically homologous to twice $C \cap \partial X_0$.*
- *The homology from the preceding point with any choice of an extension of κ to a section of K over X_0 can be used with C to canonically define a closed, oriented subvariety in X_0.*
- *The class in $H_2(X_0, \mathbb{Z})$ of the subvariety just mentioned is independent of the extension of κ off of ∂X_0, and its dual maps to $c_1(L)$ in $H^2(X_0; \mathbb{Z})$.*

Accept this lemma on faith for the moment to finish the definition of Proposition 5.1's class σ_C: Take σ_C to be the dual class to the fundamental class in $H_2(X_0; \mathbb{Z})$ of the subvariety from Lemma 5.3's fourth point. Then, the pair $(\mathfrak{s}_C, \sigma_C) \in S$ because σ_c is constructed to be the first Chern class of the complex line bundle $L_+^2 K^{-1}$ and this line bundle is $L = \det(S_+)$.

With the preceding understood, then the proof of Proposition 5.1 is completed with the

Proof of Lemma 5.3. To start, it is important to realize that the $\mathbb{R}^2$ bundle underlying K is isomorphic to the orthogonal complement in Λ_+ to ω. This is because the restriction to this complement of cl_+ produces purely off diagonal endomorphisms of S_+ with respect to the decomposition of S_+ as $L_+ \oplus L_-$. Thus, studying K means studying the orthgonal complement of ω in Λ_+; and this view of K will be used to prove the statements in Lemma 5.3.

To prove the first two statements of the lemma, it proves useful to distinguish in the discussion those components of ∂X_0 with oriented z-axis line bundle from those without. Consider first the case where a component $Y \subset \partial X_0$ has orientable z-axis line bundle. In this case, there are coordinates on Y where α is given by (2.11) and then ω on $[0, \infty) \times Y$ is given by the first line in (2.21). In particular, where $0 < \theta < \pi$, the line bundle K is spanned by the forms

- $dt \wedge \sin^2 \theta d\varphi - g^{-2} df \wedge dh$.
- $dt \wedge dh - \sin^2 \theta d\varphi \wedge df$.

(5.1)

Take the first form above for the section κ of K. With κ understood, note that $\kappa^{-1}(0)$ on ∂X_0 is the set $S^1 \times \{0, \pi\}$; it is left to the reader to check that the orientations are such that $\kappa^{-1}(0)$ is twice the generator of $H^1(S^1 \times S^2; \mathbb{Z})$. These last observations justify the first two points of Lemma 5.3 for Y.

To obtain the homology in question, use the points in Lemma 2.9 to conclude that when s is large, then C's intersection with $\{s\} \times Y \subset [0, \infty) \times Y$ is a very small distance push-off of the parameterized loop given by (2.17) where the the left hand side of (2.18) is an integer and has positive denominator. (Note that the natural orientations are opposite, however.) Thus, for large s, there is a canonical homotopy between C's intersection with $\{s\} \times Y$ and the orientation reversed version of the loop in (2.17). Meanwhile, the orientation reversed version of this same loop from (2.17) is canonically homotopic to $S^1 \times \{\theta = 0\}$: Indeed, simply decrease the θ coordinate in (2.17) from its given value of θ_0 to 0. Likewise, this loop is canonically homotopic to $S^1 \times \{\theta = \pi\}$ via the homotopy which increases the θ coordinate from θ_0 to π.

Now consider the case where the given component $Y \subset \partial X_0$ has unoriented z-axis line bundle. In this case, there are coordinates on the non-trivial double cover of Y where α pulls back to give the form in (2.11) and ω pulls back to give the form in the top line of (2.21). In particular, where $0 < \theta < \pi$ the pull-back of K is spanned by the forms in (5.1). Save these last observation.

To exploit the preceding, introduce

$$(x_1 = \sin\theta\cos\varphi, x_2 = \sin\theta\sin\varphi, x_3 = \cos\theta)$$

on the non-trivial double cover of Y and use them to define a set of functions (t', x'_1, x'_2, x'_3) on Y itself via (2.28). It then follows from (2.28) and the remarks in the preceding paragraph that the Poincaré dual to the first Chern class of K's restriction to Y is the parameterized curve

$$\tau \to (t' = 2\tau, x'_1 = 0, x'_2 = \sin(\tau), x'_3 = \cos(\tau)). \tag{5.2}$$

Meanwhile, it follows from the final point of Lemma 2.9 that when s is large, the intersection of C with $\{s\} \times Y$ is a very small distance push-off of the circle where $x'_1 = 1$. Thus, this intersection is canonically homotopic to the $x'_1 = 1$ circle. Meanwhile, the circle in (5.2) is canonically homotopic to twice the $x'_1 = 1$ circle. Indeed, consider the homotopy which replaces the right side of (5.2) which sends $(\tau, r) \in S^1 \times [0, 1]$ to $(t' = 2\tau, x'_1 = (1 - r^2)^{1/2}, x2' = r\sin(\tau), x'_3 = r\cos(\tau))$.

With the first two points of Lemma 5.3 understood, the third point of the lemma follows now from the preceding discussion in an absolutely straightforward manner.

The fourth point of Lemma 5.3 immediately from the first three points.

b) SW-admissable HWZ subvarieties and the Seiberg-Witten invariants

Take X_0 as in the previous subsection and reintroduce the Seiberg-Witten invariant for X_0, $SW : S \to Z \oplus H^1(X_0;\mathbb{Z}) \oplus \Lambda^2 H^1(X_0;\mathbb{Z}) \oplus \cdots$. The following theorem is the key result in this article:

Theorem 5.4. *Let $(\mathfrak{s}, \sigma) \in S$ be a class on which SW is non-zero. Then there exists an SW-admissable, generalized HWZ subvariety c with $\mathfrak{s}_c = \mathfrak{s}$.*

Note that there is no assertion here that about the existence of an SW-admissable, HWZ subvariety C which obeys the four points in Lemma 2.9, and has both $\mathfrak{s}_C = \mathfrak{s}$ and $\sigma_C = \sigma$. Indeed, it is not clear that this must be the case even with reasonable, additional assumptions.

Theorem 5.4 is a generalization of the main theorem in [17], [18] which gives the $\partial X_0 = \emptyset$ version. Theorem 5.4 should also be compared with Theorem 1.2 in [16] which proves an analgous theorem in the more restrictive context of Riemannian pseudoholomorphic geometry. Moreover, the proof of Theorem 5.4 here borrows heavily from that of Theorem 1.2 in [16], while the latter follows many of the lines of the main theorem in [17], [18]. On the other hand, Theorem 1.2 in [16] can be viewed as a corollary to the Theorem 5.4 and Proposition 4.5. Conversely, a special case of Theorem 5.4 can be deduced from Theorem 1.2 in [16].

c) A generalization of Theorem 5.4

With a metric on X as described at the beginning of this section, fix $r \geq 1$ and consider the version of (3.7) where $\mu = 2^{-1} r\omega$. If Ψ in this version (3.7) is replaced by $\sqrt{r}\Psi$, then (3.7) reads

- $D_A\Psi \equiv \underline{cl}(\nabla_A\Psi) = 0$,
- $F_A^+ = r(cl^+(\Psi \otimes \Psi^\dagger) - i2^{-1}\omega)$.

(5.3)

Proposition 4.6 implies that the $((\mathfrak{s}, \sigma), r)$ version of (5.3) has a solution for every $r \geq 1$ when $SW(\mathfrak{s}, \sigma) \neq 0$. Thus, Theorem 5.4 is a corollary of:

Theorem 5.5. *Fix $(\mathfrak{s}, \sigma) \in S$ and suppose that there exists an unbounded, increasing sequence $\{r_n\} \in (0, \infty)$ with the property that for each index n, the $r = r_n$ version of (5.3) has a solution, $(A_n, \Psi_n) \in \mathrm{Conn}_e(L) \times C_e^\infty(S_+)$. Then there exists an SW-admissable, generalized HWZ subvariety c with $\mathfrak{s}_C = \mathfrak{s}$. In addition, there is a subsequence* *of* $\{(A_n, \Psi_n)\}$ *(hence relabled by consecutive integers) with the following property: For each* n, *let* α_n *denote the orthogonal projection of* Ψ_n *into eigenspace of* $cl_+(\omega)$ *in* S_+ *with eigenvalue* $i \cdot \sqrt{2}|\omega|$. *Introduce the HWZ subvariety* $C' \equiv \cup_{(C,m)\in c} C$ *Let* $Q \subset X$ *be any compact set and then*

$$\lim_{n\to\infty} [\sup_{x\in C'\cap Q} \operatorname{dist}(x, \alpha_n^{-1}(0)) + \sup_{x\in\alpha_n^{-1}(0)\cap Q} \operatorname{dist}(C', x)] \tag{5.4}$$

exists and equals zero. Finally, there is a constant $\wp$ *which depends solely on the symplectic form and the Riemannian metric of* X_0, *and is such that*

$$\sum_{(C,m)\in c} m \int_C \omega = 2^{-1}[\omega] \bullet c_1(L) + \wp \tag{5.5}$$

6. Estimates for the proof of Theorem 5.5

The argument for Theorem 5.5 is begun in this section and completed in the next. However, before starting, take note of the fact that the argument presented here is a modified version of the proof in [16] of Theorem 2.2 in [16]. (The latter asserts an analog of Theorem 5.5's existence result in the context of Riemannian pseudoholomorphic geometry which plays the role of Theorem 5.5 here.) The proof of Theorem 2.2 in [16] and that given below of Theorem 5.5 can be viewed as having three distinct parts. The first part derives global bounds for various measures of Ψ and F_A. The second part uses the global bounds to obtain stronger estimates on compact domains. Note that these first two parts are more conceptually distinct than chronologically distinct. In any event, the first two parts of the proof occupy Section 6. The final part of the proof occupies Section 7. In the third part of the proof, the bounds on compact subsets of X from this section are used in conjunction with various arguments from [17], [18] to complete the proof of Theorem 5.5. In this regard, note that [17], [18] proves Theorem 5.5 in the case where $\partial X_0 = \emptyset$.

As indicated, the discussion here follows closely the proof of Theorem 2.2 in [16], and so referrals to [16] are frequent.

In the remainder of Section 6 and in Section 7, the implicit assumption is that a pair $(\mathfrak{s}, \sigma) \in S$ has been fixed, that r is large and that (A, Ψ) is a solution to the $((s, \sigma), r)$ version of (5.3).

a) Integral bounds for $|\Psi|^2$.

As might be expected from the title, the purpose of this subsection is to obtain integral bounds for $|\Psi|^2$. The statement of these bounds requires a brief digression to introduce some notation. To start the digression, introduce the characteristic number

$$\varepsilon_\omega(s) \equiv [\omega] \bullet \sigma. \tag{6.1}$$

Here, $[\omega]$ is the class of the symplectic form in $H^2(X_0, \partial X_0)$. In this regard, remember that ω is exponentially decaying along $[0,\infty) \times \partial X_0$ and so canonically defines a class in $H^2(X_0, \partial X_0)$. Also, note that the right hand side of (6.3), though written in terms of $\sigma \in H^1(X_0, \partial X_0)$, depends only on σ's image, $c_1(L)$, in $H^2(X_0)$.

To continue the digression, let g denote the chosen Riemannian metric and let R_g denote g's scalar curvature function and W_g^+ the self-dual part of the Weyl curvature. Then, let R_{g-} denote the minimum of zero and R_g. Thus, R_{g-} has compact support in X_0. Finally, let dvolg denote the volume form of the metric g.

With the digression now over, consider:

Lemma 6.1. *There is a universal constant c with the following significance: Let $(\mathfrak{s}, \sigma) \in S$ be given. Now, suppose that (A, Ψ) solves (5.3) for the given $(\mathfrak{s}, \sigma)$ and for some choice of $r \geq 1$. Then*

- $\int_X (2^{-1/2}|\omega| - |\Psi|^2)^2 d\,\mathrm{vol}_g \leq cr^{-1}(\varepsilon_\omega(s)$
  $+ \int_X (|R_g| + r^{-1}|R_g|^2)|\omega| d\,\mathrm{vol}_g$.
- $\int_X |2^{-1/2}|\omega| - |\Psi|^2| d\,\mathrm{vol}_g \leq cr^{-1}(\varepsilon_\omega(s)$
  $+ \int_X (|R_g| + |W_g^+|^2)|\omega| + r^{-1}|R_{g-}|) d\,\mathrm{vol}_g$.

Proof of Lemma 6.1. The argument here is almost an exact copy of that which proves Lemma 3.1 in [16]. The only difference occurs in the modification of a particular term which appears in a differential equation for $|\Psi|^2$, the latter being implied by the Weitzenboch formula which writes $D_A^\dagger D_A$ in terms of the Laplacian $\nabla_A^\dagger \nabla_A$. To be precise, note that the Weitzenboch formula used in the proof of Lemma 3.1 of [16] implies that

$$2^{-1}d^*d|\Psi|^2 + |\nabla_A \Psi|^2 + 4^{-1}r|\Psi|^2(|\Psi|^2 - 2^{-1/2}|\omega| + r^{-1}R_g) \leq 0, \tag{6.2}$$

which is the same equation as (3.4) from [16]. Now, the point is that this last inequality holds with R_g replaced by R_{g-} :

$$2^{-1}d^*d|\Psi|^2 + |\nabla_A \Psi|^2 + 4^{-1}r|\Psi|^2(|\Psi|^2 - 2^{-1/2}|\omega| + r^{-1}R_{g-}) \leq 0, \tag{6.3}$$

With (6.3) understood, the subsequent arguments for Lemma 3.1 in [16] can be imported verbatim to prove Lemma 6.1 here. (Note that

R_g can be replaced by R_{g-} in the corresponding Equation (3.4) from [16], but in the latter equation, the distinction is superfluous because [16] considers these equations on compact manifolds.)

b) Pointwise bounds for $|\Psi|^2$

The purpose of this subsection is to obtain pointwise bounds for $|\Psi|^2$. These bounds come via (6.3) with the help of the maximum principle. In particular, since both $|\Psi|$ and $|\omega|$ decay to zero (exponentially fast) as the parameter $s \in [0, \infty)$ gets large on $[0, \infty) \times \partial X_0$, and since $|\omega| \leq \sqrt{2}$, the maximum principle with (6.3) immediately gives the bound

$$|\Psi|^2 \leq 1 + r^{-1}|R_{g-}|. \tag{6.4}$$

The following lemma gives some fine structure to the pointwise behavior of $|\Psi|^2$:

Lemma 6.2. *There is a consant ξ which depends only on the Riemannian metric and which has the following significance: Given $(\mathfrak{s}, \sigma) \in S$ and $r \geq 1$, let (A, Ψ) be a solution to the corresponding version of (6.1). Let $\underline{s} \equiv 6^{-1/2} \ln r$. Then,*

- $|\Psi|^2 \leq 2^{-1/2}|\omega| + \xi r^{-1}$ *where $s \leq \underline{s}$.*
- $|\Psi|^2 \leq \zeta r^{-1} e^{-(s-\underline{s})/\sqrt{2}}$ *where $s \geq \underline{s}$.*

(6.5)

The remainder of this subsection contains the

Proof of Lemma 6.2. First, remember that ω is both self-dual and closed, and so $(d^*d\omega)^+ = 0$. The Bochner-Weitzenboch formula for this last equation (see, e.g. Appendix C in [3]) implies that

$$d^*d|\omega| + |\omega|^{-1}|\nabla\omega|^2 \geq -k^+|\omega|, \tag{6.6}$$

where k^+ has compact support on X_0 and is bounded by a universal multiple of $|R_g|$ and $|W_g^+|$. With this last equation understood, introduce $u \equiv |\Psi|^2 - 2^{-1/2}|\omega|$ and then (6.3) and (6.6) together imply that

$$2^{-1}d^*du + (4\sqrt{2})^{-1}r|\omega|u \leq \zeta_1 e^{-\sqrt{6}s}, \tag{6.7}$$

where ζ_1 is a constant which depends only on the Riemannian metric. Here, s has been extended as a smooth function to the whole of X from is original domain of definition, $[0, \infty) \times X_0$. (This extension of the domain of s will be implicit in the subsequent appearances of the function s.)

By the way, the derivation of (6.7) uses (6.4) and the fact that $|\omega|$ and $|\nabla\omega|$ obey bounds on X of the form

- $|\omega| \geq m^{-1}e^{-\sqrt{6}s}$,
- $|\omega| + |\nabla\omega| \leq me^{-\sqrt{6}s}$,

(6.8)

where $m \geq 1$ is a constant.

The next step in the proof of Lemma 6.2, starts with the following observation: Equation (6.7) implies that there exists a constant ξ which depends solely on the Riemannian metric on X and is such that $\underline{u} \equiv u - \xi r^{-1}$ obeys the differential inequality $2^{-1}d^*d\underline{u} + (4\sqrt{2})^{-1}r|\omega|\underline{u} \leq 0$. Since u is negative where s is very large on X, the maximum principle can be invoked with this last equation to prove that $\underline{u} \leq 0$ everywhere on X. That is,

$$|\Psi|^2 \leq 2^{-1/2}|\omega| + \zeta r^{-1}. \tag{6.9}$$

This last bound gives the first point in (6.5).

To obtain the second point, first note that (6.9) and (6.2) together imply that $|\Psi|^2$ obeys

$$d^*d|\Psi|^2 + 2^{-1}|\Psi|^2 \leq \zeta(e^{-\sqrt{6}s} + r^{-1})e^{-\sqrt{6}s}. \tag{6.10}$$

where $s \geq 0$. Here, ζ is a constant which depends only on the Riemannian metric. At the same time, $|\Psi|^2 \leq \zeta r^{-1}$ where $s \geq \underline{s}$, and so the comparison principle can be invoked for (6.10) to establish that

$$|\Psi|^2 \leq \zeta r^{-1}e^{-(s-\underline{s})/\sqrt{2}} \tag{6.11}$$

where $s \geq \underline{s}$. This is the second point in (6.7).

c) Writing $\Psi = (\alpha, \beta)$ and estimates for $|\beta|^2$.

As in the proof of Theorem 2.2 of [16], the next step Theorem 5.5's proof requires the introduction of the components (α, β) of Ψ as follows:

- $\alpha \equiv 2^{-1}(1 + i(\sqrt{2}|\omega|)^{-1}cl_+(\omega))\Psi$,
- $\beta \equiv 2^{-1}(1 - i(\sqrt{2}|\omega|)^{-1}cl_+(\omega))\Psi$.

(6.12)

The claim now is that $|\beta|$ is uniformly small over X. Here is the precise statement:

Proposition 6.3. *There are constant $\xi_1, \xi_2 \geq 1$ which depends only on the Riemannian metric chosen for X and which have the following significance: Let (A, Ψ) be a solution to (5.3) as defined by the chosen* *pair* $(\mathfrak{s}, \sigma) \in S$ *and* $r > (\xi_2)^4$. *Let* $R \in (r/\zeta_1, \zeta_1)$. *Then, the* β *component of* Ψ *obeys*

$$|\beta|^2 \leq \xi_2 R^{-1}(2^{-1/2}|\omega| - |\alpha|^2) + \xi_1 R^{-2} \tag{6.13}$$

where $s \leq \underline{s} - 6^{-1/2} \ln R$. *(This is where* $re^{-\sqrt{6}s} \geq R$.*)*

The remainder of this subsection is occupied with the

Proof of Proposition 6.3. Modulo some notational changes, the proof of Proposition 3.3 in [16] proves this proposition. Indeed, the arguments in the latter proof can be followed with minor notational changes to establish the existence of constants ζ_3, ζ_4 which are independent of the data $r, R, (\mathfrak{s}, \sigma)$, and (A, Ψ), and which have the following significance: Set $w \equiv (2^{-1/2}|\omega| - |\alpha|^2)$ and then let $u \equiv |\beta|^2 - \zeta_1 R^{-1} w - \zeta_2 R^{-2}$. Also, let u_+ denote the maximum of u and 0. Note that u_+ may only be Lipschitz where it is zero. In any event, where $re^{-\sqrt{6}(s-1)} \geq R$, this u_+, viewed as a distribution, obeys the differential inequality

$$d^* du_+ + \xi^{-1} R u_+ \leq 0. \tag{6.14}$$

where $\xi \geq 1$ is independent of $R, r, (\mathfrak{s}, \sigma)$ and (A, Ψ). (Note that (6.8) was used to derive (6.14).)

Meanwhile, where $re^{-\sqrt{6}(s-1)} = R$, the first point in (6.5) and (6.8) imply that $u_+ \leq \zeta_3 r^{-1} R$ with $\zeta_3 \geq 1$ a constant which is independent of $r, R, (\mathfrak{s}, \sigma)$ and (A, Ψ). Given this last observation and (6.14), the comparison principle implies that

$$u_+ \leq r^{-1} R \zeta_4 \exp(-\sqrt{R}(\sqrt{6}\ln(r/R) + 1 - s)/\sqrt{\xi}), \tag{6.15}$$

where $re^{-\sqrt{6}s} \geq R$ and where $\zeta_4 \geq 1$ is independent of $r, R, (\mathfrak{s}, \sigma)$ and (A, Ψ). This last inequality implies the lemma since $r^{-1} R e^{-\sqrt{R}/\sqrt{\xi}} \leq \zeta R^{-2}$ where ζ can be taken to be independent of r and R.

d) Bounds for the curvature

The purpose of this subsection is to exhibit bounds on the curvature 2-form of the connection A. In this regard, the arguments for these bounds are essentially the same as those which appear in Section 3d of [16] so the discussion will be fairly brief.

The discussion here begins with the self-dual projection, F_A^+, of the curvature. In particular, the second line of (5.3) implies that

$$\begin{aligned} |F_A^+| = r(2\sqrt{2})^{-1}((2^{-1/2}|\omega| - |\alpha|^2)^2 \\ + 2|\beta|^2(2^{-1/2}|\omega| + |\alpha|^2) + |\beta|^4)^{1/2}. \end{aligned} \tag{6.16}$$

This last equality and (6.13) provide the following useful lemma:

Lemma 6.4. *Fix $k \geq 1$ and there is a constant, $\zeta_k \geq 1$, which is independent of the data, $r(\mathfrak{s}, \sigma)$ and (A, Ψ) and which has the following significance: If $r \geq \zeta_k$, then*

$$|F_A^+| \leq r(2\sqrt{2})^{-1}(2^{-1}|\omega| - |\alpha|^2) + \zeta_k \tag{6.17}$$

at all points where $s \leq k$.

Bounds for the anti-self dual part, F_A^-, of the curvature of A are obtained, as in [16], by exploiting a differential equation for the latter which is implied by the fact that the total curvature is a closed 2-form. The following proposition summarizes these bounds:

Proposition 6.5. *Fix the* Spin^C *structure and fix $m \geq 1$. Then, there are constants, $\zeta_m, \zeta_m' \geq 1$, which are independent of the data r, σ and (A, Ψ) and which have the following significance: Take $r \geq \zeta_m'$, and then*

$$|F_A^-| \leq r(2\sqrt{2})^{-1}(1 + \zeta_m r^{-1/2})(2^{-1}|\omega| - |\alpha|^2) + \zeta_m' \tag{6.18}$$

at all points where $s \leq m$.

Proof of Proposition 6.5. Except for some minor notational changes, the proof is essentially identical to the proof of Proposition 3.4 in [16] to which the reader is referred. Note that this argument for Proposition 6.5 provides along the way the amusing integral inequalities given in (6.19), below. Both involve a constant $\zeta \geq 1$ which depends on the given Spin^C structure $\mathfrak{s}$, but which is independent of r, σ, and (A, Ψ). Moreover both inequalities hold only when $r \geq \zeta$. Here are the inequalities:

- $\int_X |F_A|^2 \leq \zeta r$.
- $\int_X (1 + \operatorname{dist}(x, \cdot)^{-2})(|\nabla_A \Psi|^2 + r^{-1}|F_A^+|^2) \leq \zeta$ for any point $x \in X$.

(6.19)

(The proof of the preceding two inequalities is the same as the proof of (3.29) in [16].)

e) Bounds for $\nabla_A \alpha$ and $\nabla_A \beta$

The required bounds for these derivatives are summarized by

Proposition 6.6. *Fix the* Spin^C *structure and fix $m \geq 1$. Then, there are constants, $\zeta_m, \zeta_m' \geq 1$, which are independent of the data r, σ*

and (A, Ψ) and which have the following significamce: Take $r \geq \zeta'_m$, and then

$$|\nabla_A \alpha|^2 + r|\nabla_A \beta|^2 \leq \zeta_m r(2^{-1/2}|\omega| - |\alpha|^2) + \zeta'_m. \tag{6.20}$$

at all points where $s \leq m$.

Proof of Proposition 6.6. Except for some minor notational changes, the argument is the same as that for Proposition 3.7 in [16].

f) A summary of conclusions from [16] which now apply

The next series of arguments for Theorem 5.5 are borrowed virtually verbatim from the proof of Theorem 2.2 of [16]. The results of these arguments are summarized below, while the reader is referred to the appropriate place in [16] for the proof.

To begin, suppose that $B \subset X$ is a compact set, and consider the energy of B:

$$e_B \equiv (4\sqrt{2})^{-1} r \int_B |\omega||2^{-1/2}|\omega| - |\Psi|^2| d\,\mathrm{vol}_g\,. \tag{6.21}$$

The key feature of e_B is summarized by

Proposition 6.7. *There is a constant $\zeta \geq 1$, and given $m \geq 1$, there is a constant $\zeta_m \geq 1$; and these constants have the following significance:*

Suppose that $r \geq \zeta_m$ and let (A, Ψ) be a solution to the $((\mathfrak{s}, \sigma), r)$ version of (5.3). Let $B \subset X$ be a geodesic ball with center x on which $s \leq m$. Let ρ denote the radius of B and require that $1/\zeta_m \geq \rho \geq 2^{-1} r^{-1/2}$. Then

- *$e_B \leq \zeta \rho^2$.*
- *If $|\alpha(x)| \leq (2\sqrt{2})^{-1}|\omega|$, then $e_B \geq \zeta_m^{-1} \rho^2$.*

This last proposition has various collaries, the most immediate being:

Lemma 6.8. *Given $m \geq 1$, there is a constant $\zeta_m \geq 4$ with the following significance: Fix $r \geq \zeta_m$ and let (A, Ψ) be a solution to the $((\mathfrak{s}, \sigma), r)$ version of (5.3). Let $\rho \in (\zeta_m r^{-1/2}, \zeta_m)$. Then,*

- *Let Λ be any set of disjoint balls of radius ρ whose centers lie on $\alpha^{-1}(0)$ and lie where $s \leq m$. Then Λ has less then $\zeta_m^{-1} \rho^{-2}$ elements.*
- *The set of points in $\alpha^{-1}(0)$ which lie where $s \leq m$ has a cover by a set Λ of no more that $\zeta_m \rho^{-2}$ balls of radius ρ. Moreover, each ball in this set has its center on $\alpha^{-1}(0)$. Finally, the set of concentric balls of radius $\rho/2$ is disjoint.*

The preceding lemma can then be used to prove the following refinement of Proposition 6.5:

Proposition 6.9. *Given $m \geq 1$, there are constants $\zeta_m, \zeta'_m \geq 1$ with the following significance: Fix $r \geq \zeta_m$ and let (A, Ψ) be a solution to the $((\mathfrak{s}, \sigma), r)$ version of (5.3). Then, at points of X where $s \leq m$,*

$$|F_A^-| \leq r(2\sqrt{2})^{-1}(2^{-1/2}|\omega| - |\alpha|^2) + \zeta'_m. \tag{6.22}$$

Proof of Propositions 6.7 and 6.9, and Lemma 6.8. These are the respective analogs of Propositions 4.1 and 4.3, and Lemma 4.2 in [16], and the proofs of the latter in Section 4 of [16] carry over with only small notational changes.

The next step in the proof of Theorem 5.5 is also borrowed from [16], this being a description of (A, Ψ) at distances from $\alpha^{-1}(0)$ which are $o(r^{-1/2})$. In particular,the assertion of Proposition 5.2 of [16] holds here with the obvious changes: First, (A, Ψ) is a solution on X to (5.3). Second, instead of choosing $\delta > 0$, choose $m \geq 1$ and restrict the point x to lie where $s \leq m$. Finally, the constant ζ_δ in the statement of Proposition 5.2 of [16] is replaced by a constant $\zeta_m \geq 1$.

With the structure of (A, Ψ) near $\alpha^{-1}(0)$ understood, consider now the behavior from Section 6 of [16] at larger distance from $\alpha_{-1}(0)$. Here, the assertions of Proposition 6.1 and Lemma 6.2 from [16] can be borrowed with only notational changes. The notationally modified assertions are summarized in

Proposition 6.10. *Given $m \geq 1$, there is a constant $\zeta_m \geq 4$ with the following significance: Fix $r \geq \zeta_m$ and let (A, Ψ) be a solution to the $((\mathfrak{s}, \sigma), r)$ version of (5.3). If $x \in X$ is such that $s \leq m$, then*

$$\begin{aligned} r|2^{-1}|\omega| - |\alpha|^2| + r^2|\beta|^2 + |\nabla_A\alpha|^2 + r|\nabla_A\beta|^2 \\ \leq \zeta_m(1 + r\exp[-\sqrt{r}\,\mathrm{dist}(x, \alpha^{-1}(0))/\zeta_m]). \end{aligned} \tag{6.23}$$

Proof of Proposition 6.10. Mimic the proof of Lemma 6.2 in [16].

This last result facilitates the identification of the connection A at distances which are uniformly far from $\alpha^{-1}(0)$. Indeed, at distances from $\alpha^{-1}(0)$ which are $o(1)$, the bounds in (6.17), (6.22) and (6.23) imply that the curvature F_A has an r independent upper bound. This suggests that when r is large, the connection A is close to some fiducial connection, A^0, at such distances from $\alpha^{-1}(0)$. This is indeed the case. To describe this canonical connection, introduce $K \subset \Lambda^+$ to denote the orthogonal complement to the span of ω. As Λ^+ is oriented, so K is oriented by writing $\Lambda^+ = \mathbb{R}\omega \oplus K$. Moreover, Λ^+ has a natural inner product, so does K and thus K can be viewed in a canonical way as a complex line bundle over X. Furthermore, the Levi-Civita connection on TX induces a connection on Λ^+ and thus, by orthogonal projection, a connection on K. The latter is hermitian with respect to the aforementioned complex line bundle structure. Use $\underline{A}^0$ to denote the dual connection on K^{-1}.

To proceed with the defintion of A^0, reintroduce the line bundle $L_+ \to X$ from the proof of Proposition 5.1. The line bundle L_+ enters because the determinant line bundle L for the Spin^C structure is naturally isomorphic to $L = K^{-1}L_+^2$. With this point understood, note that α is a section of L_+ and so α^2 can be viewed as a section of $\mathrm{Hom}(K^{-1}, L)$. In particular, where α is not zero, $\alpha^2/|\alpha|^2$ defines a hermitian identification between K^{-1} and L.

With the previous two paragraphs understood, it can now be stated that the canonical connection A^0 on L is the image of the Levi-Civita induced connection $\underline{A}^0$ on K^{-1} under the identification via $\alpha^2/|\alpha|^2$ of these two bundles.

Having now defined A^0, consider:

Proposition 6.11. *Given $m \geq 1$, there is a constant $\zeta_m \geq 4$ with the following significance: Fix $r \geq \zeta_m$ and let (A, Ψ) be a solution to the $((\mathfrak{s}, \sigma), r)$ version of (5.3). If $x \in X$ is such that $s \leq m$, and* $\mathrm{dist}(x, \alpha^{-1}(0)) \geq r^{-1/2}$, *then*

$$(6.24)\quad |A - A_0| + |F_A - F_{\underline{A}^0}| \leq \zeta_m r^{-1} + \zeta_m r \exp[-\sqrt{r}\,\mathrm{dist}(x, \alpha^{-1}(0))/\zeta_m].$$

Proof of Proposition 6.12. Copy the proof of Proposition 6.1 in [16].

7. Completion of the proof of Theorem 5.5

The proof of Theorem 5.5 is completed here with an analysis of the $n \to \infty$ limit of the sets $\alpha_n^{-1}(0)$ which appear in the statement of Theorem 5.5. The analysis of this limit follows closely the discussion in Section 7 of [16].

a) The curvature as a current

In this section, let $\{r_n\}_{n=1,2,\ldots}$ be an unbounded, increasing sequence of positive numbers such that for each n, the $((\mathfrak{s}, \sigma), r = r_n)$ version of (5.3) has a solution $\{(A_n, \Psi_n)\}$. The difference between the curvature

2-form of the connection A_n and that of the canonical connection $\underline{A}^0$ on K^{-1} defines a current on X, which is to say, a linear functional on the Frechet space of compactly supported, smooth 2-forms. To be precise, the current in question assigns to a smooth, 2-form ν with compact support the number

$$f_n(\nu) \equiv 2^{-1} \int_X i/(2\pi)(F_{A_n} - F_{\underline{A}^0}) \wedge \nu. \tag{7.1}$$

Note that if $m > 0$ is given, then (6.17), (6.22) and (6.23) together provide $\zeta_m > 1$ such that

$$|f_n(\nu)| \leq \zeta_m \sup_X |\nu| \tag{7.2}$$

when $s \leq m$ on the support of ν. This implies, in particular, that the sequence $\{f_n(\cdot)\}$ of linear functionals on the space of compactly supported 2-forms has weak limits and any such limit defines a bounded, linear functional on the space of 2-forms with support on some fixed compact subdomain in X. Choose one such weak limit, denote it by f, and renumber the subsequence of $\{f_n\}$ which converges to f consecutively from $n = 1$.

The current f is integral in the following sense: If ν is a closed 2-form with compact support and with integral periods on $H_2(X; Z)$, then

$$f(\nu) = c_1(L_+) \bullet [\nu] \in Z, \tag{7.3}$$

where $[\nu]$ is the class of ν in $H^2(X, \partial X; Z)$.

b) The support of f

The support of f is described by Lemma 7.1, below. Note that except for notational changes, the proofs of Lemmas 7.1 here and Lemma 7.1 in [16] are the same.

Lemma 7.1. *There is a closed subspace $C' \subset X$ with the following properties:*

- *$f(\nu) = 0$ if ν has compact support on $X - C'$.*
- *Let $B \subset X$ be an open set which intersects C'. Then, there is a 2-form ν with compact support on B and with $f(\nu) \neq 0$.*
- *Fix $m \geq 0$ and the set of points in C' where $s \leq m$ has finite 2-dimensional Hausdorf measure.*

- *Fix $m \geq 0$ and there is a constant ζ_m with the following significance: Let $B \subset X$ be a ball of radius $\rho \leq \zeta_m^{-1}$ and center on C'. Then, the 2-dimensional Hausdorf measure of $C \cap B$ is greater than $\zeta_m^{-1}\rho^2$.*
- *There is a subsequence of $\{(A_n, \Psi_n)\}$ such that the corresponding sequence $\{\alpha_n^{-1}(0)\}$ converges to C' in the following sense: If $Q \subset X$ is any compact set, the following limit exists and is zero:*

$$\lim_{n\to\infty}[\sup_{x\in C'\cap Q} \operatorname{dist}(x, \alpha_n^{-1}(0)) + \sup_{x\in\alpha_n^{-1}(0)\cap Q} \operatorname{dist}(x, C')]. \tag{7.4}$$

With Lemma 7.1 understood, the arguments in Section 7c,d of [16] can be transferred here essentially verbatim to prove

Proposition 7.2. *The set C' from Lemma 7.1 is the image of a smooth, complex curve, C_0, via a proper, pseudoholomorphic map $f : C_0 \to X$. Thus, C' is a pseudoholomorphic subvariety and so an HWZ subvariety. Moreover, there is a positive integer assigned to each irreducible component of C' such that the following is true: Let c denote the corresponding generalized HWZ subvariety. Then c and the current f are related in the following sense: For any compactly supported 2-form ν,*

$$f(\nu) = \sum_{(C,m)\in c} m \int_C \nu. \tag{7.5}$$

(Note that the conclusion here that C' is an HWZ subvariety follows from Lemma 2.1.)

c) An SW-admissable, generalized subvariety

The purpose of this subsection is to prove that the generalized, HWZ pseudoholomorpic variety, c, in Proposition 7.2 is SW-admissable. In this regard, note that (7.3) and (7.5) imply that $e_c = c_1(L)$.

A proof that $\sum_{(C,m)\in c} m[C]$ in $H_2(X_0, \partial X_0; \mathbb{Z})$ maps to the sum of the oriented generators of $H_1(\partial X_0; \mathbb{Z})$ proves that c is admissable. For this purpose, introduce a function χ on $\mathbb{R}$ with total integral equal to 1 and with compact support in $[0, 1]$. Then, for $R \geq 0$, introduce the function χ_R of the parameter s via the formula $\chi_R(s) \equiv \chi(s - R)$. Thus χ_R is a function on X with support where $s \in [R, R + 1]$. With χ_R now defined, set $\nu \equiv (2\pi)^{-1}\chi_R ds \wedge dt$.

Now, consider first the case where $Y \subset \partial X_0$ is a component with oriented z-axis line bundle. Identify Y as $S^1 \times S^2$ via coordinates where the contact form is given by (2.11) and ω by (2.21). Then, the image of $\sum_{(C,m)\in c} m[C]$ in $H_1(Y;\mathbb{Z})$ is the number $q \equiv \sum_{(C,m)\imath nc} m \int_C \nu$ times the oriented generator.

Now, the point is that (7.5) identifies this number q as equal to $f(\nu)$, and thus it follows from the definition of f that the number q is also given by (7.1) in the case where n is large. In particular, since the curvature of A_n is exponentially decreasing to zero as $s \to \infty$ on $[0,\infty) \times M$, it follows that the the image of $\sum_{(C,m)\in c} m[C]$ in $H_1(Y;\mathbb{Z})$ is q times the oriented generator, where q is equal to

$$-i/(4\pi) \int_X F_{\underline{A}^0} \wedge \nu. \tag{7.6}$$

On the other hand, this last intergral computes the evaluation of $-2^{-1}c_1(K^{-1})$ on the oriented 2-sphere $\{\text{point}\} \times S^2$ in Y. The latter is half of the evaluation of $c_1(K)$ on this same 2-sphere, and it follows by considering the zeros of the sections of K in (5.1) that this number equals 1, which is the required answer.

A similar argument proves the case for those components of ∂X_0 with unoriented z-axis line bundle.

d) The symplectic area of C

The assertion in (5.5) follows directly from (7.3) and (7.5). In particular, the constant $\wp$ is given by

$$\wp = -2^{-1} \int_X i/(2\pi) F_{\underline{A}^0} \wedge \omega. \tag{7.7}$$

(This integral converges since $F_{\underline{A0}}$ is bounded on $[0,\infty) \times \partial X_0$ while $|\omega|$ decays exponentially fast.)

References

[1] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. **98** (1989) 623–637.

[2] ———, *Filling by holomorphic discs and its applications*, London Math. Soc. Lect. Notes Ser. **151** (1991) 45–67.

[3] D. S. Freed & K. K Uhlenbeck, *Instantons and four-manifolds,* Springer, 1984.

[4] R. Gompf, private communication.

[5] H. Hofer, *Pseudoholomorphic curves in symplectization with applications to the Weinstein conjecture in dimension-3*, Invent. Math. **114** (1993) 515–563.

[6] ———, *Dynamics, topology and holomorphic curves*, Proc. Internat. Congr. Math., Berlin 1998, Vol I, Documenta Math. Extra Volume ICM, 1998, 255–280.

[7] K. Honda, *Harmonic forms for generic metrics*, Preprint.

[8] ———, *Local properties of self-dual harmonic 2-forms on a 4-manifold*, Preprint.

[9] H. Hofer, K. Wysocki & E. Zehnder, *Properties of pseudoholomorphic curves in symplectizations. I*, Ann. Inst. Henri Poincaré **13** (1996) 337–379.

[10] ———, *Properties of pseudoholomorphic curves in symplectizations. II*, Geom. Funct. Anal. **5** (1995) 270–328.

[11] ———, *Properties of pseudoholomorphic curves in symplectizations. III*, Preprint.

[12] J. W. Morgan, *The Seiberg-Witten equations and applications to the topology of smooth 4-manifolds*, Math. Notes, Vol. 44, Princeton Univ. Press, Princeton, 1996.

[13] J. W. Morgan, Z. Szabo & C. H. Taubes, *A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture*, J. Differential Geom. **44** (1996) 706–788.

[14] J. W. Morgan, T. S. Mrowka & Z. Szabo, *Product formulas along T^3 for Seiberg-Witten invariants*, Math. Res. Lett. **4** (1997) 915–929.

[15] P. Osvath & Z. Szabo, *Higher type adjunction inequalities in Seiberg-WItten theory*, Preprint.

[16] C. H. Taubes, *Seiberg-Witten invariants and pseudoholomorphic subvarieties for self- dual, harmonic 2-forms*, J. Top. Geom.

[17] ———, *SW => Gr: From the Seiberg-Witten equations to pseudoholomorphic curves*, J. Amer. Math. Soc. **9** (1996) 845–918.

[18] ———, *SW => Gr: From the Seiberg-Witten equations to pseuooholomorphic curves*, in Proc. First IP Lecture Ser., Vol II, (R. Wentworth ed.), Internat. Press, 2000.

[19] ———, *The Seiberg-Witten and the Gromov invariants*, Math. Res. Lett. **2** (1995) 221–238.

[20] ———, *SW = GR: Counting curves and connections*, J. Differential Geom, to appear, and reprinted in Proc. First IP Lecture Ser., Vol II, (R. Wentworth ed.), Internat. Press, 2000.

[21] ———, *Counting pseudoholomorphic submanifolds in dimension-4*, J. Differential Geom. **44** (1996) 818-893, and reprinted in Proc. First IP Lecture Ser., Vol II, (R. Wentworth ed.), Internat. Press, 2000.

[22] ———, *The geometry of the Seiberg-Witten invariants*, Proc. Internat. Congr. Math., Berlin 1998, Vol II, Documenta Math. Extra Volume ICM 1998, 493–504.

[23] ———, *The structure of pseudo-holomorphic subvarieties for a degenerate almost complex structure and symplectic form on* $S^1 \times B^3$, Geom. Top. **2** (1998) 221–332.

[24] E. Witten, *Monopoles and 4-manifolds*, Math. Res. Lett. **1** (1994) 769–796.

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