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WHICH SINGER IS THAT?

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Most of us learn to crawl and walk; a few of us learn to run swiftly. Is Singer is one of those people — he ran swiftly and still does, as this is written. For those who doubt that, I invite them to try a set of tennis with him! Singer was visiting us (at Penn from MIT) for a few days while he gave us some lectures. Walking, with one of our graduate students, in a department hallway that had a large glass window at one end, I reached that window and looked out at some university tennis courts below. Singer was there doing what he often does as he travels: playing a set with a local pro. That student (now a famous homological algebraist) and I watched as Singer fired aces to one side of the court and the other. I asked my young, student friend, "Who is that fellow out there?" He didn't know. "That's I. M. Singer," I said. "Gosh! All that and the Atiyah–Singer Index Theorem, too!" he exclaimed. We were both impressed. As Singer runs, he also takes mathematics, and, often, physics with him along the paths he follows.

Everyone acquainted with the major developments of research mathematics in the last third of the twentieth century has had contact with the Atiyah–Singer Index Theorem. There are people who have referred to it as the "best" or "most important" theorem of the twentieth century. I have called it that, but I have heard that from other people, as well. Such declarations may not have very clear meanings; my own is based on some "absolute" feeling for depth, scope, and applicability. A good guess is that the Atiyah–Singer Index Theorem would appear on virtually every broadly educated, research mathematician's list of the ten most important theorems of the twentieth century. Each such list will contain personal favorites. Mine might include the Murray–von Neumann theorems on existence and uniqueness of the additive trace in factors of type II_1 and the uniqueness of the hyperfinite II_1 factor, as well as Tomita's Main Theorem (in the Tomita–Takesaki Theory) — to which many of the readers might respond "Huh?" My list would also include the Spectral Theorem, and in an *ecumenical spirit*, The Allendorfer, Chern–Weil versions of Werner Fenchel's original extension of the Gauss–Bonnet theorem to higher dimensions. "Huh?" would probably not be heard in connection with these latter choices, and certainly not with the Atiyah–Singer Index Theorem. The Atiyah–Singer Index Theorem is at the core of a vast body of work, created by Atiyah and Singer, which has touched and influenced most of current mathematics and much of theoretical physics.

Since pure mathematics is, in my view, the poetry of basic science, it's not surprising that its results and advances are less accessible to the general public than those of, say, biology and chemistry. Nevertheless, if we were seeking a result in those areas with an effect analogous to the discovery of the Atiyah–Singer Index, we might point to the Crick–Watson discovery of the double helix nature of DNA. Where the Crick–Watson work discloses something of the fundamental biology of life, the Atiyah–Singer Index Theorem reveals something as fundamental as the interplay among the topological, geometric and analytic patterns in the fabric of our universe. There is also something analogous in the process by which both results were discovered. Two superbly talented scientists, with a very clear view of where they are headed, a sure knowledge of what they want to achieve, and a firm grasp of the techniques they need, organize the large scientific enterprise needed to arrive at their goal and orchestrate the results and methods used from the many component subdisciplines into a powerful solution of their major problems.

The Atiyah–Singer Index Theorem extended the algebraic Riemann–Roch Theorem to complex manifolds using analytic techniques in the tradition of Hodge, Weyl, and Kodaira. It produced new topological invariants that topologists are still challenged to describe by traditional methods, thirty years later. Atiyah and Singer provided a unified treatment of the Riemann–Roch, Hirzebruch Signature, and Gauss–Bonnet Theorems by their use of the Dirac operator. The Index Theorem removed many barriers between algebraic geometry, differential geometry, topology, and analysis.

They gave two proofs. The first proof involved a cobordism argument that required the solution to an elliptic boundary value problem. The methods they discovered remain valuable today. Their second proof was an axiomatic treatment of the topological and analytic index. This approach lends itself to a natural extension to operator algebras and noncommutative geometry. A third proof, by P. Gilkey, is based on the heat equation approach to index theory, initiated by H. McKean and Singer. There is a masterful presentation by Atiyah, Bott, and Patodi. The Feynman-Kac formula expresses the heat kernel as a path integral. Using supersymmetric path integrals, Witten derived the index formula for the Dirac operator in a simple, elegant way .

Because the Index Theorem, its statement and proofs, encompasses so much mathematics, it has had a great impact on virtually every area of modern mathematics. Among the topics to which it has had crucial application, one can list: invariants of actions of groups on manifolds — the fixed point formula, families of elliptic operators and the determinant line bundle, the value of Hecke L-series at 0 in number theory, spectral flow and the theory of anomalies in physics, K-homology in operator theory, gauge theories as applied to three- and four-dimensional topology, the non-existence of spaces of positive scalar curvature in differential geometry.

The last paragraph of a book review (N. Hitchin, BAMS, Vol. 15, 1986, pp. 243-245) sums up, very nicely, what many of us feel on the subject of the Atiyah–Singer Index Theorem. "Like Stonehenge, the theorem stands there as an immovable edifice, with each generation giving its own interpretation. For one it is a computational device, for another a more mystical representation of supersymmetry. Either way, it has created a bridge between mathematics and physics and has given mathematicians and physicists a deeper, or at least more sympathetic, understanding of each other's work. The Dirac operator will never be reinvented a third time!"

Given the monumental nature of the work just mentioned, it is surprising to realize that Singer has run swiftly in other directions. He is recognized as one of the great geometers of our time. Few of the younger geometers are aware that the geometry of higher-dimensional manifolds was anything but the smoothly functioning apparatus that they know today. Of course, E. Cartan was the great initial developer and set the theory on paper in a form that contained most of the important beginning ideas; but that account had a noticeably different form from what one sees today. Chern understood, in the deepest sense, what Cartan was saying and taught it to us, along with his own deep contributions, in our graduate student days (1949) at the University of Chicago. It was a wonderful experience. Cartan's largely descriptive account (collected works) now found itself in a working mathematical form, though still far from the precise style available to us today. It must be remembered that Norman Steenrod was in the process of developing and writing his celebrated book on fiber bundles (Princeton University Press 1951), a theory implicit in Elie Cartan's treatment. We must also recall that there were no copying machines available, so preliminary copies of vital manuscripts did not "cover" the mathematical landscape as they do today (in both paper and electronic form). In that environment, Singer and his more senior colleague at MIT, Warren Ambrose, armed with the notes from the Chern course, undertook to make precise mathematical sense of connections, holonomy, parallel translation and all the other key concepts of differential geometry that were cloaked in mystery for all but a handful of "initiates." Independently, Ch. Ehresmann and they produced the form of the subject substantially as it is today. From my own personal observation of that process, it was an heroic effort. Their paper, "A Theorem on Holonomy" (TAMS Vol. 75, 1953, pp. 428–443), remains a classic account of that theory. At the same time, their Ph. D. students have written several "best sellers" based on the lecture notes of his courses (among them, Bishop-Crittenden, Hicks, and Warner).

Singer's current research in string theory and other differential geometric aspects of quantum physics is well known and highly respected in a large segment of the mathematics and theoretical physics communities. His work in that area earned him the Wigner Medal (1988) for contributions to theoretical physics. It is unusual for a pure mathematician to have students (*e.g.*, D. Freed, D. Friedan, and J. Lott) who have contributed significantly to high energy theoretical physics.

Even less known is Singer's powerful influence on the mid-century development of functional analysis. In the early fifties, during his association with UCLA, he teamed with Richard Arens to usher in a new era in the study of commutative Banach Algebras. This study broadens the scope of the theory of several complex variables and recasts it in the framework of functional analysis. Some of Singer's Ph. D. students and postdocs, notably Hugo Rossi and Ken Hoffman, became leading researchers in this area of analysis.

The background for the title of this article is an incident that occurred when Jacques Dixmier was revising his von Neumann algebra book. I was visiting Paris and spent an afternoon with Dixmier discussing some of the additions he wanted to make. In particular, I told him about Singer's early contribution to the subject of derivations of operator algebras (more about that, shortly). He interrupted me during that description to ask, "Which Singer is that?" I was puzzled and asked him what he meant. On an earlier occasion, I had mentioned to him that I often "wrote up" our joint papers for publication "since Singer had trouble writing them." He replied that there were the Singers who did differential geometry, commutative Banach algebras, operator algebras (and others). I don't know how, but I managed to display no more than a smile, and responded, "They are all the same Singer." Dixmier mused for a moment and said, "No wonder he has trouble writing up papers!"

One evening, at a 1953 conference, Irving Kaplansky asked Is Singer what he thought the derivations of $C(X)$ (the algebra of all continuous, complex-valued, functions on the compact Hausdorff space X under pointwise operations and supremum norm) were. The next day, Is showed us a sweet little argument that each such derivation is 0. I can't resist giving it here! To recall, a linear mapping δ of $C(X)$ into itself satisfying the Leibnitz rule (for differentiation of products), $\delta(fg) = \delta(f)g + f\delta(g)$, is called a *derivation* (of $C(X)$ into itself). Since each f in $C(X)$ is the sum of a "real" and "purely imaginary" function in $C(X)$ it suffices to show that $\delta(f) = 0$ for each real f in $C(X)$. Of course, $\delta(1) = \delta(1^2) = 2\delta(1)$, where '1' denotes the function whose value is 1 at each point of X. Thus $\delta(1) = 0$; by linearity, $\delta(a) = 0$ for each constant function a. Given a point x in X, $\delta(f - f(x)) = \delta(f)$. Let h be $f - f(x)$, h_+ be $\frac{1}{2}(|h| + h)$, and h_- be $\frac{1}{2}(|h| - h)$. Then h_+ and h_- are positive functions in $C(X)$, $h_+ - h_- = h$, $h_+h_- = 0$, and $h_+(x) = h_-(x) = 0$. Now, $h_+ = g^2$ for some (positive) g in $C(X)$. Then $g(x) = 0$, and $\delta(h_+)(x) = 2g(x)\delta(g)(x) = 0$. Similarly, $\delta(h_-)(x) = 0$. Thus $\delta(f)(x) = \delta(h)(x) = 0$. Since x was arbitrarily chosen in X, $\delta(f) = 0$. Hence $\delta = 0$. Kaplansky went on from there to write his famous paper [11] showing (among other things) that all derivations of type I von Neumann algebras are inner. Singer and Wermer [19] brought Singer's argument into a commutative Banach algebra context and extended it. A veritable army of researchers took the theory of derivations of operator algebras to dizzying heights — producing a theory of cohomology of operator algebras as well as much information about automorphisms of operator algebras. It all started with Kaplansky's thoughts and Singer's argument nearly fifty years ago.

Along with their efforts to put global differential geometry on a firm foundation and make it broadly accessible, Ambrose and Singer concentrated on understanding the basic structure of the Murray–von Neumann factors of type II_1. They were trying to display such a factor as a "matrix algebra" relative to an appropriate "orthonormal basis." In this instance there are complex entries at each "point" of the matrix — the rows and columns are thought of as the unit interval $[0, 1]$ and each has an appropriate measure on it. Multiplication of elements in the factor becomes matrix multiplication with the row-colunm product being integrated rather than summed. The mathematical problems they encountered were daunting; one of their most baffling questions was answered just a few years ago — more than 45 years after it was posed. What remains in print is a single-sentence abstract, "L_2-matrices are studied," and some reference to the project in [18].

It is Singer's role in the early development of the theory of operator algebras that is the primary focus of this article; it is a role that is too little known outside the field of operator algebras. I had better advise the reader that, at this point, the account turns into what has become known as a "research-expository" article. Let's pause, first, to establish some notation and background information. With $\mathcal{H}$ a Hilbert space over the complex numbers $\mathbb{C}$, we denote by '$\mathcal{B}(\mathcal{H})$' the family of all linear transformations of $\mathcal{H}$ into itself ("operators") continuous relative to the metric topology on $\mathcal{H}$ induced by the norm that assigns to each x in $\mathcal{H}$ its length $(\|x\| =) < x, x >^{\frac{1}{2}}$, where $< u, v >$ is the inner product of u and v in $\mathcal{H}$. If T is in $\mathcal{B}(\mathcal{H})$, $\sup\{\|Tx\| : \|x\| \leq 1\}$ $(= \|T\|)$ is finite (T is "bounded" with "bound" or "norm" $\|T\|$). The metric topology on $\mathcal{B}(\mathcal{H})$ associated with the norm $T \to \|T\|$ is called the *norm* or *uniform* topology. Equipped with this norm, $\mathcal{B}(\mathcal{H})$ is a complete normed space, a Banach space.

The usual operations of addition, multiplication by a scalar, and multiplication (iteration) of linear transformations of a vector space into itself provide $\mathcal{B}(\mathcal{H})$ with the structure of an associative algebra. It has a unit element I, the *identity* operator, that assigns x to each x is $\mathcal{H}$. Each T in $\mathcal{B}(\mathcal{H})$ has associated with it a unique operator T^*, called its *adjoint*, characterized by the equality $< Tx, y > = < x, T^*y >$ for all x and y in $\mathcal{H}$.

The properties, $(aA + B)^* = \bar{a}A^* + B^*$, $(AB)^* = B^*A^*$, $(A^*)^* = A$, $\|T\| = \|T^*\|$, and $\|T^*T\| = \|T\|^2$ are established by simple computations. If $\mathcal{F}$ is a subset of $\mathcal{B}(\mathcal{H})$, we denote by '$\mathcal{F}^*$' the family $\{T^* : T \in \mathcal{F}\}$ and say that $\mathcal{F}$ is *self-adjoint* when $\mathcal{F} = \mathcal{F}^*$. In particular, a self-adjoint subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ (called a *self-adjoint operator algebra*) is a *C^*-algebra* when it is norm closed in $\mathcal{B}(\mathcal{H})$. The topology on $\mathcal{B}(\mathcal{H})$ corresponding to *strong-operator convergence* ($A_n \to A$ when $A_nx \to Ax$, in the metric of $\mathcal{H}$, for each x in $\mathcal{H}$) is *the strong-operator topology* on $\mathcal{B}(\mathcal{H})$.

The *von Neumann algebras* are the self-adjoint operator algebras on a Hilbert space, containing I, that are strong-operator closed. Those whose centers consist of just the scalar multiples of I are called *factors.* The von Neumann algebras were introduced in [16] (as "rings of operators"), where it is proved that for each such $\mathcal{R}$, $\mathcal{R} = \mathcal{R}''$. (The *commutant* $\mathcal{F}'$ of a subset $\mathcal{F}$ of $\mathcal{B}(\mathcal{H})$ is $\{T : T \in \mathcal{B}(\mathcal{H}),\, TA = AT$ for all A in $\mathcal{F}\}$.)

The factors were studied, intensively, in a series of papers [12], [13], [14] and [15] published between 1936 and 1943. Of course, $\mathcal{B}(\mathcal{H})$ itself is an example of a self-adjoint operator algebra, C*-algebra, von Neumann algebra, and factor. The operators that "project" a vector in $\mathcal{H}$, orthogonally, onto a given (closed) subspace of $\mathcal{H}$ are called *projections.* Each projection E lies in $\mathcal{B}(\mathcal{H})$, $\|E\| = 1$ (0 when E is 0), $E = E^*$, and $E^2 = E$. The last two properties characterize the projections in $\mathcal{B}(\mathcal{H})$. An operator A in $\mathcal{B}(\mathcal{H})$ *commutes* with a projection E if and only if A and A^* leave the space on which E projects (its range) invariant. Each von Neumann algebra is the norm closure of the linear span of its projections. In particular, there are many projections in a von Neumann algebra, while a C*-algebra may have no projections other than 0 and I. The projections are ordered by the size of their ranges: $E \leq F$ when $E(\mathcal{H}) \subseteq F(\mathcal{H})$. This is equivalent to the equality, $EF = E$ (and agrees with their ordering as self-adjoint operators). If E lies in a von Neumann algebra $\mathcal{R}$, is non-zero, and no smaller projection in $\mathcal{R}$ distinct from it is non-zero, we call E a *minimal projection* in $\mathcal{R}$. One of the earliest results proved by Murray and von Neumann classifies factors that have a minimal projection.

Theorem. *A factor that has a minimal projection is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Murray and von Neumann found examples of factors without minimal projections. The first construction of such examples [12] employ a countable group G, with unit e, of one-to-one, measurability-and-measure-zero-preserving transformations of a (countably separated, σ-finite) measure space $\mathcal{S}$ (with measure μ). The action is *free* (each transformation, other than e, has a fixed-point set of measure 0). Using the Radon–Nikodým derivative of μ, transformed by a group element g of G, relative to μ, we can associate with g a unitary operator U_g on the Hilbert space $L_2(\mathcal{S},\mu)$ $(= \mathcal{H})$. If the transformations are measure preserving, the Radon–Nikodým derivatives are (1 and) not needed.

Let $\mathcal{K}$ be the linear space of functions ϕ from G to $\mathcal{H}$, under pointwise addition and multiplication by scalars, for which $\sum_{g\in G} \|\phi(g)\|^2 < \infty$. Provided with the inner product $< \phi, \psi >= \sum_{g\in G} < \phi(g), \psi(g) >$, $\mathcal{K}$ becomes a Hilbert space. Of course, $\mathcal{K}$ may be identified with the direct sum of copies $\mathcal{H}_g$ of $\mathcal{H}$, g in G. Thus operators in $\mathcal{B}(\mathcal{K})$ have representations as matrices with rows and columns indexed by the elements of G and entries from $\mathcal{B}(\mathcal{H})$. With T in $\mathcal{B}(\mathcal{H})$, let $\Phi(T)$ be the operator on $\mathcal{K}$ whose matrix has T at each diagonal entry and 0 at all other entries (so $(\Phi(T)(\phi))(g) = T(\phi(g))$ $(\phi \in \mathcal{K},\ g \in G)$).

With f a bounded measurable function on $\mathcal{S}$, we denote by 'M_f' the (*multiplication*) operator that assigns $f \cdot h$ to h in $\mathcal{H}$. The family $\mathcal{A}_0$ of all such multiplication operators is an abelian von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$. Each T in $\mathcal{B}(\mathcal{H})$ commuting with all the elements of $\mathcal{A}_0$ is a multiplication operator, whence $\mathcal{A}_0$ is a *maximal abelian, self-adjoint subalgebra* of $\mathcal{B}(\mathcal{H})$ (a *masa*). With f a function on $\mathcal{S}$ and g in G, let $f_g(s)$ be $f(g^{-1}(s))$, where '$g(s)$' denotes the result of g acting on s in $\mathcal{S}$. Assuming that g acts by measure-preserving transformations, we note that

$$\begin{aligned}(U_g M_f U_g^{-1} h)(s) &= (M_f U_{g^{-1}} h)(g^{-1}(s)) = f(g^{-1}(s))(U_{g^{-1}} h)(g^{-1}(s)) \\ &= f_g(s) h(g(g^{-1}(s))) = (M_{f_g} h)(s),\end{aligned}$$

for each g in G, h in $\mathcal{H}$, s in $\mathcal{S}$, and bounded measurable f on $\mathcal{S}$. Thus $U_g M_f U_g^{-1} = M_{f_g}$. It follows that G acts by automorphisms ($M_f \to U_g M_f U_g^{-1} = M_{f_g}$) on $\mathcal{A}_0$.

Let $(V_g \phi)(g')$ be $U_g \phi(g^{-1} g')$ for all g and g' in G and ϕ in $\mathcal{K}$. The mapping $g \to V_g$ is a unitary representation of G on $\mathcal{K}$. Moreover,

$$\begin{aligned}[(V_g \Phi(M_f) V_g^{-1})\phi](g') &= U_g([\Phi(M_f) V_{g^{-1}}(\phi)](g^{-1} g')) \\ &= U_g[M_f(V_{g^{-1}}(\phi)(g^{-1} g'))] \\ &= U_g[M_f(U_{g^{-1}} \phi(g g^{-1} g'))] \\ &= M_{f_g} \phi(g') = (\Phi(M_{f_g})\phi)(g')\end{aligned}$$

for all g and g' in G, ϕ in $\mathcal{K}$, and bounded measurable f on $\mathcal{S}$. Thus $V_g \Phi(M_f) V_g^{-1} = \Phi(M_{f_g})$, and G acts by automorphisms ($\Phi(M_f) \to V_g \Phi(M_f) V_g^{-1} = \Phi(M_{f_g})$) of the abelian von Neumann algebra $\Phi(\mathcal{A}_0)$ on $\mathcal{K}$. The von Neumann algebra $\mathcal{R}$ generated by $\Phi(\mathcal{A}_0)$ on $\mathcal{K}$ and $\{V_g : g \in G\}$ provides us with the example we want. The von Neumann algebra $\mathcal{R}$ is a factor if and only if G acts "ergodically" on $\mathcal{S}$ (that is, $\cup_{g \in G} g(\mathcal{S}_0)$ or $\mathcal{S} \setminus \cup_{g \in G} g(\mathcal{S}_0)$ has measure 0 for each measurable set $\mathcal{S}_0$). When $\mathcal{S}$ has no atoms (no sets of positive measure without subsets of smaller positive measure), the factor has no minimal projections. If $\mu(\mathcal{S}) < \infty$, the factor is one of type II_1 — the factors that most of us studied in the early days of the subject. A specific example is given by the group of rotations of the circle, with Lebesgue measure, generated by a single rotation through an irrational multiple of π. A description of $\mathcal{R}$ in matrix terms follows. Let $(T_{p,q})$ be the matrix of T in $\mathcal{B}(\mathcal{K})$.

Theorem. *An operator T in $\mathcal{B}(\mathcal{K})$ lies in $\mathcal{R}$ if and only if there is a mapping $g \to A(g)$ from G into $\mathcal{A}_0$ such that $T_{p,q} = U_{pq^{-1}} A(pq^{-1})$. An* *operator T' lies in $\mathcal{R}'$ if and only if there is a mapping $g \to A'(g)$ of G into $\mathcal{A}_0$ such that $T'_{p,q} = U_p A'(q^{-1}p) U_{p^{-1}}$.*

Note that the diagonal entries $T_{p,p}$ of each T in $\mathcal{R}$ are equal to a single element $A(e)$ of $\mathcal{A}_0$, while those $T'_{p,p}$ of T' are the transforms $U_p A'(e) U_{p^{-1}}$ of a single element $A'(e)$ of $\mathcal{A}_0$. If we use the mapping $g \to A(g)$ that assigns 0 to each g other than e, the resulting operator T in $\mathcal{R}$ is $\Phi(A(e))$, the diagonal operator with $A(e)$ at each diagonal entry. It is not hard to show that the abelian von Neumann subalgebra $\Phi(\mathcal{A}_0)$ of $\mathcal{R}$ is a masa in $\mathcal{R}$, by using the assumption of free action of G on $\mathcal{S}$.

Let us suppose that G acts ergodically, $\mathcal{S}$ has no atoms, and $\mu(\mathcal{S}) = 1$. The algebra $\mathcal{M}$ we construct is a factor of type II_1 in this case. If u is the element of $\mathcal{K}$ that assigns the constant function 1 on $\mathcal{S}$ to e and 0 to each other g in G, the linear functional τ on $\mathcal{M}$ that takes the value $< Tu, u >$ ($= \int f\, d\mu$, where $A(e)$ in $\mathcal{A}_0$ is $\mathcal{M}_f$ in our preceding notation) has special properties. To begin with, it is a *state* of $\mathcal{M}$. A *state* on a C*-algebra is a linear functional on the algebra that takes non-negative, real values on positive operators and is 1 at I. States arising from vectors (as τ does from u) are called *vector states*. Using the facts that $U_g(1) = 1$, $U_g \mathcal{A}_0 U_g^* = \mathcal{A}_0$, and $\mathcal{A}_0$ is abelian, we also have that $\tau(AB) = \tau(BA)$. In this case, we call τ a *tracial state* of $\mathcal{M}$. In the explicit example of factors of type II_1, just described, we exhibit a tracial state in terms of the special construction. It is a deep fact (proved in [13]) that such a state exists for each factor of type II_1 and is unique. For present purposes, we may define the factors of type II_1 as those factors that have no minimal projections and satisfy the condition that $VV^* = I$ when $V^*V = I$ for some V in the factor. As innocent and insignificant as this condition may seem, it is a simple expression of the property that leads to a stunningly rich structure. The factors of type II_1 are at least as natural a replacement of the finite-dimensional matrix algebra $M_n(\mathbb{C})$, in the infinite dimensional case, as $\mathcal{B}(\mathcal{H})$ is, with $\mathcal{H}$ infinite dimensional. For one thing, the factors of type II_1 are simple algebras. For another, they have the all-important tracial state. If we divide the trace of a matrix in $M_n(\mathbb{C})$ by n, we arrive at the unique tracial state τ_n on $M_n(\mathbb{C})$. It assigns to projections in $M_n(\mathbb{C})$ one of the values $0, \frac{1}{n}, \frac{2}{n}, \dots, 1$ and all these values are assumed. For a factor of type II_1, the tracial state assigns all values in $[0,1]$ to the projections in the factor. "Discrete" dimensionality has been replaced by "continuous" dimensionality in the infinite-dimensional case; it makes serious sense to speak of projections with dimensions $\sqrt{2}/2$ and $1/e$ in factors of type II_1.

As remarked, earlier, Ambrose and Singer undertook to represent a factor of type II_1 as a "matrix algebra" with complex entries at each point of a square and measures associated with each column and row. To go from transformations to matrices requires the choice of a basis — an orthonormal basis, when the adjoint operation is part of the structure considered. In view of "non-atomicity" of factors of type II_1, we cannot allow ourselves the luxury of an orthonormal basis of vectors. An orthonormal basis $e_1, e_2, \ldots$, for a (separable) Hilbert space $\mathcal{H}$ is determined, up to a phase factor (that is, a c in $\mathbb{C}$ of modulus 1), by the family of one-dimensional projections $E_1, E_2, \ldots$, there the range $E_j(\mathcal{H})$ of E_j is spanned by e_j. The family $\{E_j\}$ generates an abelian von Neumann algebra in which $\{E_j\}$ is precisely the family of minimal projections. Relative to the basis $\{e_j\}$, the operators in the algebra are precisely those with diagonal matrices. That algebra is a masa in $\mathcal{B}(\mathcal{H})$. In this sense, we may speak of each masa $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ as a "generalized orthonormal basis." In this instance, some (or all) of the "vectors" in the basis may correspond to "Dirac delta functions." In the same way, an "orthonormal basis," where the factor $\mathcal{M}$ of type II_1 replaces $\mathcal{B}(\mathcal{H})$, is a masa $\mathcal{A}$ in $\mathcal{M}$.

Applying a special process (the "GNS construction") to the tracial state τ of a factor $\mathcal{M}$ of type II_1, we "represent" $\mathcal{M}$ as a factor of type II_1 acting on a Hilbert space $\mathcal{H}$ in such a way that τ is the vector state of $\mathcal{M}$ corresponding to a (unit) vector u whose transforms under the elements of $\mathcal{M}$ form a dense submanifold of $\mathcal{H}$. (We say that u is a *cyclic* or *generating* vector for $\mathcal{M}$.) In this situation, $\mathcal{M}'$ is also a factor of type II_1 and the tracial state of $\mathcal{M}'$ is the vector state corresponding to u. We say that $\mathcal{M}$ is in *standard form* in this case. For each T in $\mathcal{M}$, there is a unique T' in $\mathcal{M}'$ such that $Tu = T'u$. The mapping $T \to T'$ is an adjoint-preserving, anti-isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$. In particular, the image $\mathcal{B}$ of a masa $\mathcal{A}$ in $\mathcal{M}$ under this mapping is a masa in $\mathcal{M}'$. Of course, the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$ is abelian. Is it a masa in $\mathcal{B}(\mathcal{H})$? In the parallel situation of $\mathcal{M}_n(\mathbb{C})$ acting on $\mathcal{H}$ in such a way that its commutant is also (isomorphic to) $\mathcal{M}_n(\mathbb{C})$, there is a unit vector giving rise to the tracial states on each of the algebras and an adjoint-preserving, anti-isomorphism of the algebra onto its commutant. The abelian algebra generated by a masa and its image is a masa in $\mathcal{B}(\mathcal{H})$, in this case. As we shall show, shortly, the algebra generated algebraically by $\mathcal{A}$ and $\mathcal{B}$ is maximal abelian in the algebra generated algebraically by $\mathcal{M}$ and $\mathcal{M}'$. We shall also give an example showing that the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$ need not be a masa in $\mathcal{B}(\mathcal{H})$ (the von Neumann algebra generated by the factors $\mathcal{M}$ and $\mathcal{M}'$). As Ambrose and Singer discovered, when $\mathcal{A}$ and $\mathcal{B}$ generate a masa in $\mathcal{B}(\mathcal{H})$, $\mathcal{A}$ serves as a particularly useful "orthonormal basis" for their matrix representation. (It could be effected in terms of "single-sheeted," rather than "multiple-sheeted" matrices.) They called such a masa in $\mathcal{M}$ *simple* and knew that the masa $\Phi(\mathcal{A}_0)$ of the measure-theoretic construction we discussed is simple. Returning to that example, we note that if T in $\mathcal{M}$ is positive, then $T_{e,e}$ $(= A(e))$ is positive (in $\mathcal{B}(\mathcal{H})$). If $< Tu, u >= 0$, and $A(e) = M_f$, then $f(s) \geq 0$ for almost all s in $\mathcal{S}$ and $\int f\,d\mu = 0$. Hence f is 0 almost everywhere on $\mathcal{S}$, and $A(e) = 0$. The diagonal entries $T_{p,p}$ of T are all 0 and $T = 0$ (since T is assumed positive). It follows that S in $\mathcal{M}$ is 0 when $Su = 0$. We say that u is a *separating vector* for $\mathcal{M}$, in this case. This implies that $\mathcal{M}'u$ is dense in $\mathcal{K}$, that is u is generating for $\mathcal{M}'$. Suppose $< T'u, u >= 0$ for some positive T' in $\mathcal{M}'$. If $T'_{p,q} = U_pA'(q^{-1}p)U_{p^{-1}}$ and $A'(e) = M_h$, then $T'_{e,e} = A'(e) = M_h \geq 0$ and $h(s) \geq 0$ for almost all s in $\mathcal{S}$. Moreover, $0 =< T'u, u >= \int h\,d\mu$, and h is 0 almost everywhere. Thus $0 = A'(e) = U_pA'(e)U_{p^{-1}} = T'_{p,p}$ and $T' = 0$. Hence u is separating for $\mathcal{M}'$. It follows that u is generating for $\mathcal{M}$ $(= \mathcal{M}'')$. When a *trace vector* u is generating for a II_1 factor $\mathcal{M}$, u is also a generating trace vector for $\mathcal{M}'$.

The mapping $g \to A'(g)$ that assigns 0 to each g other than e, produces the operator T' in $\mathcal{M}'$ with matrix whose diagonal entry $T'_{p,p}$ is $U_pA'(e)U_{p^{-1}}$ and all off-diagonal entries are 0. If $T = \Phi(A(e))$, $A(e) = M_f = A'(e)$, then Tu and $T'u$ are the vector in $\mathcal{K}$ that assigns f to e and 0 to each other g in G. In particular, $Tu = T'u$. The adjoint-preserving, anti-isomorphism of $\mathcal{M}$ onto $\mathcal{M}'$ corresponding to u maps T to T' and transforms $\Phi(\mathcal{A}_0)$ onto the algebra $\mathcal{B}$ of diagonal matrices in $\mathcal{M}'$. Thus $\mathcal{B}$ is a masa in $\mathcal{M}'$. We note, finally, that $\Phi(\mathcal{A}_0)$ and $\mathcal{B}$ generate a masa in $\mathcal{B}(\mathcal{K})$. If S, with matrix $(S_{p,q})$, commutes with $\Phi(\mathcal{A}_0)$, then each $S_{p,q}$ is a multiplication operator since $\mathcal{A}_0$, the algebra of multiplication operators on $L_2(\mathcal{S}, \mu)$, is maximal abelian in $\mathcal{B}(\mathcal{H})$. If, in addition, $\mathcal{S}$ commutes with all diagonal matrices in $\mathcal{M}'$, then $S_{p,q}U_pEU_p^* = S_{p,q}U_qEU_q^*$ for each projection E in $\mathcal{A}_0$. Replacing E by $U_p^*EU_p$, we have that $S_{p,q}E = S_{p,q}U_gEU_g^*$, where $g = qp^{-1}$. If $p \neq q$, then $g \neq e$, and there is a non-zero subprojection E_0 of E in $\mathcal{A}_0$ such that $E_0U_gE_0U_g^* = 0$ (from the freeness of the action of G on $\mathcal{S}$ [8, Lemma 8.6.5]). With E_0 in place of E, $S_{p,q}E_0 = S_{p,q}U_gE_0U_g^*$, whence $S_{p,q}E_0 = S_{p,q}E_0^2 = S_{p,q}U_gE_0U_g^*E_0 = 0$. Each non-zero projection E in $\mathcal{A}_0$ has a non-zero subprojection E_0 such that $S_{p,q}E_0 = 0$ when $p \neq q$. By using Zorn's lemma, we find a maximal orthogonal family $\{E_a\}$ of projections in $\mathcal{A}_0$ such that $S_{p,q}E_a = 0$ for each a. If $I - \sum_a E_a$ were not 0, there would be a non-zero subprojection E_0 of it in $\mathcal{A}_0$ such that $S_{p,q}E_0 = 0$, contradicting the maximality of $\{E_a\}$. It follows that $S_{p,q} = S_{p,q}\sum_a E_a = \sum_a S_{p,q}E_a = 0$ (with convergence in the strong-operator topology), when $p \neq q$. Thus the algebra $\tilde{\mathcal{A}}'$ of operators in $\mathcal{B}(\mathcal{K})$ commuting with both $\Phi(\mathcal{A}_0)$ and $\mathcal{B}$ (hence, with the von Neumann algebra $\tilde{\mathcal{A}}$ they generate) is a subalgebra of the abelian algebra $\mathcal{D}$ of diagonal matrices with diagonal entries from $\mathcal{A}_0$. As $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ are abelian, $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}' \subseteq \tilde{\mathcal{A}}'' = \tilde{\mathcal{A}}$. Thus $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}'$, $\tilde{\mathcal{A}}$ is maximal abelian in $\mathcal{B}(\mathcal{K})$, and $\Phi(\mathcal{A}_0)$ is a simple masa in $\mathcal{M}$. For our example of a masa that is not simple, we turn to a construction of factors of type II_1 introduced in [14]. We start with a countably infinite, discrete group and construct an operator-algebra group algebra. Let G be a countable (discrete) group and $\mathcal{H}$ be the separable Hilbert space $l_2(G)$, that is

$$\{\phi : \sum_{g\in G} |\phi(g)|^2 < \infty\}, \qquad < \phi, \psi >= \sum_{g\in G} \phi(g)\overline{\psi(g)}),$$

Let $(L_g\phi)(g')$ be $\phi(g^{-1}g')$ and $(R_g\phi)(g')$ be $\phi(g'g)$ for ϕ in $\mathcal{H}$. Then L_g and R_g are unitary operators. Let $\mathcal{L}_G$ and R_G (the *left and right-von Neumann-group algebras of* G) be the von Neumann algebras generated by $\{L_g\}$ and $\{R_g\}$, respectively.

Theorem. *$\mathcal{L}_G$ and $\mathcal{R}_G$ are factors iff all conjugacy classes in G but $\{e\}$ are infinite. In this case, $\mathcal{L}_G$ and $\mathcal{R}_G$ are factors of type* II_1, *$\mathcal{L}_G = \mathcal{R}'_G$, and $\mathcal{R}_G = \mathcal{L}'_G$.*

The free (non-abelian) group $\mathcal{F}_n$ on $n(> 1)$ generators and Π, the group of "finite" permutations of the integers, are examples of these *i.c.c groups.*

Murray and von Neumann took a crucial step, proving [14] that there are factors of type II_1 acting on separable Hilbert spaces that are not isomorphic. We now have examples of uncountably many non-isomorphic factors of type II_1 (as was to be expected after the Murray–Von Neumann result that follows).

Theorem. *$\mathcal{L}_{\mathcal{F}_n}$ is not isomorphic to $\mathcal{L}_\Pi$.*

As this is written, we do not know if $\mathcal{L}_{\mathcal{F}_2}$ is isomorphic to $\mathcal{L}_{\mathcal{F}_3}$, but deep work of Voiculescu, laying the foundations of a non-commutative, free probability theory, has given us such results as:

Theorem(Voiculescu). *The* II_1 *factors $\mathcal{L}_{\mathcal{F}_2}$ and $M_2(\mathcal{L}_{\mathcal{F}_5})$, the algebra of 2×2 matrices with entries from $\mathcal{L}_{\mathcal{F}_5}$, are isomorphic.*

This same work of Voiculescu provides the technical basis for a brilliant proof [6] that the factors $\mathcal{L}_{\mathcal{F}_n}$, among others, do not have simple masas. (This came after the proof of a difficult intermediate result by Voiculescu [20] that such factors do not possess a "Cartan subalgebra.")

Let $x_g(h)$ be 1 when $h = g$ and 0 otherwise. Then $\{x_g : g \in G\}$ is an orthornormal basis for $\mathcal{H}$ and each x_g is a (unit) trace vector for $\mathcal{L}_G$ and for $\mathcal{R}_G$ (that is, $< ABx_g, x_g >=< BAx_g, x_g >$ when $A, B \in \mathcal{L}_G$ or $A, B \in \mathcal{R}_G$). In general, each element of $\mathcal{L}_G$ ($\mathcal{R}_G$) is uniquely representable as $\sum_{g\in G} \eta_g L_g$ ($\sum_{g\in G} \eta_g R_g$), where the sum converges in the strong-operator topology over the net of finite subsums. Defining $\eta(g)$ to be η_g, $\eta \in l_2(G)$, but not each η in $l_2(G)$ appears in this way. Since $L_g x_e = x_g = R_{g^{-1}} x_e$, the anti-isomorphism $A \to A'$ of $\mathcal{L}_G$ onto $\mathcal{R}_G$ (*reflection about the trace vector* x_e) maps $\sum_{g\in G} \eta_g L_g$ onto $\sum_{g\in G} \eta'_g R_g$, where $\eta'_g = \eta_{g^{-1}}$

To complete our construction of a masa that is not simple, we choose $\mathcal{L}_{\mathcal{F}_2}$ for G. Let a and b be (free) generators of $\mathcal{F}_2$. We show that the algebra $\mathcal{A}$ generated by L_a in $\mathcal{L}_{\mathcal{F}_2}$ is a masa in $\mathcal{L}(\mathcal{F}_2)$. In any case, it consists of elements representable as $\sum \eta_g L_g$, where $\eta_g = 0$ unless $g = a^m$ for some integer m. Suppose $A = \sum \eta_g L_g$ and $L_a A = AL_a$. Then $\sum \eta_g L_{ag} = \sum \eta_g L_{ga}$. Thus $\eta_g = \eta_{aga^{-1}}$ for each g in G. So $\eta_g = \eta_{aga^{-1}} = \eta_{a^2ga^{-2}} = \ldots = \eta_{a^nga^{-n}}$ for each integer n. If $g \notin \{a^m : m \in \mathbb{Z}\}$, then $\{a^nga^{-n} : n \in \mathbb{Z}\}$ is an infinite subset of $\mathcal{F}_2$. Since $\eta \in l_2(G)$, $\eta(g) = \eta_g = 0$, in this case. Thus $AL_a = L_a A$ if and only if $\eta_g = 0$ unless $g = a^m$ for some integer m. It follows that $\mathcal{A}$ is a masa in $\mathcal{L}_{\mathcal{F}_2}$. Of course, this argument and conclusion applies to the von Neumann subalgebra generated by any one of the (free) generators of $\mathcal{L}_{\mathcal{F}_n}$.

Theorem. *The masa generated by* L_a *in* $\mathcal{L}_{\mathcal{F}_n}$ *is not simple.*

Proof. With A in $\mathcal{B}(l_2(G))$, if $Ax_d = \sum_c \alpha_{c,d} x_c$, then $\alpha_{c,d}$ is the entry in row c and column d of the matrix for A relative to $\{x_g\}$. Since $L_a x_c = x_{ac}$, the matrix for L_a has a 1 in row ac at column c and 0 at all other entries of that column, for each c in $\mathcal{F}_2$. Similarly, $R_a x_c = x_{ca^{-1}}$, and the matrix for R_a has a 1 in row ca^{-1} at column c and 0 at all other entries of that column. Hence, with A as before, if $L_a A = AL_a$, then $\alpha_{c,d} = \alpha_{ac,ad}$ for all c, d in G. If $R_a A = AR_a$, then $\alpha_{c,d} = \alpha_{ca,da}$. Conversely, these conditions on the matrix of A imply commutativity with L_a and R_a.

Let $\mathcal{A}$ and $\mathcal{B}$ be the von Neumann subalgebras of $\mathcal{L}(\mathcal{F}_2)$ and $\mathcal{R}(\mathcal{F}_2)$ generated by L_a and R_a, respectively. Since $L_a x_e = x_a = R_{a^{-1}} x_e$, $\mathcal{B}$ is the reflection of $\mathcal{A}$ about the trace vector x_e. Let $\sum_{n,m=-k}^{k} \alpha_{n,m} L_{a^n} R_{a^m}$ be C. Such sums C form a weak-operator dense subalgebra of $\mathcal{A}_0$, the (abelian) von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{B}$. Moreover, $< Cx_b, x_{b^2} >= 0$ for each such sum C. Thus $< Tx_b, x_{b^2} >= 0$ for each T in $\mathcal{A}_0$. Let A be the linear operator that maps $x_{a^nba^m}$ to $x_{a^nb^2a^m}$, for $n, m = 0, \pm 1, \pm 2, \ldots$ and x_c to 0 for each other c in $\mathcal{F}_2$. Then A is the product of a "permutation unitary" (relative to the basis $\{x_g\}$) and the projection onto the subspace generated by $\{x_{a^n ba^m} : n, m \in \mathbb{Z}\}$. Thus $A \in \mathcal{B}(l_2(\mathcal{F}_2))$.

The matrix for A satisfies $\alpha_{c,d} = 1$ if $c = a^n b^2 a^m$ and $d = a^n b a^m$ for some n and m in $\mathbb{Z}$, otherwise, $\alpha_{c,d} = 0$. If $\alpha_{c,d} = 1$, then $\alpha_{ac,ad} = 1$. If $\alpha_{c,d} = 0$, then $\alpha_{ac,ad} = 0$. Similarly, $\alpha_{c,d} = \alpha_{ca,da}$ for all c and d in $\mathcal{F}_2$. Thus $AL_a = L_a A$, $AR_a = R_a A$, and $A \in \mathcal{A}_0'$. But $< Ax_b, x_{b^2} >= \alpha_{b^2,b} = 1$. Since $< Tx_b, x_{b^2} >= 0$ for all T in $\mathcal{A}_0$, $A \notin \mathcal{A}_0$. It follows that $\mathcal{A}_0$ is not maximal abelian in $\mathcal{B}(l_2(G))$ and $\mathcal{A}$ is not a simple masa in $\mathcal{L}_{\mathcal{F}_2}$. q.e.d.

This same argument applies to the abelian von Neumann subalgebra generated by L_a for each (free) generator of $\mathcal{F}_n$; each is a masa in $\mathcal{L}_{\mathcal{F}_n}$ but none is simple. To what extent does the finite-dimensional situation (where a masa in a factor and its reflection about a trace vector generate a masa in the algebra of all linear transformations on the finite-dimensional space) carry over to infinite dimensions? The theorem that follows shows that it does transfer in the algebraic sense. It is proved in fairly general terms. For the case we have been discussing, $\mathcal{R}$ and $\mathcal{S}$ should be taken to be the same factor of type II_1, and $\mathcal{R}'$ and $\mathcal{T}$ to be its commutant.

Theorem. *Let $\mathcal{R}$ be a von Neumann algebra, with center $\mathcal{Z}$, acting on a Hilbert space $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{T}$ be von Neumann subalgebras, containing $\mathcal{Z}$, of $\mathcal{R}$ and $\mathcal{R}'$, respectively, and $\mathcal{A}$ and $\mathcal{B}$ be masas in $\mathcal{S}$ and $\mathcal{T}$, respectively. Then the algebra $\mathcal{C}$ generated by $\mathcal{A}$ and $\mathcal{B}$ is maximal abelian in the algebra $\mathcal{D}$ generated by $\mathcal{S}$ and $\mathcal{T}$.*

Proof. Let D be an element of $\mathcal{D}$ commuting with $\mathcal{C}$. Then $D = S_1T_1 + \cdots + S_nT_n$, for some $S_1, \dots, S_n$ in $\mathcal{S}$ and some $T_1, \dots, T_n$ in $\mathcal{T}$. Let $\tilde{S}$ be the $n \times n$ matrix whose first row is $\{S_1, \dots, S_n\}$ and all of whose other entries are 0. Let $\tilde{\mathcal{H}}$ be the n-fold direct sum of $\mathcal{H}$ with itself and a be the norm of $\tilde{S}$ acting on $\tilde{\mathcal{H}}$. We wish to show that $D \in \mathcal{C}$ and, thence, that $\mathcal{C}$ is maximal abelian in $\mathcal{D}$. If $a = 0$, then $D = 0$, and $D \in \mathcal{C}$. We may assume that $a > 0$ and that $\|\tilde{S}\| = 1$, after multiplying each S_j by a^{-1} and T_j by a.

Let $\{A_1, \dots, A_m\}$ be a finite subset of $\mathcal{A}$. Since $\mathcal{A} \subseteq \mathcal{C}$ and D commutes with $\mathcal{C}$, we have that

$$0 = A_1D - DA_1 = (A_1S_1 - S_1A_1)T_1 + \cdots + (A_1S_n - S_nA_1)T_n.$$

(Note, too, for this that each $A_j \in \mathcal{A} \subseteq \mathcal{S} \subseteq \mathcal{R}$, and each $T_j \in \mathcal{R}'$.) From [7, Theorem 5.5.4], there are operators C_{jk} in $\mathcal{Z}$ $(j, k \in \{1, \dots, n\})$

such that the $n \times n$ matrix $\tilde{C}$, acting on $\tilde{\mathcal{H}}$, with C_{jk} as j, k entry, is an orthogonal projection and

$$\sum_{j=1}^{n}(A_1S_j - S_jA_1)C_{jk} = 0 \quad (k \in \{1, \dots, n\}),$$

$$\sum_{k=1}^{n} C_{jk}T_k = T_j \quad (j \in \{1, \dots, n\}).$$

Hence

$$A_1\left(\sum_{j=1}^{n} S_jC_{jk}\right) = \left(\sum_{j=1}^{n} S_jC_{jk}\right)A_1 \qquad (k = 1, \dots, n)$$

and

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n} S_jC_{jk}\right)T_k = \sum_{j=1}^{n} S_j\left(\sum_{k=1}^{n} C_{jk}T_k\right) = \sum_{j=1}^{n} S_jT_j.$$

Thus $\sum_{j=1}^{n} S_jT_j = \sum_{k=1}^{n} S_{k1}T_k$, where $S_{k1} = \sum_{j=1}^{n} S_jC_{jk}$ for k in $\{1, \dots, n\}$, and each $S_{k1} \in \mathcal{S}$ (since $\mathcal{Z} \subseteq \mathcal{S}$).

The matrix $\tilde{S}_1$ with first row $\{S_{11}, \dots, S_{n1}\}$ and all other entries 0 is $\tilde{S}\tilde{C}$. Thus $\|\tilde{S}\| \leq 1$. With A_2 in place of A_1 and S_{k1} in place of S_k, proceeding as before, we find operators C'_{jk} in $\mathcal{Z}$ such that the $n \times n$ matrix with j, k entry C'_{jk} is an orthogonal projection on $\tilde{\mathcal{H}}$, $\sum_{j=1}^{n} S_{j1}C'_{jk}$ $(= S_{k2})$ commutes with A_2 and lies in $\mathcal{S}$, and the $n \times n$ matrix $\tilde{S}_2$ with first row $\{S_{12}, \dots, S_{n2}\}$ and all other entries 0 has norm not exceeding 1. In addition, $D = \sum_{k=1}^{n} S_{k2}T_k$ and each S_{k2} commutes with A_1, as well as A_2, since each S_{j1} and each C'_{jk} commute with A_1.

Continuing in this way, we construct operators $\{S_{1m}, \dots, S_{nm}\}$ in $\mathcal{S}$ such that each S_{jm} commutes with all of $A_1, \dots, A_m$, $D = \sum_{k=1}^{n} S_{km}T_k$, and the $n \times n$ matrix $\tilde{S}_m$ with $\{S_{1m} \dots, S_{nm}\}$ as first row and all other entries 0 has norm 1 or less. In general, if '$\mathcal{F}$' denotes the finite subset $\{A_1, \dots, A_m\}$ of $\mathcal{A}$, we write '$S_{j\mathcal{F}}$' in place of 'S_{jm}' and '$\tilde{S}_{\mathcal{F}}$' in place of '$\tilde{S}_m$.' Since $\|\tilde{S}_{\mathcal{F}}\| \leq 1$ for each finite subset $\mathcal{F}$ of $\mathcal{A}$, each $S_{j\mathcal{F}}$ lies in $(\mathcal{S})_1$, the closed unit ball of $\mathcal{S}$.

The net $\{S_{j\mathcal{F}}\}_{\mathcal{F} \in \mathbb{A}}$, indexed by the family $\mathbb{A}$ of finite subsets of $\mathcal{A}$ ordered by inclusion has a weak-operator convergent cofinal subnet since $(\mathcal{S})_1$ is weak-operator compact. Starting with a convergent subnet $\{S_{1\mathcal{F}(1)}\}$ of $\{S_{1\mathcal{F}}\}$, passing to a convergent cofinal subnet $\{S_{2\mathcal{F}(2)}\}$ of $\{S_{2\mathcal{F}(1)}\}$ and, successively, to a convergent, cofinal subnet $\{S_{m\mathcal{F}(m)}\}$, we have that each cofinal subnet $\{S_{j\mathcal{F}(m)}\}$ of $\{S_{j\mathcal{F}}\}$ converges, in the weak-operator toplogy, to some A'_j in $\mathcal{S}$. We shall show that each $A'_j \in \mathcal{A}$ and that $D = A'_1T_1 + \cdots + A'_nT_n$.

If $A \in \mathcal{A}$, the terms of $\{S_{j\mathcal{F}(m)}\}$ such that $A \in \mathcal{F}(m)$ form a cofinal subnet of it, and each of these terms commutes with A, by construction of $S_{j\mathcal{F}(m)}$ (with A in $\mathcal{F}(m)$). Hence the weak-operator limit A'_j of this cofinal subnet commutes with A. Thus each A'_j commutes with $\mathcal{A}$. Since $A'_j \in \mathcal{S}$ and $\mathcal{A}$ is maximal abelian in $\mathcal{S}$, each $A'_j \in \mathcal{A}$.

For each finite subset $\mathcal{F}$ of $\mathcal{A}$, we have, by construction, that $\sum_{j=1}^{n} S_{j\mathcal{F}} T_j = D$. Thus

$$\sum_{j=1}^{n} < S_{j\mathcal{F}} T_j x, y > = < Dx, y > \qquad (x, y \in \mathcal{H}).$$

Passing to weak-operator limits over the appropriate subnet, we conclude that

$$\left\langle \left(\sum_{j=1}^{n} A'_j T_j \right) x, y \right\rangle = < Dx, y > \qquad (x, y \in \mathcal{H}).$$

Thus $D = A'_1 T_1 + \cdots + A'_n T_n$.

Applying what we have just proved, with $\mathcal{S}$ and $\mathcal{T}$ interchanged, $\mathcal{A}$ and $\mathcal{B}$ interchanged, and A'_j in place of S_j, we see that there are operators $B'_1, \ldots, B'_n$ in $\mathcal{B}$ with the property that $D = A'_1 B'_1 + \cdots + A'_n B'_n \in \mathcal{C}$. Hence $\mathcal{C}$ is maximal abelian in $\mathcal{D}$. q.e.d.

If we limit the scope of the preceding theorem by assuming that $\mathcal{H}$ is separable, then $\mathcal{A}$ is generated (as a von Neumann algebra) by a single self-adjoint operator A. With A in place of A_1, we conclude that $\sum_{j=1}^{n} S_j C_{jk} \in \mathcal{A}$, for each k in $\{1, \ldots, n\}$. Letting A'_k be $\sum_{j=1}^{n} S_j C_{jk}$, we arrive at the equality $\sum_{k=1}^{n} A'_k T_k = D$ without the need to introduce nets.

Is $\mathcal{C}^=$, the norm closure of $\mathcal{C}$, (the C*-algebra generated by $\mathcal{A}$ and $\mathcal{B}$) maximal abelian in $\mathcal{D}^=$, the C*-algebra generated by $\mathcal{S}$ and $\mathcal{T}$?

While almost nothing of the Ambrose–Singer project for representing a II_1 factor as a matrix algebra appeared in print, it still had an important influence on the development of the theory of operator algebras. In one way or another, word of it reached the ears of capable people over the years. Among other routes, I included the question of whether all factors of type II_1 possess a simple masa in my Baton Rouge list of problems (from the 1967 conference at LSU in honor of Jacques Dixmier).

A paper [18] of Singer's, that appeared in 1955, makes reference to the Ambrose-Singer project in a footnote on p. 121. The talk that Singer gave at the 1953 conference (mentioned earlier in connection with his derivation result) was based on the results in [18]. In that paper, Singer analyzes special automorphisms of a factor $\mathcal{M}$ of type II_1 arising from a countable group G acting as measure-preserving transformations of a measure space $(\mathcal{S}, \mu)$ $(\mu(\mathcal{S}) = 1)$ that we discussed before. We use the notation of that discussion. Singer studies the group $\mathrm{Aut}_1(\mathcal{M})$ of automorphisms α of $\mathcal{M}$ that map the masa $\Phi(\mathcal{A}_0)$ onto itself. Each such automorphism α gives rise to a measure-preserving transformation α' of $\mathcal{S}$ onto itself. He characterizes the elements of $\mathrm{Aut}_1(\mathcal{M})$ in terms of the action of α' on $\mathcal{S}$.

Theorem. *A measure-preserving transformation α' of $\mathcal{S}$ is induced by an automorphism α in $\mathrm{Aut}_1(\mathcal{M})$ if and only if there are measurable sets X_h^g in $\mathcal{S}$ $(g, h \in G)$ such that*

(*i*) $\mu(X_h^g \cap X_k^g) = 0$ *when* $h \neq k$;

(*ii*) $\mu(\cup_{h\in G} X_h^g) = 1$;

(*iii*) $(\alpha'^{-1} h^{-1} \alpha')(x) = g^{-1}(x)$ *for almost every* x *in* $\alpha'(X_h^g)$.

Ambrose [1] developed a framework for studying groups of measure-preserving transformations, his H-systems, that is roughly equivalent to the Murray-von Neumann, group-measure-space construction. In [18], Singer passes freely between both formulations, using the one he found better suited to a particular situation. This probably led to the article [18] not receiving as much attention as it deserved. In section 6 of [18], the last section, consisting of two brief paragraphs, Singer notes that the Murray-von Neumann construction (in our terminology) could be effected without assuming ergodicity of G on $\mathcal{S}$. The resulting von Neumann algebra would not, then, be a factor. He remarks, that the resulting operator algebra can be studied in terms of factors through the then-recently-published "direct integral theory" [17]. He notes that that is not his main interest. He was concerned, primarily, with the factor case.

In the second paragraph of that section, he notes that the Murray-von Neumann construction really occurs *algebraically* in terms of the multiplication algebra $\mathcal{A}_0$ and G acting by automorphisms of $\mathcal{A}_0$. He suggests that this construction can be carried out with another algebra in place of $\mathcal{A}_0$, and notes that it would probably lead to different and interesting examples of factors. Of course, Singer is anticipating the "crossed product" construction in this comment (compare [8, Chapter 13]). It has, indeed, become one of the basic constructions of the subject of operator algebras, leading to new and vital aspects of the theory.

Singer and I have several joint articles. The question of what an orthonormal basis is has been a dominant theme in most of that research. At first glance, every trained mathematician will think that the construction and properties of such bases form one of the less strenuous and most completely understood chapters in twentieth century mathematics! Is there really anything left to say? Certainly, the question of the existence of a simple masa in a factor of type II_1, needed as a "preferred basis" for the Ambrose-Singer project of assigning a "matrix" to each of the elements of that factor, is one aspect of that question. It led us on a merry chase for nearly fifty years!

A good way to start thinking of the meaning of orthonormal bases is to consider the uses to which we put these bases. In one instance, if we are given an especially interesting basis for the topic we are studying, we may want to expand all or some of the elements of $\mathcal{H}$ in terms of that basis. We recognize the L_2-theory of Fourier series as one aspect of that use of orthonormal bases.

We can turn that process around — instead of having an interesting basis given to us, we may want to *find* a particularly appropriate basis for some purpose, say, one that diagonalizes a self-adjoint operator on $\mathcal{H}$ or a commuting family of such operators. Let's phrase this example in a more physical way. Given a compatible family of observables, we want to find a complete set of simultaneous eigenstates for them. Dirac speaks of finding a "representation" and even presents an agenda for this. The following is quoted from pp. 74-75 of the Third Edition of his famous "Quantum Mechanics." Oxford University Press, London 1947

"To introduce a representation in practice

(i) We look for observables which we would like to have diagonal either because we are interested in their probabilities or for reasons of mathematical simplicity;

(ii) We must see that they all commute — a necessary condition since diagonal matrices always commute;

(iii) We then see that they form a complete commuting set, and if not we add some more commuting observables to them to make them into a complete commuting set;

(iv) We set up an orthogonal representation with this complete commuting set diagonal."

The representation is then completely determined except for arbitrary phase factors. For most purposes the arbitrary phase factors are unimportant and trivial, so that we may count the representation as being completely determined by the observables that are diagonal ... "

The emphasis, above, is mine. What would that say if it were put down in precise mathematical form? For one thing, Dirac talks about finding a basis that diagonalizes a self-adjoint operator, and while that is always possible when $\mathcal{H}$ is finite dimensional, there are perfectly respectable self-adjoint operators on infinite-dimensional Hilbert space that do not have a single eigenvector, in the strict sense. Still, we do have a "spectral resolution" of such operators. Again, Dirac's way of going at that problem is inspiring. On pp. 57-58, he writes:

> We have not yet considered the lengths of the basic vectors. With an orthogonal representation, the natural thing to do is to normalize the basic vectors, rather than leave their lengths arbitrary, and so introduce a further stage of simplification into the representation. However, it is possible to normalize them only if the parameters which label them all take on discrete values. If any of these parameters are continuous variables that can take on all values in a range, the basic vectors are eigenvectors of some observable belonging to eigenvalues in a range and are of infinite length..."

Dirac's "ranges" are "intervals" and his "continuous variables" are points in the interval. At this stage, Dirac introduces his δ-functions and develops their formalism. But without eigenstates that are vectors in $\mathcal{H}$, there are problems with what we mean by a "diagonalizing orthonormal basis" — especially, if we are "representing" *families* of compatible observables.

Let's see what this means in the case of a classical basis $\{e_1, e_2, \ldots\}$. If $\mathcal{A}_d$ is the family of all bounded operators on $\mathcal{H}$ that are diagonal relative to that basis, then $\mathcal{A}_d$ is abelian, as Dirac notes, and it is "complete" in his sense — that is "maximal abelian" in $\mathcal{B}(\mathcal{H})$. We have noted that $\mathcal{A}_d$ is a "masa" in $\mathcal{B}(\mathcal{H})$. Of course, there is no difficulty, here, in identifying the "simultaneous eigenstates" for our "complete commuting" family of observables; they are the vectors e_n of our basis. But what are they when our observables have "ranges" in their spectra. Dirac has his δ-functions, his vectors of "infinite length." This is a bit cumbersome, from the rigorous mathematical point of view. What we want to do is to replace the vectors e_n by some acceptable mathematical construct that is effectively the same as the vector, when there is one, and gives us something precise and usable when there is only a δ-function. Something that works very well is the vector state ω_{e_n} corresponding to e_n ($\omega_{e_n}(T) = < Te_n, e_n >$ for each T in $\mathcal{B}(\mathcal{H})$). With this notation, ω_{e_n} is "the expectation functional" of the state, in physical terminology, corresponding to the vector being replaced. The value $\omega_{e_n}(T)$, the expectation value of T in the state corresponding to e_n, is what is measured in the laboratory. If the observable corresponding to T is measured many times with the physical system in the state corresponding to e_n and those measurements are averaged, the resulting number is (close to) $\omega_{e_n}(T)$.

Of course, ω_{e_n} is a state of $\mathcal{B}(\mathcal{H})$. We are not there as yet; the states of $\mathcal{B}(\mathcal{H})$ are not quite the "replacement" for the (unit) vectors of $\mathcal{H}$. The states ω_x of $\mathcal{B}(\mathcal{H})$ corresponding to unit vectors x in $\mathcal{H}$ have another crucial property; they are "pure." A state ω is a *pure state* when $\omega = \frac{1}{2}(\omega_1+\omega_2)$ only if $\omega = \omega_1 = \omega_2$. In physical language, ω is pure when it is not a proper mixture of other states. The pure states of $\mathcal{B}(\mathcal{H})$ are the "generalized unit vectors in $\mathcal{H}$," the smoothly functioning replacement for the δ-function in this quantum-measurement context.

We can certainly speak of states of operator algebras other than $\mathcal{B}(\mathcal{H})$ — and pure states of those algebras — states that are not proper mixtures of other states of the algebra. As luck would have it, the pure states of $\mathcal{A}_{\rm d}$ are precisely the (non-zero) multiplicative linear functionals on $\mathcal{A}_{\rm d}$. More generally, the pure states of each abelian operator algebra are the (non-zero) multiplicative functionals on the algebra.

For each unit vector x in $\mathcal{H}$, ω_x is a pure state of $\mathcal{B}(\mathcal{H})$. But there are others — many! If there weren't, we wouldn't have succeeded at including all the δ-functions, the "eigenstates" of observables with "ranges" in their spectra. Even in the case of the classical basis $\{e_n\}$, there are "simultaneous eigenstates" of $\mathcal{A}_{\rm d}$ other than the states ω_{e_n} — again, many! When we try to deal with the non-vector eigenstates of a system in a rigorous mathematical fashion, we open a large Pandora's Box. But it's one that we must open, as we shall soon note.

When we speak of an "orthonormal basis," or as Dirac does, "a representation," shall we talk about all the pure states of the masa $\mathcal{A}$ or just those that correspond to unit vectors in $\mathcal{H}$? As remarked, $\mathcal{A}_{\rm d}$ has many other pure states. The vector states are the only ones that are "normal" (that is, strong-operator continuous on the unit ball of $\mathcal{B}(\mathcal{H})$). If we want to deal with the system (masa) $\mathcal{A}_{\rm c}$ generated by an observable whose spectrum is the "range" $[0, 1]$, for example, the position observable of a particle oscillating back and forth on the unit interval, there are no normal eigenstates, and we want to talk about eigenstates of that masa. We can say that the "generalized orthonormal basis" "representing" a masa $\mathcal{A}$ is the set of all simultaneous eigenstates of $\mathcal{A}$, and wind up with a "few" more eigenstates than we need in the case of $\mathcal{A}_{\rm d}$. If we insist on *normal* eigenstates in the case of $\mathcal{A}_{\rm c}$, we wind up with nothing — there are no normal pure states. In the end, the best approach is to say that $\mathcal{A}$, itself, is the (generalized) orthonormal basis.

Definition. A generalized orthonormal basis for $\mathcal{H}$ is a masa on $\mathcal{B}(\mathcal{H})$.

We do know all these generalized bases.

Theorem. *Each masa on a separable Hilbert space is unitarily equivalent to one of* $\mathcal{A}_d$*, where the underlying Hilbert space can have any finite dimension or* $\aleph_0$*, to* $\mathcal{A}_c$*, or to* $\mathcal{A}_d \oplus \mathcal{A}_c$.

There are, however, a number of basic things about generalized orthonormal bases that we do not know. Of course, each unit vector x in $\mathcal{H}$ is contained in an orthonormal basis — so, ω_x is multiplicative on some masa. Is each generalized unit vector "contained" in a masa? That is, if ω is a pure state of $\mathcal{B}(\mathcal{H})$, is it multiplicative on some masa $\mathcal{A}$? That question has been with us for more than fifty years. There's still no answer. In [4] it is proved that, for a countably generated C*-algebra, each pure state is multiplicative (pure) on some masa. In [3], it is proved that the restriction of that pure state to the masa has unique state extension.

What becomes of Dirac's statement in this framework: "so that we may count the representation as being completely determined by the observables that are diagonal ..."? First, we must interpret it in our rigorous language. If two generalized unit vectors (pure states of $\mathcal{B}(\mathcal{H})$) ω_1 and ω_2 give rise to the same eigenstate (pure state) of a masa $\mathcal{A}$, are ω_1 and ω_2 equal? Put in another way, can a pure state (multiplicative linear functional) of $\mathcal{A}$ have distinct pure state extensions to $\mathcal{B}(\mathcal{H})$? This is the problem of "uniqueness of pure state extension" (from a masa to $\mathcal{B}(\mathcal{H})$).

In [9], Singer and I showed that answer is "No!" in general in the case of pure states of $\mathcal{A}_c$. We proved something stronger. Using a technique of von Neumann [16], we defined and produced a "diagonalization process" for $\mathcal{B}(\mathcal{H})$ relative to a masa $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$. This "process" is a module mapping Φ of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}$, where $\mathcal{B}(\mathcal{H})$ is a two-sided module over $\mathcal{A}$ (under left and right multiplication by elements of $\mathcal{A}$) that takes positive operators to positive operators and I to I. (It is a "conditional expectation" of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}$, in present day terminology.) If ρ is a state of $\mathcal{A}$, then $\rho \circ \Phi$ is a state of $\mathcal{B}(\mathcal{H})$. We proved that there are distinct diagonalization processes for $\mathcal{A}_c$. If Φ_1 and Φ_2 are two such and T is an operator in $\mathcal{B}(\mathcal{H})$ such that $\Phi_1(T) \neq \Phi_2(T)$ then there is a pure state ρ of $\mathcal{A}$ such that $\rho(\Phi_1(T)) \neq \rho(\Phi_2(T))$ (the pure states of $\mathcal{A}$ "separate" the elements of $\mathcal{A}$). Let ρ_1 be $\rho \circ \Phi_1$ and ρ_2 be $\rho \circ \Phi_2$. With A in $\mathcal{A}$, $\Phi_1(A) = A\Phi_1(I) = A = \Phi_2(A)$. Thus $\rho_1(A) = \rho(A) = \rho_2(A)$, and ρ_1, ρ_2 are distinct state extensions of ρ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. The set of all extensions of ρ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is convex and compact in a special "weak" topology, whence, it is the closed convex hull of its extreme points (from the Krein-Milman theorem). Each of these extreme points extends ρ and is a pure state of $\mathcal{B}(\mathcal{H})$, since ρ is a pure state of $\mathcal{A}$. Since the set of state extensions of ρ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ does not consist of a single element, there are distinct *pure* state extensions of ρ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$.

We showed that each ω_{e_n} has a unique (pure) state extension from $\mathcal{A}_\mathrm{d}$ to $\mathcal{B}(\mathcal{H})$. We raised the question of whether or not the other pure states of $\mathcal{A}_\mathrm{d}$ have unique extension. The techniques we developed in proving the non-uniqueness of conditional expectations from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{A}$ make it possible to reduce this problem to inequalities with matrices. Some of these matrix problems have arisen in other contexts. While much work has been done on this set of (equivalent) problems, they remain open.

The discussion of orthonormal bases, to this point, has focussed on their "general meaning" and the nature of the "vectors" in those bases. There is another aspect of an orthonormal basis, inherent in the way we usually use such bases, that is less recognized. That aspect is an *ordering* of the basis. Typically, we are dealing with a separable Hilbert space $\mathcal{H}$ and we choose our orthonormal basis as $e_1, e_2, \ldots$. In terms of this basis, it is easy to describe the "one-way-shift" operator V that maps each e_n onto e_{n+1}. The operator V is a non-unitary isometry of $\mathcal{H}$ into itself with spectrum the closed unit disk in $\mathbb{C}$. If we want to describe the "two-way-shift," a unitary operator U on $\mathcal{H}$ with spectrum all complex numbers of modulus 1, it's convenient to choose our orthonormal basis labeled by all the integers $\{e_n\}_{n\in\mathbb{Z}}$. With this basis, U is the unitary operator that maps e_n to e_{n+1}. Of course, we are using bases labeled by a linearly ordered set: the ordering type of the positive integers, with a smallest element but no largest element, in the first case, and the ordering type of all integers with no smallest element and no largest element, in the second case. There are other ordered sets that will serve as labels for an orthonormal basis, for example, the set $\mathbb{Q}_1$ of rationals in the interval $[0, 1]$. The basis so labeled can be used to confound "the unsuspecting." Recalling an earlier quote of Dirac, "However, it is possible to normalize them only if the parameters which label them all take on discrete values. If any of these parameters are continuous variables that can take on all values in a range, the basic vectors are eigenvectors of some observable belonging to eigenvalues in a range and are of infinite length...," we can form the bounded self-adjoint operator A that assigns re_r to e_r, for each basis element e_r of an orthonormal basis labeled by $\mathbb{Q}_1$, for a separable Hilbert space $\mathcal{H}$. Then A is diagonalized by the basis $\{e_r\}$, each e_r is an eigenvector for A ("normalized" to have length 1) corresponding to the eigenvalue r, and $\|A\| = 1$. Since the spectrum of A is a closed subset of $[0, 1]$ containing $\mathbb{Q}_1$, that spectrum is $[0, 1]$. Each point of $\mathbb{Q}_1$ lies in the "range" $[0, 1]$ and is an eigenvalue corresponding to an eigenvector of finite length 1.

Ordered bases serve many purposes; it is well worth understanding what an ordered basis is. Singer and I studied that question in a paper [10] that appeared in 1960. Work on that paper began while I was visiting MIT during the academic year 1956-57. My permanent job was at Columbia University, at that time. On occasion, I shared Is Singer's office with him at MIT. A large part of our joint work was done sitting and talking together, in the office, at home, and while driving; we traded ideas, thought about them, and then commented to one another about them. Of course, a good deal of work was done privately — trying to make computations and lemmas "go." At first, the guiding question was what it meant to put an operator on a Hilbert space in "triangular form" — that is, to view it as part of the algebra of, say, upper triangular matrices. So, we tried to isolate what it should mean to say that an algebra of bounded operators on a Hilbert space is the algebra of all upper triangular matrices. Of course, we thought first of the algebra of upper triangular matrices of finite order. We see this algebra as upper triangular matrices only after we have chosen an appropriate basis *and* put that basis in an appropriate order. We knew that we didn't want to be too literal in our interpretation of "basis" when dealing with infinite-dimensional Hilbert space $\mathcal{H}$, and we knew what a generalized orthonormal basis should be in that case, namely, a masa on $\mathcal{H}$. From the algebra of finite matrices of a given order, a good working definition seemed to be: $\mathcal{T}$ is the algebra of all triangular matrices when $\mathcal{T} \cap \mathcal{T}^*$ is a given masa $\mathcal{A}$ and $\mathcal{T}$ is maximal with respect to that property. So, we tried that in infinite dimensions. Zorn's lemma gave us maximal algebras $\mathcal{T}$ for a given $\mathcal{A}$. We called these algebras *maximal triangular* and $\mathcal{A}$ the *diagonal* of the algebras. The important question at the earliest stage of our work was whether there is a family of projections in $\mathcal{A}$, totally ordered, generating $\mathcal{A}$ as a von Neumann algebra, each member of the family invariant under the operators of $\mathcal{T}$. We called such a projection a *hull* and the von Neumann algebra $\mathcal{C}$ generated by these projections, the hulls, the *core* of $\mathcal{T}$. If $\mathcal{S}$ is a set of vectors in $\mathcal{H}$, the closure of the linear span of $\{Tx : x \in \mathcal{S}, T \in \mathcal{T}\}$ is invariant under each operator in $\mathcal{T}$, in particular, under the operators in $\mathcal{A}$. Thus the projection E with this closure as range commutes with $\mathcal{A}$. Since $\mathcal{A}$ is maximal abelian, $E \in \mathcal{A}$. Since the range of E is invariant under each operator in $\mathcal{T}$, E is a hull in $\mathcal{T}$. With F a projection in $\mathcal{T}$, we denote by '$h(F)$' the projection constructed in this way when the range of F is taken for $\mathcal{S}$. We call $h(F)$ the *hull of F*. There was no difficulty in showing that $\mathcal{C}$ is contained in $\mathcal{A}$. With some effort, we showed that the set of hulls of $\mathcal{T}$

is totally ordered (by the usual ordering on self-adjoint operators). At that point, we knew that we had a theory before us, and there was no turning back.

Is it the case that the core is always $\mathcal{A}$? That was the next question that we tackled. In a short while, we knew that the algebra generated by $\mathcal{A}$ and a unitary operator U that induces an ergodic automorphism of $\mathcal{A}$ (no projection E in $\mathcal{A}$ such that $UEU^* = E$ other than 0 and I) is triangular; Zorn's lemma then gives us maximal triangular algebras $\mathcal{T}$ containing it. Of course, the core of such a $\mathcal{T}$ is just the scalars. We called those triangular algebras (with core the scalars) *irreducible.* A specific example of an irreducible maximal triangular algebra is obtained by choosing the multiplication algebra of the unit circle in $\mathbb{C}$ with Lebesgue measure for $\mathcal{A}$ and the unitary operator induced by a rotation of that circle through an irrational multiple of π radians for U. The maximal triangular algebras whose core is the diagonal we called *hyperreducible.* We proved several general results about the hyperreducible maximal triangular algebras and then classified them completely algebraically and with respect to their action on the underlying Hilbert space. We did not get much further than establishing the existence of the irreducible maximal triangular algebras. The main problem was that the final passage to the full algebra through the use of Zorn's lemma did not give us much of a handle on the elements in the final algebra. Although we had found examples of such algebras, we had not constructed examples in which we had any control over the general element in the algebra. In the case of a von Neumann algebra, our examples were usually arrived at as the strong-operator closures of a self-adjoint algebra whose operators could be easily described — we could approach the general operator in the algebra with nets or sequences of the operators in that self-adjoint algebra. That gave us a handle, though not necessarily an easy path to a proof. There certainly are (uncountably many) non-isomorphic irreducible maximal triangular algebras but that hasn't been proved as this is written.

Theorem. *If $\{E_a\}$ is a totally-ordered family of projections that generates a maximal abelian algebra $\mathcal{A}$, then $\mathcal{T}$, the set of all bounded operators that leave each E_a invariant, is maximal triangular with core and diagonal $\mathcal{A}$. Each hyperreducible algebra arises in this way.*

Theorem. *If $\mathcal{T}$ is a maximal triangular algebra with diagonal $\mathcal{A}$ generated by its family $\{E_a\}$ of minimal projections, then $\mathcal{T}$ is hyperreducible. If we order $\{E_a\}$ by the relation $\precsim$, where $E_a \precsim E_b$ precisely when $h(E_a) \le h(E_b)$ then $\precsim$ is a total ordering. Two maximal triangular* *algebras with totally-atomic diagonals are unitarily equivalent if and only if their sets of minimal projections are order isomorphic. Corresponding to each total-ordering type there is a maximal triangular algebra with a totally-atomic diagonal whose set of minimal projections has this order type.*

Theorem. *If $\mathcal{T}$ is hyperreducible, its diagonal $\mathcal{A}$ has no minimal projections, and $\mathcal{H}$ is separable, then $\mathcal{T}$ is unitarily equivalent to the algebra of all bounded operators on $L_2([0,1],\mu)$, where μ is Lebesgue measure, leaving each F_λ invariant, where F_λ is the multiplication operator corresponding to the characteristic function of* $[0,\lambda]$.

Singer and I felt that our maximal triangular algebras played roughly the role for the theory of non-self-adjoint operator algebras that von Neumann algebras played in the self-adjoint theory. In any event, the theory of non-self-adjoint operator algebras was initiated by [10]. It has developed into a flourishing subject with a large number of very talented research workers. Some of the original questions that we asked are still open as this is written.

As we began to develop an intuition for the subject, we felt that the irreducible case corresponds to factors and the hyperreducible case corresponds to maximal abelian von Neumann algebras. Of course, we understood that a masa is a generalized orthonormal basis — and we realized that we should add "unordered orthonormal basis" to that understanding. It was at a fairly early stage, certainly during that academic year, 1956-1957, that we knew that the hypperreducible maximal triangular algebra was precisely what should be meant by a generalized ordered basis. The ordering of the hulls corresponds to the ordering of the basis and the maximal abelian algebra that serves as the diagonal is the unordered basis. We called these hyperreducible algebras (*generalized*) *ordered bases.*

After that initial development, the main thrust of our paper was classifying the ordered bases — the hyperreducible case — roughly, the equivalent of handling the abelian case in the self-adjoint theory. We were able to do that completely. Each ordered basis corresponds to a closed subset of $[0,1]$ containing 0 and 1 up to what we called Lebesgue order isomorphism — that is a homeomorphism of $[0,1]$ onto itself preserving orientation and Lebesgue null sets. Given such an equivalence class of closed sets, there is a canonically constructed ordered basis that corresponds to it. Two ordered bases are unitarily equivalent if and only if they correspond to the same equivalence class of closed sets.

The most difficult technical lemma we had to prove in connection with this classification is the following. If $\mathfrak{X}$ and $\mathfrak{Y}$ are two dense denumerable subsets of $[0,1]$ containing 0 and 1 and m, M are two numbers such that $0 < m < M < 1$, then there is a homeomorphism f of $[0,1]$ onto itself such that $f(0) = 0$, $f(1) = 1$, f maps $\mathfrak{X}$ onto $\mathfrak{Y}$, and $m(x - y) \leq f(x) - f(y) \leq M(x - y)$ when $y \leq x$ and x, y are in $[0,1]$.

The work leading to [9], described before, grew out of the project with triangular operator algebras. A year after that MIT work, Singer was visiting me at Columbia. We were sitting together trading ideas on some of the problems we still had with triangular operator algebras. Singer suggested something. I thought about it and said, "To carry that out, we would have to settle the question of uniqueness of pure state extension from maximal abelian algebras." That was a problem that Is and I had discussed on occasion over the nine preceding years. At that point, Singer said, "OK, let's settle it!" Two to three weeks later we had settled it. You may ask, with some justice, "And how about the ones that got away?" There were plenty of those — but that's another story!

Toward the end of my year at MIT, Singer and I were sitting in his office — at about 1 AM — each reading material that the other had written on our project. We were at desks against opposite walls with our backs to one another. Suddenly, Singer began to laugh uncontrollably. I turned around, smiling, and began to laugh, as well — it was catching, and we were both slightly giddy after a long day of work. Singer asked, "Dick, are you trying to become the William Faulkner of mathematics?" He had just been reading some particularly complex prose I had written — the syntax was correct, but required an oscilloscope for its analysis. Well, the years have gone by; I can't say anything about my becoming the William Faulkner of mathematics, but I know who has become the Pavarotti–Sinatra! Those two gentlemen have a duet on the popular hit, "My Way." Singer could teach them each something on that topic, and he'd have his usual standing-room-only audience while doing it.

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