SURVEYS IN DIFFERENTIAL GEOMETRY, 2000
Vol. VII ©2000, INTERNATIONAL PRESS
pp. 313-326

THE ATIYAH-BOTT-SINGER FIXED POINT THEOREM AND NUMBER THEORY

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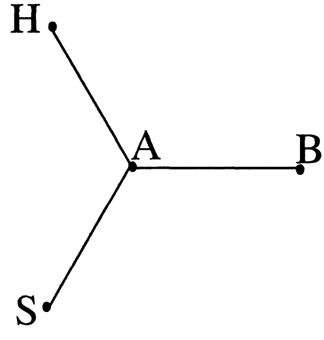
1. Introductory remarks

It was a great idea of Shing-Tung Yau to organize a meeting sponsored by the Journal of Differential Geometry and dedicated to the "Gang of Four," Atiyah, Bott, Singer, and myself. The four members of the Gang were not supposed to lecture at the meeting, they were to give dinner speeches. Perhaps the ability to lecture decreases with age, whereas the willingness to give dinner speeches increases. But nevertheless the out-of-town members of the Gang, Atiyah and I, lectured just before the opening of the meeting. I talked at MIT in the joint colloquium on May 13, 1999. My lecture had the above title and was of course dedicated to three members of the Gang. The lecture was of a rather elementary character (if one knows the ABS-theorem), but I hope ABS enjoyed it. In the dinner speech I pointed out that ABHS make up the following graph of type D_4



where two vertices are connected by an edge if and only if they have a joint paper. But I also said that BHS have many relations. There was much cooperation not represented by joint papers to which I look back with great pleasure and gratitude. All this began in the fifties. All four of us are good friends since more than four decades and influenced each other mathematically through all these years.

My paper is also dedicated to Michael Atiyah on the occasion of his 70th birthday. I owe him very much mathematically (we have nine joint papers) and in many other ways. It is impossible to thank him here in a proper way. I would have to write many pages. But let me mention two facts:

1. The thirty Arbeitstagungen in Bonn under my direction (1957 – 1991) were the backbone of the mathematical activity I tried to build up and to keep in Bonn (Sonderforschungsbereich Theoretische Mathematik 1969 – 1985, Max-Planck-Institut für Mathematik since 1982.) At these Arbeitstagungen Michael lectured 32 times. Very often it was the opening lecture. Everybody can see how much Bonn owes him. On July 16, 1962, Michael gave the Arbeitstagung lecture "Harmonic Spinors and Elliptic Operators". He reported on joint work with Iz Singer and on their conjecture that the $\widehat{A}$-genus equals the index of the Dirac Operator on Spin-manifolds. Here the story of the index and fixed point theorems begins. This is the origin of the line which led to the "Gang of Four"-meeting now 37 years later.

2. Michael worked for the foundation of the European Mathematical Society (EMS) through the European Mathematical Council for many years. He proposed me as the first president of the EMS when the society was finally founded in 1990.

On July 3, 1999, I received an honorary degree of the University of Konstanz. I have many connections with the mathematicians there. Konstanz was founded in 1966. It is now in a process of reform. For some time it looked as if mathematics would be reformed down to a pure service institution. In my acceptance speech of the degree I tried to make it clear that a University without mathematics hardly deserves the name University. In the Konstanz mathematical colloquium I gave a lecture in the same spirit as my MIT lecture. The manuscript was translated into English by Dr. Bruce Hunt whom I thank very much. This is the present paper. The ABS-theorem and its relation to number theory show the strength and beauty of mathematics and the unity of mathematics independently of applications and of service to other fields. But it has applications. The role of Atiyah, Bott and Singer in Mathematical Physics shows what I mean.

I thank the University of Konstanz for preparing the German TEX file and International Press for producing the English version.

Last but not least many thanks again to Shing-Tung Yau for his great energy and enthusiasm in organizing the "Gang of Four" meeting.

2. Lecture

We will first apply the Atiyah-Bott-Singer fixed point theorem ([1], page 473) to a compact connected Riemann surface X.

Let $\boldsymbol{a}$ be an automorphism of finite order of X, not equal to the identity. Then $\boldsymbol{a}$ has finitely many fixed points x and at each a rotation angle α_x with $0 < \alpha_x < 2\pi$. The automorphism $\boldsymbol{a}$ induces by lifting an action on the finite-dimensional $\mathbb{C}$-vector space $H^{1,0}(X)$ of holomorphic one-forms on X. According to ABS, one has for the trace of $\boldsymbol{a}$, denoted $\chi(\boldsymbol{a})$ (and which will also be referred to as the character), the formula

$$\chi(\boldsymbol{a}) - \overline{\chi(\boldsymbol{a})} = i \sum_{\substack{x \in X \\ ax = x}} \cot \frac{\alpha_x}{2}. \tag{1}$$

(In ABS one has a $-i$ on the right-hand side instead of i; we are using slightly different notations). According to the Lefschetz fixed point theorem of topology, we have

$$\chi(\boldsymbol{a}) + \overline{\chi(\boldsymbol{a})} = 2 - \text{ the number of fixed points,} \tag{2}$$

so that we can calculate $\chi(\boldsymbol{a})$ from these two equations.

Example. Consider the lattice $\mathbb{Z}i + \mathbb{Z}$ in $\mathbb{C}$ (with coordinate z) and the automorphism $\boldsymbol{a}$ of order 4 which is given by multiplication by i. There are two fixed points which are represented by 0 and $\frac{1}{2} + \frac{i}{2}$, both of which have rotation angle $\frac{\pi}{2}$. Hence

$$\begin{aligned} \chi(\boldsymbol{a}) - \overline{\chi(\boldsymbol{a})} &= 2i \cot \frac{\pi}{4} = 2i \\ \chi(\boldsymbol{a}) + \overline{\chi(\boldsymbol{a})} &= 0 \\ \chi(\boldsymbol{a}) &= i, \end{aligned}$$

which is the same as saying $\boldsymbol{a}^* dz = i dz$.

For the following considerations, which will lead to a theorem of HECKE (1928), compare [7] and the literature given there (E. HECKE, Mathematische Werke, page 549).

We now make the following basic assumption, without worrying at the moment whether it can be satisfied:

(*) Let p be an odd prime number and $\boldsymbol{a}$ an automorphism of X of order p with $\frac{p-1}{2}$ fixed points, with rotation angles $2\pi\frac{k}{p}$, $1 \leq k \leq p-1$, where k a quadratic residue modulo p, i.e., $\left(\frac{k}{p}\right) = 1$.

The assumption (*) implies

(**)

$$\chi(\boldsymbol{a}) - \overline{\chi(\boldsymbol{a})} = i \sum_{\substack{1\leq k\leq p-1 \\ \left(\frac{k}{p}\right)=1}} \cot \pi \frac{k}{p}$$

$$\chi(\boldsymbol{a}) + \overline{\chi(\boldsymbol{a})} = -\frac{p-5}{2}.$$

We first consider the case $p \equiv 1 \bmod 4$. then we have $\chi(\boldsymbol{a}) - \overline{\chi(\boldsymbol{a})} = 0$, since -1 is a quadratic residue, and

(3) $$\chi(\boldsymbol{a}) = -\frac{p-5}{4}, \qquad \text{(if (*) holds and } p \equiv 1 \bmod 4).$$

For $p = 3$ we would have

$$\chi(\boldsymbol{a}) - \overline{\chi(\boldsymbol{a})} = i \cot \frac{\pi}{3} = \frac{i}{\sqrt{3}}.$$

This is impossible, as $\chi(\boldsymbol{a})$ is an algebraic integer. We now assume $p \equiv 3 \bmod 4$ and $p > 3$. Then by Gauß' theorem, we have

(4) $$\sum_{\substack{1\leq k\leq p-1 \\ \left(\frac{k}{p}\right)=1}} \cot \pi \frac{k}{p} = \sqrt{p}\, h(-p),$$

where $h(-p)$ is the class number of the field $\mathbb{Q}(\sqrt{-p})$ of discriminant $-p$. Using the formula

(5) $$\pi \cot \pi z = \sum_{n\in\mathbb{Z}} \frac{1}{z-n}$$

with the summation which collects the summands for n and $-n$, one can show that (4) is equivalent to the following formula

(6) $$\sum_{k>0} \left(\frac{k}{p}\right) \frac{1}{k} = \frac{\pi}{\sqrt{p}} h(-p).$$

For more details compare the beautiful book [8].

From (**) and (4), we have the formula

$$\chi(\boldsymbol{a}) = \frac{1}{2}\left(-\frac{p-5}{2} + i\sqrt{p}\,h(-p)\right), \tag{7}$$

which also holds for $p \equiv 1 \bmod 4$, if one sets $h(-p) = 0$, since $-p$ is not a discriminant.

Can the fundamental assumption (*) be realized? The answer is yes for $p > 3$.

Consider the modular group $PSL_2(\mathbb{Z})$, which we will denote by Γ. This group acts on the upper half-plane $\mathbb{H}$ by means of fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}.$$

The principal congruence subgroup $\Gamma(p)$ consists of those integral unimodular matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, which modulo p are equal to the identity matrix. The group $\Gamma(p)$ acts freely on $\mathbb{H}$.We obtain a non-compact Riemann surface $\Gamma(p)\backslash\mathbb{H}$, which covers $\Gamma\backslash\mathbb{H}$ finite-to-one. The Galois group of the covering is

$$PSL_2(\mathbb{F}_p) = \Gamma/\Gamma(p),$$

of order $N := \frac{1}{2}p(p^2-1)$. The Riemann surface $\Gamma\backslash\mathbb{H}$ can be identified with the complex plane $\mathbb{C}$. There are two special points, as Γ does not act freely on $\mathbb{H}$; these are representatives of the fixed points of orders 2 and 3, respectively, which one takes to be i and $\rho := \frac{1}{2}(-1+i\sqrt{3})$. The covering

$$\Gamma(p)\backslash\mathbb{H} \longrightarrow \Gamma\backslash\mathbb{H}$$

is branched at these two special points and has there $\frac{N}{2}$ and $\frac{N}{3}$ inverse images, respectively. We obtain for the Euler-Poincaré characteristic

$$e(\Gamma(p)\backslash\mathbb{H}) = -N + \frac{N}{2} + \frac{N}{3} = -\frac{N}{6} = -\frac{1}{12}p(p^2-1).$$

The Riemann surface $\Gamma\backslash\mathbb{H}$ can be compactified to the Riemannian sphere S^2 by adding one cusp. The surface $\Gamma(P)\backslash\mathbb{H}$ can be compactified in the same manner by adding $\frac{N}{p}$ cusps, yielding a compact Riemann surface $X(p)$, on which $PSL_2(\mathbb{F}_p)$ acts with quotient S^2. The isotropy group of a cusp is cyclic of order p. We denote by $i\infty$ the standard cusp whose stabilizer is generated by the transformation $\boldsymbol{a} : z \mapsto z+1$. The action of $PSL_2(\mathbb{F}_p)$ on $X(p)$ has three exceptional orbits of orders $N/2$, $N/3$ and N/p. The Euler-Poincaré characteristic of $X(p)$ is

$$e(X(p)) = -\frac{1}{12}p(p^2-1) + \frac{1}{2}(p^2-1).$$

The genus g is equal to the dimension of $H^{1,0}(X(p))$ and satisfies

$$2 - 2g = e(X(p)).$$

One has $g = 0, 3, 26, \ldots$ for $p = 5, 7, 11, \ldots$.

The element $\boldsymbol{a}$ mentioned above of order p generates a cyclic group U, which has $\frac{p-1}{2}$ fixed points acting on the set of $\frac{1}{2}(p^2-1)$ cusps, and $\frac{p-1}{2}$ orbits of p elements each (cyclic permutations). It has no further fixed points for $p > 3$. The cusps correspond to the cosets $PSL_2(\mathbb{F}_p)/U$, the fixed points to the cosets $N(U)/U$, where $N(U)$ is the normalizer of U, which consists of all maps $z \mapsto az + b$, where a is a quadratic residue modulo p. The quotient $PSL_2(\mathbb{F}_p)/N(U)$ is the projective line $\mathbb{F}_p \cup \infty$, on which $\boldsymbol{a} : z \mapsto z + 1$ acts with ∞ as sole fixed point and cyclicly permutes $\mathbb{F}_p$. Each point of the projective line represents $\frac{p-1}{2}$ cusps. Consider the case $p = 5$. Then, as is well known, $PSL_2(\mathbb{F}_5)$ is the automorphism group A_5 of an icosahedron. The 12 corners of the latter correspond to the 12 cusps of $\Gamma(5)$.

For the element $\boldsymbol{a} : z \mapsto z + 1$ the fundamental assumption $(*)$ is satisfied. It is easy to check that indeed the rotation angles at the $\frac{p-1}{2}$ fixed points are equal to $2\pi\frac{k}{p}$, with quadratic residues k modulo p.

Now we want to investigate for $p \equiv 3(4)$ the representation of $PSL_2(\mathbb{F}_p)$ on $H^{1,0}(X(p))$. Since one has the rule $(\boldsymbol{ab})^* = \boldsymbol{b}^*\boldsymbol{a}^*$ for liftings of differential forms under automorphisms $\boldsymbol{a}$ and $\boldsymbol{b}$, we may pass to the transposed representation and get a homomorphism

$$PSL_2(\mathbb{F}_p) \longrightarrow \mathrm{End} H^{1,0}(X(p)). \tag{8}$$

According to F. G. FROBENIUS and I. SCHUR the irreducible representations of $PSL_2(\mathbb{F}_p)$ for $p \equiv 3(4)$ are classified in the following manner. There is the trivial representation of degree 1, the representation of degree p which is obtained from the permutation representation on the projective line over $\mathbb{F}_p$ by splitting off the trivial representation, and there are $\frac{1}{4}(p-3)$ representations each of degrees $p-1$ and $p+1$, all of which are real, and in addition there are two conjugate representations χ^+ and χ^- of degree $\frac{1}{2}(p-1)$ with

$$\chi^+(\boldsymbol{a}) = \frac{1}{2}(-1 + i\sqrt{p}), \quad \chi^-(\boldsymbol{a}) = \frac{1}{2}(-1 - i\sqrt{p}). \tag{9}$$

These are the traces for the element

$$\boldsymbol{a} : z \mapsto z + 1, \quad \boldsymbol{a} \in PSL_2(\mathbb{F}_p)$$

mentioned above. It is interesting to recall the Gaußian sums

$$\chi^+(\boldsymbol{a}) = \sum_{\substack{1\le k\le p-1\\ \left(\frac{k}{p}\right)=1}} \alpha^k, \quad \text{where } \alpha = e^{2\pi\imath/p},$$

and

$$\chi^-(\boldsymbol{a}) = \sum_{\substack{1\le k\le p-1\\ \left(\frac{k}{p}\right)=-1}} \alpha^k = \overline{\chi^+(\boldsymbol{a})},$$

and that these relations characterize the splitting of the representations χ^+ and χ^- when these are restricted to the cyclic subgroup of order p generated by $\boldsymbol{a}$.

Let m and n be the multiplicities of χ^+ and χ^-, respectively, in the representation (8).

Theorem of Hecke. *Let $p > 3$ and $p \equiv 3(4)$. Then for the multiplicities m and n, we have*

$$m - n = h(-p). \tag{10}$$

This follows immediately from (7) and (9). By the way, one can also calculate $m + n$:

$$m + n = \frac{1}{2}\left(\frac{p-1}{6} + (-1)^{\frac{p+1}{4}}\right) + \frac{2}{3}\text{iff } p \equiv 2(3).$$

As is well-known, there are exactly 7 prime numbers $p \equiv 3(4)$ with $h(-p) = 1$, namely 3, 7, 11, 19, 43, 67, 163 (HEEGNER 1952, STARK).

p	m	n
7	1	0
11	1	0
19	1	0
43	2	1
67	3	2
163	7	6

Complex dimension two

Preliminary report (in preparation with DON ZAGIER)

We consider a compact connected Kählerian surface X, for example a complex algebraic surface. There are two fundamental topological invariants, the Euler-Poincaré characteristic $e(X)$ and the signature sign(X).

If $\boldsymbol{a}$ is an automorphism of X, then the equivariant Euler characteristic $e(X, \boldsymbol{a})$ and the equivariant signature $\mathrm{sign}(X, \boldsymbol{a})$ are well-defined [1]. If $\boldsymbol{a}$ has finite order and isolated fixed points, then one has

$$\tag{11} \begin{aligned} e(X, \boldsymbol{a}) &= \text{number of fixed points of } \boldsymbol{a} \\ \mathrm{sign}(X, \boldsymbol{a}) &= -\sum_{\substack{x \in X \\ \boldsymbol{a}x = x}} \cot \frac{\alpha_x}{2} \cot \frac{\beta_x}{2}. \end{aligned}$$

Note that (1) is the equivariant signature for the one-dimensional case (which vanishes for $\boldsymbol{a} = \mathrm{Id}$).

The first formula in (11) is the classical fixed point theorem of Lefschetz, the second is the ABS-fixed point theorem for the signature. Of course α_x, β_x denote the rotation angles of $\boldsymbol{a}$ at the fixed point x. See also [3] and [5].

As is well known, $\frac{1}{4}(e(X) + \mathrm{sign}(X))$ is the arithmetic genus of the surface X. This fact can also be applied equivariantly. This leads to a formula for the character $\chi(\boldsymbol{a})$ of the action of $\boldsymbol{a}$ in the vector space $H^{2.0}(X)$ of holomorphic two-forms on X, as long as we make the following assumption.

Assumption 1. The first Betti number of X vanishes. The representation of $\boldsymbol{a}$ in $H^{2,0}$ is real, that is, equivalent to its complex conjugate representation in $\overline{H^{2.0}(X)}$.

The vanishing of the first Betti number implies that the arithmetic genus is equal to $1 + \dim H^{2.0}(X)$. Because the representation is real, we also get the relation

$$1 + \chi(\boldsymbol{a}) = \frac{1}{4}(e(X, \boldsymbol{a}) + \mathrm{sign}(X, \boldsymbol{a})) = \frac{1}{4} \sum_{\substack{x \in X \\ \boldsymbol{a}x = x}} \left(1 - \cot \frac{\alpha_x}{2} \cot \frac{\beta_x}{2}\right).$$

We now make an assumption which is analogous to $(*)$ in the one-dimensional case, which, however, we shall only be able to realize in very special cases.

Assumption 2. Let p be an odd prime number > 3 and $\boldsymbol{a}$ an automorphism of X of order p with $\frac{p-1}{2}$ fixed points with rotation angles $2\pi\frac{k}{p}$, $-2\pi\frac{kd}{p}$, where $1 \leq k \leq p-1$ and k is a quadratic residue modulo p. In this formula d denotes a given fixed coset modulo p, which is relatively prime to p.

The assumptions 1 and 2 imply

$$(12)\qquad 1+\chi(\boldsymbol{a})=\frac{1}{4}\left(\frac{p-1}{2}+\sum_{\substack{1\le k\le p-1\\ \left(\frac{k}{p}\right)=1}}\cot\frac{\pi k}{p}\cot\frac{\pi kd}{p}\right).$$

The sum of cotangents lies in the field of pth roots of unity and in fact lies in the quadratic subfield $\mathbb{Q}(\sqrt{-p})$ for $p\equiv 3(4)$ and in $\mathbb{Q}(\sqrt{p})$ for $p\equiv 1(4)$. This can be seen with the help of Galois theory: the sum is invariant upon replacing k by rk, where r is a quadratic residue modulo p.

For $p\equiv 3(4)$, -1 is a quadratic non residue. Hence the sum of cotangents is invariant under the Galois automorphisms of $\mathbb{Q}(\sqrt{-p})$ and lies in $\mathbb{Q}$, which is clear anyway since the sum is real. One has

$$(13)\qquad 1+\chi(\boldsymbol{a})=\frac{1}{8}\left(p-1+\sum_{1\le k\le p-1}\cot\frac{\pi k}{p}\cot\frac{\pi kd}{p}\right).$$

Here one of the usual Dedekind sums appears:

$$\operatorname{ded}(p,d)=\sum_{1\le k\le p-1}\cot\frac{\pi k}{p}\cot\frac{\pi kd}{p}$$

(see [3], [5]). The expression on the right-hand side of (13) is a half-integer, and an integer if and only if d is a quadratic residue modulo p ([3], formula (39), and [5]). The character must be integer-valued. This is no contradiction to our assumptions, if we assume in addition that $\left(\frac{d}{p}\right)=1$.

For $p\equiv 1(4)$, $\chi(\boldsymbol{a})$ is of the form

$$\chi(\boldsymbol{a})=\frac{u+v\sqrt{p}}{2}.$$

Because of our assumptions, $\chi(\boldsymbol{a})$ is an algebraic integer in $\mathbb{Q}(\sqrt{p})$. Hence:

$$u,\ v\in\mathbb{Z}\quad\text{and}\quad u\equiv v \text{ modulo } 2.$$

The Galois automorphism of $\mathbb{Q}(\sqrt{p})$ will be denoted by $\rho\mapsto\widetilde{\rho}$ ($\rho\in\mathbb{Q}(\sqrt{p})$). Then according to (12), we have

$$(14)\qquad 2+\chi(\boldsymbol{a})+\widetilde{\chi(\boldsymbol{a})}=2+u=\frac{1}{4}(p-1+\operatorname{ded}(p,d)),$$

while at the same time

$$
\begin{aligned}
v &= \frac{\chi(a) - \widetilde{\chi(a)}}{\sqrt{p}} \\
&= \frac{1}{4\sqrt{p}} \sum_{1 \le k \le p-1} \left(\frac{k}{p}\right) \cot\frac{\pi k}{p} \cot\frac{\pi k d}{p} \qquad (15) \\
&\overset{\text{DEF}}{=} f(p,d).
\end{aligned}
$$

One can show that (14) and (15) for $p > 5$ determine u, v as integers with $u \equiv v$ modulo 2. For $p > 5$ and $p \equiv 1(4)$, our assumptions again do not lead to a contradiction. Both numbers u and v are even if and only if $\left(\frac{d}{p}\right) = 1$. For $p = 5$, one has $f(p,1) = \frac{2}{5}$ and $f(p,2) = \frac{1}{5}$ and moreover $\operatorname{ded}(p,1) = -4$ and $\operatorname{ded}(p,2) = 0$.

We have been led to the introduction of the twisted Dedekind sums $f(p,d)$, which have very interesting properties, for example (for $p \equiv 1(4)$):

$$
\begin{aligned}
f(p,1) &= \frac{2}{5} \sum_{\substack{1 \le r < \sqrt{p} \\ r \in \mathbb{Z},\ r \text{ odd}}} \sigma_1\left(\frac{p - r^2}{4}\right) \\
f(p,1) - 2f(p,2) &= 0 \\
f(p,1) - 3f(p,3) &= \tfrac{1}{2} h(-3p) \\
f(p,1) - 4f(p,4) &= h(-4p) \\
f(p,1) - 6f(p,6) &= 5h(-3p) \quad \text{for } p \equiv 1 \bmod 8 \\
&= -h(-3p) \quad \text{for } p \equiv 5 \bmod 8 \\
f(p,1) - 8f(p,8) &= 3h(-4p) + 2h(-8p) \quad \text{for } p \equiv 1 \bmod 8 \\
&= -h(-4p) + 2h(-8p) \quad \text{for } p \equiv 5 \bmod 8
\end{aligned}
$$

The first formula is related to the value of the Dedekind zeta function of the field $\mathbb{Q}(\sqrt{p})$ at 2 (see [4], page 192), which can be seen with the help of formula (5). The other formulas can be proved with the help of (5) and formulas of the type (6).

Naturally we would like to have examples of surfaces X with an action of $PSL_2(\mathbb{F}_p)$ so that the action of $a \in PSL_2(\mathbb{F}_p)$ with $a : z \mapsto z+1$ satisfies our assumptions. All representations for $p \equiv 1(4)$ are real anyhow. There are irreducible representations of degrees $1,\ p, p-1,\ p+1$ and two exceptional representations χ^+, χ^- of degree $\frac{p+1}{2}$ with

$$
\chi^+(a) = \frac{1+\sqrt{p}}{2}, \quad \chi^-(a) = \frac{1-\sqrt{p}}{2}.
$$

Once again, the Gaußian sums are interesting. For example one has

$$\chi^+(\boldsymbol{a}) = 1 + \sum_{\substack{1\le k\le p-1 \\ \left(\frac{k}{p}\right)=1}} \alpha^k, \quad \text{where } \alpha = e^{\frac{2\pi i}{p}}.$$

All irreducible representations except for χ^+ and χ^- have characters whose values at $\boldsymbol{a}$ are in $\mathbb{Z}$. For the multiplicities m and n of χ^+ and χ^- in the representation of $PSL_2(\mathbb{F}_p)$ on $H^{2,0}(X)$, we have

$$m - n = f(p, d). \tag{16}$$

With the help of the congruence subgroups of the Hilbert modular group for real quadratic fields, we can obtain many examples, which however in general do not fulfill our simple assumptions. There is a general theory ([6] and the paper cited in that reference by H. Saito). In [6] the twisted Dedekind sums only occur implicitly. We would like to develop the theory of these sums independently and derive some new properties of the usual Dedekind sums. For the theory of the Hilbert modular group see [4] and [2].

Examples which satisfy the assumptions 1 and 2:

Consider the field $K = \mathbb{Q}(\sqrt{5})$ and the ring of integers in K:

$$\mathcal{O} = \mathbb{Z}\cdot 1 + \mathbb{Z}\frac{1+\sqrt{5}}{2}.$$

The Hilbert modular group $\Gamma = PSL_2(\mathcal{O})$ acts on the product $\mathbb{H}\times\mathbb{H}$ of two half planes via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z_1, z_2) = \left(\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'}\right)$, where the Galois automorphism of K is being denoted by $x \mapsto x'$. The surface $\Gamma\backslash\mathbb{H}^2$ has six quotient singularities, two each of the orders 2, 3 and 5. The prime numbers $p \equiv \pm 1(5)$ can be split

$$p = \pi\pi' \quad \text{with } \pi > 0,$$

where (π) is a prime ideal for which $\mathcal{O}/(\pi) = \mathbb{F}_p$. Now let $\Gamma(\pi)$ be the principal congruence subgroup of matrices in $SL_2(\mathcal{O})$, which are equivalent to the identity modulo π.

The subgroup $\Gamma(\pi)$ of Γ acts freely on $\mathbb{H}^2$. We have a Galois cover

$$\Gamma(\pi)\backslash\mathbb{H}^2 \longrightarrow \Gamma\backslash\mathbb{H}^2 \tag{17}$$

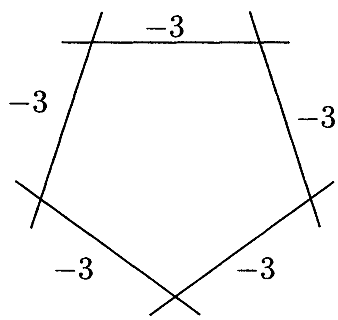
with Galois group $PSL_2(\mathbb{F}_p)$. Under the action on $\Gamma(\pi)\backslash\mathbb{H}^2$, there are only fixed points of orders 2, 3 and 5.

We must compactify the surface and consider from the start the resolution of the cusp singularities:

The surface $\Gamma\backslash\mathbb{H}^2$ is compactified by adding a rational curve with a double point. According to (17) there lie on $\Gamma(\pi)\backslash\mathbb{H}^2$ over this curve $\frac{p^2-1}{2}$ smooth rational curves, each of self-intersection number -3, which split into $p+1$ times $\frac{p-1}{2}$ curves, corresponding to the points of the projective line $\mathbb{P}^1(\mathbb{F}_p)$. Each set of $\frac{p-1}{2}$ curves forms $\frac{p-1}{2}/t$ cycles, each consisting of t curves, where t is determined as follows.

The fundamental unit $\varepsilon = \frac{1+\sqrt{5}}{2}$ as well as $\varepsilon^2 = \frac{3+\sqrt{5}}{2}$ may be viewed as cosets modulo p, as we have a $\sqrt{5}$ in $\mathbb{F}_p$ (just choose a root). The order t of the element ε^2 in $\mathbb{F}_p^*$ is a divisor of $\frac{p-1}{2}$. For $p = 11$, we have $t = 5$ ($\varepsilon^2 \equiv 5 \bmod 11$).

Hence we have 12 cycles of the form



where each of the 60 curves obtained in this manner has self-intersection -3.

The element $\boldsymbol{a} : z \mapsto z+1$ of $PSL_2(\mathbb{F}_p)$ permutes p of the points of $\mathbb{P}^1(\mathbb{F}_p)$ cyclically, while fixing the point ∞. Indexed by this latter point there are $\frac{p-1}{2}$ curves, which are ordered in cycles. There are $\frac{p-1}{2}$ intersection points which are the fixed points of $\boldsymbol{a}$. Our assumptions 1 and 2 are satisfied, with $d \equiv \varepsilon^2 \bmod p$.

One can show that

$$\text{(18)} \qquad \operatorname{ded}(p, \varepsilon^2) = 1 - p \quad \text{for } p \equiv \pm 1(5),$$

for example one has $\operatorname{ded}(11, 5) = -10$. Using this, relations (13), (14), (15) and (16) combined yield the following

Theorem. *Let p be a prime with $p \equiv \pm 1(5)$ and $X(p)$ the compact smooth Hilbert modular surface for the congruence subgroup $\Gamma(\pi)$ of the Hilbert modular group Γ for the field $\mathbb{Q}(\sqrt{5})$. Here $p = \pi\pi'$ (with $\pi > 0$) and (π) is a prime ideal of norm p. The group $PSL_2(\mathbb{F}_p)$ acts on $X(p)$ and hence on $H^{2,0}(X(p)$, the space of holomorphic two-forms on $X(p)$*

(cusp forms of weight 2 for $\Gamma(\pi)$). The character of $\boldsymbol{a} : z \mapsto z+1$ under this action is

$$\chi(\boldsymbol{a}) = -1 + \frac{f(p,d)}{2}\sqrt{p}, \tag{19}$$

where $d \equiv \varepsilon^2 = \frac{3+\sqrt{5}}{2}$ modulo p. Here one has $f(p,d) = 0$ if $p \equiv 3(4)$. For the multiplicities m, n of χ^+, χ^- in the representation of $PSL_2(\mathbb{F}_p)$ we have

$$m - n = f(p,d).$$

From [6] and the paper of H. SAITO cited in that paper we deduce that for $p \equiv \pm 1 \bmod 5$ and $p \equiv 1 \bmod 4$, $f(p, \varepsilon^2)$ vanishes for $\left(\frac{\varepsilon}{p}\right) = 1$, and for $\left(\frac{\varepsilon}{p}\right) = -1$ its value is

$$f(p,\varepsilon^2) = -2h(\mathbb{Q}(\sqrt{5}, \sqrt{-\pi})).$$

Hence the expression $\frac{1}{2}|m-n|$ is the class number of a biquadratic number field. For $p = 41, 61, 109, 149, 241, 269, 281, 389$ the value of this class number is $1, 1, 1, 1, 3, 1, 3, 1$.

It follows from (19) that for $p \equiv \pm 1 \bmod 5$, in case $f(p, \varepsilon^2) = 0$, the representation of the cyclic subgroup of order p generated by $\boldsymbol{a}$ acting on $H^{2,0}(X(p))$ (plus the trivial representation) consists of the direct sum of $\frac{p^2-1}{120}$ cyclic permutation representations of p elements. For this, we note that for the arithmetic genus we have

$$1 + \dim H^{2,0}(X(p)) = p\frac{p^2-1}{120}.$$

(See [4]. One must multiply the Euler-volume of $\Gamma\backslash\mathbb{H}^2$ (which is $\frac{1}{15}$) with the order of $PSL_2(\mathbb{F}_p)$, which gives the Euler-Poincaré characteristic of $\Gamma(\pi)\backslash\mathbb{H}^2$, then divide by 4 to get the arithmetic genus of $X(p)$).

By the way, one can also show that for $f(p,\varepsilon^2) = 0$, the representation of $PSL_2(\mathbb{F}_p)$ on $H^{2,0}(X(p))$ (plus the trivial representation) is equivalent to the permutation representation of $PSL_2(\mathbb{F}_p)$ on the set of cosets $PSL_2(\mathbb{F}_p)/A_5$, using any embedding of A_5 in $PSL_2(\mathbb{F}_p)$.

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